Ex 1. Suppose $A = \Bbbk Q/I$ is a bounded path algebra.

1. Find an example of finite acyclic quiver Q (with $I = 0$) so that
   - every indecomposable projective A-module is uniserial, but
   - there exists a non-uniserial indecomposable injective A-module.
2. Let e_x be the primitive idempotent associated to a vertex $x \in Q_0$. Describe (with reason) the bounded quiver (Q'', I'') so that $\Bbbk Q''/I'' \cong A/Ae_xA$.
3. Consider the quiver
$$Q = \left(\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \arrow[r, "\beta"] & 2 \end{tikzcd} \right) \qquad \text{with } I = \langle \alpha^2 \rangle.$$
   Describe the radical series and socle series of each indecomposable projective A-module.
4. Show that if $M \twoheadrightarrow N$ is a sujrective A-module homomorphism, then the length of the radical series of M is at most that of N.

Ex 2. Consider the Kronecker quiver $Q = (1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2)$.

1. Let A be the algebra given by
$$\begin{pmatrix} \Bbbk & \Bbbk^2 \\ 0 & \Bbbk \end{pmatrix} = \left\{ \begin{pmatrix} a & (b_1, b_2) \\ 0 & c \end{pmatrix} \mid a, b_1, b_2, c \in \Bbbk \right\}$$
   with multiplication
$$\begin{pmatrix} a & (b_1, b_2) \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & (b_1', b_2') \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & (ab_1' + b_1c', ab_2' + b_1c') \\ 0 & cc' \end{pmatrix}.$$
   Show that $A \cong \Bbbk Q$.
2. Consider the algebra B given by
$$\left\{ \begin{pmatrix} a & (b_1, b_2) \\ 0 & a \end{pmatrix} \mid a, b_1, b_2 \in \Bbbk \right\},$$
   i.e. the subring of A where the diagonal entries are required to be the same. Find bound quiver (Q', I') such that $B \cong \Bbbk Q'/I'$.
3. Find an indecomposable B-module M of length 4 and of Loewy length 3.
4. Explain why every A-module has a natural B-module structure and find an indecomposable B-module that does not arise from an indecomposable A-module in this way.

Ex 3. Consider the truncated polynomial ring $B = \Bbbk[x]/(x^2)$ and let S be its unique simple module $S = \Bbbk y$.

1. Find a basis for the Hom-spaces $\mathrm{Hom}_B(X, Y)$ for $X, Y \in \{B, S\}$. *Note*: One of these spaces have dimension 2, and all other have dimension 1.
2. Show that $\mathrm{End}_B(S \oplus B)$ is isomorphic to the algebra given by
$$\left\{ \begin{pmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, x, y \in \Bbbk \right\}$$
with usual matrix multiplication.
3. Find the bound path algebra presentation of $A := \mathrm{End}_B(S \oplus B)$, i.e. a $\Bbbk$-algebra isomorphism $A \cong \Bbbk Q/I$.

Ex 4. Let A be a finite-dimensional $\Bbbk$-algebra and I an ideal of A. Let $\pi : A \to A/I$ be the natural projection.

1. Show that $\pi_*(M) := M$ defines a functor $\mathrm{mod} A/I \to \mathrm{mod} A$.
2. Explain why π_* is fully faithful, but not dense in general.

Consider a category $\mathcal{C}$ with objects $\{a, a'\}$ and

- $\mathcal{C}(x, x) = \{\mathrm{id}_x\}$ for both $x \in \{a, a'\}$.
- $\mathcal{C}(a, a') = \{f\}$ and $\mathcal{C}(a', a) = \{g\}$
- such that $gf = \mathrm{id}$ and $fg = \mathrm{id}$.

Consider also a category $\mathcal{D}$ with object $\{b\}$ and $\mathcal{D}(b, b) = \{\mathrm{id}_b\}$.

3. Show that $\mathcal{C}$ and $\mathcal{D}$ are equivalent but not isomorphic.

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