Topics in Mathematics Science VII

Spring 2026

Introduction to Koszul Duality

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Convention

Throughout the course, $\Bbbk$ is a field. All rings are unital and associative. All modules are right modules unless otherwise specified. When we discuss dimensions, we mean dimensions as $\Bbbk$-vector spaces.

There are a lot of contents not covered in lecture, which will be written in coloured blocks:

> This is supplementary material: proofs and routine verifications of statements made in the main text, often 'unnecessarily long or pedantic check of definitions' – things that are only there for rigour.

Supplementary material are often (but not always) QA'ed.

> This is advanced material; things that I personally would like to add, from personal experience in the theory, but definitely not necessary for lecture purpose.

Advanced material is mostly produced by AI and most of them has not been rigorously checked, so the reader should read it with extra care.

The main content of the notes are drafted by AI and QA'ed (heavily edited) by myself.

1 Graded algebras, modules, and duals

1.1 Algebras and modules

Definition 1.1. *A $\Bbbk$-algebra is a $\Bbbk$-vector space with an associative unital multiplication that is bilinear, meaning linear in each variable.*

A right A-module is understood to be a $\Bbbk$-vector space M with a bilinear action satisfying $(ma)b = m(ab)$ and $m1_A = m$ for $m \in M$ and $a, b \in A$. A module homomorphism $f : M \to N$ is linear and satisfies $f(ma) = f(m)a$. A homomorphism of $\Bbbk$-algebras is a linear map preserving multiplication and the identity.

Example 1.2. *(1) The matrix ring* $\mathrm{Mat}_n(\Bbbk)$ *is a* $\Bbbk$*-algebra. The space of* $1 \times n$ *row vectors is a right* $\mathrm{Mat}_n(\Bbbk)$*-module by matrix multiplication.*

(2) Let $\Bbbk A_n$ *be the upper triangular matrix ring, which is also a* $\Bbbk$*-algebra.*

$$\Bbbk A_n = \begin{pmatrix} \Bbbk & \Bbbk & \cdots & \Bbbk \\ 0 & \Bbbk & \cdots & \Bbbk \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \Bbbk \end{pmatrix}$$

For those familiar with path algebras, $\Bbbk A_n$ *is the path algebra of a linearly oriented* A_n *quiver, which explains the notation. We use matrix calculations, so no knowledge of quivers is required. Write* e_{ij} *for the matrix with* 1 *in entry* (i, j) *and zero elsewhere. Our convention composes paths from left to right:* e_{ij} *represents* $i \to j$*, and* $e_{ij}e_{j\ell} = e_{i\ell}$*.*

1.2 Categories and functors

We organise modules together with their homomorphisms. A category records both: specifying the objects alone does not specify a category.

Definition 1.3. *A category* $\mathcal{C}$ *consists of objects, a set* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *of morphisms* $f : X \to Y$ *for each pair of objects, and composition sending* $f : X \to Y$ *and* $g : Y \to Z$ *to* $g \circ f : X \to Z$*. Each object* X *has an identity* $\mathrm{id}_X : X \to X$*, and*

$$h \circ (g \circ f) = (h \circ g) \circ f, \qquad \mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X \quad \text{for all composable } f, g, h.$$

A morphism $f : X \to Y$ is an *isomorphism* if some $g : Y \to X$ satisfies $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$; we then write $X \cong Y$. A *full subcategory* of $\mathcal{C}$ consists of chosen objects of $\mathcal{C}$, with all morphisms between them and the same composition.

Example 1.4. *For a* $\Bbbk$*-algebra* A*, we use the notation*

$$\begin{aligned} \mathrm{Mod}\, A &:= \textit{category of right } A\textit{-modules and } A\textit{-linear maps}, \\ \mathrm{mod}\, A &:= \textit{full subcategory of finitely generated right } A\textit{-modules}. \end{aligned}$$

In either category, $\mathrm{Hom}_A(M, N)$ *is the space of* A*-linear maps* $M \to N$*; composition is the usual composition of maps. For* $A = \Bbbk$*, these are the categories of all and of finite-dimensional vector spaces, respectively.*

Note that, if $\dim_{\Bbbk} A < \infty$*, then* $M \in \mathrm{mod}\, A \Leftrightarrow M$ *is finite-dimensional (*$\because$ *finite generation implies that* M *is a quotient of* $A^{\oplus n}$ *some* n*, and a finite* $\Bbbk$*-basis of* M *is clearly a finite generating set.)*

Example 1.5. *For* $M \in \mathrm{mod}\, A$*, let* $\mathrm{add}\, M := \mathrm{add}_{\mathrm{mod}\, A}(M)$ *be the full subcategory with*

$$N \in \mathrm{add}\, M \iff N \oplus N' \cong M^{\oplus r} \quad \text{for some } N' \in \mathrm{mod}\, A \text{ and integer } r \geq 0.$$

Thus $\mathrm{add}\, M$ *consists of direct summands of finite direct sums of* M*.*

In modern representation theory, remembering A and $\mathrm{add}\, A$ holds the same categorical information is crucial in utilising cateogircal technique in representation theory; this allows one to push many results to a more abstract framework.

Definition 1.6. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *assigns an object* $F(X)$ *to each object* X *and a morphism* $F(f) : F(X) \to F(Y)$ *to each* $f : X \to Y$*, such that*

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \qquad F(g \circ f) = F(g) \circ F(f) \quad \text{for all composable } f, g.$$

For example, forgetting the A-action gives a functor $\operatorname{Mod} A \to \operatorname{Mod} \Bbbk$: it keeps the underlying vector spaces and linear maps. The *identity functor* $\mathrm{id}_{\mathcal{C}}$ leaves every object and morphism unchanged; functors compose by applying one after the other. Every functor preserves isomorphisms, since $F(f^{-1})$ is inverse to $F(f)$.

The *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, with $\operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(Y,X)$ and composition reversed. A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ means a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Thus it sends $f : X \to Y$ in $\mathcal{C}$ to $F(f) : F(Y) \to F(X)$ and satisfies $F(g \circ f) = F(f) \circ F(g)$. Duals will provide our main examples.

Definition 1.7. *For functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\eta : F \to G$ is a family of morphisms $\eta_X : F(X) \to G(X)$ such that*

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array} \qquad G(f) \circ \eta_X = \eta_Y \circ F(f) \quad \text{for every } f : X \to Y. \tag{1.1}$$

If every η_X is an isomorphism, η is a natural isomorphism, written $F \cong G$.

The square in (1.1) is said to *commute*: its two routes from $F(X)$ to $G(Y)$ give the same morphism. We often say a family of morphisms (resp. isomorphisms) in a category $\mathcal{C}$ is *natural in $M \in \mathcal{C}$* to mean that it defines a natural transformation (resp. isomorphism).

For example, for fixed $c \in \Bbbk$, the maps $c\,\mathrm{id}_M : M \to M$ define a natural transformation $\mathrm{id}_{\operatorname{Mod} A} \to \mathrm{id}_{\operatorname{Mod} A}$ because $f(cm) = cf(m)$ for every module homomorphism $f : M \to N$. It is a natural isomorphism when $c \neq 0$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $GF \cong \mathrm{id}_{\mathcal{C}}$ and $FG \cong \mathrm{id}_{\mathcal{D}}$. A *duality* between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Thus an equivalence recovers objects and maps up to natural isomorphism; a duality also reverses the direction of maps.

Definition 1.8. *Let A be a $\Bbbk$-algebra, and let $L, M, N \in \operatorname{Mod} A$, with $i : L \to M$ and $p : M \to N$ module homomorphisms. The sequence $L \xrightarrow{i} M \xrightarrow{p} N$ is exact at M if $\operatorname{Im} i = \operatorname{Ker} p$. A sequence of module homomorphisms is exact if it is exact at every term with an incoming and an outgoing map. A short exact sequence is an exact sequence*

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0, \qquad \operatorname{Ker} i = 0, \quad \operatorname{Im} i = \operatorname{Ker} p, \quad \operatorname{Im} p = N. \tag{1.2}$$

Definition 1.9. *Let A, B be $\Bbbk$-algebras. A functor $F : \operatorname{Mod} A \to \operatorname{Mod} B$ is exact if $F(0) = 0$ and every short exact sequence (1.2) gives a short exact sequence*

$$0 \longrightarrow F(L) \xrightarrow{F(i)} F(M) \xrightarrow{F(p)} F(N) \longrightarrow 0.$$

An exact functor preserves injective and surjective maps: apply it to $0 \to L \xrightarrow{i} M \to M/i(L) \to 0$ and $0 \to \operatorname{Ker} p \to M \xrightarrow{p} N \to 0$, respectively. For a functor $F : (\operatorname{Mod} A)^{\mathrm{op}} \to \operatorname{Mod} B$, exactness instead means that (1.2) gives

$$0 \longrightarrow F(N) \xrightarrow{F(p)} F(M) \xrightarrow{F(i)} F(L) \longrightarrow 0.$$

Thus an exact contravariant functor exchanges injective and surjective maps.

Example 1.10 ($\Bbbk$-linear duality). *For a vector space V, define its $\Bbbk$-linear dual by $D(V) := \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$. For a linear map $u : V \to W$, set*

$$D(u) : D(W) \longrightarrow D(V), \qquad \lambda \longmapsto \lambda \circ u.$$

The identities $D(\mathrm{id}_V) = \mathrm{id}_{D(V)}$ and $D(v \circ u) = D(u) \circ D(v)$ make $D : (\operatorname{Mod} \Bbbk)^{\mathrm{op}} \to \operatorname{Mod} \Bbbk$ a functor. The functor D is exact.

For a short exact sequence (1.2) *of vector spaces,* $D(p)$ *is injective because* p *is surjective, and* $D(i)$ *is surjective because a functional on a subspace extends to the whole vector space. If* $\lambda \in D(M)$ *satisfies* $\lambda \circ i = 0$*, then*

$$\mu : N \longrightarrow \Bbbk, \qquad p(m) \longmapsto \lambda(m)$$

is well-defined since $\operatorname{Ker} p = i(L)$*, and* $\lambda = \mu \circ p = D(p)(\mu)$*. Conversely,* $p \circ i = 0$ *gives* $D(i) \circ D(p) = 0$*, so* $\operatorname{Ker} D(i) = \operatorname{Im} D(p)$*. Thus* D *exchanges injective and surjective linear maps.*

For every vector space V*, evaluation defines a natural injective map*

$$\mathrm{ev}_V : V \longrightarrow D(D(V)), \qquad v \longmapsto (\lambda \mapsto \lambda(v)). \tag{1.3}$$

If V *is finite-dimensional, choose a basis* $v_1, \dots, v_n$ *and its dual basis* $v_1^*, \dots, v_n^*$*, defined by* $v_i^*(v_j) = \delta_{ij}$*. Then*

$$\mathrm{ev}_V^{-1} : D(D(V)) \longrightarrow V, \qquad \Phi \longmapsto \sum_{i=1}^{n} \Phi(v_i^*) v_i.$$

Consequently, $D : (\operatorname{mod} \Bbbk)^{\mathrm{op}} \to \operatorname{mod} \Bbbk$ *is a duality.*

If $v \neq 0$*, choose* $\lambda \in D(V)$ *with* $\lambda(v) \neq 0$*; hence* ev_V *is injective. For* $u : V \to W$*,* $v \in V$*, and* $\lambda \in D(W)$*,*

$$D(D(u))(\mathrm{ev}_V(v))(\lambda) = \lambda(u(v)) = \mathrm{ev}_W(u(v))(\lambda),$$

which proves naturality.

In finite dimension, writing $\lambda = \sum_{i=1}^{n} \lambda(v_i) v_i^*$ *gives* $\Phi(\lambda) = \lambda(\sum_{i=1}^{n} \Phi(v_i^*) v_i)$*. Thus the displayed formula is the inverse of* (1.3)*, and is independent of the chosen basis.*

1.3 Semisimple algebras

Definition 1.11. *A module* L *is simple if* $L \neq 0$ *and its only submodules are* 0 *and* L*. A module is semisimple if it is a direct sum of simple modules. An algebra* B *is semisimple if it is semisimple as a right module* B_B *over itself. The Jacobson radical* $\operatorname{rad} B$ *is the intersection of all maximal right ideals of* B*. For a right* B*-module* M*, the radical* $\operatorname{rad} M$ *is the intersection of all maximal submodules of* M*, with* $\operatorname{rad} M = M$ *if there are none.*

We use the following standard ring-theoretic facts without proof. For a finite-dimensional $\Bbbk$-algebra B and any right B-module M,

$$\operatorname{rad} M = M \operatorname{rad} B.$$

Recall that a *division* $\Bbbk$*-algebra* is a non-zero $\Bbbk$-algebra whose non-zero elements are invertible.

Theorem 1.12 (Artin–Wedderburn). *For a non-zero finite-dimensional* $\Bbbk$*-algebra* B*,*

$$B \text{ is semisimple} \iff \operatorname{rad} B = 0 \iff B \cong \prod_{i=1}^{n} \mathrm{Mat}_{n_i}(D_i),$$

where $n, n_i \geq 1$ *and each* D_i *is a finite-dimensional division* $\Bbbk$*-algebra. If* $\Bbbk$ *is algebraically closed, then* $D_i = \Bbbk$ *for every* i*. For semisimple* B*, every left or right* B*-module is semisimple, and every submodule has a complement. In particular, the definition is independent of the chosen side.*

For proofs of $\operatorname{rad} M = M \operatorname{rad} B$ and of Theorem 1.12, see for example [**?**]; [**?**] gives a short account of semisimple modules and Wedderburn's theorem.

Corollary 1.13. *Every finite-dimensional left or right module over a finite-diemnsional semisimple algebra S is a direct summand of $S^{\oplus n}$ for some n, i.e. $M \in \operatorname{add}(S)$ for all $M \in \operatorname{mod} S$.*

Proof Choose a surjection $p : S^{\oplus n} \twoheadrightarrow M$. By Theorem 1.12, there is a submodule L such that $S^{\oplus n} = \operatorname{Ker} p \oplus L$. The restriction $p|_L : L \to M$ has kernel $L \cap \operatorname{Ker} p = 0$ and image $p(L) = p(S^{\oplus n}) = M$. Hence $L \cong M$, so $S^{\oplus n} \cong \operatorname{Ker} p \oplus M$. □

Example 1.14. *The algebras $\Bbbk^{\oplus n}$ and $\operatorname{Mat}_n(\Bbbk)$ are semisimple by Theorem 1.12. Upper triangular matrices need not form a semisimple algebra: for $A = \Bbbk A_2$,*

$$\operatorname{rad} A = \Bbbk e_{12} \neq 0, \qquad A/\operatorname{rad} A \cong \Bbbk^{\oplus 2}.$$

Put $J = \Bbbk e_{12}$, so $J^2 = 0$. For a simple right module L, simplicity gives $LJ = 0$ or $LJ = L$; the second alternative implies $L = LJ^2 = 0$, a contradiction. Hence J is contained in every maximal right ideal. The quotient $A/J \cong \Bbbk^{\oplus 2}$ has exactly two maximal right ideals, whose inverse images have intersection

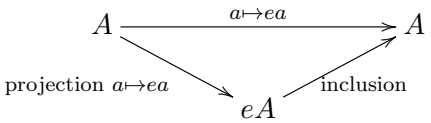
$$(\Bbbk e_{11} \oplus \Bbbk e_{12}) \cap (\Bbbk e_{12} \oplus \Bbbk e_{22}) = J.$$

Definition 1.15. *An element $e \in A$ is idempotent if $e^2 = e$. Idempotents e, f are orthogonal if $ef = fe = 0$. A non-zero idempotent is primitive if it is not the sum of two non-zero orthogonal idempotents. A finite family $e_1, \dots, e_r$ of pairwise orthogonal idempotents is complete if $\sum_{i=1}^r e_i = 1_A$.*

Remark 1.16. For orthogonal idempotents $e, f \in A$,

$$(e + f)A = eA \oplus fA.$$

In particular, we always have $A \cong eA \oplus (1 - e)A$. This strengthen the viewpoint of idempotent-as-local-identity: left-multiplying e defines a canonical projection onto the direct summand eA of A, and yields a commutative diagram



So multiplying e means we work inside the direct summand eA. The corner subalgebra $eAe = \{eae \mid a \in A\}$ has identity e.[1] For $A = \prod_{i=1}^r A_i$, the coordinate idempotents $e_i = (0, \dots, 1_{A_i}, \dots, 0)$ are a complete orthogonal family and are the identities of the individual factors. In $A = \Bbbk A_n$, the diagonal matrix units $e_{11}, \dots, e_{nn}$ are primitive, since $e_{ii} A e_{ii} = \Bbbk e_{ii}$. For $n \geq 2$, the sum $e = e_{11} + \cdots + e_{n-1,n-1}$ gives the corner

$$eAe = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \;\middle|\; B \in \Bbbk A_{n-1} \right\} \cong \Bbbk A_{n-1}.$$

Idempotents also describe maps out of the summands of A_A.

Lemma 1.17 (Yoneda lemma). *For an idempotent $e \in A$ and a right A-module M, evaluation at e gives a natural (in M) vector space isomorphism*

$$\operatorname{Hom}_A(eA, M) \xrightarrow{\sim} Me, \qquad f \longmapsto f(e), \quad \text{with inverse} \quad m \longmapsto (ea \mapsto ma).$$

[1] note that this is not a unital subalgebra since the identity element changes; a more proper way to think about this is as a subcategory $\operatorname{add}_{\operatorname{mod} A}(eA) \simeq \operatorname{add}_{\operatorname{mod} eAe}(eAe)$.

Proof We have $f(e) = f(e)e$ and $f(ea) = f(e)a$. If $m = me$, the map $ea \mapsto ma$ is well-defined because $ea = ea'$ implies $ma = mea = mea' = ma'$. These formulas prove that the maps are inverse; postcomposition by a module map $M \to N$ proves naturality. □

1.4 Graded algebras

Definition 1.18. *Let G be a group with identity e. A G-graded $\Bbbk$-algebra is a $\Bbbk$-algebra A with $1_A \in A_e$ and a decomposition*

$$A = \bigoplus_{g \in G} A_g \quad \text{such that} \quad A_g A_h \subset A_{gh} \quad \text{for all } g, h \in G.$$

An element of A_g is homogeneous of degree g.

Convention. *Unless stated otherwise, a graded algebra is $\mathbb{Z}$-graded and concentrated in non-negative degrees.*

Note that the decomposition is into a direct sum, so every element of a graded algebra A can always be written as a *finite* sum of homogeneous elements.

Example 1.19. *For the upper triangular matrix ring $\Bbbk A_n$ of Example 1.2, the decomposition*

$$(\Bbbk A_n)_d = \operatorname{span}_\Bbbk\{e_{ij} \mid j - i = d\}$$

makes $\Bbbk A_n$ a graded algebra.

Example 1.20. *The algebra $\Bbbk A_3$ also has a $\mathbb{Z}^{\oplus 2}$-grading with*

$$\deg e_{ii} = (0,0), \qquad \deg e_{12} = (1,0), \qquad \deg e_{23} = (0,1), \qquad \deg e_{13} = (1,1).$$

The homomorphism $\mathbb{Z}^{\oplus 2} \to \mathbb{Z}$, $(p,q) \mapsto p+q$, gives the grading of Example 1.19. Alternatively, setting $\deg e_{ij} = 2(j-i)$ gives a $\mathbb{Z}$-grading with $A_1 = 0$ but $A_2 \neq 0$.

Definition 1.21. *A homomorphism of graded algebras $f : A \to B$ is a $\Bbbk$-algebra homomorphism with $f(A_i) \subset B_i$ for all $i \in \mathbb{Z}$.*

Definition 1.22. *A graded algebra A is locally finite-dimensional (lfd) if $\dim_\Bbbk A_i < \infty$ for every $i \in \mathbb{Z}$.*

1.5 Graded modules

Definition 1.23. *Let $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra. A G-graded right A-module is a right A-module M with a direct sum decomposition*

$$M = \bigoplus_{g \in G} M_g \quad \text{such that} \quad M_g A_h \subset M_{gh} \quad \text{for all } g, h \in G.$$

An element of M_g is called homogeneous of degree g.

A G-graded A-module $M = \bigoplus_{g \in G} M_g$ is locally finite-dimensional (lfd) if $\dim_\Bbbk M_g < \infty$ for all $g \in G$.

Unless specified otherwise, graded modules are $\mathbb{Z}$-graded; their degrees need not be non-negative.

For an algebra A, the *opposite algebra* A^{op} has the same vector space as A, with multiplication $a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}$. A left A-module is a right A^{op}-module by $m \cdot a^{\mathrm{op}} = am$. Thus $\mathrm{Mod}(A^{\mathrm{op}})$ is the

category of left A-modules and left A-linear maps; not to be confused with the opposite category $(\operatorname{Mod} A)^{\mathrm{op}}$ though. For a $\mathbb{Z}$-graded algebra $A = \bigoplus_{i\geq 0} A_i$,

$$\text{the opposite algebra is also graded with } (A^{\mathrm{op}})_i = A_i.$$

Definition 1.24. *Let M be a graded A-module and let $d \in \mathbb{Z}$. The degree shift, or grading shift, $M(d)$ is the graded A-module whose underlying A-module is M, but whose grading is*

$$M(d)_j = M_{j+d}.$$

By Definition 1.24, $1 \in A_0 = A(-d)_d$, so $A(-d)$ has a generator in degree d. If M is concentrated in degree 0, then $M(d)$ is concentrated in degree $-d$.

Example 1.25. *For $A = \Bbbk A_3$ with the grading of Example 1.19, the module $P_1 := e_{11}A$ consists of matrices supported on the first row. Its basis e_{11}, e_{12}, e_{13} has degrees $0, 1, 2$, respectively; in $P_1(-2)$, the same basis has degrees $2, 3, 4$.*

Definition 1.26. *Let M, N be graded A-modules. A homomorphism $f : M \to N$ is homogeneous of degree d if $f(M_j) \subset N_{j+d}$ for all $j \in \mathbb{Z}$. A homomorphism of graded modules means a homogeneous A-module homomorphism of degree 0, also said to preserve degrees.*

Example 1.27. *For $A = \Bbbk A_3$, left multiplication by e_{12} yields a map*

$$\begin{aligned} e_{22}A &\longrightarrow e_{11}A, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & v \\ 0 & 0 & 0 \end{pmatrix} &\longmapsto \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (u, v \in \Bbbk). \end{aligned}$$

The u and v entries have degrees $0, 1$ in the source and $1, 2$ in the target, and so the map has degree 1.

Notation. *For graded right A-modules M, N, the space of homomorphisms preserving degrees is*

$$\operatorname{Hom}_A^{\mathbb{Z}}(M, N) = \{f \in \operatorname{Hom}_A(M, N) \mid f(M_j) \subset N_j \text{ for all } j \in \mathbb{Z}\}. \tag{1.4}$$

Thus $\operatorname{Hom}_A^{\mathbb{Z}}(M, N(d))$ consists of homomorphisms $M \to N$ of degree d, since $N(d)_j = N_{j+d}$. Note that, like the grading shift functor, different literature has different notation for this (there does not seem to be any widely agreed notation).

We write

$$\begin{aligned} \operatorname{GrMod} A &:= \text{category of graded right } A\text{-modules with morphism spaces } \operatorname{Hom}_A^{\mathbb{Z}}(M, N), \\ \operatorname{grmod} A &:= \text{full subcategory of } \operatorname{GrMod} A \text{ consisting of locally finite-dimensional modules.} \end{aligned}$$

Thus $\operatorname{GrMod}(A^{\mathrm{op}})$ is the category of graded left A-modules and degree-preserving maps. Forgetting the grading gives a functor $\operatorname{GrMod} A \to \operatorname{Mod} A$.

Remark 1.28. $\operatorname{grmod} A$ is somewhat analogous to $\operatorname{mod} A$ but not really the same: If A is locally finite-dimensional, every finitely generated graded A-module M is locally finite-dimensional. ($\because$ a finite homogeneous generating set of a graded A-module M gives a surjection from a finite direct sum of shifts of A to M, so every M_i has to be finite-dimensional.) However, a locally finite-dimensional M need not be finitely generated.

For $d \in \mathbb{Z}$, the shift of Definition 1.24 is a functor

$$(d) : \operatorname{GrMod} A \to \operatorname{GrMod} A.$$

It sends $f : M \to N$ to the same underlying map $f(d) : M(d) \to N(d)$. Since $M(d)(-d) = M$ and $f(d)(-d) = f$, the shift is an equivalence with inverse $(-d)$; it restricts to an equivalence $\operatorname{grmod} A \to \operatorname{grmod} A$.

Remark 1.29. The notion of exactness makes sense for graded module categories – a sequence in $\operatorname{GrMod} A$ is exact if the underlying sequence in $\operatorname{Mod} A$ is exact. In particular, we talk about functors invovle such categories being exact.

> For $L, M, N \in \operatorname{GrMod} A$ with maps $i : L \to M$ and $p : M \to N$, the sequence $0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ is exact if and only if
>
> $$0 \longrightarrow L_j \xrightarrow{i|_{L_j}} M_j \xrightarrow{p|_{M_j}} N_j \longrightarrow 0 \quad \text{is exact for every } j \in \mathbb{Z}.$$
>
> Indeed, i and p preserve the direct sum decompositions, so their kernels and images decompose into the kernels and images of the restrictions.

Example 1.30. *Let $A = \Bbbk A_2$. Consider $P_1 := e_{11}A$ which has $\operatorname{rad} P_1 = \Bbbk e_{12}$ and $L_1 := P_1/\operatorname{rad} P_1$. Then we have a short exact sequence in $\operatorname{GrMod} A$:*

$$0 \longrightarrow \operatorname{rad} P_1 \xrightarrow{i} P_1 \xrightarrow{p} P_1/\operatorname{rad} P_1 \longrightarrow 0, \qquad i(ce_{12}) = ce_{12}, \quad p(m) = \overline{m}.$$

Note that $\deg \operatorname{rad} P_1 = 1$ and $\deg L_1 = 0$. Let $L_2 := \Bbbk e_{22} = e_{22}A$, which we treat as a graded A-module concentrated in degree 0. Note that $L_2 \cong \operatorname{rad} P_1$ as ungraded modules but we only have $L_2 \cong (\operatorname{rad} P_1)(1)$, which yields the following short exact sequence in $\operatorname{GrMod} A$:

$$0 \longrightarrow L_2(-1) \to P_1 \to P_1/\operatorname{rad} P_1 \longrightarrow 0,$$

Lemma 1.31. *For graded right A-modules M, N, there is an inclusion*

$$\bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_A^{\mathbb{Z}}(M, N(d)) \subset \operatorname{Hom}_A(M, N), \text{ which is equality when } M \text{ is finitely generated.}$$

Proof For $f \in \operatorname{Hom}_A(M, N)$, define $f_d(m) = f(m)_{j+d}$ when $m \in M_j$, and extend linearly. For $a \in A_i$, we have $f_d(ma) = f(ma)_{j+i+d} = f(m)_{j+d}a$, so f_d is A-linear of degree d. The sum of the homogeneous Hom spaces is direct: on M_j, distinct degrees have values in distinct summands of N.

If M is finitely generated, the homogeneous components of a finite generating set form a finite homogeneous generating set. Only finitely many f_d are non-zero on these homogeneous generators, hence on M, giving $f = \sum_d f_d$. □

1.6 Homogeneous ideals and quotients

Definition 1.32. *Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a graded algebra. A right ideal $I \subset A$ is homogeneous if*

$$I = \bigoplus_{i\in\mathbb{Z}} (I \cap A_i).$$

Equivalently, whenever $a \in I$ and $a = a_0 + \cdots + a_N$ is its decomposition into homogeneous components, each a_i also belongs to I.

For quotient algebras we need two-sided ideals. If I is a homogeneous two-sided ideal, then

$$A/I = \bigoplus_{i\in\mathbb{Z}} A_i/(I \cap A_i)$$

is again a graded algebra.

Example 1.33. *In $A = \Bbbk A_3$ with the grading of Example 1.19, the two-sided ideal $I = (e_{13}) = \Bbbk e_{13}$ is homogeneous. Writing $\overline{e}_{ij}$ for the image of e_{ij},*

$$(A/I)_0 \cong \Bbbk^{\oplus 3}, \qquad (A/I)_1 = \Bbbk\overline{e}_{12} \oplus \Bbbk\overline{e}_{23}, \qquad (A/I)_d = 0 \quad (d \geq 2).$$

In particular, $\overline{e}_{12}\overline{e}_{23} = 0$.

Example 1.34. *Let $A = \Bbbk[x]$ with $\deg x = 1$, and let $I = (x - x^2)$. The ideal I is two-sided because A is commutative. Every element of I vanishes at $x = 1$, whereas neither x nor x^2 does. Thus $x - x^2 \in I$ but $x, x^2 \notin I$, so I is not homogeneous. Writing $\overline{f} = f + I$, we have $0 \neq \overline{x} = \overline{x^2}$. The images of A_1 and A_2 therefore intersect non-trivially, so A/I does not inherit the quotient grading. As an ungraded algebra, the quotient is semisimple:*

$$\Bbbk[x]/(x - x^2) \xrightarrow{\sim} \Bbbk \oplus \Bbbk, \qquad \overline{f} \longmapsto (f(0), f(1)).$$

The inverse sends (a, b) to $a + (b - a)\overline{x}$.

Definition 1.35. *The positive degree ideal of a graded algebra A is $A_+ := \bigoplus_{i>0} A_i$.*[2]

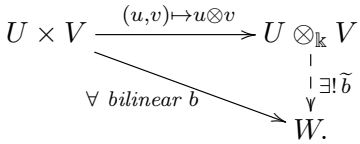
The subspace A_+ is a homogeneous two-sided ideal.

1.7 Tensor products of vector spaces

Definition 1.36. *Choose a basis $\{u_i \mid i \in I\}$ of U and a basis $\{v_j \mid j \in J\}$ of V. The tensor product $U \otimes_{\Bbbk} V$ is the vector space with basis $\{u_i \otimes v_j \mid i \in I,\ j \in J\}$.*

The assignment $(u_i, v_j) \mapsto u_i \otimes v_j$ extends uniquely to a bilinear map $U \times V \to U \otimes_{\Bbbk} V$ sending $(u, v) \mapsto u \otimes v$. Note that this means that elements like $(au_1 + bu_2) \otimes (cv_1 + dv_2)$ in $U \otimes_{\Bbbk} V$ is by definition the linear combination $ac(u_1 \otimes v_1) + ad(u_1 \otimes v_2) + bc(u_2 \otimes v_1) + bd(u_2 \otimes v_2)$.

Proposition 1.37 (Universal property). *Every bilinear map $b : U \times V \to W$ has a unique linear factorisation (commutative diagram)*



Consequently, $U \otimes_{\Bbbk} V$ is independent of the chosen bases up to a unique isomorphism preserving the bilinear map.

Proof The formula $\widetilde{b}(u_i \otimes v_j) = b(u_i, v_j)$ determines a unique linear map $\widetilde{b} : U \otimes_{\Bbbk} V \to W$. Bilinearity gives $\widetilde{b}(u \otimes v) = b(u, v)$ for all $u \in U$ and $v \in V$.

Suppose $t : U \times V \to T$ and $t' : U \times V \to T'$ both satisfy the universal property. They give unique linear maps $a : T \to T'$ and $b : T' \to T$ with $a \circ t = t'$ and $b \circ t' = t$. Both $b \circ a : T \to T$ and $\mathrm{id}_T : T \to T$ factor t through itself, since

$$(b \circ a) \circ t = b \circ t' = t = \mathrm{id}_T \circ t.$$

Uniqueness for (T, t) gives $b \circ a = \mathrm{id}_T$. Likewise, $(a \circ b) \circ t' = a \circ t = t' = \mathrm{id}_{T'} \circ t'$, so uniqueness for (T', t') gives $a \circ b = \mathrm{id}_{T'}$. □

For finite bases with $|I| = r$ and $|J| = s$, we obtain $\dim_{\Bbbk}(U \otimes_{\Bbbk} V) = rs$. We use the canonical identifications

$$(U \otimes_{\Bbbk} V) \otimes_{\Bbbk} W \xrightarrow{\sim} U \otimes_{\Bbbk} (V \otimes_{\Bbbk} W), \quad (u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w),$$
$$\Bbbk \otimes_{\Bbbk} U \xrightarrow{\sim} U, \qquad\qquad c \otimes u \longmapsto cu,$$

and set $U^{\otimes_{\Bbbk} 0} := \Bbbk$.

[2] It is also called the augmentation ideal; we avoid that terminology.

1.8 Direct sums and direct products

For an arbitrary family $(M_i)_{i\in I}$ of right A-modules, indexed by a set I, the *direct product* and *direct sum* are

$$\prod_{i\in I} M_i = \{(m_i)_{i\in I} \mid m_i \in M_i \text{ for all } i \in I\},$$
$$\bigoplus_{i\in I} M_i = \{(m_i)_{i\in I} \in \prod_{i\in I} M_i \mid m_i = 0 \text{ for all but finitely many } i\}.$$

Addition and the A-action are componentwise; the same definitions apply to vector spaces by taking $A = \Bbbk$. Thus $\bigoplus_{i\in I} M_i \subset \prod_{i\in I} M_i$, with equality for finite I. For example, $(1, 1, \ldots) \in \prod_{i\geq 0} \Bbbk$ does not belong to $\bigoplus_{i\geq 0} \Bbbk$. In a direct sum, an element is a finite sum of its components; no notion of convergence is involved. For instance, ordinary polynomials use non-negative powers, whereas Laurent polynomials allow integer powers, but both are finite sums:

$$\Bbbk[x] = \bigoplus_{n\geq 0} \Bbbk x^n, \qquad \Bbbk[x, x^{-1}] = \bigoplus_{n\in\mathbb{Z}} \Bbbk x^n.$$

The *formal power series ring* allows arbitrary coefficients:

$$\Bbbk[[x]] = \left\{ \sum_{n\geq 0} a_n x^n \;\middle|\; a_n \in \Bbbk \text{ for every } n \geq 0 \right\} \cong \prod_{n\geq 0} \Bbbk x^n \quad \text{as vector spaces.}$$

Addition is coefficientwise, and multiplication is

$$\Big(\sum_{n\geq 0} a_n x^n\Big)\Big(\sum_{n\geq 0} b_n x^n\Big) = \sum_{n\geq 0} \Big(\sum_{j=0}^{n} a_j b_{n-j}\Big) x^n.$$

Tensor products distribute over direct sums in either variable. For vector spaces U, V and arbitrary families $(U_i)_{i\in I}$, $(V_j)_{j\in J}$ indexed by sets I, J, there are natural linear isomorphisms

$$\begin{aligned} \Big(\bigoplus_{i\in I} U_i\Big) \otimes_{\Bbbk} V &\xrightarrow{\sim} \bigoplus_{i\in I} (U_i \otimes_{\Bbbk} V), \quad (u_i)_{i\in I} \otimes v \longmapsto (u_i \otimes v)_{i\in I}, \\ U \otimes_{\Bbbk} \Big(\bigoplus_{j\in J} V_j\Big) &\xrightarrow{\sim} \bigoplus_{j\in J} (U \otimes_{\Bbbk} V_j), \quad u \otimes (v_j)_{j\in J} \longmapsto (u \otimes v_j)_{j\in J}. \end{aligned} \tag{1.5}$$

The displayed bilinear assignments induce these linear maps by Proposition 1.37. On the ith summand, the inverse of the first map sends $u_i \otimes v$ to the element with u_i in coordinate i, tensored with v. On the jth summand, the inverse of the second sends $u \otimes v_j$ to u tensored with the element having v_j in coordinate j. Extending over the direct sums gives the two inverses.

For direct products, the corresponding map need not be an isomorphism.

Lemma 1.38. *For an arbitrary family $(U_i)_{i\in I}$ of vector spaces, indexed by a set I, and a vector space V, the coordinate projections define a linear map*

$$\Big(\prod_{i\in I} U_i\Big) \otimes_{\Bbbk} V \longrightarrow \prod_{i\in I} (U_i \otimes_{\Bbbk} V), \qquad (u_i)_i \otimes v \longmapsto (u_i \otimes v)_i. \tag{1.6}$$

This map is injective, and is an isomorphism if I is finite or V is finite-dimensional. The same statements hold with the two tensor factors reversed.

Proof The bilinear map $\big(\prod_{i\in I} U_i\big) \times V \to \prod_{i\in I} (U_i \otimes_{\Bbbk} V)$, $((u_i)_i, v) \mapsto (u_i \otimes v)_i$, induces

(1.6) by Proposition 1.37. Every element can be written $\sum_{\ell=1}^{r} u^{(\ell)} \otimes v_\ell$ with the v_ℓ linearly independent: choose a basis of the span of its finitely many second factors. If its image is zero, then $\sum_{\ell=1}^{r} u_i^{(\ell)} \otimes v_\ell = 0$ for every i. The tensor-product basis construction gives $u_i^{(\ell)} = 0$ for all i, ℓ, proving injectivity. If $v_1, \dots, v_r$ is a basis of V, the inverse is

$$\prod_{i\in I}(U_i \otimes_{\Bbbk} V) \longrightarrow \Big(\prod_{i\in I} U_i\Big) \otimes_{\Bbbk} V, \qquad \Big(\sum_{\ell=1}^{r} u_i^{(\ell)} \otimes v_\ell\Big)_i \longmapsto \sum_{\ell=1}^{r} (u_i^{(\ell)})_i \otimes v_\ell.$$

For finite I, products equal sums. Reversing the tensor factors gives the other statements, since $U \otimes_{\Bbbk} V \to V \otimes_{\Bbbk} U$, $u \otimes v \mapsto v \otimes u$, is an isomorphism. □

Example 1.39. *The comparison in* (1.6) *need not be surjective when V is infinite-dimensional. Take $U_i = \Bbbk$ for $i \geq 1$ and $V = \bigoplus_{j\geq 1} \Bbbk v_j$. Under $\Bbbk \otimes_{\Bbbk} V \cong V$, the image of the product comparison in* (1.6) *is*

$$\left\{ (w_i)_i \in \prod_{i\geq 1} V \;\middle|\; \dim_{\Bbbk} \operatorname{span}_{\Bbbk}\{w_i \mid i \geq 1\} < \infty \right\}.$$

Indeed, the image of $\sum_{\ell=1}^{r} (c_i^{(\ell)})_i \otimes w_\ell$ has every entry in $\operatorname{span}_{\Bbbk}\{w_1, \dots, w_r\}$. Conversely, expand all entries in a basis of their common finite-dimensional span. Since the v_i are linearly independent, $(v_i)_i$ is not in the image.

1.9 Bimodules and enveloping algebras

Definition 1.40. *For $\Bbbk$-algebras B, C, a bimodule ${}_B M_C$ is a vector space with $\Bbbk$-bilinear left B- and right C-module actions satisfying $(bm)c = b(mc)$ for all $b \in B$, $c \in C$, and $m \in M$. The subscripts indicate the left and right acting algebras; when $B = C = A$, we call M an A-bimodule. A bimodule homomorphism $f : {}_B M_C \to {}_B N_C$ is a linear map satisfying $f(bmc) = bf(m)c$ for all $b \in B$, $c \in C$, and $m \in M$. For graded B, C, a bimodule M is graded if $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and $B_i M_j C_k \subset M_{i+j+k}$ for all $i, j, k \in \mathbb{Z}$. A graded bimodule homomorphism is a bimodule homomorphism $f : M \to N$ satisfying $f(M_i) \subset N_i$ for all $i \in \mathbb{Z}$.*

Convention.

- *Bilinearity implies that $\Bbbk$ acts centrally: $(c1_B)m = cm = m(c1_C)$ for $c \in \Bbbk$. From the perspective of general module theory over general rings (algebras over non-field commutative rings), we should have said that our bimodules are $\Bbbk$-central bimodule. Since we alawys work over a field, all bimodules we considered are always $\Bbbk$-central and we will not write such adjective every time.*
- *Instead of saying (B, C)-bimodule, we usually just write ${}_B M_C$. It would be natural to write $B\operatorname{Mod} C$ for the category of (B, C)-bimodules, but since $\Bbbk$-centrality comes for free from the assumption of $\Bbbk$ being a field, we can use $\operatorname{Mod} C \otimes_{\Bbbk} B^{\mathrm{op}}$ instead.*

Example 1.41.

- *Multiplication makes A_A a graded right module and ${}_A A_A$ a graded bimodule; each degree component is a bimodule ${}_{A_0}(A_i)_{A_0}$.*
- *For a bimodule ${}_A M_A$ and idempotents $e, f \in A$, the space $eMf := \{enf \mid n \in M\}$ is a bimodule ${}_{eAe}(eMf)_{fAf}$.*

Definition 1.42. *For a $\Bbbk$-algebra A, its enveloping algebra is $A^{\mathrm{en}} := A \otimes_{\Bbbk} A^{\mathrm{op}}$, with multiplication extended bilinearly from*

$$(a \otimes b^{\mathrm{op}})(a' \otimes (b')^{\mathrm{op}}) = aa' \otimes (b'b)^{\mathrm{op}}, \qquad 1_{A^{\mathrm{en}}} = 1_A \otimes 1_A^{\mathrm{op}} \quad (a, b, a', b' \in A).$$

A bimodule ${}_AM_A$ becomes a right A^{en}-module by

$$m \cdot (a \otimes b^{\mathrm{op}}) = bma \quad (a, b \in A,\ m \in M). \tag{1.7}$$

The first tensor factor records the right action, and the second the left action:

$$\big(m \cdot (a \otimes b^{\mathrm{op}})\big) \cdot (a' \otimes (b')^{\mathrm{op}}) = b'bmaa' = m \cdot \big(aa' \otimes (b'b)^{\mathrm{op}}\big).$$

Conversely, for $M \in \operatorname{Mod} A^{\mathrm{en}}$, define $ma = m \cdot (a \otimes 1)$ and $bm = m \cdot (1 \otimes b^{\mathrm{op}})$. The two actions commute because $a \otimes 1$ and $1 \otimes b^{\mathrm{op}}$ commute in A^{en}. For bimodules ${}_AM_A$, ${}_AN_A$, a linear map $u : M \to N$ satisfies

$$u(bma) = bu(m)a \quad (a, b \in A,\ m \in M) \quad \Longleftrightarrow \quad u \in \operatorname{Hom}_{A^{\mathrm{en}}}(M, N).$$

Thus $A \operatorname{Mod} A$ is identified with $\operatorname{Mod} A^{\mathrm{en}}$. More generally, for $\Bbbk$-algebras B, C, the vector space $C \otimes_{\Bbbk} B^{\mathrm{op}}$ is an algebra with

$$(c \otimes b^{\mathrm{op}})(c' \otimes (b')^{\mathrm{op}}) = cc' \otimes (b'b)^{\mathrm{op}}, \qquad 1 = 1_C \otimes 1_B^{\mathrm{op}}.$$

The same construction identifies the categories

$$B \operatorname{Mod} C \simeq \operatorname{Mod}(C \otimes_{\Bbbk} B^{\mathrm{op}}), \qquad m \cdot (c \otimes b^{\mathrm{op}}) = bmc.$$

Grant–Iyama [**?**, §2, p. 2591] identify bimodules with left modules over A^{en}, using $(a \otimes b^{\mathrm{op}})m = amb$. Our right module convention is (1.7): $m \cdot (a \otimes b^{\mathrm{op}}) = bma$. Both describe the same bimodules; the side on which A^{en} acts determines the formula.

Remark 1.43. For a finite-dimensional semisimple $\Bbbk$-algebra S, semisimplicity of S^{en} requires an additional condition. For a field extension $K/\Bbbk$, give $K \otimes_{\Bbbk} S$ the multiplication $(c \otimes s)(d \otimes t) = cd \otimes st$. The algebra S is called *separable over* $\Bbbk$ if $K \otimes_{\Bbbk} S$ is semisimple for every field extension $K/\Bbbk$. By [**?**, Theorem 6.1.2, p. 105],

$$S^{\mathrm{en}} \text{ is semisimple} \quad \Longleftrightarrow \quad S \text{ is separable over } \Bbbk.$$

For a product of matrix algebras, every field extension $K/\Bbbk$ gives

$$K \otimes_{\Bbbk} \prod_{j=1}^{n} \operatorname{Mat}_{n_j}(\Bbbk) \xrightarrow{\sim} \prod_{j=1}^{n} \operatorname{Mat}_{n_j}(K), \qquad c \otimes (a_j)_j \longmapsto (ca_j)_j.$$

The target is semisimple by Theorem 1.12. Thus products of matrix algebras over $\Bbbk$ are separable; in particular, every finite-dimensional semisimple algebra over an algebraically closed field is separable. Semisimplicity of S alone does not imply semisimplicity of S^{en} over an arbitrary field.

Example 1.44. *Let $\Bbbk = \mathbb{F}_2(t)$, where t is an indeterminate, and let $S = \Bbbk[u]/(u^2 - t)$, also writing u for its residue class. The element t is not a square in $\Bbbk$: an equality $p(t)^2 = tq(t)^2$ for non-zero polynomials would give $2 \deg p = 1 + 2 \deg q$. Thus $u^2 - t$ is irreducible and S is a field, hence semisimple. However, since S is commutative and has characteristic 2,*

$$S^{\mathrm{en}} = S \otimes_{\Bbbk} S \cong S[v]/(v^2 - t) = S[v]/((v - u)^2) \cong S[\varepsilon]/(\varepsilon^2).$$

Here the first tensor factor supplies the coefficients, v corresponds to $1 \otimes u$, and $\varepsilon = 1 \otimes u - u \otimes 1$. The ideal $S\varepsilon$ is non-zero, has square zero, and is the unique maximal ideal, so $\operatorname{rad}(S^{\mathrm{en}}) = S\varepsilon \neq 0$.

Thus S^{en} is not semisimple, and the field extension $S/\Bbbk$ already shows that S is not separable over $\Bbbk$.

1.10 Graded $\Bbbk$-linear duals

A linear functional on a graded vector space can be non-zero on infinitely many degrees. To obtain a graded space, retain only finite sums of homogeneous functionals.

Definition 1.45. *For a graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$, the graded $\Bbbk$-linear dual is the graded vector space*

$$\mathbb{D}(V) = \bigoplus_{i\in\mathbb{Z}} \mathbb{D}(V)_i, \qquad \mathbb{D}(V)_i = D(V_{-i}) = \mathrm{Hom}^{\mathbb{Z}}_{\Bbbk}(V, \Bbbk(i)).$$

Note that here $\Bbbk$ is regarded as a graded module concentrated in degree 0.

Linear duality reverses the side of a module action. For a right A-module M and a left A-module N, set

$$(af)(m) = f(ma), \qquad (ga)(n) = g(an) \quad \text{for } a \in A,\ f \in D(M),\ g \in D(N),\ m \in M,\ n \in N. \tag{1.8}$$

For graded modules these actions restrict to the graded linear duals: if $a \in A_p$ and $f \in \mathbb{D}(M)_q$, then af vanishes outside M_{-p-q}.

Example 1.46. *Let $A = \Bbbk A_3$ and $P_1 = e_{11}A$ as in Example 1.25. The functionals $\varepsilon_{1j}(e_{1\ell}) = \delta_{j\ell}$ for $1 \le j, \ell \le 3$ form a basis of $\mathbb{D}(P_1)$, with*

$$\deg \varepsilon_{1j} = 1 - j, \qquad e_{12}\varepsilon_{12} = \varepsilon_{11}, \qquad e_{23}\varepsilon_{13} = \varepsilon_{12}, \qquad e_{jj}\varepsilon_{1j} = \varepsilon_{1j}.$$

The actions follow from (1.8).

For any $u \in \mathrm{Hom}_A(M, N)$, we have a dual map

$$D(u) : D(N) \to D(M), \qquad f \mapsto f \circ u,$$

which is a left A-linear map, i.e. $D(u) \in \mathrm{Hom}_{A^{\mathrm{op}}}(D(N), D(M))$. This gives rise to four exact (contravariant) functors (without any finiteness assumption):

$$\begin{aligned} D : (\mathrm{Mod}\, A)^{\mathrm{op}} &\longrightarrow \mathrm{Mod}(A^{\mathrm{op}}), & D : (\mathrm{Mod}(A^{\mathrm{op}}))^{\mathrm{op}} &\longrightarrow \mathrm{Mod}\, A, \\ \mathbb{D} : (\mathrm{GrMod}\, A)^{\mathrm{op}} &\longrightarrow \mathrm{GrMod}(A^{\mathrm{op}}), & \mathbb{D} : (\mathrm{GrMod}(A^{\mathrm{op}}))^{\mathrm{op}} &\longrightarrow \mathrm{GrMod}\, A. \end{aligned} \tag{1.9}$$

Exactness follows from Example 1.10, degree by degree for $\mathbb{D}$.

For these to be duality, we need to recover a module from its bidual; this requires some finiteness conditions.

Lemma 1.47. *Let A be a $\Bbbk$-algebra.*

(1) *The functor D defines a duality between finite-dimensional right and left A-modules. In particular, if A is finite-dimensional, then we have a duality on finitely generated modules*

$$D : (\mathrm{mod}\, A)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{mod}(A^{\mathrm{op}}), \qquad D : (\mathrm{mod}(A^{\mathrm{op}}))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{mod}\, A.$$

(2) *Suppose that A is graded. For locally finite-dimensional $M \in \mathrm{grmod}\, A$, evaluation is a natural graded module isomorphism*

$$\mathrm{ev}_M : M \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(M)), \qquad m \longmapsto (f \mapsto f(m)). \tag{1.10}$$

In particular, we have a duality between locally finite-dimensional graded right and left A-modules:

$$\mathbb{D} : (\mathrm{grmod}\, A)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{grmod}(A^{\mathrm{op}}), \qquad \mathbb{D} : (\mathrm{grmod}(A^{\mathrm{op}}))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{grmod}\, A. \tag{1.11}$$

Proof First we show that ev_M is A-linear. Indeed,

$$\mathrm{ev}_M(ma)(f) = f(ma) = \mathrm{ev}_M(m)(af) = (\mathrm{ev}_M(m)a)(f) \quad (m \in M,\ a \in A,\ f \in D(M)).$$

For (1), the claim follows from Example 1.10. For (2), apply (1) degree-wise. □

For a locally finite-dimensional graded vector space V, restriction gives

$$\mathbb{D}(V) = \bigoplus_{i \in \mathbb{Z}} D(V_{-i}) \subset D(V) \cong \prod_{i \in \mathbb{Z}} D(V_i).$$

The inclusion is equality exactly when only finitely many V_i are non-zero. For an arbitrary vector space, evaluation $V \to D(D(V))$ is injective but need not be surjective; hence the functors in (1.9) are not dualities on all modules.

Example 1.48. *Graded linear duality need not preserve finite generation, even for a polynomial ring. Let $A = \Bbbk[x]$ with $\deg x = 1$ and let $\varepsilon_n(x^m) = \delta_{nm}$ for $n, m \geq 0$. Then $\deg \varepsilon_n = -n$, $x\varepsilon_0 = 0$, and $x\varepsilon_n = \varepsilon_{n-1}$ for $n \geq 1$. Any finite set of elements of $\mathbb{D}(A)$ lies in the span of $\varepsilon_0, \ldots, \varepsilon_N$ for some N, and multiplication by x preserves that span. Thus $\mathbb{D}(A)$ is locally finite-dimensional but not finitely generated.*

For vector spaces V, W, a functional $f \in D(V)$ and a vector $w \in W$ define a linear map $V \to W$, $v \mapsto f(v)w$. Finite sums of these maps give the following description of Hom.

Proposition 1.49. *For vector spaces V, W, there is a natural linear map*

$$W \otimes_{\Bbbk} D(V) \longrightarrow \mathrm{Hom}_{\Bbbk}(V, W), \qquad w \otimes f \longmapsto \big(v \mapsto f(v)w\big). \tag{1.12}$$

It is an isomorphism if either V or W is finite-dimensional.

Proof If V has a finite basis $\{v_i \mid 1 \leq i \leq n\}$ with dual basis $\{v_i^* \mid 1 \leq i \leq n\}$, the inverse of (1.12) is

$$\mathrm{Hom}_{\Bbbk}(V, W) \longrightarrow W \otimes_{\Bbbk} D(V), \qquad F \longmapsto \sum_{i=1}^{n} F(v_i) \otimes v_i^*,$$

since $v = \sum_{i=1}^{n} v_i^*(v)v_i$. If instead W has a finite basis $\{w_j \mid 1 \leq j \leq m\}$, write $F(v) = \sum_{j=1}^{m} f_j(v)w_j$; the uniquely determined coordinate functionals $f_j \in D(V)$ give the inverse $F \mapsto \sum_{j=1}^{m} w_j \otimes f_j$. The formulas commute with precomposition and postcomposition, so the comparison is natural. □

Assumption 1.50. *For simplicity, from now on and unless stated otherwise, we assume that all graded algebras are $\mathbb{Z}$-graded with $A_{<0} = 0$, and*

$$A_0 = S = \bigoplus_{i=1}^{n} \Bbbk e_i$$

is semisimple with coordinate idempotents $e_1, \ldots, e_n$.

This simplifies many cumbersome check when dealing with dual modules – the general setting requires using S-dual, which is naturally isomorphic to $\Bbbk$-dual via a choice of 'symmetrising form'; we will try to deal with this general setting in the advanced sections.

For a bimodule ${}_S M_S$, the $\Bbbk$-linear dual retains both actions:

$$(s\lambda t)(m) = \lambda(tms) \qquad (s, t \in S,\ \lambda \in D(M),\ m \in M). \tag{1.13}$$

With precomposition on morphisms, we obtain functors

$$D : (\operatorname{Mod} S^{\mathrm{en}})^{\mathrm{op}} \longrightarrow \operatorname{Mod} S^{\mathrm{en}}, \qquad \mathbb{D} : (\operatorname{GrMod} S^{\mathrm{en}})^{\mathrm{op}} \longrightarrow \operatorname{GrMod} S^{\mathrm{en}},$$

whose restrictions to finite-dimensional bimodules, and to locally finite-dimensional graded bimodules, are dualities by Lemma 1.47.

Evaluation preserves both actions: for $m \in M$, $\lambda \in D(M)$, and $s, t \in S$,

$$(s \operatorname{ev}_M(m)\, t)(\lambda) = \operatorname{ev}_M(m)(t\lambda s) = (t\lambda s)(m) = \lambda(smt) = \operatorname{ev}_M(smt)(\lambda).$$

Applying this in each degree gives the graded statement.

In particular, dualising reverses the two idempotents: for $1 \le i, j \le n$,

$$e_j D(M) e_i = \{\lambda \in D(M) \mid \lambda(e_k M e_\ell) = 0 \text{ unless } (k,\ell) = (i,j)\} \cong D(e_i M e_j), \tag{1.14}$$

since $(e_j \lambda e_i)(m) = \lambda(e_i m e_j)$ by (1.13).

1.11 S-dual

Under Assumption 1.50, the $\Bbbk$-linear dual is all we need. For an arbitrary finite-dimensional semisimple S, the choice-free substitutes are the one-sided S-duals below; they are the duals used in [**?**, §2.7]. Under Assumption 1.50 they are canonically isomorphic to D; see Remark 1.58.

One-sided S-duals.

Definition 1.51. *For a right S-module M and a left S-module N, define their S-duals by*

$$M^* := \operatorname{Hom}_S(M, S) \in \operatorname{Mod}(S^{\mathrm{op}}), \quad (sf)(m) := sf(m),$$
$$^*N := \operatorname{Hom}_{S^{\mathrm{op}}}(N, S) \in \operatorname{Mod} S, \qquad (gs)(n) := g(n)s.$$

Here $s \in S$, $m \in M$, $n \in N$, $f \in M^*$, and $g \in {}^*N$.

Similar to $\Bbbk$-lienar duality, this definition extends to exact contravariant functors: for a right module map $u : M \to N$, define $u^* : N^* \to M^*$ by precomposition $u^*(f) = f \circ u$; for a left module map $u : M \to N$, the same recipe defines ${}^*u : {}^*N \to {}^*M$. This yields the desired functors

$$(-)^* : (\operatorname{Mod} S)^{\mathrm{op}} \longrightarrow \operatorname{Mod}(S^{\mathrm{op}}), \qquad {}^*(-) : (\operatorname{Mod}(S^{\mathrm{op}}))^{\mathrm{op}} \longrightarrow \operatorname{Mod} S.$$

We also have graded analogue similar to Definition 1.45.

Definition 1.52. *For graded right and left S-modules M, respectively, define their graded S-duals by*

$$M^{(*)} := \bigoplus_{i \in \mathbb{Z}} (M_{-i})^* = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}^{\mathbb{Z}}_S(M, S(i)), \qquad {}^{(*)}M := \bigoplus_{i \in \mathbb{Z}} {}^*(M_{-i}) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}^{\mathbb{Z}}_{S^{\mathrm{op}}}(M, S(i)),$$

where each indicated summand has degree i.

This also extends to a functor: for a degree 0 right S-module map $u : M \to N$, define $u^{(*)} : N^{(*)} \to M^{(*)}$ by

$$(u^{(*)})_i : (N_{-i})^* \longrightarrow (M_{-i})^*, \qquad f \longmapsto f \circ u|_{M_{-i}} \quad (i \in \mathbb{Z}).$$

For a degree 0 left module map $u : M \to N$, the same formula defines ${}^{(*)}u : {}^{(*)}N \to {}^{(*)}M$. With the actions from Definition 1.51, we obtain functors

$$(-)^{(*)} : (\operatorname{GrMod} S)^{\mathrm{op}} \longrightarrow \operatorname{GrMod}(S^{\mathrm{op}}), \qquad {}^{(*)}(-) : (\operatorname{GrMod}(S^{\mathrm{op}}))^{\mathrm{op}} \longrightarrow \operatorname{GrMod} S. \tag{1.15}$$

Each functor in (1.15) restricts to locally finite-dimensional modules: replace GrMod by grmod in its source and target. Indeed, for $R = S$ or S^{op} and $M \in \operatorname{grmod} R$,

$$\dim_{\Bbbk} \operatorname{Hom}_R(M_{-i}, R) \leq (\dim_{\Bbbk} M_{-i})(\dim_{\Bbbk} R) < \infty \quad (i \in \mathbb{Z}). \tag{1.16}$$

Lemma 1.53. *We have dualities*

$$\begin{aligned} (-)^* : (\operatorname{mod} S)^{\mathrm{op}} &\xrightarrow{\sim} \operatorname{mod}(S^{\mathrm{op}}), & {}^*(-) : (\operatorname{mod}(S^{\mathrm{op}}))^{\mathrm{op}} &\xrightarrow{\sim} \operatorname{mod} S, \\ (-)^{(*)} : (\operatorname{grmod} S)^{\mathrm{op}} &\xrightarrow{\sim} \operatorname{grmod}(S^{\mathrm{op}}), & {}^{(*)}(-) : (\operatorname{grmod}(S^{\mathrm{op}}))^{\mathrm{op}} &\xrightarrow{\sim} \operatorname{grmod} S. \end{aligned}$$

Proof We show that evaluation gives natural isomorphisms of functors

$$\mathrm{id}_{\operatorname{mod} S} \xrightarrow{\sim} {}^*((-)^*), \qquad \mathrm{id}_{\operatorname{mod}(S^{\mathrm{op}})} \xrightarrow{\sim} ({}^*(-))^*,$$

with components

$$\begin{aligned} \mathrm{ev}_M : M &\xrightarrow{\sim} {}^*(M^*), \quad m \longmapsto (f \mapsto f(m)) \quad (M \in \operatorname{mod} S), \\ \mathrm{ev}_N : N &\xrightarrow{\sim} ({}^*N)^*, \quad n \longmapsto (g \mapsto g(n)) \quad (N \in \operatorname{mod}(S^{\mathrm{op}})). \end{aligned}$$

For $u : M \to N$ in $\operatorname{mod} S$, $m \in M$, and $f \in N^*$,

$$\big({}^*(u^*)(\mathrm{ev}_M(m))\big)(f) = (u^* f)(m) = f(u(m)) = \mathrm{ev}_N(u(m))(f).$$

Hence ${}^*(u^*) \circ \mathrm{ev}_M = \mathrm{ev}_N \circ u$. For $M = S$, evaluation is an isomorphism: its inverse sends $F \in {}^*(S^*)$ to $F(\mathrm{id}_S)$. Indeed, every $f \in S^*$ satisfies $f(s) = f(1)s$, so $F(f) = f(1)F(\mathrm{id}_S) = f(F(\mathrm{id}_S))$. Restriction to summands identifies the dual of a finite direct sum with the direct sum of the duals; evaluation therefore is an isomorphism for $S^{\oplus n}$. By Corollary 1.13, $M \oplus L \cong S^{\oplus n}$ for some L. Under the direct sum identifications, $\mathrm{ev}_{M\oplus L} = \mathrm{ev}_M \oplus \mathrm{ev}_L$, so ev_M is an isomorphism. Replacing S by S^{op} proves the left module statement.

For the graded cases, apply the ungraded case degree-wise. Note that since S is finite-dimensional, finitely generated S-module is the same as finite-dimensional S-modules. □

For a bimodule ${}_S M_S$, the linear and one-sided duals retain both actions:

$$(s\lambda t)(m) = \lambda(tms), \qquad (sft)(m) = sf(tm), \qquad (sgt)(m) = g(ms)t, \tag{1.17}$$

where $s, t \in S$, $m \in M$, $\lambda \in D(M)$, $f \in \operatorname{Hom}_S(M, S)$, and $g \in \operatorname{Hom}_{S^{\mathrm{op}}}(M, S)$. The functionals still take values in $\Bbbk$ or S. With precomposition on morphisms, we obtain

$$\begin{aligned} D,\ (-)^*,\ {}^*(-) &: (\operatorname{Mod} S^{\mathrm{en}})^{\mathrm{op}} \longrightarrow \operatorname{Mod} S^{\mathrm{en}}, \\ \mathbb{D},\ (-)^{(*)},\ {}^{(*)}(-) &: (\operatorname{GrMod} S^{\mathrm{en}})^{\mathrm{op}} \longrightarrow \operatorname{GrMod} S^{\mathrm{en}}. \end{aligned} \tag{1.18}$$

The restrictions to finite-dimensional modules, and to locally finite-dimensional graded modules, are dualities by Lemmas 1.47 and 1.53. Evaluation preserves both actions: for $m \in M$, $f \in M^*$, and $s, t \in S$,

$$(s\,\mathrm{ev}_M(m)\,t)(f) = f(sm)t = f(smt) = \mathrm{ev}_M(smt)(f).$$

The corresponding checks for D and the left dual use the same evaluation formula; applying them in each degree gives the graded statements.

Recall from Definition 1.42 and (1.7) that $S^{\mathrm{en}} = S \otimes_{\Bbbk} S^{\mathrm{op}}$ and S-bimodules are the same thing as S^{en}-modules via $m \cdot (s \otimes t^{\mathrm{op}}) = tms$.

Definition 1.54. *The two-sided dual of a bimodule ${}_SM_S$ is*

$$M^{*_{\mathrm{en}}} := \mathrm{Hom}_{S^{\mathrm{en}}}(M, S^{\mathrm{en}}), \qquad (sFt)(m) := (s \otimes t^{\mathrm{op}})F(m) \quad (s,t \in S,\ F \in M^{*_{\mathrm{en}}},\ m \in M).$$

In Definition 1.54, M has the right action (1.7), so $F(tms) = F(m)(s \otimes t^{\mathrm{op}})$. The output is a left S^{en}-module by multiplication on the values, which gives the displayed bimodule action. The two-sided dual takes values in S^{en}, not in S. On bimodule morphisms it acts by precomposition, as do D, $(-)^*$, and ${}^*(-)$; each is a contravariant functor on the category of S-bimodules. The two-sided dual records both module actions in the target algebra S^{en}.

For a graded bimodule ${}_SM_S$, define its *graded two-sided dual* by

$$M^{(*_{\mathrm{en}})} := \bigoplus_{i \in \mathbb{Z}} (M_{-i})^{*_{\mathrm{en}}}, \qquad (M^{(*_{\mathrm{en}})})_i = (M_{-i})^{*_{\mathrm{en}}}.$$

Definition 1.54 gives a functor

$$(-)^{(*_{\mathrm{en}})} : (\mathrm{GrMod}\, S^{\mathrm{en}})^{\mathrm{op}} \longrightarrow \mathrm{GrMod}\, S^{\mathrm{en}}. \tag{1.19}$$

On a degree 0 bimodule map $u : M \to N$, its degree i component is

$$(u^{(*_{\mathrm{en}})})_i : (N_{-i})^{*_{\mathrm{en}}} \longrightarrow (M_{-i})^{*_{\mathrm{en}}}, \qquad F \longmapsto F \circ u|_{M_{-i}} \quad (i \in \mathbb{Z}).$$

The functor restricts to locally finite-dimensional bimodules by (1.16) with $R = S^{\mathrm{en}}$.

The four duals above follow the distinction in Grant–Iyama [**?**, §2.1, pp. 2592–2593], with the following notation changes:

construction	Grant–Iyama	these notes
$\Bbbk$-linear dual	M^*	$D(M)$
left S-linear dual	M^{*_ℓ}	${}^*M$
right S-linear dual	M^{*_r}	M^*
two-sided dual	$M^\vee$	$M^{*_{\mathrm{en}}}$

The last row also changes the convention for enveloping modules. Their $F : M \to S^{\mathrm{en}}$ is left S^{en}-linear. Define

$$\sigma : S^{\mathrm{en}} \longrightarrow S^{\mathrm{en}}, \qquad a \otimes b^{\mathrm{op}} \longmapsto b \otimes a^{\mathrm{op}}.$$

Since $\sigma(xy) = \sigma(y)\sigma(x)$ for $x, y \in S^{\mathrm{en}}$, the map $\sigma \circ F : M \to S^{\mathrm{en}}$ satisfies

$$(\sigma \circ F)(tms) = (\sigma \circ F)(m)(s \otimes t^{\mathrm{op}}) \quad (s,t \in S,\ m \in M),$$

and belongs to our $M^{*_{\mathrm{en}}}$. Thus $M^\vee \to M^{*_{\mathrm{en}}}$, $F \mapsto \sigma \circ F$, translates their two-sided dual into our convention. In particular, their star denotes the linear dual; our star denotes the right S-dual.

1.12 Comparing the duals

The one-sided S-duals and the $\Bbbk$-linear dual are naturally isomorphic after choosing a symmetrising form on S.

Definition 1.55. *A symmetrising form on a finite-dimensional $\Bbbk$-algebra S is a linear map $\tau : S \to \Bbbk$ satisfying*

$$\tau(st) = \tau(ts) \quad \text{for all } s,t \in S, \qquad \big(\tau(st) = 0 \text{ for all } t \in S\big) \implies s = 0.$$

An algebra admitting such a form is symmetric.

The second condition in Definition 1.55 says that the bilinear form $(s,t) \mapsto \tau(st)$ is *non-degenerate*: every non-zero $s \in S$ pairs non-trivially with some $t \in S$.

Every finite-dimensional semisimple $\Bbbk$-algebra admits a symmetrising form. For products of matrix algebras over $\Bbbk$, the forms are explicit:

Example 1.56. *The following maps are symmetrising forms:*

$$\tau : \Bbbk^{\oplus n} \longrightarrow \Bbbk, \quad (a_1, \dots, a_n) \longmapsto \sum_{j=1}^{n} a_j,$$

$$\tau : \mathrm{Mat}_n(\Bbbk) \longrightarrow \Bbbk, \qquad a \longmapsto \mathrm{Tr}(a) := \sum_{i=1}^{n} a_{ii},$$

and more generally, $$\tau : \prod_{j=1}^{n} \mathrm{Mat}_{n_j}(\Bbbk) \longrightarrow \Bbbk, \quad (a_1, \dots, a_n) \longmapsto \sum_{j=1}^{n} \mathrm{Tr}(a_j).$$

Generally, any finite-dimensional semisimple algebra has a symmetrising form, by Eilenberg–Nakayama.[a] For the graded analogue, see [**?**]. For example, let $\lambda_j : \Delta_j \to \Bbbk$ be a symmetrising form on a finite-dimensional division $\Bbbk$-algebra Δ_j for $j = 1, \dots, n$. Then the semisimple algebra $S = \prod_{j=1}^{n} \mathrm{Mat}_{n_j}(\Delta_j)$ has a symmetrising form given by

$$\tau : S \longrightarrow \Bbbk, \qquad (a_1, \dots, a_n) \longmapsto \sum_{j=1}^{n} \sum_{i=1}^{n_j} \lambda_j((a_j)_{ii}).$$

Theorem 1.57 (Comparison of duals). *Choose a symmetrising form $\tau : S \to \Bbbk$. For $M \in \operatorname{mod} S$ and $N \in \operatorname{mod}(S^{\mathrm{op}})$, postcomposition with τ gives S-module isomorphisms*

$$\alpha_M : M^* \longrightarrow D(M), \quad f \longmapsto \tau \circ f, \qquad \beta_N : {}^*N \longrightarrow D(N), \quad g \longmapsto \tau \circ g, \tag{1.20}$$

natural in M and N. Applied in each degree, the same maps give natural isomorphisms $M^{(*)} \cong \mathbb{D}(M)$ and ${}^{(*)}N \cong \mathbb{D}(N)$ for $M \in \operatorname{grmod} S$ and $N \in \operatorname{grmod}(S^{\mathrm{op}})$; we again write α_M and β_N.

We can spell this out more precisely:

Ungraded duals. The isomorphisms (1.20) give natural isomorphisms

$$\begin{aligned} &\alpha : (-)^* \xrightarrow{\sim} D \quad \text{of functors } (\operatorname{mod} S)^{\mathrm{op}} \longrightarrow \operatorname{mod}(S^{\mathrm{op}}), \\ &\beta : {}^*(-) \xrightarrow{\sim} D \quad \text{of functors } (\operatorname{mod}(S^{\mathrm{op}}))^{\mathrm{op}} \longrightarrow \operatorname{mod} S. \end{aligned} \tag{1.21}$$

Graded duals. Composition with τ in each degree gives natural isomorphisms

$$\begin{aligned} \alpha &: (-)^{(*)} \xrightarrow{\sim} \mathbb{D} \quad \text{of functors } (\operatorname{grmod} S)^{\mathrm{op}} \longrightarrow \operatorname{grmod}(S^{\mathrm{op}}), \\ \beta &: {}^{(*)}(-) \xrightarrow{\sim} \mathbb{D} \quad \text{of functors } (\operatorname{grmod}(S^{\mathrm{op}}))^{\mathrm{op}} \longrightarrow \operatorname{grmod} S. \end{aligned} \tag{1.22}$$

For $M \in \operatorname{grmod} S$ and $N \in \operatorname{grmod}(S^{\mathrm{op}})$, the degree i components are $\alpha_{M_{-i}}$ and $\beta_{N_{-i}}$, respectively.

Proof We prove the right module statement, i.e. taking $\alpha_M(f) := \tau \circ f$ yields

$$\text{ungraded case } \alpha : (-)^* \xrightarrow{\sim} D, \quad \text{and} \quad \text{graded case } \alpha : (-)^{(*)} \xrightarrow{\sim} \mathbb{D}.$$

First, non-degeneracy of τ gives an isomorphism $S \to D(S)$, $a \mapsto (s \mapsto \tau(as))$. Hence, for $\lambda \in D(M)$, there is a unique linear map $f : M \to S$ satisfying

$$\tau(f(m)s) = \lambda(ms) \quad \text{for all } m \in M,\ s \in S. \tag{1.23}$$

For $m \in M$ and $s, t \in S$,

$$\tau(f(mt)s) = \lambda(mts) = \tau(f(m)ts),$$

so non-degeneracy implies $f(mt) = f(m)t$. Taking $s = 1$ in (1.23) gives $\tau \circ f = \lambda$; conversely, every right S-linear preimage of λ must satisfy (1.23). Thus α_M is bijective.

Both M^* and $D(M)$ are left S-modules, so the component $\alpha_M : M^* \to D(M)$ must preserve the left action:

$$\alpha_M(sf)(m) = \tau(sf(m)) = \tau(f(m)s) = \tau(f(ms)) = (s\alpha_M(f))(m) \quad (s \in S,\ f \in M^*,\ m \in M).$$

For a right module map $u : M \to N$ and $f \in N^*$, associativity of composition gives $\alpha_M(f \circ u) = \alpha_N(f) \circ u$, proving naturality.

For the graded comparisons, apply the ungraded natural isomorphism degree-wise.

Same argument applies dually to S^{op} and β. □

Remark 1.58. The natural isomorphisms in Theorem 1.57 depend on the chosen symmetrising form. For example, every choice $c_1, \dots, c_n \in \Bbbk \setminus \{0\}$ gives a form $\tau(a_1, \dots, a_n) = \sum_{i=1}^n c_i a_i$ on $\Bbbk^{\oplus n}$. Under Assumption 1.50 we always use the *standard symmetrising form* $\tau_S(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i$, which involves no choice; then α_M and β_N are canonical isomorphisms.

Remark 1.59 (General S). For an arbitrary finite-dimensional semisimple S, identifying an S-dual with a $\Bbbk$-linear dual requires choosing a symmetrising form. If τ' is another one, then $\tau'(s) = \tau(zs)$ for a central unit $z \in S$, and the two identifications differ by multiplication by z; see Exercise 1.66(3).

Remark 1.60. Let A be a graded algebra with $A_0 = S$. The space $\operatorname{Hom}_S(M, S)$ has no natural left A-module structure on its own, but acquires one after *picking a symmetrising form τ of S*; under Assumption 1.50 we pick $\tau = \tau_S$. Indeed, the graded comparison α_M of Theorem 1.57 transports the canonical left A-action on $\mathbb{D}(M)$ to the graded S-dual of a locally finite-dimensional module:

$$af = \alpha_M^{-1}(a\alpha_M(f)) \quad \text{for right } M, \qquad ga = \beta_N^{-1}(\beta_N(g)a) \quad \text{for left } N,$$

where $a \in A$, $f \in M^{(*)}$, and $g \in {}^{(*)}N$. Similarly for right action on left dual.

The comparison theorem has a bimodule version:

Theorem 1.61. *With the actions in* (1.17)*, we obtain natural isomorphisms of functors*

$$\begin{aligned}(-)^* \xrightarrow[\alpha]{\sim} D \xleftarrow[\beta]{\sim} {}^*(-) \quad &: (\operatorname{mod} S^{\mathrm{en}})^{\mathrm{op}} \longrightarrow \operatorname{mod} S^{\mathrm{en}},\\ (-)^{(*)} \xrightarrow[\alpha]{\sim} \mathbb{D} \xleftarrow[\beta]{\sim} {}^{(*)}(-) \quad &: (\operatorname{grmod} S^{\mathrm{en}})^{\mathrm{op}} \longrightarrow \operatorname{grmod} S^{\mathrm{en}}.\end{aligned} \tag{1.24}$$

There is also a natural isomorphism $\omega : (-)^{*_{\mathrm{en}}} \xrightarrow{\sim} (-)^*$ *given by*

$$\omega_M : M^{*_{\mathrm{en}}} \to M^*, \qquad F \mapsto \big(m \mapsto \sum_j a_j \tau(b_j)\big) \quad \text{for} \quad F(m) = \sum_j a_j \otimes b_j^{\mathrm{op}}, \tag{1.25}$$

which also depends on τ*, but does not require* S^{en} *to be semisimple.*

Consequently, on finite-dimensional S*-bimodules and lfd graded* S*-bimodules we have natural isomorphisms:*

Ungraded: $(-)^{*_{\mathrm{en}}} \xrightarrow[\omega]{\sim} (-)^* \xrightarrow[\alpha]{\sim} D \xleftarrow[\beta]{\sim} {}^*(-)$ *as functors* $(\operatorname{mod} S^{\mathrm{en}})^{\mathrm{op}} \to \operatorname{mod} S^{\mathrm{en}}$,

Graded: $(-)^{(*_{\mathrm{en}})} \xrightarrow[\omega]{\sim} (-)^{(*)} \xrightarrow[\alpha]{\sim} \mathbb{D} \xleftarrow[\beta]{\sim} {}^{(*)}(-)$ *as functors* $(\operatorname{grmod} S^{\mathrm{en}})^{\mathrm{op}} \to \operatorname{grmod} S^{\mathrm{en}}$.

Proof The form $\tau_{\mathrm{en}}(a \otimes b^{\mathrm{op}}) = \tau(a)\tau(b)$ is symmetrising on S^{en}. Indeed, $\tau_{\mathrm{en}}((a \otimes b^{\mathrm{op}})(c \otimes d^{\mathrm{op}})) = \tau(ac)\tau(db) = \tau(ca)\tau(bd)$, and its pairing is the tensor product of two non-degenerate pairings. The formula (1.25) satisfies

$$\omega_M(F)(ms) = \omega_M(F)(m)s, \qquad \omega_M(sFt) = s\omega_M(F)t, \qquad \alpha_M(\omega_M(F)) = \tau_{\mathrm{en}} \circ F.$$

The argument (1.23), applied to S^{en} and τ_{en}, makes $F \mapsto \tau_{\mathrm{en}} \circ F$ an isomorphism. Thus ω_M is an isomorphism. For a bimodule map $u : M \to N$ and $F \in N^{*_{\mathrm{en}}}$, the formula (1.25) gives $\omega_M(F \circ u) = \omega_N(F) \circ u$, proving naturality. Taking the direct sum of $\omega_{M_{-i}}$ over $i \in \mathbb{Z}$ gives the graded two-sided comparison. Each component is a bimodule isomorphism, and naturality follows in each degree from the ungraded comparison. □

[a] S. Eilenberg and T. Nakayama, *On the dimension of modules and algebras, II. Frobenius algebras and quasi-Frobenius rings*, Nagoya Math. J. **9** (1955), 1–16, §2, Propositions 3–5, pp. 4–5; doi:10.1017/S0027763000023229.

1.13 Exercises

Exercise 1.62 (Matrix calculations). *Let* $n \geq 2$ *and* $A = \Bbbk A_n$*. Verify the grading in Example 1.19 using* $e_{ij}e_{pq} = \delta_{jp}e_{iq}$*. Compute* $\operatorname{rad} A$ *from its definition as the intersection of all maximal right ideals.*

Hint. *The strictly upper triangular ideal* J *satisfies* $J^n = 0$*. Show that* $LJ = 0$ *for every simple right* A*-module* L*, and use the diagonal maps* $A \to \Bbbk$*,* $X \mapsto X_{ii}$*, for* $1 \leq i \leq n$*.*

Exercise 1.63 (Shifts and duals). *Let* $A = \Bbbk A_3$ *and* $M = e_{22}A$ *with the inherited grading.*

(1) *Find a homogeneous basis of* $\mathbb{D}(M)$*, its degrees, and all non-zero actions of matrix units on the basis vectors.*

(2) *Show that* $\Bbbk e_{13} \subset Ae_{33}$ *is a graded left submodule and that*

$$\mathbb{D}(M) \cong (Ae_{33}/\Bbbk e_{13})(1)$$

as graded left A*-modules.*

(3) *Deduce that* $\mathbb{D}(M) \ncong (Ae_{jj})(r)$ *for every* $j \in \{1, 2, 3\}$ *and* $r \in \mathbb{Z}$*.*

Hint for (3). *Compare dimensions and the actions of* e_{11}, e_{22}, e_{33}.

Exercise 1.64 (Homogeneous maps). *The finite-generation hypothesis in Lemma 1.31 is needed. Let* $S = \Bbbk e_1 \oplus \Bbbk e_2$ *with componentwise multiplication, let* $N = e_1 S$ *have degree* 0*, and put*

$$M = \bigoplus_{i \geq 0} (e_1 S)(-i) = \bigoplus_{i \geq 0} \Bbbk m_i, \qquad \deg m_i = i, \quad m_i e_1 = m_i, \quad m_i e_2 = 0.$$

For the S*-linear map* $f : M \to N$ *with* $f(m_i) = e_1$ *for every* $i \geq 0$*, determine all homogeneous components and prove that* f *is not a finite sum of homogeneous maps.*

Exercise 1.65 (Dependence on the form). *Let* $S = (\Bbbk A_2)_0 = \Bbbk e_{11} \oplus \Bbbk e_{22}$*,* $V = (\Bbbk A_2)_1 = \Bbbk e_{12}$*, and* $c_1, c_2 \in \Bbbk \setminus \{0\}$*. Set* $\tau(a e_{11} + b e_{22}) = c_1 a + c_2 b$ *and let* $\varepsilon : V \to \Bbbk$ *satisfy* $\varepsilon(e_{12}) = 1$*. Find the right and left* S*-linear maps corresponding to* ε *under* (1.20)*, and determine how they depend on* c_1, c_2*.*

Exercise 1.66 (Dependence of the action on the form). *The transported* A*-action can depend on the symmetrising form, but its graded isomorphism class does not. Let* $A = \Bbbk A_2$*,* $S = \Bbbk e_{11} \oplus \Bbbk e_{22}$*, and* $M = e_{11} A$*, with the inherited gradings. For* $c = (c_1, c_2) \in (\Bbbk \setminus \{0\})^2$*, define*

$$\tau_c : S \longrightarrow \Bbbk, \qquad a e_{11} + b e_{22} \longmapsto c_1 a + c_2 b \quad \text{for all } a, b \in \Bbbk. \tag{1.26}$$

Write $\alpha_c : M^{(*)} \to \mathbb{D}(M)$*,* $f \mapsto \tau_c \circ f$*, and let* $a \cdot_c f = \alpha_c^{-1}(a \alpha_c(f))$ *be the transported action of Remark 1.60. Consider the right* S*-linear maps*

$$\begin{aligned} &\varphi_0 : M \longrightarrow S, \quad a e_{11} + b e_{12} \longmapsto a e_{11}, \\ &\varphi_1 : M \longrightarrow S, \quad a e_{11} + b e_{12} \longmapsto b e_{22} \quad \text{for all } a, b \in \Bbbk. \end{aligned} \tag{1.27}$$

(1) *Verify that* τ_c *is symmetrising and that* φ_0, φ_1 *form a homogeneous basis of* $M^{(*)}$*. Determine their degrees.*

(2) *Calculate the actions of* e_{11}, e_{22}, e_{12} *on* φ_0, φ_1*. For* $c, c' \in (\Bbbk \setminus \{0\})^2$*, prove that* $\cdot_c = \cdot_{c'}$ *if and only if* $c' = \rho c$ *for some* $\rho \in \Bbbk \setminus \{0\}$*.*

(3) *More generally, let* A *be a locally finite-dimensional graded algebra with semisimple* $S = A_0$*, and let* M *be a locally finite-dimensional graded right* A*-module. Let* τ, τ' *be symmetrising forms on* S*, with comparison maps* $\alpha_\tau, \alpha_{\tau'} : M^{(*)} \to \mathbb{D}(M)$ *given by composition with* τ, τ'*, respectively, and transported actions* $\cdot_\tau, \cdot_{\tau'}$*. Prove that there is an invertible* $z \in S$ *satisfying* $\tau'(s) = \tau(zs)$ *for every* $s \in S$*, and that*

$$\gamma : (M^{(*)}, \cdot_{\tau'}) \longrightarrow (M^{(*)}, \cdot_\tau), \qquad (\gamma f)(m) = z f(m)$$

is an isomorphism of graded left A*-modules. Deduce that the modules in (2) are isomorphic for every* c, c'*, and give the isomorphism on* φ_0, φ_1*.*

Hint for (3). *Compare* $\alpha_{\tau'}$ *with* α_τ*.*

2 Tensor and quadratic algebras

2.1 Tensor products over an algebra

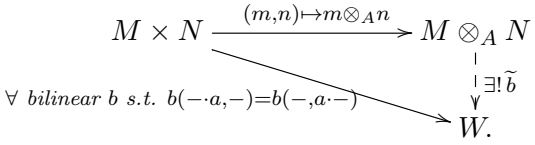
Definition 2.1. *Let A be any $\Bbbk$-algebra, M a right A-module, and N a left A-module. The tensor product of M and N over A is the vector space*

$$M \otimes_A N := M \otimes_{\Bbbk} N / \operatorname{span}_{\Bbbk}\{ma \otimes n - m \otimes an \mid m \in M,\ n \in N,\ a \in A\}.$$

Write $m \otimes_A n$, or simply $m \otimes n$, for the image of $m \otimes n$ in this quotient. Thus $ma \otimes_A n = m \otimes_A an$ for all $m \in M$, $n \in N$, and $a \in A$.

Lemma 2.2 (Basic properties of tensor products). *Let A be any $\Bbbk$-algebra, M a right A-module, and N a left A-module.*

(1) *(Universal property) For a vector space W and a bilinear map $b : M \times N \to W$ satisfying $b(ma, n) = b(m, an)$ for all $m \in M$, $a \in A$, and $n \in N$, there is a unique linear factorisation (commutative diagram)*



(2) *If B, C are $\Bbbk$-algebras and ${}_BM_A$, ${}_AN_C$ are bimodules, then $M \otimes_A N$ is a bimodule ${}_B(M \otimes_A N)_C$ with*

$$b(m \otimes_A n)c = bm \otimes_A nc \qquad (b \in B,\ c \in C,\ m \in M,\ n \in N).$$

For right A-module maps $u : M \to M'$ and left A-module maps $v : N \to N'$, the assignment $(m, n) \mapsto u(m) \otimes v(n)$ induces

$$u \otimes_A v : M \otimes_A N \longrightarrow M' \otimes_A N', \qquad m \otimes n \longmapsto u(m) \otimes v(n).$$

These define functors

$$(-) \otimes_A N : \operatorname{Mod} A \longrightarrow \operatorname{Mod} C, \qquad M \otimes_A (-) : \operatorname{Mod}(A^{\mathrm{op}}) \longrightarrow \operatorname{Mod}(B^{\mathrm{op}}).$$

On bimodule maps, they retain the outer actions:

$$(-) \otimes_A N : B\operatorname{Mod} A \longrightarrow B\operatorname{Mod} C, \qquad M \otimes_A (-) : A\operatorname{Mod} C \longrightarrow B\operatorname{Mod} C.$$

(3) *For $\Bbbk$-algebras A, B, a right A-module U, a bimodule ${}_AV_B$, and a left B-module W, there is a natural linear isomorphism*

$$(U \otimes_A V) \otimes_B W \xrightarrow{\sim} U \otimes_A (V \otimes_B W), \qquad (u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w).$$

There are also natural module isomorphisms

$$M \otimes_A A \xrightarrow{\sim} M, \quad m \otimes a \longmapsto ma, \qquad A \otimes_A N \xrightarrow{\sim} N, \quad a \otimes n \longmapsto an.$$

Moreover, these maps are linear with respect to any additional bimodule action on U, W, M, N.

(4) *Tensor products distribute over direct sums in either variable. For arbitrary families $(M_i)_{i \in I}$ of right A-modules and $(N_j)_{j \in J}$ of left A-modules, indexed by sets I, J,*

$$\Big(\bigoplus_{i \in I} M_i\Big) \otimes_A N \xrightarrow{\sim} \bigoplus_{i \in I} (M_i \otimes_A N), \quad (m_i)_{i \in I} \otimes n \longmapsto (m_i \otimes n)_{i \in I},$$
$$M \otimes_A \Big(\bigoplus_{j \in J} N_j\Big) \xrightarrow{\sim} \bigoplus_{j \in J} (M \otimes_A N_j), \quad m \otimes (n_j)_{j \in J} \longmapsto (m \otimes n_j)_{j \in J}.$$

Proof For (1), Proposition 1.37 gives a unique linear map $M \otimes_{\Bbbk} N \to W$, $m \otimes n \mapsto b(m,n)$. The map vanishes on the defining subspace precisely when $b(ma,n) = b(m,an)$.

For (2), the outer actions preserve that subspace: for $b \in B$, $c \in C$, and $a \in A$,

$$b(ma \otimes n - m \otimes an)c = (bm)a \otimes nc - bm \otimes a(nc).$$

The module axioms follow from those on M and N. For module maps u, v,

$$u(ma) \otimes v(n) = u(m)a \otimes v(n) = u(m) \otimes av(n) = u(m) \otimes v(an),$$

so the displayed formula induces $u \otimes_A v$. The functor identities follow from $(u' \otimes v')(u(m) \otimes v(n)) = u'u(m) \otimes v'v(n)$ and $(\mathrm{id}_M \otimes \mathrm{id}_N)(m \otimes n) = m \otimes n$. If u, v preserve the outer actions, then $u \otimes_A v$ does as well.

For (3), both parenthesisations impose

$$(ua) \otimes v \otimes w = u \otimes (av) \otimes w, \qquad u \otimes (vb) \otimes w = u \otimes v \otimes (bw) \quad (a \in A,\ b \in B).$$

Thus the displayed associativity map and the reverse assignment are well-defined and inverse. The unit inverses are $M \to M \otimes_A A$, $m \mapsto m \otimes 1$, and $N \to A \otimes_A N$, $n \mapsto 1 \otimes n$.

For (4), the bilinear map $((m_i)_{i\in I}, n) \mapsto (m_i \otimes n)_{i\in I}$ satisfies the identity in (1), so it induces the displayed linear map. For each $i \in I$, let $\iota_i : M_i \to \bigoplus_{j\in I} M_j$ be the inclusion. The inverse sends $m \otimes n$ in the ith summand to $\iota_i(m) \otimes n$ and extends over the direct sum. Both composites fix every element $m \otimes n$ from a summand. The other variable is handled in the same way. □

Lemma 2.3 (Hom and finite direct sums). *For finite sets I, J and right A-modules M_j, N_i, there is a natural isomorphism*

$$\begin{aligned} \mathrm{Hom}_A\Big(\bigoplus_{j\in J} M_j, \bigoplus_{i\in I} N_i\Big) &\xrightarrow{\sim} \bigoplus_{\substack{i\in I\\ j\in J}} \mathrm{Hom}_A(M_j, N_i), \\ f &\longmapsto (\pi_i f \iota_j)_{i\in I,\ j\in J}, \end{aligned} \tag{2.1}$$

where $\iota_j : M_j \to \bigoplus_{j'\in J} M_{j'}$ and $\pi_i : \bigoplus_{i'\in I} N_{i'} \to N_i$ are the inclusions and projections.

Proof Write $\pi'_j : \bigoplus_{j'\in J} M_{j'} \to M_j$ and $\iota'_i : N_i \to \bigoplus_{i'\in I} N_{i'}$ for the remaining projections and inclusions. The inverse sends $(f_{ij})_{i\in I,\ j\in J}$ to $\sum_{i\in I}\sum_{j\in J} \iota'_i f_{ij} \pi'_j$. Both composites are identities because $\sum_{j\in J} \iota_j \pi'_j$ and $\sum_{i\in I} \iota'_i \pi_i$ are identity maps, while $\pi'_j \iota_{j'} = \delta_{jj'} \mathrm{id}_{M_j}$ and $\pi_i \iota'_{i'} = \delta_{ii'} \mathrm{id}_{N_i}$. The same formulas commute with pre- and postcomposition, proving naturality. □

Lemma 2.4 (Hom, direct sums, and direct products). *Let M, N be right A-modules, and let $(M_i)_{i\in I}$ and $(N_i)_{i\in I}$ be arbitrary families of right A-modules, indexed by a set I. Write $\iota_i : M_i \to \bigoplus_{j\in I} M_j$ for the inclusions and $\pi_i : \prod_{j\in I} N_j \to N_i$ for the projections. There are*

natural linear isomorphisms

$$\operatorname{Hom}_A\Big(\bigoplus_{i\in I} M_i, N\Big) \xrightarrow{\sim} \prod_{i\in I} \operatorname{Hom}_A(M_i, N), \quad f \longmapsto (f\circ \iota_i)_{i\in I},$$
$$\operatorname{Hom}_A\Big(M, \prod_{i\in I} N_i\Big) \xrightarrow{\sim} \prod_{i\in I} \operatorname{Hom}_A(M, N_i), \quad f \longmapsto (\pi_i \circ f)_{i\in I}.$$

There is also a natural injection

$$\bigoplus_{i\in I} \operatorname{Hom}_A(M, N_i) \longrightarrow \operatorname{Hom}_A\Big(M, \bigoplus_{i\in I} N_i\Big), \qquad (f_i)_{i\in I} \longmapsto \big(m \mapsto (f_i(m))_{i\in I}\big),$$

which is an isomorphism when M is finitely generated.

Proof The inverse of the first map sends $(f_i)_{i\in I}$ to $(m_i)_{i\in I} \mapsto \sum_{i\in I} f_i(m_i)$; the sum is finite because $(m_i)_{i\in I}$ has finite support. The inverse of the second sends $(f_i)_{i\in I}$ to $m \mapsto (f_i(m))_{i\in I}$; a direct product imposes no support condition. For the third map, coordinate projections recover each f_i, proving injectivity. If $m_1, \dots, m_r$ generate M, the images of these generators under $f : M \to \bigoplus_{i\in I} N_i$ are supported on a common finite subset $J \subset I$. Then $f(M) \subset \bigoplus_{i\in J} N_i$, so f comes from a finite family of maps. All three constructions commute with precomposition and postcomposition. □

Proposition 2.5 (Tensor products and direct products). *Let A be a $\Bbbk$-algebra, M, M_i right A-modules, and N, N_i left A-modules, with $i \in I$. The coordinate projections define linear maps*

$$\begin{aligned} \kappa_N &: \Big(\prod_i M_i\Big) \otimes_A N \longrightarrow \prod_i (M_i \otimes_A N), \quad (m_i)_i \otimes n \longmapsto (m_i \otimes n)_i, \\ \lambda_M &: M \otimes_A \Big(\prod_i N_i\Big) \longrightarrow \prod_i (M \otimes_A N_i), \quad m \otimes (n_i)_i \longmapsto (m \otimes n_i)_i. \end{aligned} \tag{2.2}$$

(1) *If I is finite, both maps are isomorphisms.*

(2) *The map κ_N is an isomorphism if N is a direct summand of $A^{\oplus r}$ as a left module for some finite r. The corresponding condition on the right module M makes λ_M an isomorphism.*

(3) *If A is finite-dimensional semisimple, both maps are injective. Moreover, κ_N is an isomorphism when N is finite-dimensional, and λ_M is an isomorphism when M is finite-dimensional.*

Proof The formulas in (2.2) are bilinear. For $s \in A$, the component identities $(m_i s) \otimes n = m_i \otimes sn$ and $(ms) \otimes n_i = m \otimes sn_i$ show that both formulas give well-defined maps from the indicated tensor products. For finite I, products equal sums, and Lemma 2.2(4) proves (1).

For $N = A$, the map κ_A is the identity on $\prod_i M_i$ under $m \otimes s \mapsto ms$. For $N = A^{\oplus r}$, both sides are $(\prod_i M_i)^{\oplus r}$, since a finite direct sum commutes with a direct product. If $N \oplus N' \cong A^{\oplus r}$, the comparison for $N \oplus N'$ is $\kappa_N \oplus \kappa_{N'}$. Its bijectivity implies the bijectivity of κ_N. This proves the assertion about κ_N in (2).

Now let A be finite-dimensional semisimple. For finite-dimensional N, Corollary 1.13 gives (2). For arbitrary N, take $z = \sum_{\ell=1}^{r} m^{(\ell)} \otimes n_\ell$ in $\operatorname{Ker} \kappa_N$ and set $N_0 = \sum_{\ell=1}^{r} An_\ell$. The submodule N_0 is finite-dimensional and has a complement in N by Theorem 1.12. Thus the inclusions

$$\Big(\prod_i M_i\Big) \otimes_A N_0 \longrightarrow \Big(\prod_i M_i\Big) \otimes_A N, \qquad M_i \otimes_A N_0 \longrightarrow M_i \otimes_A N$$

are injective, by the direct sum formula in Lemma 2.2(4). Regard z as an element with second factors in N_0. The equality $\kappa_N(z) = 0$ then gives $\kappa_{N_0}(z) = 0$. Aince κ_{N_0} is an isomorphism, $z = 0$.

For λ_M, apply the same arguments over A^{op}, using $N \otimes_{A^{\mathrm{op}}} M \cong M \otimes_A N$, $n \otimes m \mapsto m \otimes n$. □

The finite-dimensional hypothesis on the fixed factor cannot be dropped from the isomorphism assertion: Example 1.39 gives a non-surjective comparison even for vector spaces. Thus tensor products always distribute over direct sums, but not over arbitrary direct products.

Remark 2.6. For infinite families, Hom need not preserve direct sums in its second variable. For $M = \bigoplus_{i\geq 1} \Bbbk v_i$, the identity $M \to M$ does not come from $\bigoplus_i \mathrm{Hom}_{\Bbbk}(M, \Bbbk v_i)$: its projection onto every $\Bbbk v_i$ is non-zero. Thus the injection in Lemma 2.4 need not be surjective.

A product in the source of Hom behaves differently from a product in the target. For example, a linear map $\prod_{i\geq 1} \Bbbk \to \Bbbk$ need not be determined by its restrictions to the coordinate copies of $\Bbbk$. Indeed, $(1, 1, \ldots) \notin \bigoplus_{i\geq 1} \Bbbk$. Define a functional on $\bigoplus_{i\geq 1} \Bbbk + \Bbbk(1, 1, \ldots)$ to vanish on the direct sum and take value 1 at $(1, 1, \ldots)$, and extend it to the product by extending a basis. The resulting non-zero functional vanishes on every coordinate copy.

For a bimodule ${}_AV_A$, define

$$V^{\otimes_A 0} := A, \qquad V^{\otimes_A i} := \underbrace{V \otimes_A \cdots \otimes_A V}_{i \text{ factors}} \quad (i \geq 1).$$

Note that there is no parenthesis in the definition of $V^{\otimes_A i}$ since Lemma 2.2(3) says that tenso product is an associatitive operation.

Let A, B, C be $\Bbbk$-algebras. For bimodules ${}_AM_B$, ${}_CX_B$, we have $\mathrm{Hom}_B(M, X) \in C\,\mathrm{Mod}\,A$, with

$$(cfa)(m) = cf(am) \quad (f \in \mathrm{Hom}_B(M, X),\ c \in C,\ a \in A,\ m \in M). \tag{2.3}$$

If $X \in \mathrm{Mod}\,B$ has no specified left action, the same formula defines the right A-action $(fa)(m) = f(am)$. The left-module version follows by passing to opposite algebras.

Proposition 2.7 (Tensor and Hom). *Consider bimodules ${}_AM_B$, ${}_BN_C$, a right C-module X, and a left A-module Y. There are maps, natural in M, N,*

$$\begin{aligned} \theta_r : \mathrm{Hom}_C(N, X) \otimes_B \mathrm{Hom}_B(M, B) &\longrightarrow \mathrm{Hom}_C(M \otimes_B N, X), \\ \theta_r(g \otimes f)(m \otimes n) &= g(f(m)n), \\ \theta_\ell : \mathrm{Hom}_{B^{\mathrm{op}}}(N, B) \otimes_B \mathrm{Hom}_{A^{\mathrm{op}}}(M, Y) &\longrightarrow \mathrm{Hom}_{A^{\mathrm{op}}}(M \otimes_B N, Y), \\ \theta_\ell(g \otimes f)(m \otimes n) &= f(mg(n)). \end{aligned} \tag{2.4}$$

The maps θ_r and θ_ℓ are right A-linear and left C-linear, respectively.

(1) *The map θ_r is an isomorphism if $M \in \mathrm{add}\,B$ as a right B-module.*

(2) *The map θ_ℓ is an isomorphism if ${}_BN \in \mathrm{add}_{\mathrm{mod}\,B^{\mathrm{op}}}(B)$.*

Here $\mathrm{add}\,B = \mathrm{add}_{\mathrm{mod}\,B}(B)$, as in Example 1.5. If ${}_CX_C$ and ${}_AY_A$ are bimodules, the respective maps also preserve the additional actions and are morphisms in $C\,\mathrm{Mod}\,A$.

Proof *Well-definedness of θ_r.* For $f \in \mathrm{Hom}_B(M, B)$, $g \in \mathrm{Hom}_C(N, X)$, and $b \in B$,

$$g(f(mb)n) = g(f(m)bn), \qquad (gb)(f(m)n) = g(bf(m)n) = g((bf)(m)n).$$

These identities respect the defining identifications in both tensor products. Right C-linearity follows from $g(f(m)nc) = g(f(m)n)c$ for $c \in C$. For $a \in A$,

$$\theta_r(g \otimes fa)(m \otimes n) = g(f(am)n) = (\theta_r(g \otimes f)a)(m \otimes n).$$

If X also has a commuting left C-action, replacing g by cg multiplies the value by c, so θ_r preserves that action as well.

Bijectivity of θ_r. Regard M only as a right B-module. For $M = B$, evaluation at 1 identifies $\mathrm{Hom}_B(B, B) \cong B$, while multiplication identifies $B \otimes_B N \cong N$. Under these isomorphisms, θ_r becomes

$$\mathrm{Hom}_C(N, X) \otimes_B B \longrightarrow \mathrm{Hom}_C(N, X), \qquad g \otimes b \longmapsto gb,$$

which is an isomorphism by Lemma 2.2(3). Finite direct sums commute with both constructions by Lemma 2.2(4) and (2.1). Thus θ_r is bijective for $M = B^{\oplus r}$, where $r \geq 0$ is finite. If $M \in \operatorname{add} B$, choose M' with $M \oplus M' \cong B^{\oplus r}$. The comparison for $M \oplus M'$ is the direct sum of the comparisons for M and M', so both summand maps are bijective.

The left comparison. For $f \in \mathrm{Hom}_{A^{\mathrm{op}}}(M, Y)$ and $g \in \mathrm{Hom}_{B^{\mathrm{op}}}(N, B)$, well-definedness follows from

$$f(mb\, g(n)) = f(mg(bn)), \qquad f(m(gb)(n)) = f(mg(n)b) = (bf)(mg(n)) \quad (b \in B).$$

Left A-linearity of f makes the resulting functional left A-linear. For $c \in C$,

$$\theta_\ell(cg \otimes f)(m \otimes n) = f(mg(nc)) = (c\theta_\ell(g \otimes f))(m \otimes n).$$

If Y also has a commuting right A-action, replacing f by fa multiplies the value by a, so θ_ℓ preserves that action as well. For $N = B$ as a left module, evaluation at 1 identifies $\mathrm{Hom}_{B^{\mathrm{op}}}(B, B) \cong B$ and turns θ_ℓ into

$$B \otimes_B \mathrm{Hom}_{A^{\mathrm{op}}}(M, Y) \longrightarrow \mathrm{Hom}_{A^{\mathrm{op}}}(M, Y), \qquad b \otimes f \longmapsto bf.$$

The same finite-sum and direct-summand argument proves (2).

Naturality. Let $p : M \to M'$ and $q : N \to N'$ be bimodule maps. For $f \in \mathrm{Hom}_B(M', B)$ and $g \in \mathrm{Hom}_C(N', X)$, applying θ_r before or after precomposition gives $g(f(p(m))q(n))$ at $m \otimes n$. For $f \in \mathrm{Hom}_{A^{\mathrm{op}}}(M', Y)$ and $g \in \mathrm{Hom}_{B^{\mathrm{op}}}(N', B)$, the two orders give $f(p(m)g(q(n)))$ for θ_ℓ. Thus both comparisons are natural. □

For an arbitrary finite-dimensional semisimple S, the one-sided S-duals satisfy an analogue of Proposition **??** without any choice. Consider bimodules ${}_SV_S$, ${}_SW_S$ finitely generated on the right and left, respectively. Exercise **??** shows, using only Corollary 1.13, that there are natural bimodule isomorphisms

$$\theta_r : W^* \otimes_S V^* \xrightarrow{\sim} (V \otimes_S W)^*, \qquad \theta_\ell : {}^*W \otimes_S {}^*V \xrightarrow{\sim} {}^*(V \otimes_S W), \tag{2.5}$$

$$\theta_r(g \otimes f)(v \otimes w) = g(f(v)w), \qquad \theta_\ell(g \otimes f)(v \otimes w) = f(vg(w)). \tag{2.6}$$

We refer to (2.5) as *anti-distributivity of the S-dual over $\otimes_S$*[a]: dualising reverses the order of the factors.

Example 2.8. *Finite generation alone does not suffice in Proposition 2.7 over a general algebra. Take $A = C = \Bbbk$, $B = \Bbbk[x]$, and $X = Y = \Bbbk$. Consider bimodules ${}_\Bbbk M_B$, ${}_B N_\Bbbk$ with $M = N =$*

$B/(x) \cong \Bbbk$ *and the outer* $\Bbbk$*-actions given by scalar multiplication. Every* $f \in \mathrm{Hom}_B(M,B)$ *satisfies* $f(\overline{1})x = f(\overline{1}x) = 0$*, hence* $f = 0$*. Consequently*

$$\mathrm{Hom}_{\Bbbk}(N,\Bbbk) \otimes_B \mathrm{Hom}_B(M,B) = 0, \qquad \mathrm{Hom}_{\Bbbk}(M \otimes_B N, \Bbbk) \cong \Bbbk.$$

Both M_B *and* ${}_BN$ *are generated by* $\overline{1}$*, so neither direct-summand hypothesis can be replaced by finite generation alone. For the left comparison,* $\mathrm{Hom}_{B^{\mathrm{op}}}(N,B) = 0$*, whereas* $\mathrm{Hom}_{\Bbbk}(M \otimes_B N, \Bbbk) \cong \Bbbk$*.*

For S, V, W as in (2.5), take $A = B = S$, $C = \Bbbk$, $M = V$, $N = W$, and $X = \Bbbk$ in Proposition 2.7(1). Since $\mathrm{Hom}_{\Bbbk}(W,\Bbbk) = D(W)$, we obtain the natural bimodule isomorphism [**?**, §2.1]

$$\theta_D : D(W) \otimes_S V^* \xrightarrow{\sim} D(V \otimes_S W),$$
$$\theta_D(\lambda \otimes f)(v \otimes w) = \lambda(f(v)w).$$

The actions on $D(W)$ are those in (1.17). For $s \in S$, naturality with respect to $W \to W$, $w \mapsto ws$, gives left S-linearity of θ_D.

Suppose now that S is equipped with a symmetrising form $\tau : S \to \Bbbk$, as in Definition 1.55. The two-sided dual of Definition 1.54 satisfies

$$\theta_{\mathrm{en}} : W^* \otimes_S V^{*\mathrm{en}} \xrightarrow{\sim} (V \otimes_S W)^{*\mathrm{en}},$$
$$\theta_{\mathrm{en}}(g \otimes F)(v \otimes w) = \sum_j g(a_j w) \otimes b_j^{\mathrm{op}}, \qquad F(v) = \sum_j a_j \otimes b_j^{\mathrm{op}}. \tag{2.7}$$

We check that the formula defines a map on the two indicated tensor products and preserves both outer actions. For example, $F(vs) = F(v)(s \otimes 1)$ replaces a_j by $a_j s$, while $(Fs)(v) = (1 \otimes s^{\mathrm{op}})F(v)$ replaces b_j by $b_j s$. These give, respectively, compatibility with $vs \otimes w = v \otimes sw$ and with the right outer action. Compatibility with $(gs) \otimes F = g \otimes (sF)$ follows from $(gs)(a_j w) = g(sa_j w)$. The other actions follow from right S-linearity of g and $F(tv) = F(v)(1 \otimes t^{\mathrm{op}})$.

The comparison maps ω_M in (1.25) give

$$\omega_{V \otimes_S W} \circ \theta_{\mathrm{en}} = \theta_r \circ (\mathrm{id}_{W^*} \otimes \omega_V),$$

since both sides send $g \otimes F$, evaluated at $v \otimes w$, to $\sum_j g(a_j w)\tau(b_j)$. Thus θ_{en} is an isomorphism by Theorem 1.57. Precomposition shows that θ_D and θ_{en} are natural. No assumption that S^{en} is semisimple is used; compare Remark 1.43.

For the maps $\alpha_M(f) = \tau \circ f$ in (1.20),

$$\alpha_{V \otimes_S W} \circ \theta_r = \theta_D \circ (\alpha_W \otimes \mathrm{id}_{V^*}),$$

because both sides send $g \otimes f$ to the functional $v \otimes w \mapsto \tau(g(f(v)w))$.

[a] this is not an common phrase in the literature, I only use it as a shorthand for clarifications here-and-there.

For an arbitrary finite-dimensional semisimple S, the one-sided S-duals also describe Hom spaces and give evaluation and coevaluation maps [**?**, §2.7].

Lemma 2.9. *Let* S *be finite-dimensional semisimple and* ${}_SV_S$ *a finite-dimensional bimodule.*

There are bimodule isomorphisms

$$\begin{aligned} V \otimes_S V^* &\xrightarrow{\sim} \mathrm{End}_S(V), \qquad v \otimes f \longmapsto \big(u \mapsto vf(u)\big), \\ {}^*V \otimes_S V &\xrightarrow{\sim} \mathrm{End}_{S^{\mathrm{op}}}(V), \quad g \otimes v \longmapsto \big(u \mapsto g(u)v\big). \end{aligned} \tag{2.8}$$

The source actions are those of Lemma 2.2(2). The target actions are defined, for $s, t \in S$ and $u \in V$, by

$$\begin{aligned} (sFt)(u) &= sF(tu) \quad \text{for } F \in \mathrm{End}_S(V), \\ (sGt)(u) &= G(us)t \quad \text{for } G \in \mathrm{End}_{S^{\mathrm{op}}}(V). \end{aligned} \tag{2.9}$$

Proof For the first comparison, $(vs)f(u) = v(sf)(u)$ and $vf(us) = vf(u)s$ give a well-defined map to $\mathrm{End}_S(V)$. By Corollary 1.13, identify V_S with a direct summand of $S^{\oplus m}$ for some integer $m \geq 0$, and let $p : S^{\oplus m} \to V$ be the right S-linear projection. For $1 \leq j \leq m$, let $\varepsilon_j \in S^{\oplus m}$ be the vector with 1 in coordinate j and zero elsewhere, and let $q_j : S^{\oplus m} \to S$ be the coordinate projection. Put $v_j = p(\varepsilon_j)$ and $f_j = q_j|_V$. Then $u = \sum_{j=1}^m v_j f_j(u)$ for every $u \in V$. The inverse is

$$\mathrm{End}_S(V) \longrightarrow V \otimes_S V^*, \qquad F \longmapsto \sum_{j=1}^m F(v_j) \otimes f_j.$$

Indeed, one composite sends F to $u \mapsto \sum_{j=1}^m F(v_j)f_j(u) = F(u)$, and the other sends

$$v \otimes f \longmapsto \sum_{j=1}^m vf(v_j) \otimes f_j = v \otimes \sum_{j=1}^m f(v_j)f_j = v \otimes f.$$

Applying this construction over S^{op} proves the second isomorphism, using $V \otimes_{S^{\mathrm{op}}} {}^*V \cong {}^*V \otimes_S V$, $v \otimes g \mapsto g \otimes v$.

The commuting left and right actions on V make (2.9) bimodule actions. For example, $(sFt)(ur) = sF(tur) = sF(tu)r$ for $r \in S$, so sFt is right S-linear. The images of $sv \otimes ft$ and $sg \otimes vt$, evaluated at u, are $svf(tu)$ and $g(us)vt$, respectively. These are exactly the actions in (2.9), proving bimodule linearity of both comparisons. □

Evaluation applies a functional to an element. Coevaluation expresses the identity endomorphism as an element of a tensor product.

Proposition 2.10. *For a bimodule ${}_SV_S$, where S is finite-dimensional semisimple, evaluation gives bimodule maps*

$$\begin{aligned} V^* \otimes_S V &\longrightarrow S, \quad f \otimes v \longmapsto f(v), \\ V \otimes_S {}^*V &\longrightarrow S, \quad v \otimes g \longmapsto g(v). \end{aligned} \tag{2.10}$$

If V is finite-dimensional, the coevaluation maps are

$$\begin{aligned} S &\longrightarrow V \otimes_S V^* \xrightarrow{\sim} \mathrm{End}_S(V), \qquad s \longmapsto \big(v \mapsto sv\big), \\ S &\longrightarrow {}^*V \otimes_S V \xrightarrow{\sim} \mathrm{End}_{S^{\mathrm{op}}}(V), \quad s \longmapsto \big(v \mapsto vs\big). \end{aligned} \tag{2.11}$$

The isomorphisms are those in (2.8)*. Both coevaluation maps are bimodule maps, and each image of* 1 *corresponds to* id_V*.*

Proof The identities $(fs)(v) = f(sv)$ and $(sg)(v) = g(vs)$ from (1.17) make evaluation well-defined on the two tensor products; the other two actions give its bimodule linearity. For coevaluation, left multiplication is right S-linear and right multiplication is left S-linear because

the two actions on V commute. Under the endomorphism actions in (2.9), both multiplication assignments preserve the two S-actions. The isomorphisms in (2.8) therefore give the stated bimodule maps. □

For a finite-dimensional graded bimodule ${}_SV_S$, use $V^{(*)}$ and ${}^{(*)}V$ in Proposition 2.10. With the total grading of Definition **??**, evaluation and coevaluation preserve degree: evaluation pairs degrees i and $-i$, and id_V has degree 0.

The evaluation and coevaluation maps in Proposition 2.10 are the constructions in the last two paragraphs of [**?**, §2.7, pp. 484–485], translated to our notation.

For a locally finite-dimensional graded bimodule ${}_SV_S$, (2.11) applies to each V_d; a coevaluation for the whole module need not exist. For example, take $S \neq 0$ and $V = \bigoplus_{d\geq 0} S(-d)$. Every element $\sum_{i=1}^{n} v_i \otimes f_i$ of $V \otimes_S V^{(*)}$ defines an endomorphism with image contained in the finite-dimensional submodule $\sum_{i=1}^{n} v_i S$, so it cannot represent id_V.

2.2 Generated algebras

Definition 2.11. *Let A be graded with $S = A_0$. Multiplication makes A_1 an S-bimodule and gives*

$$\mu_i : A_1^{\otimes_S i} \longrightarrow A_i, \qquad a_1 \otimes \cdots \otimes a_i \longmapsto a_1 \cdots a_i \quad (i \geq 1), \qquad \mu_0 = \mathrm{id}_S\,. \tag{2.12}$$

The algebra A is generated in degree 1 (over S) if every μ_i in (2.12) is surjective.

Equivalently, S and A_1 generate A as an algebra.

Example 2.12. *The algebra $\Bbbk A_n$ of Example 1.19 is generated in degree 1 over $S = \bigoplus_i \Bbbk e_{ii}$.*

Example 2.13. *The polynomial algebra $\Bbbk[x_1, \ldots, x_n]$, with each $\deg x_i = 1$ for all i, is generated in degree 1 over $\Bbbk$. Indeed, every monomial of total degree d is a product of d variables, and these monomials span the degree d component.*

Example 2.14. *Let*

$$A = \begin{pmatrix} \Bbbk[x] & \Bbbk[x] \\ 0 & \Bbbk[x] \end{pmatrix}, \qquad \deg(x^m e_{ij}) = m + j - i \quad (m \geq 0,\ 1 \leq i \leq j \leq 2).$$

Then

$$A_0 = \Bbbk e_{11} \oplus \Bbbk e_{22}, \qquad A_d = \Bbbk x^d e_{11} \oplus \Bbbk x^d e_{22} \oplus \Bbbk x^{d-1} e_{12} \quad (d \geq 1).$$

It is generated in degree 1 over A_0, since

$$x^d e_{11} = (x e_{11})^d, \qquad x^d e_{22} = (x e_{22})^d, \qquad x^{d-1} e_{12} = (x e_{11})^{d-1} e_{12} \quad (d \geq 1).$$

Here A_0 has two simple summands, $A_d \neq 0$ for every $d \geq 0$, and A is non-commutative: $e_{11}e_{12} = e_{12}$ whereas $e_{12}e_{11} = 0$.

2.3 Tensor algebras and the universal property

Definition 2.15. *Let S be a $\Bbbk$-algebra and consider a bimodule ${}_SV_S$. The tensor algebra of V over S is the graded $\Bbbk$-algebra*

$$T_S(V) := \bigoplus_{i\in\mathbb{Z}} T_S(V)_i, \quad \text{where} \quad T_S(V)_i = \begin{cases} V^{\otimes_S i}, & i \geq 1, \\ S, & i = 0, \\ 0, & i < 0, \end{cases}.$$

For $v_1, \ldots, v_{p+q} \in V$, multiplication is determined $\Bbbk$-bilinearly by

$$(v_1 \otimes_S \cdots \otimes_S v_p)(v_{p+1} \otimes_S \cdots \otimes_S v_{p+q}) = v_1 \otimes_S \cdots \otimes_S v_{p+q} \qquad \text{for all } p, q \geq 1,$$

and by the left and right S-actions when a factor has degree 0.

The displayed decomposition is the tensor grading: $S = T_S(V)_0$ and $V = T_S(V)_1$. Multiplication satisfies $T_S(V)_p T_S(V)_q \subset T_S(V)_{p+q}$ for all $p, q \in \mathbb{Z}$. We also call $T_S(V)$ the tensor algebra generated by V over S. Every element has finitely many non-zero homogeneous components. The unit is 1_S, and associativity follows from Lemma 2.2(3). Write $v_1 \cdots v_d$ for $v_1 \otimes_S \cdots \otimes_S v_d$ in $T_S(V)$. In a quotient algebra (including $T_S(V)$ itself) of $T_S(V)$, the notation $v_1 \cdots v_d$ denotes the image of $v_1 \otimes_S \cdots \otimes_S v_d$.

Last lecture ends here