Ex 1. Let $A = A_1 \times A_2$ be a direct product of two $\Bbbk$-algebras A_1, A_2. Note that there are algebra homomorphisms $\pi_i : A \to A_i$. Consider the idempotents $e_1 := (1_{A_1}, 0), e_2 := (0, 1_{A_2})$ of A.

1. For $M \in \mod A$, show that $M = M_1 \oplus M_2$ with $M_i = Me_i$ for both $i \in \{1, 2\}$, i.e. $M = Me_1 + Me_2$ with $Me_1 \cap Me_2 = 0$.
2. Show that π_i induces a simple A-module structure on the simple A_i-modules for both $i \in \{1, 2\}$.
3. Show that there exists a natural A_i-action on the direct summand M_i of M (in the notation of (1)). In particular, classify the simple A-modules.

Ex 2. Let A be the ring

$$\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a \in \mathbb{R}, b, c \in \mathbb{C} \right\}.$$

1. Show that A is an $\mathbb{R}$-algebra and find the dimension of A over $\mathbb{R}$.
2. There are two simple A-modules. Describe them.
3. Write down a composition series (i.e. filtration whose subquotients are simples) of the free A-module A_A.

Ex 3. Let A be the following $\Bbbk$-algebra:

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, x, y \in \Bbbk \right\}$$

Note that for every matrix in A, the (1,1)-entry and the (3,3)-entry must be the same. Let e_1 be the idempotent of A given by the matrix with 1 in the (1,1)- and (3,3)-entry and 0 everywhere else; and $e_2 = 1_B - e_1$.

1. Show that both e_1A and e_2A have a unique simple submodules and they are isomorphic.
2. Let S_1 be the simple module in (1). Show that $S_2 := e_2A/S_1 \ncong S_1$.
3. Find the composition series of e_1A and of e_2A.

Ex 4.

1. Consider the linearly oriented $\vec{\mathbb{A}}_5$-quiver:

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5$$

   and the following representation M of $\vec{\mathbb{A}}_5$:

$$\Bbbk \xrightarrow{1} \Bbbk \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \Bbbk^2 \xrightarrow{\left(\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right)} \Bbbk^2 \xrightarrow{(1,1)} \Bbbk$$

   Find the indecomposable decompositions of M in the cases when the matrix $\left(\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right)$ is of rank 1 and of rank 2. Explain in details your reasoning.

   You may use the fact that there are indecomposable modules of the form $U_{i,j}$ for $1 \leq i \leq j \leq 5$ such that

$$U_{i,j}e_x = \begin{cases} \Bbbk & \text{if } i \leq x \leq j; \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } U_{i,j}\alpha_k = \begin{cases} \mathrm{id} & \text{if } i \leq k < j; \\ 0 & \text{otherwise.} \end{cases}$$

2. Consider $A = \Bbbk Q/I$ given by

$$\begin{tikzcd} Q: 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\delta"'] & 2 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] & 4, \end{tikzcd} \quad I = \langle \alpha\beta\gamma - \delta\gamma \rangle$$

   (a) Find a basis for the socle (i.e. maximal semisimple submodule) of the indecomposable projective P_1.
   Hint: The socle is 2-dimensional.

   (b) Show that the radical $\mathrm{rad}P_1 := \mathrm{Ker}(P_1 \twoheadrightarrow S_1)$ of P_1 is not indecomposable. Details your reason.

Deadline: 24th October, 2025
Submission: E-mail to (replace at by @) aaron.chan at math.nagoya-u.ac.jp