You may assume all algebras are finite-dimensional over a field $\Bbbk$. You may attempt the exercises with the additional assumption of $\Bbbk$ being algebraically closed.

Throughout, unadorned tensor product over assumed to be taken over a field, i.e. $\otimes = \otimes_{\Bbbk}$. For a module $X$ over some algebra, denote by $\mathsf{add}(X)$ the full subcategory of the module category consisting of finite direct sums of direct summands of $X$ (up to isomorphism).

**Ex 1.** Suppose that $M, N \in \mathsf{mod}\, A$. Recall that the **projective dimension** of a module $M$ is

$$\operatorname{pdim} M := \inf\{d \geq 0 \mid 0 \to P^d \to \cdots \to P^0 \to M \to 0 \text{ if a projective resolution of } M\}$$

(1) Use induced long exact sequence (from ses) to show that $\operatorname{pdim} M = d$ if, and only if, $\operatorname{Ext}_A^{d+1}(M, X) = 0$ for all $X \in \mathsf{mod}\, A$ and $\operatorname{Ext}_A^d(M, X) \neq 0$ for some $X \in \mathsf{mod}\, A$.

(2) Show that, when $0 \to X \to Y \to Z \to 0$ is a ses, then $\operatorname{pdim} Z \leq \max(\operatorname{pdim} Y, 1 + \operatorname{pdim} X)$, and equality holds when $\operatorname{pdim} Y \neq \operatorname{pdim} X$.

(3) Show that if $\operatorname{Ext}_A^i(M, A) = 0$ for all $i \geq 1$, then $\operatorname{Ext}_A^i(M, N) = 0$ if $N$ is of finite projective dimension.

(4) Let $I$ be a two-sided ideal of $A$ such that $I$ is projective as a right $A$-module, and take $B := A/I$. Show that $\operatorname{Tor}_d^A(A/I, A/I) = 0$ for all $d \geq 1$.

**Ex 2.** Suppose that $M, N \in \mathsf{mod}\, A$.

(1) Show that when $M$ is simple, we have $\operatorname{Hom}_A(\Omega(M), N) \cong \operatorname{Ext}_A^1(M, N) \cong \operatorname{Hom}_A(M, \Omega^{-1}(N))$.

(2) Show that the same holds when $N$ is simple.

(3) Show that when every projective $A$-module is injective, then $\operatorname{Ext}_A^1(M, N) \cong \underline{\operatorname{Hom}}_A(M, N)$ where $\underline{\operatorname{Hom}}_A(M, N)$ is the quotient of $\operatorname{Hom}_A(M, N)$ by all homomorphisms factoring through a projective module.

Let $\underline{\mathsf{mod}} A$ be the category whose objects are the same as that of $\mathsf{mod}\, A$ and morphism sets from $X$ to $Y$ given by $\underline{\operatorname{Hom}}_A(X, Y)$.

(4) Show that we have $M \oplus P \cong M$ in $\underline{\mathsf{mod}} A$ for all projective $A$-module $P$.

(5) Suppose that every projective $A$-module is injective, and vice versa. By considering the syzygy $\Omega(N)$ and an appropriate pullback square, show that every $f \in \operatorname{Hom}_A(M, N)$ corresponds to some $f' \in \operatorname{Hom}_A(M', N)$ such that

- $M \cong M'$ in $\underline{\mathsf{mod}} A$,
- $f'$ is surjective (as $A$-module homomorphism),
- $f$ and $f'$ are in the same class in $\underline{\operatorname{Hom}}_A(M, N)$.

**Ex 3.** Let $C_2$ be the cyclic group of order 2, and $V_4$ be the Klein-4 group. For a finite group $G$, the group algebra $\Bbbk G$ over $\Bbbk$ is the algebra whose underlying $\Bbbk$-space has basis $G$, and the multiplication is given by extending that of $G$.

(1) Compute (with reasoning) bound path algebra that is Morita equivalent to $\mathbb{C}C_2$.

(2) Compute (with reasoning) (with reasoning) bound path algebra that is Morita equivalent to $\Bbbk C_2$, where $\Bbbk$ is an algebraically closed field of characteristic 2.

(3) For a finite-dimensional algebra $A$ and $n \geq 1$ a positive integer, show that the matrix algebra $M_n(A)$ consisting of $n$-by-$n$ matrices with entries in $A$ is Morita equivalent to $A$.

(4) Find an example of an algebra $A$ over a non-algebraically closed field such that $A$ is not Morita equivalent to a bound path algebra. Explain your reasoning.
*Hint*: You may find a previous assignment useful.

(5) Let $\Bbbk$ be a field of characteristic 2. Then $\Bbbk V_4$ has a unique simple $\Bbbk$ (up to isomorphism) where every $g \in V_4$ acts as identity, and $\dim_{\Bbbk} \operatorname{Ext}^1_{\Bbbk V_4}(\Bbbk, \Bbbk) = 2$. Find the Ext-quiver of $\Bbbk V_4$.

(6) Let $\Bbbk$ be a field of characteristic 2. Let $M = M_1 \underset{M_\beta}{\overset{M_\alpha}{\rightrightarrows}} M_2$ be a representation of the Kronecker quiver $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. Find a construction that equip $M$ with a structure of $\Bbbk V_4$-module that preserves indecomposability. (You may use the classification of indecomposable representation over the Kronecker quiver if you want.)

Deadline: 23rd January, 2026
Submission: In lecture or E-mail to (replace at by @) aaron.chan at math.nagoya-u.ac.jp