Complete at least 2 exercises.

Throughout, $\Bbbk$ is a field, and all modules are right modules unless a left action is specified.

Exercise 1.

(1) Let $A = \bigoplus_{d\geq 0} A_d$ be a graded algebra with $S = A_0$ finite-dimensional semisimple.
   i. Prove that $A/A_+ \cong S$ and $\operatorname{rad} A \subset A_+$.
   ii. Prove that if $A_+^N = 0$ for some $N \geq 1$ (for example if $\dim_\Bbbk A < \infty$), then $\operatorname{rad} A = A_+$.
   iii. Show that for $A = \Bbbk[x]$, $\deg x = 1$, one has $\operatorname{rad} A = 0 \neq A_+$.
   iv. Suppose that $\dim_\Bbbk A < \infty$ and that A is generated in degree 1 over S. Prove that

$$\operatorname{rad}^k A = \bigoplus_{d\geq k} A_d, \qquad \operatorname{rad}^k A/\operatorname{rad}^{k+1} A \cong A_k \qquad (k \geq 0).$$

(2) Let $A = \Bbbk A_n$, $P_i := e_{ii}A$ with the inherited grading, and $L_j := e_{jj}A/e_{jj}A_+$ as graded module concentrated in degree 0.
   i. Prove that for $k \geq 0$, $\operatorname{rad}^k P_i = \operatorname{span}_\Bbbk\{e_{im} \mid m \geq i + k\}$.
   ii. Prove that for $0 \leq k \leq n - i$, $\operatorname{rad}^k P_i/\operatorname{rad}^{k+1} P_i \cong L_{i+k}(-k)$ as graded right A-modules, and deduce that

$$P_i \cong P_j(d) \text{ as graded modules} \iff (i, d) = (j, 0).$$

Exercise 2.

(1) Let $A = \Bbbk A_3$, $M = e_{22}A$, and consider $\mathbb{D}(M)_d = \operatorname{Hom}_\Bbbk(M_{-d}, \Bbbk)$ with $(af)(m) = f(ma)$.
   i. Find a homogeneous basis of $\mathbb{D}(M)$, its degrees, and all non-zero actions of matrix units on it.
   ii. Show that $\Bbbk e_{13} \subset Ae_{33}$ is a graded left submodule and that $\mathbb{D}(M) \cong (Ae_{33}/\Bbbk e_{13})(1)$ as graded left A-modules.
   iii. Deduce that $\mathbb{D}(M)$ is not isomorphic to any $(Ae_{jj})(r)$, $j \in \{1, 2, 3\}$, $r \in \mathbb{Z}$.

(2) Let $A = \Bbbk[x, y]/(x^2, y^3)$ with $\deg x = \deg y = 1$, and let $V = A_1$.
   i. Compute the Hilbert series $h_A(t)$.
   ii. Let $R = \ker(V \otimes_\Bbbk V \to A_2)$ and $Q = T_\Bbbk(V)/(R)$. Show that

$$Q \cong \Bbbk[x, y]/(x^2), \qquad h_Q(t) = \frac{1 + t}{1 - t}.$$

   iii. Prove that A is not quadratic, and find the smallest degree in which A and Q differ.

Exercise 3.

(1) For $n \geq 3$ and $Z \subset \{1, \ldots, n - 2\}$ let

$$A_Z = \Bbbk A_n/(e_{k,k+2} \mid k \in Z)$$

with the quotient grading.
   i. Show that

$$\{e_{ii}\}_i \cup \{e_{ij} \mid i < j,\ \nexists\, k \in Z :\ i \leq k \leq j - 2\}$$

   is a basis of A_Z.

ii. Write $A_Z = T_S(V)/(R)$ with explicit S, V, and $R \subset V \otimes_S V$.

(2) Let $S = \Bbbk e_1 \oplus \Bbbk e_2$ and $V = \Bbbk a \oplus \Bbbk b$ with

$$a \in e_1 V e_2, \qquad b \in e_2 V e_1,$$

and $e_i V e_j = 0$ otherwise.

i. Find a basis of $T_S(V)_d$ for all $d \geq 0$, and compute $h_{T_S(V)}(t)$.

ii. Find a basis of $A = T_S(V)/(ab)$ and show that $\dim_{\Bbbk} A < \infty$.

(3) Let A be a locally finite-dimensional graded algebra generated in degree 1 over the finite-dimensional semisimple algebra $S = A_0$.

i. Fix a bimodule ${}_S V_S$ and an S-bimodule isomorphism $\varphi : V \xrightarrow{\sim} A_1$. Prove that

$$h_A(t) = h_{T_S(V)}(t) \quad \Longrightarrow \quad T_S(V) \xrightarrow{\sim} A,$$

where the isomorphism of graded algebras on the right-hand side is induced by id_S and φ.

ii. Consider $\Bbbk[x, z]/(x^2)$ with $\deg x = 1$ and $\deg z = 2$. Show that

$$h_B(t) = h_{T_{\Bbbk}(B_1)}(t) \qquad \text{but} \qquad B \ncong T_{\Bbbk}(B_1)$$

as graded algebras.