Topics in Mathematical Science VIII

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Introduction to quiver representations and homological algebras

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

1 Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M, +)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r, s \in R$ and $m, n \in M$, we have*

- $m \cdot 1 = m$,
- $(m + n) \cdot r = m \cdot r + n \cdot r$,
- $m \cdot (r + s) = m \cdot r + m \cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i\in I}$ such that for all $x \in F$, $x = \sum_{i\in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $\theta(R)$ being in the center $Z(A) := \{z \in A \mid za = az\ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_\Bbbk A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a $\mathbb{Z}$-algebra.*

The matrix ring $M_n(R)$ given by n-by-n matrices with entries in R is an R-algebra.

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. Most of the time, we will also assume $\Bbbk$ is algebraically closed for simplicity. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the *characteristic* of R, denoted by $\operatorname{char} R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R-module with basis G, i.e. every $a \in A$ can be written as the formal R-linear combination $\sum_{g \in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R-linearly) to a ring multiplication on $R[G]$, making A an R-algebra.*

Example 1.6. *Recall that the direct product of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A\times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R-algebras, then $A \times B$ is also an R-algebra.*

Example 1.7. *Suppose that A is a $\Bbbk$-algebra and B is a $\Bbbk$-subspace of A containing 1_A and closed under multiplication. Then B is also a $\Bbbk$-algebra. We call such a B a subalgebra of A. For a concrete example, the space of diagonal matrices forms a subalgebra of $M_n(\Bbbk)$.*

Definition 1.8. *A map $f : M \to N$ between right R-modules M, N is a homomorphism if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R-action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\operatorname{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. We also write $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$.*

Lemma 1.9. *$\operatorname{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \operatorname{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\operatorname{Hom}_R(M, N)$ is an R-module, namely, for a homomorphism $f : M \to N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.*

Definition 1.10. *$\operatorname{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M. We call this the endomorphism ring of M.*

Lemma 1.11. *If A is an R-algebra over a commutative ring R, then any right A-module is also an R-module, and $\operatorname{Hom}_A(M, N)$ is also an R-module (hence, $\operatorname{End}_R(M)$ is an R-algebra).*

Example 1.12. *$A \cong \operatorname{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R-algebras if A is an R-algebra). Note that if we work with left modules, then $A \cong \operatorname{End}_A({}_AA)^{\mathrm{op}}$, where $(-)^{\mathrm{op}}$ denotes the opposite ring given by the same underlying set with reverse direction of multiplication, i.e. $a \cdot_{\mathrm{op}} b := b \cdot a$.*

Recall that an R-module M is *finitely generated* if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x\in X} x r_x$ for some $r_x \in R$.

Notation. *We write $\mathsf{mod}\, A$ for the collection of all finitely generated right A-modules.*

2 Indecomposable modules and Krull-Schmidt property

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \dots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + N_{\hat{j}} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written uniquely as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable.

Convention. *We write $(n_1, \dots, n_r)$ instead of $n_1 + \cdots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \cdots \oplus N_r$.*

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose that R_R is indecomposable as an R-module. If F is a free R-module of rank n, then $R^{\oplus n} := R \oplus R \oplus \cdots \oplus R$ (with n copies of R) is a decomposition of F.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1 \le j \le n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A-submodule of V other than 0 or V itself. Hence, V is an indecomposable A-module. In particular, the n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.*

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \ge 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_{\Bbbk}(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

Consider a 2-dimensional space $V = \Bbbk\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By definition $(av_1 + bv_2)x = (a+b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules. If, on the contrary, V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. But v_2 must be contained in any submodule of V, hence, we have $v_2 \in U_1 \cap U_2$, i.e. $U_1 \cap U_2 \neq 0$ – a contradiction not decomposability.

Proposition 2.5. *There is a canonical R-module isomorphism*

$$\begin{array}{ccc} \mathrm{Hom}_A(\bigoplus_{j=1}^m M_j, \bigoplus_{i=1}^n N_i) & \xrightarrow{\cong} & \bigoplus_{i,j} \mathrm{Hom}_A(M_j, N_i) \\ f & \longmapsto & (\pi_i f \iota_j)_{i,j} \end{array}$$

where $\iota_j : N_j \to \bigoplus_j N_j$ is the canonical inclusion for all j and $\pi_i : \bigoplus_i M_i \to M_i$ is the canonical projection for all i.

One can think of the right-hand space above as the space of m-by-n matrix with entries in each corresponding Hom-space.

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent. From the previous proposition, we see that for a decomposition $M = N_1 \oplus N_2$, we have idempotents

$$e_i : M \xrightarrow{\pi_i} N_i \xrightarrow{\iota_i} M$$

for both $i = 1, 2$. Hence, being decomposable implies existence of multiple idempotents; this turns out characterise indecomposability completely.

Proposition 2.6. *Let A be a finite-dimensional algebra and M be a finite-dimensional non-zero A-module. Then the following hold.*

(1) *(Fitting's lemma) For any $f \in \mathrm{End}_A(M)$, there exists $n \geq 1$ such that $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.*

(2) *The following are equivalent.*
- *M is indecomposable.*
- *The endomorphism algebra $\mathrm{End}_A(M)$ does not contain any idempotents except 0 and id_M.*
- *Every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*
- *$\mathrm{End}_A(M)$ is local (see below).*

Remark 2.7. It is known that if M is only artinian or only noetherian, then Fitting's lemma (and hence part (2)) fails. Nevertheless, in general, the proposition still hold for M that is both artinian and noetherian.

Let us briefly recall various characterisation of local rings.

Definition 2.8. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.*

Remark 2.9. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit (right) inverses.

Lemma 2.10. *The following are equivalent for a finite-dimensional algebra A.*

- *A is local (i.e. has a unique maximal right ideal).*
- *Non-invertible elements of A form a two-sided ideal.*
- *For any $a \in A$, one of a or $1 - a$ is invertible.*
- *0 and 1_A are the only idempotents of A.*
- *$A/J(A) \cong \Bbbk$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.*

Example 2.11. *Consider the upper triangular 2-by-2 matrix ring*

$$A = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1\leq i\leq j\leq 2} \,\middle|\, \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Let $M = \{(x, y) \in \Bbbk^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \mathrm{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \Bbbk$. Then being an A-module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f\left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \Bbbk$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\mathrm{End}_A(M) \cong \Bbbk$ which is clearly a local algebra. Hence, M is indecomposable.

A natural question is to ask when is a decomposition of modules, if it exists, unique up to permuting the direct summands.

Definition 2.12. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^{m} M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^{n} N_j$ implies that $m = n$ and there is a permutation σ such that $M_i \cong N_{\sigma(i)}$ for all $1 \leq i \leq m$. $\mathsf{mod}\, A$ is said to be Krull-Schmidt if every (finitely generated) A-module M admits a unique indecomposable decomposition.*

Theorem 2.13. *For a finite-dimensional algebra A, $\mathsf{mod}\, A$ is Krull-Schmidt.*

Remark 2.14. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Theorem 2.15 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \leq i \leq m$, then the decomposition of M is unique.*

Remark 2.16. Some people refer to this result as Krull-Remak-Schmidt theorem.

3 Simple modules, Schur's lemma

Definition 3.1. *Let M be an R-module.*

(1) *M is simple if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 3.2. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Remark 3.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 3.4. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Then the row-space representation V is an n-dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.*

Example 3.5. *The ring of dual numbers is $A := \Bbbk[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = AxA$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by $\Bbbk v$ where $v = 1+(x)$. If A is semisimple, then the 1-dimensional space $\Bbbk v$ is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \Bbbk$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\Bbbk v \cong (x)$, a contradiction.*

Lemma 3.6. *S is a simple A-module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).*

Let us see how one can find a simple module.

Definition 3.7. *Let M be an A-module and take any $m \in M$. The annihilator of m (in A) is the set $\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\}$.*

Note that $\mathrm{Ann}_A(m)$ is a right ideal of A - hence, a right A-module.

Lemma 3.8. *For a simple A-module S and any non-zero $m \in S$, we have $S \cong A/\mathrm{Ann}_A(m)$ as A-module. In particular, if A is finite-dimensional, then every simple A-module is also finite-dimensional.*

Suppose I is a two-sided ideal of A. Then we have a quotient algebra $B := A/I$. For any B-module M, we have a canonical A-module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the *restriction of M along the algebra homomorphism $A \twoheadrightarrow A/I$*.

Lemma 3.9. *Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \mathsf{mod}\, B$ is simple, then S is also simple as A-module*

Proof This follows from the easy observation that any a B-submodule of S_B is also a A-submodule of S_A under restriction. □

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 3.10 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.11. Note that if A is an R-algebra, then the division ring appearing is also an R-algebra (since it is the endomorphism ring of an A-module). In particular, if R is an algebraically closed field $\Bbbk = \overline{\Bbbk}$, then any division $\Bbbk$-algebra is just $\Bbbk$ itself.

Proof The claim is equivalent to saying that any $f \in \mathrm{Hom}_A(S,T)$ is either zero or an isomorphism. Since $\mathrm{Im}(f)$ is a submodule of T, simplicity of T says that $\mathrm{Im}(f) = 0$, i.e. $f = 0$, or $\mathrm{Im}(f) \cong T$. In the latter case, we can consider $\mathrm{Ker}(f)$, which is a submodule of S, so by simplicity of S it is either 0 or S itself. But this cannot be S as this means $f = 0$, hence, $\mathrm{Im}(f) \cong T$ implies that $\mathrm{Ker}(f) = 0$, i.e. f is an isomorphism. □

Example 3.12. *In Example 2.11, we showed that the upper triangular 2-by-2 matrix ring A has a 2-dimensional indecomposable module $P_1 = \{(x,y) \mid x,y \in \Bbbk^2\}$ given by 'row vectors'. It is straightforward to check that there is a 1-dimensional (hence, simple) submodule given by $S_2 := \{(0,y) \mid y \in \Bbbk^2\}$.*

Consider the module $S_1 := P_1/S_2$. This is a 1-dimensional (simple) module spanned by, say, w with A-action given by

$$w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := wa.$$

Consider a homomorphism $f \in \mathrm{Hom}_A(S_1, S_2)$. This will be of the form $w \mapsto (0,y)$ for some $y \in \Bbbk$ and has to satisfy

$$(0,ya) = (0,y)a = f(wa) = f(w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = f(w) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (0,y)c = (0,yc)$$

for any $a,b,c \in \Bbbk$. Hence, we must have $y = 0$, which means that $f = 0$. In particular, by Schur's lemma $S_1 \not\cong S_2$.

Lemma 3.13. *Suppose that S is a simple A-module. Consider a semisimple A-module $M = S_1 \oplus \cdots \oplus S_n$ with $S_i \cong S$ for all i. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_n(D)$, where $D := \mathrm{End}_A(S)$.*

Proof We have canonical inclusion $\iota_j : S_j \hookrightarrow M$ and projection $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this is an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \le i,j \le r},$$

which is clearly injective. Conversely, for $(a_{i,j})_{1\le i,j \le r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \overset{a_{i,j}}{\to} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. □

Example 3.14. *For a tautological example, take $A = \Bbbk$ to be just a field. Then we have a 1-dimensional simple A-module $S = \Bbbk$ with $\mathrm{End}_A(S^{\oplus n}) = \mathrm{Mat}_n(\mathrm{End}_A(\Bbbk)) = \mathrm{Mat}_n(\Bbbk)$. Note that now we have an n-dimensional simple $\mathrm{Mat}_n(\Bbbk)$-module (given by the row vectors).*

4 Quiver and path algebra

Definition 4.1. *A (finite) quiver is a datum $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ for finite sets Q_0, Q_1. The elements of Q_0 are called vertices and those of Q_1 are called arrows. The source (resp. target)of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).*

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 4.2. *Let Q be a quiver.*

- *A trivial path on Q is a "stationary walk at i", denoted by e_i for some $i \in Q_0$.*
- *A path of Q is either a trivial path or a word $\alpha_1\alpha_2 \cdots \alpha_\ell$ of arrows with $s(\alpha_i) = t(\alpha_{i+1})$.*

The source and target functions extend naturally to paths, with $s(e_i) = i = t(e_i)$. Two paths p, q can be concatenated to a new one pq if $t(p) = s(q)$; note that our convention is to read *from left to right*.

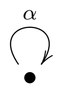
Definition 4.3. *The path algebra $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra whose underlying vector space is given by $\bigoplus_{p:\text{paths of } Q} \Bbbk p$, with multiplication given by path concatenation. That is $x \in \Bbbk Q$ is a formal linear combinations of paths on Q.*

Note that $e_ie_j = \delta_{i,j}e_i$, where $\delta_{i,j} = 1$ if $i = j$ else 0. In other words, e_i is an *idempotent* of the path algebra $\Bbbk Q$. Moreover, we have an idempotent decomposition

$$1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$$

of the unit element of $\Bbbk Q$.

Example 4.4. *Consider the one-looped quiver, a.k.a. Jordan quiver,*

$Q = ($  $)$

Then $\Bbbk Q$ has basis $\{\alpha^k \mid k \geq 0\}$ (note that the trivial path at the unique vertex is the identity element). Then $\Bbbk Q \cong \Bbbk[x]$.

An *oriented cycle* is a path of the form $v_1 \to v_2 \to \cdots v_r \to v_1$, i.e. starts and ends at the same vertex. If Q does not contain any oriented cycle, we say that it is *acyclic*.

Proposition 4.5. *$\Bbbk Q$ is finite-dimensional if, and only if, Q is finite acyclic.*

Proof If there is an oriented cycle c, then $c^k \in \Bbbk Q$ for all $k \geq 0$, and so $\Bbbk Q$ is infinite-dimensional. Otherwise, there are only finitely many paths on Q. □

Example 4.6. *Consider the linearly oriented $\vec{\mathbb{A}}_n$-quiver*

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then the path algebra $\Bbbk Q$ has basis $\{e_i, \alpha_{j,k} \mid 1 \leq i \leq n, 1 \leq j \leq k \leq n\}$, where $\alpha_{j,k} := \alpha_j\alpha_{j+1} \cdots \alpha_k$.

Consider the upper triangular n-by-n matrix ring

$$\begin{pmatrix} \Bbbk & \Bbbk & \cdots & \Bbbk \\ 0 & \Bbbk & \cdots & \Bbbk \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq n} \;\middle|\; \begin{matrix} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{matrix} \right\}.$$

Denote by $E_{i,j}$ the elementary matrix whose entries are all zero except at (i,j) where it is one. This ring is isomorphic to $\Bbbk Q$ via $E_{i,i} \mapsto e_i$ and $E_{i,j} \mapsto \alpha_{i,j-1}$ for $1 \leq j < k \leq n$.

From now on, we will focus in the following setting.

Assumption 4.7. (1) *Quivers are finite (i.e. finitely many vertices and arrows).*

(2) *Representations (equivalently, modules) are finite-dimensional.*

5 Duality

For a quiver Q, the opposite quiver Q^{op} has the same set of vertices with the reverse direction of arrows, i.e. $Q_0^{\mathrm{op}} = Q_0, Q_1^{\mathrm{op}} = Q_1, s_{Q^{\mathrm{op}}} = t_Q$, and $t_{Q^{\mathrm{op}}} = s_Q$.

Exercise 5.1. *Show that there is a canonical isomorphism $(\Bbbk Q)^{\mathrm{op}} \cong \Bbbk(Q^{\mathrm{op}})$.*

Let M be a finite-dimensional A-module. Then we have a dual space

$$D(M) := M^* := \mathrm{Hom}_{\Bbbk}(M, \Bbbk),$$

which has a natural A^{op}-module structure, namely, $(a \cdot f)(m) := f(ma)$ for any $a \in A, f \in M^*, m \in M$. Moreover, for an A-module homomorphism $\theta : M \to N$, we have also an A^{op}-module homomorphism $\theta^* : N^* \to M^*$ with $\theta^*(f)(m) = f(\theta(m))$.

Lemma 5.2. *There is a $\Bbbk$-vector space isomorphism $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$.*

Proof Just a straightforward check that $(\theta^*)^* = \theta$. □

We note as a fact that D preserves indecomposability of (finite-dimensional) modules. This can be seen using the fact that $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$ and can be upgraded to an algebra isomorphism for the case when $N = M$; then uses characterisation of indecomposable module by local endomorphism ring.

Example 5.3. *The* left *A-module ${}_AA$ yields a right A-module structure on $D(A)$. More generally, suppose we have a left ideal Ae of A for some element $e \in A$, then $D(Ae)$ is a right ideal of A.*

Remark 5.4. There is another natural duality, which we will not used, between $\mathsf{mod}\, A$ and $\mathsf{mod}\, A^{\mathrm{op}}$ given by sending M to $\mathrm{Hom}_A(M, A)$. In general, this duality is different from the $\Bbbk$-linear dual unless A is a so-called *symmetric algebra*, meaning that $A \cong DA$ as bimodule; in which case, $\mathrm{Hom}_A(-, A)$ dual is naturally isomorphic to D (as functors).

6 Representations of quiver

Definition 6.1. *A $\Bbbk$-linear* representation *of Q is a datum $(\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$ where M_i is a $\Bbbk$-vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is $\Bbbk$-linear map for each $\alpha \in Q_1$.*

Such a representation is finite-dimensional *if $\dim_{\Bbbk} M_i < \infty$ for all $i \in Q_0$.*

Notation. *For a representation M of Q, we take $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$ for a path $p = \alpha_1 \cdots \alpha_\ell$.*

It is easy to notice that every representation of Q is equivalent to a $\Bbbk Q$-module, namely,

$$\text{representation } (\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1}) \leftrightarrow \begin{array}{c} \Bbbk Q\text{-module } \prod_{i \in Q_0} M_i \\ \text{s.t. } \sum_{p:\text{path}} \lambda_p p \text{ acts as } \sum_p \lambda_p M_p. \end{array}$$

Example 6.2 (Simple). *For $x \in Q_0$, denote by S_x (or $S(x)$) the representation given by putting a 1-dimensional space on x, zero on all other vertices, and zero on all arrows. This corresponds to a 1-dimensional $\Bbbk Q$-module and so we call it the simple at x.*

Note: at this stage, it is not clear if these are all the simple $\Bbbk Q$-modules (up to isomorphism) yet.

Example 6.3 (Projective). *For $x \in Q_0$, denote by P_x (or $P(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=x,\\ t(p)=y}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p\alpha=q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the projective at x. This corresponds to the right ideal $e_x\Bbbk Q$ of $\Bbbk Q$.

Example 6.4 (Injective). *Dual to the projective module construction, for $x \in Q_0$, denote by I_x (or $I(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p:\text{path with}\\ s(p)=y,\\ t(p)=x}} \Bbbk p, \qquad \text{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p=\alpha q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\text{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the injective at x. This corresponds to the dual of the left ideal generated by e_x, i.e. $D(\Bbbk Q e_x)$.

Example 6.5. *The representation of $Q = \vec{\mathbb{A}}_n$ given by*

$$U_{i,j} := 0 \to \cdots 0 \to \Bbbk \xrightarrow{\text{id}}\to \cdots \xrightarrow{\text{id}} \Bbbk \to 0 \to \cdots \to 0$$

with a copy of $\Bbbk$ on vertices $i, i+1, \dots, j$ is the uniserial $\Bbbk Q$-module corresponding to the column space (under the isomorphism of $\Bbbk Q$ with the lower triangular matrix ring) with non-zero entries in the k-th row for $i \le k \le j$.

Example 6.6. *Let Q be the Jordan quiver with unique arrow α. Then a representation of Q is nothing but an n-dimensional vector space equipped with a linear endomorphism, equivalently, an n-by-n matrix.*

Definition 6.7. *A homomorphism $f : M \to N$ of ($\Bbbk$-linear) quiver representations $M = (M_i, M_\alpha)_{i,\alpha}$ and $N = (N_i, N_\alpha)_{i,\alpha}$ is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' actions, i.e. we have a commutative diagram*

$$\begin{array}{ccc} M_i & \xrightarrow{f_i} & N_i \\ {\scriptstyle M_\alpha}\downarrow & & \downarrow{\scriptstyle N_\alpha} \\ M_j & \xrightarrow[f_j]{} & N_j \end{array}$$

for all arrows $\alpha : i \to j$ in Q.

A homomorphism $f = (f_i)_{i\in Q_0} : M \to N$ of quiver representations is injective, resp. surjective, resp. an isomorphism, if every f_i is injective, resp. surjective, resp. an isomorphism, for all $i \in Q_0$.

Example 6.8. *Let Q be the Jordan quiver. Recall that a representation of Q is equivalent to a choice of n-by-n matrix M_α. By definition, the isomorphism class of such a representation is given by the conjugacy classes of M_α. If we assume $\Bbbk$ is algebraically closed, then a representative of the isomorphism class of M_α is given by the Jordan normal form of M_α. That is, M_α can be block-diagonalise into Jordan blocks $J_{m_1}(\lambda_1), \dots, J_{m_l}(\lambda_l)$, where $J_m(\lambda)$ is the m-by-m Jordan block with eigenvalue $\lambda \in \Bbbk$.*

Proposition 6.9. *There is an isomorphism between the category of representations of Q and* $\mathsf{mod}\,\Bbbk Q$*, where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \prod_{i \in Q_0} M_i$ with $\Bbbk Q$-action given by (linear combinations of compositions of) M_α's, and isomorphism classes of Q-representations correspond to isomorphism classes of $\Bbbk Q$-modules.*

7 Idempotents

Recall that an *idempotent* of an algebra A is an element x with $x^2 = x$.

The right A-modules of the form eA and $D(Ae)$ for an idempotent $e \in A$ are of central importance in representation theory and in homological algebra.

Lemma 7.1. *The the following hold for any idempotent $e \in A$.*

(1) (Yoneda's lemma) $\mathrm{Hom}_A(eA, M) \cong Me$ *as a $\Bbbk$-vector space for all $M \in \mathsf{mod}\, A$.*

(2) *There is an isomorphism of rings* $\mathrm{End}_A(eA) \cong eAe$.

Proof For (1), check that $\mathrm{Hom}_A(eA, M) \ni f \mapsto f(e) = f(1)e \in Me$ defines a $\Bbbk$-linear map with inverse $me \mapsto (ea \mapsto mea)$. (2) follows from (1) by putting $M = eA$ with straightforward check of correspondence of multiplication on both sides. $\square$

Remark 7.2. Under the isomorphism $A \cong \mathrm{End}_A(A)$, an idempotent e of A corresponds to the 'project to direct summand $P = eA$ endomorphism', i.e. $A \twoheadrightarrow P \hookrightarrow A$. This is compatible with Yoneda lemma (think about this!) which says that there is a vector space isomorphism $fAe \cong \mathrm{Hom}_A(eA, fA)$ for any idempotents e, f.

Lemma 7.3. *For idempotents $e, f \in A$, we have $eA \cong fA$ as right A-module if and only if $f = ueu^{-1}$ for some unit $u \in A^\times$.*

Given an idempotent $e = e^2 \in A$ in an algebra A, then eA and $(1-e)A$ are both right ideal of A. Since $e(1-e) = 0 = (1-e)e$, we have $eA \cap (1-e)A = 0$, which means that $A \cong eA \oplus (1-e)A$ as right A-module. In particular, in the setting of the above lemma, we have that $eA \cong fA$ and $(1-e)A \cong (1-f)A$ by Krull-Schmidt property.

Definition 7.4. *Two idempotents e, f are orthogonal if $ef = 0 = fe$. An idempotent e is primitive if $e \neq f + f'$ for some orthogonal (pair of) idempotents f, f'.*

It follows from the definition of primitivity that

eA and $D(Ae)$ are indecomposable A-modules for a primitive idempotent e.

Example 7.5. *The trivial paths e_x for $x \in Q_0$ is (by design) a primitive idempotent of the path algebra $\Bbbk Q$, and $1 = \sum_{x \in Q_0} e_x$ is an orthgonal decomposition of primitive idempotents. Hence, we have a decomposition*

$$\Bbbk Q \cong \bigoplus_{x \in Q_0} e_x \Bbbk Q = \bigoplus_{x \in Q_0} P_x \text{ and } D(\Bbbk Q) \cong \bigoplus_{x \in Q_0} D(\Bbbk Q e_x) \cong \bigoplus_{x \in Q_0} I_x.$$

8 Composition series, Jordan-Hölder Theorem

Definition 8.1. *Let A be a $\Bbbk$-algebra and $M \in A\,\mathsf{mod}$. A composition series of M is a finite chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that M_i/M_{i-1} is simple for all $1 \leq i \leq \ell$. The number ℓ here is the length of the composition series. The module M_i/M_{i-1} for each $1 \leq i \leq \ell$ are called the composition factors of the series.

Theorem 8.2 (Jordan-Hölder Theorem). *Any two composition series have the same length and the multi-sets of their composition factors (up to isomorphisms) are the same.*

We omit the proof. The strategy is basically by induction on the length of series.

Remark 8.3. Jordan-Hölder theorem holds as long as a module, regardless of what kind of algebra, has a (finite) composition series; this condition is actually equivalent to saying that it is noetherian and artinian.

Remark 8.4. The Jordan-Hölder theorem may not hold if one relaxes the form of composition factors from simple modules to something else. There are a few active research themes, including one related to quasi-hereditary algebras, that are stemmed from this.

Lemma 8.5. *Let M be a finite-dimensional right A-module. Then M has a composition series.*

Proof Induction on $\dim_\Bbbk M$, at each step choose a maximal submodule (i.e. a submodule whose quotient is simple). □

Example 8.6. *Let $A = \Bbbk\vec{\mathbb{A}}_n$. Then the module $U_{i,j}$ has a composition series*

$$0 \subset U_{j,j} \subset U_{j-1,j} \subset \cdots \subset U_{i+1,j} \subset U_{i,j}$$

with composition factors $S_k = U_{k,j}/U_{k+1,j}$ for $i \leq k \leq j$. Note that this composition series is unique - such kind of modules are called uniserial.

Lemma 8.7. *If $M \in \mathsf{mod}\,A$ and $N \subset M$ is a submodule, then there is a composition series $(M_i)_{0 \leq i \leq \ell}$ so that $N = M_k$ for some $0 \leq k \leq \ell$.*

Proof N has a composition series, say, of length k, so we take that as the first k terms of the required composition series of M. On the other hand, M/N also has a composition series, and since every submodule of M/N is of the form L/N (for a submodule U of M/N, take $L := \{m \in M \mid m+N \in U\}$; it is routine to check that this is an inverse operation as quotienting N on the submodules of M that contains N), a composition series of M/N is of the form $(L_i/N)_{0 \leq i \leq r}$. Now take $M_{k+i} = L_i$. □

Proposition 8.8. *Suppose A is a $\Bbbk$-algebra such that A_A has a composition series. Then there are only finitely many simple A-modules up to isomorphisms, and they all appear in the form A/I for some A-submodule I of A.*

Note that while this does not require A to be finite-dimensional, it requires A_A to be of finite length (equivalently, noetherian and artinian).

Proof The final clause of the claim is just restating Lemma 3.8: any simple S is given by $A/\operatorname{Ann}_A(m)$ for any non-zero $m \in S$. Now fix such an S and $I := \operatorname{Ann}_A(m)$. Since A has a composition series, I also have one by Lemma 8.7 so that the series ends with $I \subset A$. Since this is possible for any simple S, it follows from Jordan-Hölder theorem that all simple modules other than S must appear as composition factors of I.

Since composition series is a finite chain, there must be finitely many composition factors - hence, the simple modules of A must be finite. □

9 Semisimplicity and Artin-Wedderburn theorem

In order to obtain all (isomorphism classes of) simple A-modules - or equivalently maximal right A ideal (i.e. maximal submodules of A_A) - for a finite-dimensional $\Bbbk$-algebra A, we will use the following.

Definition 9.1. *Let A be a $\Bbbk$-algebra and $M \in \mathsf{mod}\, A$.*

(1) *The (Jacobson) radical $\operatorname{rad}(A)$ (sometimes also written as $J(A)$) of A is the intersection of all maximal right ideals (i.e. maximal A-submodules) of A.*

(2) *A is semisimple if $\operatorname{rad}(A) = 0$.*

Example 9.2. *For $A = \Bbbk Q$ of a finite quiver Q and $x \in Q_0$. The projective P_x at x contains a submodule spanned by all paths starting from x with length at least 1. This is a maximal submodule of P_x since the cokernel of the natural embedding to P_x is a one-dimensional module spanned by the coset of e_x – in particular, this simple module is isomorphic to S_x. Thus, we have $\operatorname{rad}(A) = \Bbbk Q_{\geq 1}$ the submodule of A_A spanned by all paths of length at least 1.*

Proposition 9.3. *Suppose A_A has a composition series. Then the following holds for the Jacobson radical $\operatorname{rad}(A)$.*

- *$\operatorname{rad}(A)$ is the intersection of finitely many maximal right ideals.*
- *$\operatorname{rad}(A)$ is the intersection of all two-sided ideals $\operatorname{Ann}_A(S) := \{a \in A \mid ma = 0 \forall m \in S\}$, in other words*
$$\operatorname{rad}(A) = \{a \in A \mid Sa = 0 \text{ for all simple } S\}.$$
- *$\operatorname{rad}(A)$ is a two-sided ideal of A.*
- *$\operatorname{rad}(A)^\ell = 0$ for ℓ at most the length of A_A.*
- *$(A/\operatorname{rad}(A))_{A/\operatorname{rad}(A)}$ is a semisimple (as a module).*
- *A_A is a semisimple (as a module) if, and only if, $\operatorname{rad}(A) = 0$ (i.e. A semisimple as an algebra).*

Proof omitted. We note that all of these claims do make use of the Jordan-Hölder theorem.

Example 9.4. (1) *Direct product of two semisimple algebras is semisimple.*

(2) *$A = \operatorname{Mat}_n(D)$ with D a division $\Bbbk$-algebra is a semisimple $\Bbbk$-algebra. We have decomposition $A_A \cong V^{\oplus n}$ into n copies of n-dimensional simple module*
$$V = \{(v_i)_{1 \leq i \leq n} \mid v_i \in D \;\forall i\}.$$

(3) *$A := \Bbbk[x]/(x^n)$ is not semisimple for any $n \geq 2$ as it has a non-trivial (unique) maximal ideal $\operatorname{rad}(A) = (x)$.*

Theorem 9.5 (Artin-Wedderburn theorem). *Let A be a finite-dimensional $\Bbbk$-algebra and let r be the number of isoclasses of simple A-modules, say, with representatives $S_1, \ldots, S_r$. Let $D_i := \operatorname{End}_A(S_i)$ be the division $\Bbbk$-algebra given by endomorphism of the simple module S_i. Then there is an isomorphism of $\Bbbk$-algebras*
$$A/\operatorname{rad}(A) \cong \operatorname{Mat}_{n_1}(D_1) \times \cdots \times \operatorname{Mat}_{n_r}(D_r).$$

As before, if we work over algebraically closed field $\Bbbk = \overline{\Bbbk}$, then all the D_i's are just $\Bbbk$.

Proof Let $B := A/\operatorname{rad}(A)$. By definition of $\operatorname{rad}(A)$, the A-module $A/\operatorname{rad}(A)$ is semisimple, and any A-submodule M of $A/\operatorname{rad}(A)$ satisfies $M\operatorname{rad}(A) = 0$. Hence, $M = M/M\operatorname{rad}(A)$ is naturally a B-module and $\operatorname{End}_B(M) \cong \operatorname{End}_A(M)$ (even as algebras!).

By Lemma 7.1, we have $B \cong \mathrm{End}_B(B)$. Since B is semisimple, the B_B is a semisimple B-module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B-modules. Hence, it follows from Schur's lemma and its consequence (Lemma 3.10 and Lemma 3.13) that

$$B \cong \mathrm{End}_B(B) \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r),$$

where $D_i := \mathrm{End}_B(S_i)$ for all $1 \leq i \leq r$. This completes the proof. $\square$

Corollary 9.6. *For any finite-dimensional* $\Bbbk$*-algebra* A*, let* $\mathrm{Sim}(A)$ *be the set of isomorphism-class representatives of simple* A*-modules. Then there is a one-to-one correspondence*

$$\begin{array}{ccl} \mathrm{Sim}(A) & \xleftrightarrow{1:1} & \mathrm{Sim}(A/\mathrm{rad}(A)) \\ S & \longmapsto & \overline{S} := S/S\,\mathrm{rad}(A) \\ & & (= S \text{ as underlying vector space}) \\ \mathrm{res}T & \longleftarrow & T \end{array}$$

where $\mathrm{res}T$ *is the restriction of* T *along* $A \twoheadrightarrow A/\mathrm{rad}(A)$*.*

Example 9.7. *Suppose that* Q *is finite acyclic, i.e.* $\Bbbk Q$ *is finite-dimensional. Since* $\mathrm{rad}(\Bbbk Q)$ *is spanned by all non-trivial paths,* $\Bbbk Q/\mathrm{rad}(\Bbbk Q)$ *is just the semisimple* $\Bbbk Q$*-module* $\bigoplus_{i \in Q_0} S_i$*. In particular, the Artin-Wedderburn decomposition reads*

$$\Bbbk Q \cong \Bbbk \times \cdots \times \Bbbk$$

with one copy of $\Bbbk$ *for each* $i \in Q_0$ *on the right-hand side. Moreover, every simple* $\Bbbk Q$*-module is isomorphic to one of* S_i *for* $i \in Q_0$*.*

Exercise 9.8. *Show that when* Q *is the Jordan quiver, then* $\Bbbk Q$ *has infinitely many simple modules and that* $\mathrm{rad}(\Bbbk Q) = 0$*.*

10 Radical and socle

Definition 10.1. *The radical of an A-module M is* $\mathrm{rad}(M) := M\,\mathrm{rad}(A)$. *In general, take* $\mathrm{rad}^0(M) := M$ *and denote by* $\mathrm{rad}^{k+1}(M) := \mathrm{rad}(\mathrm{rad}^k(M)) = \mathrm{rad}^k(M)\,\mathrm{rad}(A)$ *for all* $k \geq 0$.

Successively taking the radical yields a series:

$$0 \subset \mathrm{rad}^\ell(M) \subset \cdots \subset \mathrm{rad}(M) \subset M$$

This is called the radical series. The quotient $M/\mathrm{rad}(M)$ *is called the top of* M*, and is denoted by* $\mathrm{top}(M)$.

Proposition 10.2. *The following hold for* $M \in \mathsf{mod}\,A$.

(1) $\mathrm{rad}(M)$ *is the intersection of all maximal submodules of* M.

(2) $\mathrm{top}(M) := M/\mathrm{rad}(M)$ *is the maximal semisimple quotient of* M.

(3) $\mathrm{rad}(M \oplus N) = \mathrm{rad}(M) \oplus \mathrm{rad}(N)$.

(4) *If* $f : M \to N$ *is a surjective A-module homomorphism, then* $f(\mathrm{rad}\,M) = \mathrm{rad}\,N$.

(5) *(Nakayama's Lemma, special case) For a submodule* $N \subset M$, $(N + \mathrm{rad}(M) = M) \Rightarrow N = M$.

Proof omitted; this follows the same kind of arguments as in the case for $\mathrm{rad}(A)$.

Example 10.3. *Let A be a finite-dimensional algebra. Suppose that e is a primitive idempotent, i.e.* $P := eA$ *is an indecomposable A-module. Since* $A = P \oplus Q$ *(by taking* $Q := (1-e)A$*), we have*

$$\mathrm{rad}(P) \oplus \mathrm{rad}(Q) = \mathrm{rad}(P \oplus Q) = \mathrm{rad}(A).$$

Since P and Q has no common (non-trivial) submodule, we get that

$$A/\mathrm{rad}(A) = \frac{P \oplus Q}{\mathrm{rad}(P \oplus Q)} = P/\mathrm{rad}(P) \oplus \frac{Q}{\mathrm{rad}(Q)}.$$

Thus, it follows from Corollary 9.6 that $P/\mathrm{rad}(P)$ *is a simple module and that every simple A-module arises this way. In other words, let* $\mathrm{PIM}(A)$ *be the set of isoclass (=isomorphism class) representatives of indecomposable direct summands of A, then we have a correspondence*

$$\begin{array}{ccc} \mathrm{PIM}(A) & \overset{1:1}{\longleftrightarrow} & \mathrm{Sim}(A) \\ P & \longmapsto & P/\mathrm{rad}(P) \end{array} \tag{10.1}$$

For a simple A-module S, denote by P_S the corresponding direct summand P of A under the correspondence (10.1).

There is a construction dual to $\mathrm{rad}(M)$.

Definition 10.4. *The socle of an A-module M is* $\mathrm{soc}(M)$*, which is defined as the maximal semisimple submodule of M. More generally, take* $\mathrm{soc}^0(M) = 0$ *and for* $k \geq 0$*, let* $\mathrm{soc}^{k+1}(M)$ *to be the submodule of M generated by the lift of* $\mathrm{soc}(M/\mathrm{soc}^k(M)) \subset M/\mathrm{soc}^k(M)$. *This yields a series*

$$0 \subset \mathrm{soc}(M) \subset \mathrm{soc}^2(M) \subset \cdots \subset \mathrm{soc}^\ell(M) = M$$

called the socle series of M.

Example 10.5. *Consider a path algebra* $\Bbbk Q$ *of a finite acyclic (for simplicity) quiver Q, and* $x \in Q_0$. *The indecomposable injective* $I_x = D(\Bbbk Q e_x)$ *has a simple socle isomorphic to* S_x*. Essentially this can be seen by a dual argument in showing* $\mathrm{top}(P_x) \cong S_x$*. More generally, analogous to Example 10.3, for a finite-dimensional algebra A, every simple A-module appears as* $\mathrm{soc}(I)$ *for an indecomposable direct summand of* $D(A)$.

Lemma 10.6. *For $M \in \mathsf{mod}\, A$, the socle series and radical series has the same length, and this length is called the Loewy length of M, and is denoted by* $\mathrm{LL}(M)$.

Proof Let r_M (resp. s_M) denotes the length of the radical (resp. socle) series of M. First, we show that $s_M \leq r_M$ by induction on s_M. This is clearly fine if $s_M = 0$.

Suppose that $s_M > 0$. By definition we have $\mathrm{rad}^{r-1}(M)$ a semisimple submodule of M, and so $\mathrm{rad}^{r-1}(M) \subset \mathrm{soc}(M)$. This means that there is a surjective homomorphism $M/\mathrm{rad}^{r-1}(M) \twoheadrightarrow M/\mathrm{soc}(M)$, and so $r_{M/\mathrm{rad}^{r-1}(M)} \geq r_{M/\mathrm{soc}\, M}$ (EXERCISE!). In particular, we have

$$r_M = r_{M/\mathrm{rad}^{r-1}(M)} + 1 \geq r_{M/\mathrm{soc}\, M} + 1.$$

Since $s_{M/\mathrm{soc}\, M} = s_M - 1$, it follows from the induction hypothesis that $s_{M/\mathrm{soc}\, M} \leq r_{M/\mathrm{soc}\, M}$, and hence

$$s_M = s_{M/\mathrm{soc}\, M} + 1 \leq r_{M/\mathrm{soc}\, M} + 1 \leq r_M,$$

as required.

One can show that $r_M \leq s_M$ dually. □

Note that the semisimple subquotients in (between the layers of) the socle series and the radical series of a module may not coincide.

Example 10.7. *Let Q be the quiver $1 \overset{\alpha}{\leftarrow} 2 \overset{\beta}{\rightarrow} 3 \overset{\gamma}{\rightarrow} 4$ and consider the projective P_2 which has the form*

$$\Bbbk \overset{1}{\leftarrow} \Bbbk \overset{1}{\rightarrow} \Bbbk \overset{1}{\rightarrow} \Bbbk$$

Then we have radical series

$$0 \subset S_4 = \Bbbk\beta\gamma \overset{S_1 \oplus S_3}{\subset} \mathrm{rad}(P_2) = \Bbbk\alpha + \Bbbk\beta + \Bbbk\beta\gamma \overset{S_2}{\subset} P_2$$

and socle series

$$0 \subset S_2 \oplus S_4 = \Bbbk\alpha + \Bbbk\beta\gamma \overset{S_3}{\subset} \mathrm{rad}(P_2) \subset P_2.$$

11 Example: Topological data analysis

Topological data analysis concerns the "rough shape of data". Here, we regard data as just a finite discrete set X in $\mathbb{R}^d$ (with usual Euclidean metric if you like). X itself is not particular interesting space (in terms of geometry or topology) for further analysis; yet, we can often see "pattern" – whether they look more or less randomly distributed, whether they are distributed in the space in a way that avoid certain areas, etc.

A more well-known mathematical approach to addressing this issue is statistics, where we try to see if the pattern tells us correlation between different parameters. For topological data analysis (TDA) we want to just tell if the data form some 'shapes with holes' (this is where 'topology' comes in). The idea is to replace each data point $x \in X$ by a ball $B_t(x)$ of very small radius t, slowly increase the radius and observe how the topology (e.g. by looking at topological invariant such as the 'genus') of the space $X_t := \bigcup_{x\in X} B_t(x)$ changes.

Note that if $s \leq t$, then we have a subspace $X_s \subset X_t$. Moreover, in practice, it makes sense to sample t to a finite sequence $t_1 < t_2 < \cdots < t_n$ and take $X_i := X_{t_i}$. Since we only concern topology of X_t, we can replace X_t by a simplicial complex $\triangle_t$ where 0-cells (points) are x, and $\{x_1, \ldots, x_r\}$ form an r-cells if $B_t(x_1) \cap \cdots \cap B_t(x_r) \neq \emptyset$. Having a simplicial complex means that we can take (e.g. the p-th) homology group $H_p(X_t) = H_p(\triangle_t)$. The fact that we have $X_{t_i} \subset X_{t_{i+1}}$ means that we have a chain

$$H_p(X_1) \to H_p(X_2) \to \cdots \to H_p(X_n).$$

If we linearise these abelian groups to $\Bbbk$-vector spaces, then we get a chain of vector spaces and linear transformations – this is nothing but a representation of the $\vec{\mathbb{A}}_n$-quiver

$$1 \to 2 \to \cdots \to n.$$

In TDA, such a chain is called *persistence module* (of 1 parameter / finite linear poset). Understanding the indecomposable decomposition of a persistence module is an important aspect in TDA, this can even be used to characterise the nature of the data set (e.g. one may record some data from various metals, and the topological information can be used to distinguish each metal just from the data set).

An *interval module* $M_{[a,b]}$ for $1 \leq a \leq b \leq n$ is the $\vec{\mathbb{A}}_n$-quiver representation given by

$$0 \to \cdots \to 0 \to \Bbbk \xrightarrow{\text{id}} \Bbbk \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \Bbbk \to 0 \to \cdots 0$$

where the non-zero space starts at a and ends at b. This is clearly an indecomposable representation. In fact, forms *all* indecomposable representation – known by Gabriel in the 70s (this is one special case of the Gabriel's theorem). The following is then just a consequence of Krull-Schmidt theorem, but turns out to be fundamental in TDA.

Proposition 11.1. *Every persistence module can be decomposed uniquely to a direct sum of interval module.*

The above is what people call 'single parameter', or (finite) 'linear poset', case. There are other possible forms:

(1) Multi-parameter case: the quiver $\vec{\mathbb{A}}_n$ is replaced by the 'commutative cube', i.e. the bound quiver $\vec{\mathbb{A}}_{n_1} \times \cdots \times \vec{\mathbb{A}}_{n_r}$ with relation $\alpha\beta - \beta\alpha$ for arrows α, β going in different directions. In other words, a persistence module in this case is the same as an A-module, where $A = \bigotimes_{i=1}^{r} \Bbbk\vec{\mathbb{A}}$.

(2) Poset case: the quiver $\vec{\mathbb{A}}_n$ is replace by the bound quiver (Q, I) whose underlying quiver Q is the Hasse quiver of the poset P, and I includes all commutation

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z - x \xrightarrow{\alpha'} w \xrightarrow{\beta'} z$$

whenever $x \geq y \geq z$ and $x \geq w \geq z$. In other words, a persistence module in this case isthe same as a module over the *incidence algebra* of the poset P.

In these general cases, one can still define interval modules, but it is no longer true that every A-module can be decomposed into interval modules. Much of the recent algebraic and homological aspect of TDA concerns how to overcome such a problem.

For other aspects and more in-depth study of applying quiver representation to TDA, see, for exmample, book of Steve Oudot.

12 Example: Linear matrix pencil

A *linear matrix pencil* is a matrix $A + \lambda B$ with $A, B \in M_{m\times n}(\Bbbk)$ and λ being an indeterminant, i.e. $A + \lambda B \in M_{m\times n}(\Bbbk[k])$. For simplicity, we just say 'matrix pencil' and drop the adjective 'linear'. Matrix pencil is used in the study of the so-called *generalised eigenvalue problem*, and has applications to various applied mathematics like control theory, differential algebraic equations, numerical linear algebra, etc.

Two matrix pencils $A + \lambda B$ and $A' + \lambda B'$ are *strictly equivalent* if there are invertible matrices $P \in M_m(\Bbbk), Q \in M_n(\Bbbk)$ such that $A' + \lambda B' = P(A + \lambda B)Q$. This is equivalent to $A' = PAQ$ and $B' = PBQ$.

For simplicity, let us specialise $\Bbbk$ to an algebraically closed field; this means that we can use Jordan canonical form $J_m(\alpha) \in M_m(\Bbbk)$.

Let $\overline{H}_m$ be the $m \times (m + 1)$-matrix given by removing the last row of $J_m(0)$, and $(I_m|0)$ be the $m \times (m + 1)$-matrix given by adding a column of zero to the identity matrix I_m. Define

$$L_m := \lambda(I_m|0) + \overline{H}_m = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & 1 \end{pmatrix} \in M_{m\times(m+1)}(\Bbbk[\lambda]).$$

Similar to the Smith/Jordan canonical form, each matrix pencil of is equivalent to one in *Kronecker canonical form*.

Theorem 12.1. *Every matrix pencil is strictly equivalent to a block-diagonal matrix, where each block is of one of the following form:*

(1) L_m *or* L_m^{tr} *for some* $m \geq 1$.

(2) $I_m + \lambda J_m(0)$ *or* $J_m(\alpha) + \lambda I_m$, *for some* $m \geq 1$ *and* $\alpha \in \Bbbk$.

One way to prove this theorem is to observe the following.

Proposition 12.2. *For any* $m, n \in \mathbb{Z}_{\geq 0}$, *there is a one-to-one correspondence between the strictly equivalent classes of* $m \times n$*-matrix pencil and the isomorphism classes of representations over the Kronecker quiver* $K_2 := (1 \rightrightarrows 2)$ *with dimension vector* (n, m).

Proof Exercise. □

Under this correspondence, the Kronecker canonical form (the blocks appearing in the block-diagonal form) corresponds to indecomposable representations of the Kronecker quiver. In quiver representation theory, such classification can be done using a Kac's theorem.

Corollary 12.3. *Every indecomposable* $\Bbbk K_2$*-modules is isomorphic to one of the following.*

(1) *Preinjective modules, which correspond to* L_m *for* $m \geq 1$.

(2) *Preprojective modules, which correspond to* L_m^{tr} *for* $m \geq 1$.

(3) *Regular modules* $R_m(x)$ *for* $x \in \mathbb{P}^1_{\Bbbk}$ *and* $m \geq 1$, *where*

$$\begin{cases} R_m(\alpha) \text{ corresponds to } J_m(\alpha) + \lambda I_m & \text{if } \alpha = [x : 1]; \\ R_m(\infty) \text{ corresponds to } I_m + \lambda J_m(0) & \text{if } \alpha = [1 : 0] = \infty. \end{cases}$$

With this, various problems about linear matrix pencil can be transformed to problems about representations of the Kronecker quiver. There are also 'higher variation' of matrix pencils that correspond to the n-Kronecker quiver where there are $n \geq 2$ arrows between the 2 vertices (instead of just $n = 2$). Examples problem includes the "matrix subpencil" problem, which are studied by Claus Ringel, Han Yang, Ştefan Şuteu-Szöllősi.

Exercise 12.4. *Write down the indecomposable* $\Bbbk K_2$*-modules as representations.*

13 Bounded path algebra

For general quiver, we loses finite-dimensionality, and so many nice things we explained do not hold any more. To retain finite-dimensionality, we need to consider nice quotients of path algebras.

Definition 13.1. *An ideal $I \lhd \Bbbk Q$ is admissible if $(\Bbbk Q_1)^k \subset I \subset (\Bbbk Q_1)^2$ for some $k \geq 2$, i.e. I is generated by linear combinations of paths of finite length at least 2. The pair (Q, I) is sometimes called bounded quiver. A bounded path algebra or quiver algebra (with relations) is an algebra of the form $\Bbbk Q/I$ for some quiver Q and admissible ideal I.*

Remark 13.2. Admissiblity ensures there is no redundant arrows (which appears if there is a relation like, for example, $\alpha - \beta\gamma \in I$ for some $\alpha \neq \beta, \gamma \in Q_1$) and there is enough vertices (trivial paths may not be primitive if there is a loop x at a vertex with relation $x^2 - x \in I$).

Lemma 13.3. *A bounded path algebra is finite-dimensional.*

Proof There exists a surjective algebra homomorphism $\Bbbk Q/(\Bbbk Q_1)^k \twoheadrightarrow \Bbbk Q/I$; the former is finite-dimensional. □

Example 13.4. *Let Q be the Jordan quiver with unique arrow α. Let I be the ideal of $\Bbbk Q$ generated by α^k for some $k \geq 2$. Then I is an admissible ideal and $\Bbbk Q/I \cong \Bbbk[x]/(x^k)$ is a truncated polynomial ring.*

Definition 13.5. *A representation M of a bounded quiver (Q, I) is a representation $M = (M_i, M_\alpha)_{i,\alpha}$ of Q such that $M_a = 0$ for all $a \in I$; here $M_a := \sum_p \lambda_p M_p$ for $a = \sum_p \lambda_p p$ written as a linear combinations of paths p.*

A homomorphism $f : M \to N$ of representations of (Q, I) is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' action.

As before, representations are really just synonyms of modules.

Lemma 13.6. *A representation of a bounded quiver (Q, I) is equivalent to a $\Bbbk Q/I$-module, and homomorphisms between representations are equivalent to those between $\Bbbk Q/I$-modules.*

We have seen that it is easy to write down the indecomposable decomposition of the free $\Bbbk Q$-module $\Bbbk Q_{\Bbbk Q}$, we would like such nice thing to carry over to bounded path algebras.

Theorem 13.7. *(Idempotent lifting) If I is a nilpotent ideal of A (i.e. $I^n = 0$ for some $n \geq 1$) and $\overline{e} = \overline{e}^2 \in A/I$, then there is a lift $e = e^2 \in A$ of $\overline{e}$, i.e. $\overline{e} = e + I$.*

Proof omitted.

Corollary 13.8. *Let I be an nilpotent ideal in A. Suppose that*

$$1_{A/I} = f_1 + \cdots + f_n$$

for $f_i \in A/I$ are primitive orthogonal idempotents. Then we have

$$1_A = e_1 + \cdots e_n$$

where each $e_i \in A$ is a primitive orthogonal idempotent that lifts f_i.

Corollary 13.9. *Let A be a bound path algebra. The primitive orthogonal idempotents of A are given by the trivial paths.*

Proof Apply Corollary 13.8 with $I = \operatorname{rad} A$. Since $A/I = \Bbbk Q_0$ is semisimple and so we have primitive orthogonal idempotents given by the trivial paths. □

Notation. *As in the case of path algebra, denote by S_x or $S(x)$ the simple $\Bbbk Q/I$-module given by placing a one-dimensional vector space at vertex $x \in Q_0$ and zero everywhere else.*

Similarly, denote by P_x or $P(x)$ the indecomposable $\Bbbk Q/I$-module $e_x \Bbbk Q/I$. Likewise, by I_x or $I(x)$ the indecomposable $D((\Bbbk Q/I)e_x)$.

Proposition 13.10. *For a finite-dimensional quiver algebra $A = \Bbbk Q/I$, there is a decomposition of A-modules*

$$A_A = \bigoplus_{x \in Q_0} P_x, \text{ and } (DA)_A = \bigoplus_{x \in Q_0} I_x.$$

Moreover, $\{S_x \cong \operatorname{top}(P_x) \cong \operatorname{soc}(I_x) \mid x \in Q_0\}$ form the complete set of isoclasses representatives of simple A-modules.

Proof By Corollary 13.9, the trivial path e_x is a primitive idempotent of A, and so $P_x = e_x A$ and $I_x = D(Ae_x)$ are indecomposable.

The simple A-modules (up to isomorphisms) correspond to those over the semisimple quotient algebra $A/\operatorname{rad}(A)$ by Corollary 9.6. Hence, there are precisely $|Q_0|$ simple modules (up to isomorphism), given by the simple top of P_x, which is also isomorphic to the simple socle of I_x. □

We give a brief justification of why quiver representations provide a good way to construct lots of algebras.

Theorem 13.11. *Suppose $\Bbbk$ is algebraically closed. Then every finite-dimensional $\Bbbk$-algebra A is Morita equivalent to a bounded path algebra $\Bbbk Q/I$. More precisely, $\Bbbk Q/I$ is given by $\operatorname{End}_A(\bigoplus_e eA)$ where e varies over the set of representative of equivalence classes of primitive idempotents of A.*

We defer the precise meaning of Morita equivalent; it roughly translates to saying that understanding A-modules and homomorphisms between them is equivalently (but not necessarily 'equal to') to understanding modules and homomorphisms between a Morita equivalent bounded path algebra.

Example 13.12. *Let $A = \operatorname{Mat}_n(\Bbbk)$ be a matrix ring. Then the elementary matrix $e := E_{1,1}$ is a primitive idempotent and $eA \cong E_{j,j}A$ for all $1 \le j \le n$. So A is Morita equivalent to $\Bbbk \cong \Bbbk Q \cong \operatorname{End}_A(eA)$ where Q is a one-vertex-no-arrow quiver.*

Primitive idempotent decomposition, say, $1 = \sum_{i=1}^n e_i$, allows us to write an algebra A in matrix form $(e_i A e_j)_{1 \le i,j \le n}$, where the 'row spaces' form the indecomposable direct summands $e_i A$ and the dual of the 'column space' form the indecomposable direct summands $D(Ae_i)$. It could be a helpful mental exercise to think about the meaning of $eAe \cong \operatorname{End}_A(eA)$ from Yoneda lemma - this maybe a useful idea to keep in mind when one tries to understand the above theorem.

14 Some basic category theory

We briefly recall some language from category theory that are commonly used in representation theory.

A *category* $\mathcal{C} = (\text{ob}\mathcal{C}, \hom\mathcal{C}, \circ)$ consists of the following data.

(1) A collection $\text{ob}\mathcal{C}$ of *objects*. It is common to write $X \in \mathcal{C}$ instead of $X \in \text{ob}\mathcal{C}$.

(2) For any pair $X, Y \in \mathcal{C}$, there is a collection of *morphisms from X to Y*. Such an element is often written $f : X \to Y$ if context clear, or $f \in \mathcal{C}(X,Y)$ or $f \in \text{Hom}_{\mathcal{C}}(X,Y)$.

(3) A binary operation $- \circ - : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ that takes $(g : Y \to Z, f : X \to Y)$ to the *composition* $g \circ f : X \to Z$, and this operations satisfies the following:
   - Composition is *associative*, i.e. $h(gf) = (hg)f$ for all (meaningful) h, g, f.
   - There are *identity morphisms* id_X that are left and right units with respect to composition, i.e. $f\,\text{id}_X = f$ and $\text{id}_Y\, f = f$ for all (meaningful) f.

For simplicity, we assume always that $\mathcal{C}(X,Y)$ are sets.

Example 14.1. *Some common categories of interest in algebras.*

(1) *The category* $\mathsf{mod}\,A$ *of* finitely generated *A-modules.*

(2) *The category* $\mathsf{Mod}\,A$ *of* all *A-modules.*

(3) *The category $\mathcal{A}$b of abelian groups and group homomorphisms.*

(4) *Let Q be a quiver and R be a ring. Then we have a $\Bbbk$-linear path category whose objects are the vertices of Q, and morphisms are $\Bbbk$-linear combinations of paths (directed, and possibly trivial, walks) of Q. Morphism compositions are induced by path concatenation in the same way as path algebra. Indeed, what we are doing is essentially just viewing the ring $\Bbbk Q$ as a category.*

(5) *The category* $\mathsf{coh}X$ *of coherent sheaves over a scheme X.*

Suppose we have categories $\mathcal{C}$ and $\mathcal{C}'$. We say that $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$ if the following are satisfied.

(1) $\text{ob}\mathcal{C}'$ is a subcollection of $\text{ob}\mathcal{C}$.

(2) $\mathcal{C}'(X,Y) \subseteq \mathcal{C}(X,Y)$ for all X, Y.

(3) Composition and identity morphisms in $\mathcal{C}$ and in $\mathcal{C}'$ coincide.

If, moreover, $\mathcal{C}'(X,Y) = \mathcal{C}(X,Y)$, then we say that $\mathcal{C}'$ is a *full subcategory* of $\mathcal{C}$.

Example 14.2. $\mathsf{mod}\,A$ *is a full subcategory of* $\mathsf{Mod}\,A$.

Definition 14.3. *Let $\Bbbk$ be a field, a category $\mathcal{C}$ is $\Bbbk$-linear if $\mathcal{C}(X,Y)$ are $\Bbbk$-vector spaces for all $X, Y \in \mathcal{C}$ and compositions are $\Bbbk$-bilinear maps. A functor $F : \mathcal{C} \to \mathcal{D}$ between $\Bbbk$-linear categories is $\Bbbk$-linear if $F_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$ is $\Bbbk$-linear.*

Example 14.4. *When A is a $\Bbbk$-algebra. Then the module category (finitely generated or not) is $\Bbbk$-linear.*

A *covariant functor* (resp. *contravariant functor*) $F : \mathcal{C} \to \mathcal{D}$ consists of

- an assignment of object $F(X) \in \mathcal{D}$ for each $X \in \mathcal{C}$, and
- an assignment of morphism $F(f) \in \mathcal{D}(F(X),F(Y))$ (resp. $F(f) \in \mathcal{D}(F(Y),F(X))$) for each $f \in \mathcal{C}(X,Y)$, such that
- $F(\text{id}_X) = \text{id}_{F(X)}$, and
- $F(gf) = F(g)F(f)$ (resp. $F(gf) = F(f)F(g)$).

Note that, a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is the same as a convariant functor from the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. Here, $\mathcal{C}^{\mathrm{op}}$ given by the same collection of objects as $\mathcal{C}$, but the morphisms and their composition are in reverse direction, i.e. $\mathcal{C}^{\mathrm{op}}(X,Y) := \mathcal{C}(Y,X)$.

Example 14.5. *The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the functor given by mapping every object and homomorphism to itself.*

In practice (in representation theory and the like), a functor tells us how we can transform from the theory of modules over one algebra to those over another algebra.

Example 14.6. *The ($\Bbbk$-linear) duality $D = \mathrm{Hom}_{\Bbbk}(-,\Bbbk) : \mathsf{mod}\, A \to \mathsf{mod}\, A^{\mathrm{op}}$ is a contravariant functor.*

To compare two functors (or compare how a pair of functors is close/far away from the identity functor), one uses *natural transformations*. More precisely, a natural transformation $\eta : F \Rightarrow G$ of functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\eta_X : F(X) \to G(X)$ such that there is the following commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

If we say that a map $\eta_X : F(X) \to G(X)$ is *natural in X*, then we mean that $\{\eta_X\}_{X \in \mathsf{mod}\, A}$ defines a natural transformation.

A *natural isomorphism* is a natural transformation η such that η_X is an isomorphism for all X. We simply write $F \cong G$ if there is a natural isomorphism between two functors F, G.

Definition 14.7. *Let $\mathcal{C}, \mathcal{D}$ be two categories.*

- *$\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists a pair of functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{D}$ such that $1_{\mathcal{C}} \cong GF$ and $1_{\mathcal{D}} \cong GF$.*
- *A functor $F : \mathcal{C} \to \mathcal{D}$ is dense if, for all $D \in \mathcal{D}$, there is $C \in \mathcal{C}$ such that $F(C) \cong D$; i.e. "surjective on object up to isomorphism".*
- *A functor $F : \mathcal{C} \to \mathcal{D}$ is faithful (resp. full), if the induced map $F_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$ given by $f \mapsto F(f)$ is injective (resp. surjective).*

Proposition 14.8. *$F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if, and only if, F is fully faithful dense.*

Exercise 14.9. *Recall that R^{op} is the opposite ring of R, whose underlying set is the same as that of R with multiplication $(a \cdot^{\mathrm{op}} b) := b \cdot a$. A representation of R is a ring homomorphism*

$$\rho : R^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(M), \qquad r \mapsto \rho_r,$$

for some abelian group $(M,+)$. A homomorphism $f : \rho_M \to \rho_N$ of representations $\rho_M : R^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(M), \rho_N : R^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(N)$ given by an abelian group homomorphism $f : M \to N$ that intertwines R-action, i.e. $\rho_N(r) \circ f = f \circ \rho_M(r)$ for all $r \in R$.

Eplain why a representation of R is equivalent to a right R-module; and why homomorphisms correspond.

15 Extra: Additive and abelian categories

In brief, additive categories are the categories where it make sense to talk about direct products and has the same backbone as the abelian groups. Abelian categories are additive categories where it makes sense (on the categorical level) to talk about kernel and cokernel, and that these behave like what we expect in many typical situations like representation theory and algebraic geometry.

Definition 15.1. *Consider an object $O \in \mathcal{C}$ in a category $\mathcal{C}$.*

- *O is an initial object if for all $X \in \mathcal{C}$, there is a* unique *morphism $O \to X$.*
- *O is a terminal object if for all $Y \in \mathcal{C}$, there is a* unique *morphism $Y \to O$.*
- *O is a zero object if it is both initial and terminal.*

Note that, if it exists, then it is unique up to unique isomorphism. For example, the zero module $0 \in \mathsf{mod}\, A$ in the module category of a ring is a zero object.

Definition 15.2. *Let I be a(n indexing) set and $(X_i)_{i\in I}$ a family of objects in $\mathcal{C}$.*

(1) *An object P equipped with morphisms $(p_i : P \to X_i)_{i\in I}$ is a product if for all $Z \in \mathcal{C}$ and all $(f_i : Z \to X_i)_{i\in I}$, there is a unique morphism f such that $p_i f = f_i$ for all $i \in I$.*

(2) *Dually, an object C equipped with morphisms $(\iota_i : X_i \to C)_{i\in I}$ is a coproduct if for all $Z \in \mathcal{C}$ and all $(f_i : X_i \to Z)_{i\in I}$, there is a unique morphism f such that $f\iota_i = f_i$ for all $i \in I$.*

(3) *A biproduct $B \in \mathcal{C}$ is an object that is isomorphic to a product and also to a coproduct.*

When a product/coproduct exist, it is unique up to unique isomorphism. It is often written as $\prod_{i\in I} X_i$ for product, $\coprod_{i\in I} X_i$ for coproduct, and $\bigoplus_{i\in I} X_i$ for biproduct.

Product and coproduct are special instances of what-are-called limit and colimit respectively. As a consequence of abstract nonsense (category theory), there are natural bijections

$$\mathcal{C}(X, \prod_{i\in I} Y_i) \cong \prod_{i\in I} \mathcal{C}(X, Y_i), \quad \mathcal{C}(\coprod_{i\in I} X_i, Y) \cong \prod_{i\in I} \mathcal{C}(X_i, Y).$$

Note that coproduct get 'extract' out to a product here. If the indexing set I is finite, then all the $\prod$ and $\coprod$ in the above formula can be replaced by $\bigoplus$.

Definition 15.3. *A category $\mathcal{C}$ is additive if the following are satisfied.*

- *It is pre-additive, i.e. $\mathcal{C}(X, Y)$ are abelian groups for all $X, Y \in \mathcal{C}$ with bilinear composition.*
- *Finite products $\prod_{i=1}^n X_i$ (in the categorical sense) exists for all $X_1, \ldots, X_n \in \mathcal{C}$.*
- *There is a zero object $0 \in \mathcal{C}$ (in the categorical sense).*

A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is additive if $F(0) \cong 0$ and F preserves (finite) products.

In an additive category, we have (see the additive category page on nLab, or Kashiwara-Schapiro's book Corollary 8.2.4)

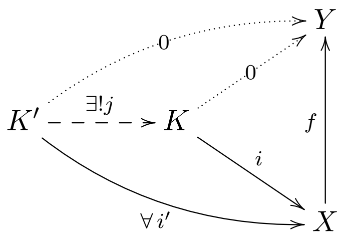
$$\text{finite product} = \text{finite direct sum} = \text{finite coproduct}.$$

Note that the terminology 'direct sum' matches the one used in module theory. Note also that being additive is a *property* not a structure, since products and zero objects are uniquely determined by universal property, there is no choice involved; c.f. Kashiwara-Schapiro's book Theorem 8.2.14.

In a pre-additive category $\mathcal{C}$, since Hom-sets are abelian groups, there is always a distinguished zero morphism $0 \in \mathcal{C}(X, Y)$ for any $X, Y \in \mathcal{C}$. This zero morphism allows us to formulate kernels and cokernels in the categorical level.

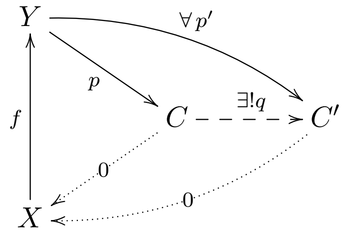
Definition 15.4. *Let $\mathcal{C}$ be a pre-additive category and $f : X \to Y$ a morphism in $\mathcal{C}$.*

(1) *A kernel of f is a morphism $i : K \to X$ such that $(K \xrightarrow{fi} Y) = 0$ and for any $i' : K' \to X$ with $fi' = 0$, there exists a unique morphism $j : K' \to K$ such that $i' = ij$.*



In the case when the kernel of f exists, then denote by $\mathrm{Ker}(f)$ the object K, and by $\ker(f)$ the morphism i, as appeared above.

(2) *Dually, a cokernel of f is a morphism $p : Y \to C$ such that $(X \xrightarrow{pf} C) = 0$ and for any $p' : Y \to C'$ with $p'f = 0$, there exists a unique morphism $q : C \to C'$ such that $p' = qp$.*



In the case when the cokernel of f exists, then denote by $\mathrm{Cok}(f)$ the object C, and by $\mathrm{cok}(f)$ the morphism p, as appeared above.

(3) *If $\ker(f)$ exists, then a coimage of f is a cokernel of $\ker(f)$. When this exists, the cokernel object is denoted by $\mathrm{Coim}(f)$.*

(4) *Dually, if $\mathrm{cok}(f)$ exists, then an image of f is a kernel of $\mathrm{cok}(f)$. When this exists, the kernel object is denoted by $\mathrm{Im}(f)$.*

Lemma 15.5. *Let $f : X \to Y$ be a morphism in a pre-additive category $\mathcal{C}$.*

(1) $\ker(f)$ *is a monomorphism, i.e. a left-cancellative morphism (meaning that, for any morphisms g_1, g_2, $\ker(f) \circ g_1 = \ker(f) \circ g_2 \Rightarrow g_1 = g_2$).*

(2) $\mathrm{cok}(f)$ *is an epimorphism, i.e. a right-cancellative morphism (meaning that, for any morphism g_1, g_2, $g_1 \circ \mathrm{cok}(f) = g_2 \circ \mathrm{cok}(f) \Rightarrow g_1 = g_2$).*

(3) *If kernel, cokernel, coimage, image all exist for f, then f can be factored uniquely as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$.*

Proof (1), (2): Follows from the universal property (definition).

(3) We have $(\mathrm{Ker}(f) \xrightarrow{\ker(f)} X \xrightarrow{f} Y) = 0$. Hence, being a cokernel of $\ker(f)$, we have a unique morphism $\mathrm{Coim}(f) \to Y$ for which f factors through.

Since coimage is a cokernel, $X \to \mathrm{Coim}(f)$ is an epimorphism, which means that any morphism g with $(X \to \mathrm{Coim}(f) \xrightarrow{g} Z) = 0$ implies $g = 0$. Now we have

$$\begin{aligned} 0 &= (X \xrightarrow{f} Y \xrightarrow{\mathrm{cok}(f)} \mathrm{Cok}(f)) \\ &= (X \to \mathrm{Coim}(f) \to Y \xrightarrow{\mathrm{cok}(f)} \mathrm{Cok}(f)). \end{aligned}$$

Hence, the morphism $(\mathrm{Coim}(f) \to Y \to \mathrm{Cok}(f)) = 0$. On the other hand, $\mathrm{Im}(f)$ is the kernel of $Y \to \mathrm{Cok}(f)$, and so $(\mathrm{Coim}(f) \to Y \to \mathrm{Cok}(f)) = 0$ implies that $\mathrm{Coim}(f) \to Y$ factors uniquely through $\mathrm{Im}(f) \to Y$. This finishes the construction of the desired morphism. □

Exercise 15.6. (1) *Show that in* $\mathsf{mod}\,A$*, injective (resp. surjective) module homomorphisms coincide with monomorphisms (resp. epimorphisms).*

(2) *Show that in the category of rings (morphisms being ring homomorphisms), the natural embedding* $\mathbb{Z} \to \mathbb{Q}$ *is an epimorphism.*

Definition 15.7. *A category* $\mathcal{C}$ *is abelian if*

- $\mathcal{C}$ *is additive, and*
- *any morphism* $f : X \to Y$ *admits a kernel* $\ker(f) : \mathrm{Ker}(f) \to X$ *and a cokernel* $\mathrm{cok}(f) : Y \to \mathrm{Cok}(f)$*, such that the induced morphism* $\mathrm{Coim}(f) := \mathrm{Cok}(\ker(f)) \to \mathrm{Ker}(\mathrm{cok}(f)) =: \mathrm{Im}(f)$ *of Lemma 15.5 (3) is an isomorphism.*

Exercise 15.8. $\mathsf{mod}\,A$ *and* $\mathsf{proj}\,A = \{$*modules isomorphic to a finite direct sum of direct summands of* $A\}$ *are* $\Bbbk$*-linear additive categories, but only the former is abelian.*

Exercise 15.9. *Consider the category* $\underline{\mathsf{mod}}A$ *whose objects are the same as those in* $\mathsf{mod}\,A$ *(i.e. finitely generated* A*-modules), but with morphism space given by*

$$\mathrm{Hom}_{\underline{\mathsf{mod}}A}(X, Y) := \mathrm{Hom}_A(X, Y)/I(X, Y),$$

where $I(X, Y)$ *consists of all morphisms that factors through some* $P \in \mathsf{proj}\,A$*. Show that*

(1) *Any* $P \in \mathsf{proj}\,A$ *is isomorphic to* 0 *in* $\underline{\mathsf{mod}}A$*.*

(2) $\underline{\mathsf{mod}}A$ *is additive but not abelian.*

Definition 15.10. *Let* $f : X \to Y$ *be a morphism in an abelian category* $\mathcal{A}$*.*

- f *is said to be injective if* $\mathrm{Ker}(f) = 0$*. In which case, we say that* X *is a subobject of* Y*.*
- f *is said to be surjective if* $\mathrm{Cok}(f) = 0$*. In which case, we say that* Y *is a quotient (object) of* X*.*

Lemma 15.11. *Let* $f : X \to Y$ *be a morphism in an abelian category* $\mathcal{A}$*.*

(1) f *is injective* $\Leftrightarrow$ f *is a monomorphism.*

(2) f *is surjetive* $\Leftrightarrow$ f *is an epimorphism.*

(3) f *is injective and surjective* $\Leftrightarrow$ f *is an isomorphism.*

Proof

$$\begin{aligned}\mathrm{Ker}(f) = 0 &\Leftrightarrow \forall i : Z \to X, (fi = 0 \Rightarrow i \text{ factors through } 0)\\ &\Leftrightarrow \forall i : Z \to X, (fi = 0 \Rightarrow i = 0)\\ &\Leftrightarrow f \text{ mono}.\end{aligned}$$

This proves (1); (2) can be proved similarly.

(3) f isom implies both mono and epi, and so by (1) and (2) we get injective and surjective. Conversely, when f is both injective and surjective, then we have $X = \mathrm{Coim}(f) \cong \mathrm{Im}(f) = Y$. □

16 Bimodule, tensor and Hom

The most typical method in constructing functors between module categories is through 'tensoring' and 'hom-ing', which we explain in details now. For simplicity, we always work over a field $\Bbbk$.

16.1 Bimodule

Definition 16.1. *Let A, B be two $\Bbbk$-algebras. An A-B-bimodule is a $\Bbbk$-vector space M that has the structure of a left A-module and also the structure of a right B-module, such that $(am)b = a(mb)$ for all $a \in A, b \in B, m \in M$. In such a case, we may write $M \in A \operatorname{mod} B$ or ${}_AM_B$ to specify M is an A-B-bimodule.*

For simplicity, we assume all bimodules are $\Bbbk$-central, i.e. $\lambda m = m\lambda$ for all $\lambda \in \Bbbk$. We will omit the adjective $\Bbbk$-central from now on.

Example 16.2. *For any algebra A, both A and $D(A)$ are naturally an A-A-bimodule. Note that the right/left module structure on $D(A)$ is induced by the left/right module structure on A. (The direction of action has swapped!)*

Example 16.3. $\operatorname{Hom}_A(X, Y)$ *is naturally a* $\operatorname{End}_A(Y)$-$\operatorname{End}_A(X)$*-bimodule with action given by composition of homomorphisms.*

16.2 Tensor product

Definition 16.4. *Let V, W be finite-dimensional $\Bbbk$-vector space with bases, say, $\mathcal{B}, \mathcal{C}$ respectively. Then the tensor product $V \otimes_\Bbbk W$ (or simplifies to $V \otimes W$ if context is clear) is the finite-dimensional $\Bbbk$-vector space with bases given by*

$$\{v \otimes w \mid v \in \mathcal{B}, w \in \mathcal{C}\}.$$

In particular, note that $\dim_\Bbbk V \otimes W = (\dim_\Bbbk V) \times (\dim_\Bbbk W)$.

Proposition 16.5. *Let A, B be $\Bbbk$-algebras. Then $A \otimes_\Bbbk B$ is also a $\Bbbk$-algebra with multiplication given by extending $(a \otimes b)(a' \otimes b') \mapsto aa' \otimes bb'$ linearly. For $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} B$, we have $M \otimes_\Bbbk N \in \operatorname{mod} A \otimes_\Bbbk B$.*

Proof Routine checking. □

Lemma 16.6. *An idempotent $e \in A \otimes_\Bbbk B$ is primitive if and only if $e = e_l \otimes e_r$ for some primitive idempotents $e_l \in A$ and $e_r \in B$. In particular, we have $\operatorname{Sim}(A \otimes_\Bbbk B) = \{S \otimes T \mid S \in \operatorname{Sim}(A), T \in \operatorname{Sim}(B)\}$.*

Note that not all $A \otimes_\Bbbk B$-module is of the form $M \otimes N$.

Example 16.7. *Let $A = \Bbbk[x]/(x^2)$ and $A' := \Bbbk[y]/(y^2)$. Then $B := A \otimes_\Bbbk A' = \Bbbk[x, y]/(x^2, y^2)$. Then we have an indecomposable 2-dimensional B-module $V = \Bbbk u + \Bbbk v$ (top $S = B/\operatorname{rad}(B)$ and socle S) where both x, y acts by $u \mapsto v$. This cannot be of the form $M \otimes N$ for some $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} A'$. Indeed, as both x, y acts non-trivially, if $V = M \otimes N$ then both M, N must have dimension at least 2, and so the tensor product has dimension at least 4; but $\dim_\Bbbk V = 2$.*

Proposition 16.8. *An $A \otimes_\Bbbk B^{\mathrm{op}}$-module is the same as a ($\Bbbk$-central) B-A-bimodule. Moreover, homomorphisms of $A \otimes B^{\mathrm{op}}$-modules correspond to ($\Bbbk$-linear) homomorphisms of B-A-bimodule.*

Definition 16.9. *Let $X \in \mathsf{mod}\, A$ be a right A-module and $Y \in \mathsf{mod}\, A^{\mathrm{op}}$ be a left A-module. Then define $X \otimes_A Y$ to be the vector space $X \otimes_{\Bbbk} Y/U$ where U is the subspace consisting of $xa \otimes y - x \otimes ay$ for all $x \in X, y \in Y, a \in A$.*

In the above, if ${}_AY_B$ is, in addition, an A-B-bimodule, then $X \otimes_A Y$ has a natural right B-module structure: $(x \otimes y)b := x \otimes (yb)$. In fact, as any left A-module is also a A-$\Bbbk$-bimodule, we can $X \otimes_A Y$ being a $\Bbbk$-vector space as a special case of this observation.

Suppose we have a homomorphism $f : M \to N$ of right A-modules. Then for an A-B-bimodule ${}_AY_B$ we get a homomorphism of

$$\begin{aligned} M \otimes_A Y_B &\xrightarrow{f \otimes_A Y} N \otimes_A Y_B \\ m \otimes y &\longmapsto f(m) \otimes y \end{aligned}$$

Note that $(gf) \otimes_A Y = (g \otimes_A Y)(f \otimes_A Y)$, that is, $- \otimes_A Y : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$ is a (covariant) *functor*. It is also $\Bbbk$*-linear additive* in the sense that $(\lambda f + \mu g) \otimes_A Y = \lambda(f \otimes_A Y) + \mu(g \otimes_A Y)$ for all homomorphisms f, g and scalar $\lambda, \mu \in \Bbbk$. Note that this naturally restricts to a functor $- \otimes_A Y : \mathsf{mod}\, A \to \mathsf{mod}\, B$ on the finitely generated modules so long as Y is so.

Likewise, if X is a bimodule, then ${}_BX \otimes_A Y$ has a left module structure; mutatis mutantis.

16.3 Hom

Suppose now that we have ${}_BX_A$ a B-A-bimodule and M a right A-module. Then the space $\mathrm{Hom}_A(X, Y)$ has a natural *right* B-module structure:

$$(f : X \to Y) \cdot b := (x \mapsto f(bx))$$

Indeed, we have

$$((f \cdot b) \cdot b')(x) = (f \cdot b)(b'x) = f(bb'x) = (f \cdot (bb'))(x),$$

and other axioms are even easier to verify.

Similarly, in the same setting, $\mathrm{Hom}_A(Y, {}_BX_A)$ also has a *left* B-module structure:

$$(b' \cdot (b \cdot f))(x) = b'((b \cdot f)(x)) = b'(bf(x)) = (b'b)f(x) = ((b'b) \cdot f)(x).$$

Exercise 16.10. *Show that $\mathrm{Hom}_A({}_BX_A, -)$ defines a $\Bbbk$-linear additive covariant functor from $\mathsf{Mod}\, A$ to $\mathsf{Mod}\, B$ (and also $\mathsf{mod}\, A \to \mathsf{mod}\, B$ when X is finitely generated). Likewise, show that $\mathrm{Hom}_A(-, {}_BX_A)$ defines a $\Bbbk$-linear additive contravariant functor on both the (big) module category and the finitely-generated-module category.*

Lemma 16.11. *For any A-module M, we have (natural) A-module isomorphisms*

$$M \otimes_A A \cong M, \quad \text{and } \mathrm{Hom}_A(A, M) \cong M$$

given by $m \otimes 1 \mapsto m$ and $f \mapsto f(1)$. Moreover, $- \otimes_A A$ and $\mathrm{Hom}_A(A, -)$ are both naturally isomorphic to the identity functor.

Proof First one follows from the construction that $ma \otimes 1 = m \otimes a$. The second one is just special case of Yoneda lemma. □

16.4 Tensor-Hom adjunction

Suppose ${}_AM_B$ is a (f.g.) A-B-bimodule, then we have two functors:

$$\mathsf{mod}\,A \underset{\mathrm{Hom}_B(M_B,-)}{\overset{-\otimes_A M}{\rightleftarrows}} \mathsf{mod}\,B.$$

These are not inverse to each other; but they form a so-called adjoint pair, which is equivalent to saying that there is the following natural isomorphisms.

Theorem 16.12 (Tensor-Hom adjunction). *Let $X \in \mathsf{mod}\,A$, $Y \in \mathsf{mod}\,B$, ${}_AM_B \in A\,\mathsf{mod}\,B$. Then there is a canonical isomorphism of $\Bbbk$-vector spaces*

$$\begin{aligned}\theta_{X,M,Y}:\quad \mathrm{Hom}_B(X \otimes_A M, Y) &\xrightarrow{\cong} \mathrm{Hom}_A(X, \mathrm{Hom}_B(M,Y))\\ f &\longmapsto (x \mapsto (m \mapsto f(x\otimes m)))\\ (x\otimes m \mapsto (g(x))(m)) &\longleftarrow\!\mid g\end{aligned}$$

that is natural in each of X, M, Y.

Proof Check that the maps written are ($\Bbbk$-linear and) mutual inverse of each other. □

In computer science, the map $\theta_{X,M,Y}$ is also called "currying".

As innocence as it looks, this isomorphism is fundamental in (homological algebra and) representation theory.

Example 16.13 (Adjoint triple (RHS)). *eA is naturally an eAe-A-bimodule. Hence, we have an adjoint pair $(-\otimes_{eAe} eA, \mathrm{Hom}_A(eA,-))$.*

On the other hand, Ae is naturally an A-eAe-bimodule, and so we have another adjoint pair $(-\otimes_A Ae, \mathrm{Hom}_{eAe}(Ae,-))$. Note that we have $\mathrm{Hom}_A(eA,-) \cong -\otimes_A Ae$ by Yoneda lemma.

Example 16.14 (Adjoint triple (LHS)). *A/I is naturally an A-A/I-bimodule for any two-sided ideal I of A, and so we have an adjoint pair $(-\otimes_A A/I, \mathrm{Hom}_{A/I}(A/I,-))$.*

A/I is also an A/I-A-bimodule, and so there is another adjoint pair $(-\otimes_{A/I} A/I, \mathrm{Hom}_A(A/I,-))$. Note that both $\mathrm{Hom}_{A/I}(A/I,-)$ and $\otimes_{A/I}A/I$ sends an A/I-module to itself (up to isomorphism) and acts identically on morphisms, i.e. $\mathrm{Hom}_{A/I}(A/I,-) \cong \mathrm{Id} \cong -\otimes_{A/I} A/I$.

Exercise 16.15. *Let $F = -\otimes_A M$ and $G = \mathrm{Hom}_B(M,-)$ for an A-B-bimodule ${}_AM_B$. Show (without appeal to category theory) that we have natural transformations $\epsilon : FG \to \mathrm{Id}$ and $\eta : \mathrm{Id} \to GF$. Show also that the following holds:*

$$\mathrm{id}_F = (\epsilon F) \circ (F\eta) \text{ and } \mathrm{id}_G = (G\epsilon) \circ (\eta G).$$

Here, ϵF (and similarly for ηG) denotes the natural transformation $\epsilon F : FGF \to F$ given by $(\epsilon F)_X = \epsilon_{FX}$, whereas $F\eta$ (and similarly for $G\epsilon$) denotes the natural transformation $F \to FGF$ given by $(F\eta)_X = F(\eta_X)$.

17 Exactness

Definition 17.1. *Consider a sequence*[1] $M_\bullet = (M_i, d_i)_{i\in\mathbb{Z}}$ *of modules and homomorphisms of modules*

$$M_\bullet : \cdots \xrightarrow{d_{i-2}} M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \xrightarrow{d_{i+1}} \cdots .$$

We say that the sequence $M_\bullet$ *is*

- *a (cochain)*[2] *complex if* $d_{i+1}d_i = 0$ *for all* $i \in \mathbb{Z}$. *In such a case, we have* $\mathrm{Im}(d_i) \subset \mathrm{Ker}(d_{i+1})$ *for all* $i \in \mathbb{Z}$ *and the* i*-th cohomology of* $M_\bullet$ *is*

$$H^i(M_\bullet) := \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i-1}).$$

- *exact at* M_k *for some* $k \in \mathbb{Z}$ *if* $\mathrm{Im}(d_{k-1}) = \mathrm{Ker}(d_k)$. *Note that this implies* $d_k \circ d_{k-1} = 0$.
- *exact if it is so at every term.*
- *short exact (often abbreviated as s.e.s. or ses) if it is a 5-term exact sequence that starts and ends at the trivial module, i.e., of the form*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{17.1}$$

such that f *is injective,* g *is surjective, and* $\mathrm{Ker}(g) = \mathrm{Im}(f)$. *In this case,* M *is also called an extension of* N *by* L.

Definition 17.2. *A (covariant) functor* $F : \mathsf{mod}\, A \to \mathsf{mod}\, B$ *is*

- *left exact if it maps short exact sequence (such as (17.1)) to an exact sequence*

$$0 \to F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N).$$

In other words, it preserves kernel.

- *right exact if it maps short exact sequence (such as (17.1)) to an exact sequence*

$$F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \to 0.$$

In other words, it preserves cokernel.

- *exact if it is both left exact and right exact, i.e. maps ses to ses.*

We define left/right exactness for contravariant functor analogously. In particular, left exact contravariant fucntor turns cokernel into kernel.

Lemma 17.3. *Let* ${}_BX_A$ *be an* A-B*-bimodule. Then the following hold.*

(1) $\mathrm{Hom}_A(X, -)$ *maps an exact sequence* $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ *to an exact sequence*

$$0 \to \mathrm{Hom}_A(X, L) \xrightarrow{f\circ -} \mathrm{Hom}_A(X, M) \xrightarrow{g\circ -} \mathrm{Hom}_A(X, N).$$

In particular, $\mathrm{Hom}_A(X, -)$ *is left exact.*

(2) $\mathrm{Hom}_A(-, X)$ *maps an exact sequence* $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *to an exact sequence*

$$0 \to \mathrm{Hom}_A(N, X) \xrightarrow{-\circ g} \mathrm{Hom}_A(M, X) \xrightarrow{-\circ f} \mathrm{Hom}_A(L, X).$$

In particular, the contravariant functor $\mathrm{Hom}_A(-, X)$ *is left exact.*

[1] Superscript/subscript indexing formalism only matters to topologist; I will be liberal in these notations.

[2] Since we do not deal with any true topological theory, cochain just means the indices increase as we go along the sequence. We always use cochain convention except perhaps when dealing with projective resolution later.

Proof We show (1) and leave (2) for the reader.

Exactness at $\mathrm{Hom}_A(X, L)$: we need $f \circ -$ to be injective. Indeed, if $f \circ \theta = 0$ for some $\theta : X \to L$, then $f(\theta(x))$ for all $x \in X$. This means that $\theta(x) \in \mathrm{Ker}(f) = 0$, and so $\theta = 0$.

$\mathrm{Im}(f \circ -) \subset \mathrm{Ker}(g \circ -)$: Suppose that $\theta : X \to M$ is given by $f \circ \phi$ for some $\phi : X \to L$. Then $g\phi(x) = g(f\phi(x)) = (gf)\phi(x) = 0$, which means that $\theta \in \mathrm{Ker}(g \circ -)$.

$\mathrm{Ker}(g \circ -) \subset \mathrm{Im}(f \circ -)$: Suppose that $g\theta = 0$ for some $\theta : X \to M$. Then for every $x \in X$, we have $\theta(x) \in \mathrm{Ker}(g) = \mathrm{Im}(f)$, and so we can write $\theta(x) = f(\phi(x))$ for some $\phi(x) \in L$. Since f is injective, $\phi(x) \in L$ is uniquely determined, and so we have a well-defined function $\phi : X \to L$. We check that $\phi \in \mathrm{Hom}_A(X, L)$:

- $f(\phi(x + x')) = \theta(x + x') = \theta(x) + \theta(x') = f(\phi(x)) + f(\phi(x')) = f(\phi(x) + \phi'(x))$. Hence, f being injective implies that $\phi(x + x') = \phi(x) + \phi(x')$.
- Suppose that $\lambda \in \Bbbk$. Then $f(\phi(\lambda x)) = \theta(\lambda x) = \lambda\theta(x) = \lambda f(\phi(x)) = f(\lambda\phi(x))$. Hence, f being injective implies that $\lambda\phi(x) = \phi(\lambda x)$.

Now we have $\theta = f\phi$ as A-module homomorphism, and so $\theta \in \mathrm{Im}(f \circ -)$. □

A similar lemma for tensor product exists, and can be proved by direct verification as in the Hom functor case. Instead, we use another trick involving tensor-Hom adjunction, but first we need one more tool.

Lemma 17.4 (Yoneda embedding reflects exactness). *Consider a sequence $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathsf{mod}\, A$. If the sequence*

$$\mathrm{Hom}_A(X, L) \xrightarrow{f\circ -} \mathrm{Hom}_A(X, M) \xrightarrow{g\circ -} \mathrm{Hom}(X, N)$$

is exact for all $X \in \mathsf{mod}\, A$, then $L \xrightarrow{f} M \xrightarrow{g} N$ is also exact. Similarly, if

$$\mathrm{Hom}_A(N, X) \xrightarrow{-\circ g} \mathrm{Hom}_A(M, X) \xrightarrow{-\circ f} \mathrm{Hom}(N, X)$$

is exact for all $X \in \mathsf{mod}\, A$, then so is the original sequence.

Proof We show the first one.

$\mathrm{Im}(f) \subset \mathrm{Ker}(g)$: Take $X = L$, then we have $gf = (g \circ -)(f \circ -)(\mathrm{id}_L) = 0$.

$\mathrm{Ker}(g) \subset \mathrm{Im}(f)$: Consider $X = \mathrm{Ker}(g)$ and inclusion $\iota : \mathrm{Ker}(g) \hookrightarrow M$. Then $(g \circ -)(\iota) = g\iota = 0$, so exactness implies that $\iota = f\phi$ for some $\phi \in \mathrm{Hom}_A(\mathrm{Ker}(g), M)$. Hence, $\mathrm{Ker}(g) = \mathrm{Im}(\iota) \subset \mathrm{Im}(f)$. □

Lemma 17.5. *$- \otimes_A X$ maps an exact sequence $L \xrightarrow{f} M \xrightarrow{g} N \to 0$ to an exact sequence*

$$L \otimes_A X \xrightarrow{f\otimes_A X} M \otimes_A X \xrightarrow{g\otimes_A X} N \otimes_A X \to 0.$$

In particular, $- \otimes_A X$ is right exact.

Proof We apply $\mathrm{Hom}_B(-, Y)$ to the sequence (after tensoring X). By the naturality of the adjoint isomorphism, we have a commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_B(N \otimes_A X, Y) & \xrightarrow{-\circ g\otimes_A X} & \mathrm{Hom}_A(M \otimes_A X, Y) & \xrightarrow{-\circ f\otimes_A X} & \mathrm{Hom}_A(L \otimes_A X, Y) \\
\| & \circlearrowleft & \downarrow\cong & \circlearrowleft & \downarrow\cong & \circlearrowleft & \downarrow\cong \\
0 & \longrightarrow & \mathrm{Hom}_A(N, \mathrm{Hom}_B(X, Y)) & \xrightarrow{-\circ g} & \mathrm{Hom}_A(M, \mathrm{Hom}_B(X, Y)) & \xrightarrow{-\circ f} & \mathrm{Hom}_A(L, \mathrm{Hom}_B(X, Y))
\end{array}$$

The second row is exact since it is given by applying the left exact functor $\mathrm{Hom}_A(-, Z)$ for $Z = \mathrm{Hom}_B(X, Y)$. Hence, (by careful diagram chasing) the first row is also exact. Since Yoneda embedding reflects exactness, we get the claimed exactness. □

18 Projective and injective modules

Definition 18.1. *An A-module P is projective if for any given surjective homomorphism $f : M \twoheadrightarrow M'$ and any homomorphism $p : P \to M$, we have p factors through f, i.e. $\exists q : P \to M'$ s.t. $fq = p$ there is the following commutative diagram*

$$\begin{array}{ccc} & & P \\ & \overset{\exists q}{\swarrow} & \downarrow{\scriptstyle \forall p} \\ M' & \overset{f}{\twoheadrightarrow} & M. \end{array}$$

In other words, $f \circ - = \mathrm{Hom}_A(P, f) : \mathrm{Hom}_A(P, M') \to \mathrm{Hom}_A(P, M)$ is surjective, i.e. $\mathrm{Hom}_A(P, -)$ is exact. Denote by $\mathsf{proj}\, A$ the category of finitely generated projective A-modules.

Dually, an A-module I is injective if for any given injective homomorphism $f : M' \hookrightarrow M$ and any homomorphism $i : M \to I$, i factors through f. This is equivalent to saying that $\mathrm{Hom}_A(f, I) : \mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(M', I)$ is surjective, i.e. $\mathrm{Hom}_A(-, I)$ is exact. Denote by $\mathsf{inj}\, A$ the category of finitely generated injective A-modules.

Example 18.2. *Take $P = A$. Then we know that $\mathrm{Hom}_A(A, Y) \cong Y$ via $\alpha \mapsto \alpha(1)$ for any $Y \in \mathsf{mod}\, A$. Hence, for any surjective $f : M' \to M$ and any $p : A \to M$, to find q we only need to show that $f(q(1)) = p(1)$, but*

$$p(1) = f({}^{\exists}x) = f({}^{\exists}q(1)).$$

That is, the free A-module A_A is projective. Note that this does not require finite-dimensionality of A. Consequently, any free A-module (of any *rank) is also projective.*

Dually, using $\mathrm{Hom}_A(X, DA) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(A, DX)$ and the same argument, we get that DA is injective. Note that this DOES *require the finite-dimensionality of A since we need to the isomorphism between the Hom-space under duality.*

Lemma 18.3. *The following are equivalent for a ses $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$.*

(1) *There is some $h : N \to N$ such that $gh = \mathrm{id}_N$.*

(2) *There is some $e : M \to L$ such that $ef = \mathrm{id}_M$.*

(3) *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \| & & {\scriptstyle u}\downarrow{\scriptstyle \cong} & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow[(1,0)^T]{} & L \oplus N & \xrightarrow[(0,1)]{} & N & \longrightarrow & 0 \end{array}$$

In the case when any of these conditions is satisfied, we say that the ses splits.

Proof See 'Splitting lemma' on Wikipedia. □

Remark 18.4. Note that (3) is strictly stronger than just having $M \cong L \oplus N$ for general modules. However, in our setting[3], having $M \cong L \oplus N$ is enough for splitness. Indeed, applying $\mathrm{Hom}_A(-, L)$ yields an exact sequence

$$0 \to \mathrm{Hom}_A(N, L) \to \mathrm{Hom}_A(L \oplus N, N) \to \mathrm{Hom}_A(N, N)$$

of left $\mathrm{End}_A(N)$-modules. Now the original ses splits is equivalent to having $hf = \mathrm{id}_N$, and so is equivalent to the last map of this induced sequence to be surjective. Since everything is finite-dimensional in our setting, and $\dim_{\Bbbk} \mathrm{Hom}_A(L \oplus N, N) = \dim_{\Bbbk} \mathrm{Hom}_A(L, N) + \dim_{\Bbbk} \mathrm{Hom}_A(N, N)$, exactness at $\mathrm{Hom}_A(L \oplus N, N)$ means that the last map must be surjective.

[3] also OK for L, N finitely generated over a Noetherian A, see `https://mathoverflow.net/questions/167701/`

The following justifies why we called eA projective before.

Lemma 18.5. *The following are equivalent of a finitely genearted A-module P.*

1. *P is projective, i.e.* $\operatorname{Hom}_A(P, -)$ *is an exact functor.*
2. *Any ses* $0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0$ *splits.*
3. *P is a direct summand of a free module of finite rank.*

Proof (1) $\Rightarrow$ (2): We have a surjective map $\operatorname{Hom}_A(P, M) \xrightarrow{f\circ -} \operatorname{Hom}_A(P, P)$, and so $\mathrm{id}_P = fq$ for some $q : P \to M$.

(2) $\Rightarrow$ (3): Since P is finitely generated, there is a surjective A-module homomorphism $\pi : A^{\oplus n} \to P$ for some n. So we have a short exact sequence

$$0 \to \operatorname{Ker}\pi \to A^{\oplus n} \xrightarrow{\pi} P \to 0.$$

Hence, it follows by (2) and Lemma 18.3 that P is a direct summand of $A^{\oplus n}$.

(3) $\Rightarrow$ (1): We have learnt that indecomposable direct summands of A_A is given by the right ideal eA of some primitive idempotent $e = e^2 \in A$. Hence, by the assumption and Krull-Schmidt property $P = \bigoplus_{i=1}^n e_i A$ with e_i primitive idempotents. Now we have a natural projection $\pi : A^{\oplus n} \to P$ given by sending the i-th identity 1_A to e_i, and a natural inclusion $\iota : P \to A^{\oplus n}$ given by $\iota|_{e_iA} = (e_iA \hookrightarrow A)$. Note that $\pi\iota = \mathrm{id}_P$.

Consider a surjective A-module homomorphism $f : M \to N$ and take any A-module homomorphism $p : P \to N$. This yields $p\pi : A^{\oplus n} \to N$, which can be lifted to some $q' : A^{\oplus n} \to M$ as $A^{\oplus n}$ is projective. Now we have

$$(fq')\iota = (p\pi)\iota = p,$$

which means that taking $q = q'\iota$ give the required lift of p. □

Remark 18.6. This result do not require finiteness anywhere, nor Krull-Schmidt; but this special case yields an easier proof. For proof of the general case, see Rotman's book Prop 3.3 and Thm 3.5.

There is a dual result under some restriction.

Lemma 18.7. *Suppose A is finite dimensional and I is a finitely generated A-module. Then the following are equivalent.*

1. *I is injective, i.e.* $\operatorname{Hom}_A(-, I)$ *is an exact functor.*
2. *An ses* $0 \to I \to M \to N \to 0$ *splits.*
3. *I is a direct summand of finite direct sum of DA.*

19 Projective cover and injective hull

By definition, any finitely generated module M comes a canonical surjective A-module homomorphism $A^{\oplus n} \twoheadrightarrow M$. One can expect the kernel of this map is 'too large', meaning that many direct summands of the domain appear in the kernel. For more efficient calculation, we often use the most optimal direct summand of $A^{\oplus n}$.

Definition 19.1. *A projective cover of an A-module M is a projective A-module P along with a surjective A-module homomorphism $p : P_M \to M$ such that the restriction $p|_Q$ for every proper submodule $Q \subset P_M$ is non-surjective.*

Dually, an injective hull of M is an injective module I along with an injective A-module homomorphism $i : M \to I_M$ such that any proper quotient $q : I_M \twoheadrightarrow J$ yields a non-injective map qi.

Lemma 19.2. *The following hold for all $M \in \mathsf{mod}\, A$.*

(1) Projective cover P_M of M exists and is unique up to isomorphism. Moreover, it is characterised by $\mathrm{top}(P_M) \cong \mathrm{top}(M)$.

(2) Injective hull I_M of M exists and is unique up to isomorphism. Moreover, it is characterised by $\mathrm{soc}(I_M) \cong \mathrm{soc}(M)$.

Proof We show (1); (2) can be shown dually.

Suppose $\mathrm{top}(M) = M/\mathrm{rad}(M) \cong S_1^{\oplus m_1} \oplus \cdots \oplus S_n^{\oplus m_n}$. By consequence of Artin-Wedderburn, we have $S_i = P_i/\mathrm{rad}\, P_i$ for each i. Take $P_M = P_1^{\oplus m_1} \oplus \cdots \oplus P_n^{\oplus m_n}$.

Since $M \twoheadrightarrow M/\mathrm{rad}(M)$, the canonical surjection $P_M \twoheadrightarrow M/\mathrm{rad}(M)$ lifts to $p : P_M \to M$. As $M \twoheadrightarrow M/\mathrm{rad}(M)$, we have $\mathrm{Im}(p)+\mathrm{rad}(M) = M$, and so it follows from Nakayama lemma (Proposition 10.2 (4)) that $\mathrm{Im}(p) = M$, meaning that p is surjective.

Let $Q \subset P_M$ be a submodule; we show that $p|_Q$ is surjective implies $Q \cong P_M$. Indeed, $p|_Q$ surjective implies that $\mathrm{top}(\mathrm{Im}(p|_Q)) = \mathrm{top}(M)$. Hence, using the definition of P_M being projective we have a commutative diagram

$$\begin{array}{ccc} & & P_M \\ & \overset{\exists q}{\swarrow} & \downarrow \overline{p} \\ Q & \overset{g}{\twoheadrightarrow} & \mathrm{top}(M). \end{array}$$

Since Q surjects onto $\mathrm{top}(M)$, for $\iota : Q \hookrightarrow P_M$ the canonical inclusion we get that $g\iota q = \mathrm{top}(M) = \mathrm{top}(P)$. Hence, we have $\mathrm{Im}(\iota q) + \mathrm{rad}(P) = P$. By Nakayama lemma, we have that $\mathrm{Im}(\iota q) = P$, which means that ι is also surjective; thus, ι is an isomorphism, as required. □

Remark 19.3. The claim for projective cover is still true for artinian algebras; but the claim for injective hull really needs finite-dimensionality of A.

20 Resolution and Ext-group

Definition 20.1. *A* projective resolution $P_\bullet$ *of an* A*-module* M *is a sequence*

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

that is exact everywhere with P_k *projective for all* $k \geq 0$*. It is* minimal *if* $P_k \twoheadrightarrow \mathrm{Ker}(d_{k-1})$ *is a projective cover for all* $k \geq 1$*. The* n-th syzygy *of* M *is* $\mathrm{Ker}(d_n)$ *for* $(P_\bullet, d_\bullet)$ *the minimal projective resolution of* M*.*

Dually, an injective coresolution $I_\bullet$ *of* M *is a sequence*

$$0 \to M \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \to \cdots$$

that is exact everywhere with I_k *injective for all* $k \geq 0$*. It is* minimal *if* $\mathrm{Cok}(d_{k-1}) \hookrightarrow I_k$ *is an injective hull for all* $k \geq 1$*. The* n-th cosyzygy *of* M *is* $\mathrm{Cok}(d_{n-1})$ *for* $(I_\bullet, d_\bullet)$ *the minimal projective resolution of* M*.*

Definition 20.2. *For* A*-modules* M, N*, let* $P_\bullet$ *be a projective resolution of* M*. Define for* $k \geq 0$

$$\begin{aligned}
\mathrm{Ext}^k_A(M,N) :=& H^k(\mathrm{Hom}_A(P_\bullet, N)) \\
=& H^k(\cdots \xleftarrow{-\circ d} \mathrm{Hom}_A(P_{k+1}, N) \xleftarrow{-\circ d} \mathrm{Hom}_A(P_k, N) \leftarrow \cdots) \\
=& \frac{\{f : P_k \to N \mid (fd : P_{k+1} \to N) = 0\}}{\{f : P_k \to N \mid f = gd \text{ some } g : P_{k-1} \to N\}}
\end{aligned}$$

Note that $\mathrm{Ext}^0_A(M,N) = \mathrm{Hom}_A(M,N)$*.*

Similar to $\mathrm{Hom}_A(-,-)$, $\mathrm{Ext}^k_A(-,-)$ also commutes with finite direct sum in both variables.

There are some other ways to calculate the Ext-groups.

Proposition 20.3. *For any* A*-modules* M, N *and any* $k \geq 0$*, we have*

$$\mathrm{Ext}^k_A(M,N) = H^k(\mathrm{Hom}_A(M, I_\bullet))$$

where $I_\bullet$ *is an injective coresolution of* N*.*

Proposition 20.4 (Dimension shifting). *For each* $k \geq 1$*, there are natural isomorphisms*

$$\mathrm{Ext}^k_A(\Omega(M), N) \cong \mathrm{Ext}^{k+1}_A(M,N) \cong \mathrm{Ext}^k_A(M, \Omega^{-1}N),$$

where $\underline{\mathrm{Hom}}_A(X,Y)$ *(resp.* $\overline{\mathrm{Hom}}_A(X,Y)$*) is the quotient of* $\mathrm{Hom}_A(X,Y)$ *by the subspace consisting of* $f : M \to N$ *that factors through a projective (resp. injective)* A*-module, i.e. there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \qquad \nearrow & \\ & Z & \end{array}$$

for some projective (resp. injective) module Z*.*

Proof Consider the space $Z_k := \{f : P_k \to N \mid f d_{k+1} = 0\}$ in the definition of $\mathrm{Ext}^k_A(M,N)$. Since we have a exact sequence $P_{k+1} \to P_k \xrightarrow{p} \Omega^k(M) \to 0$, applying $\mathrm{Hom}_A(-,N)$ yields an exact sequence

$$0 \to \mathrm{Hom}_A(\Omega^k(M), N) \xrightarrow{-\circ p} \mathrm{Hom}_A(P_k, N) \xrightarrow{-\circ d_{k+1}} \mathrm{Hom}_A(P_{k+1}, N).$$

By exactness, we have $Z_k = \mathrm{Ker}(- \circ d_{k+1}) \cong \mathrm{Hom}_A(\Omega^k(M), N)$ sending each $f \in Z_k$ to $\overline{f}$ so that $\overline{f}p = f$.

It remains to show that this isomorphisms restricts to one between $B_k := \mathrm{Im}(-\circ d_k)$ and $\mathcal{P} := \{f \in \mathrm{Hom}_A(\Omega^k(M), N)$ that factors through projective$\}$. Clearly, any $f \in B_k$ (by definition) factors through a projective P_{k-1} and so $B_k \subset \mathcal{P}$. For $\overline{f} : \Omega^k(M) \to N$ that factors through a projective, say, P, we want $\overline{f}p = gd_k$ some g. Consider $0 \to \Omega^k(M) \to P_k \to \Omega^{k-1}(M) \to 0$ and apply $\mathrm{Hom}_A(-, N)$ yields

$$0 \to \mathrm{Hom}_A(\Omega^{k-1}(M), N) \to \mathrm{Hom}_A(P_k, N) \xrightarrow{-\circ d_{k+1}} \mathrm{Hom}_A(P_{k+1}, N).$$

□

Exercise 20.5. *Suppose that $M, N \in \mathsf{mod}\, A$.*

(1) *Show that when M or N is simple, we have $\mathrm{Hom}_A(\Omega(M), N) \cong \mathrm{Ext}^1_A(M, N) \cong \mathrm{Hom}_A(M, \Omega^{-1}(N))$.*

(2) *Show that when every projective A-module is injective, then $\mathrm{Ext}^1(M, N) \cong \underline{\mathrm{Hom}}_A(M, N)$ where $\underline{\mathrm{Hom}}_A(M, N)$ is the quotient of $\mathrm{Hom}_A(M, N)$ by all homomorphisms factoring through a projective module.*

Proposition 20.6. *Consider indecomposable projective modules P_x, P_y with simple tops S_x, S_y respectively. Then we have an isomorphism of $\Bbbk$-vector spaces $\mathrm{Ext}^1_A(S_x, S_y) \cong \mathrm{Hom}_A(\mathrm{rad}(P_x)/\mathrm{rad}^2(P_x), S_y)$. Moreover, the $\Bbbk$-dimension of this space is the same as that of $e_x \frac{\mathrm{rad}(A)}{\mathrm{rad}^2(A)} e_y$.*

Proof By the previous exercise, we have

$$\mathrm{Ext}^1_A(S_x, S_y) \cong \mathrm{Hom}_A(\Omega(S_x), S_y) \cong \mathrm{Hom}_A(\mathrm{rad}(P_x), S_y) \cong \mathrm{Hom}_A(\mathrm{rad}(P_x)/\mathrm{rad}^2(P_x), S_y).$$

For the last part, first we have by Schur's lemma

$$\mathrm{Hom}_A(\mathrm{rad}(P_x)/\mathrm{rad}^2(P_x), S_y) \cong \mathrm{Hom}_A(S_y, \mathrm{rad}(P_x)/\mathrm{rad}^2(P_x))$$

as $\Bbbk$-vector space, which then yields

$$\begin{aligned} \mathrm{Hom}_A(S_y, \mathrm{rad}(P_x)/\mathrm{rad}^2(P_x)) &\cong \mathrm{Hom}_A(P_y, \mathrm{rad}(P_x)/\mathrm{rad}^2(P_x)) \\ &\cong \mathrm{Hom}_A(e_y A, e_x \frac{\mathrm{rad}(A)}{\mathrm{rad}^2(A)}) \cong e_x \frac{\mathrm{rad}(A)}{\mathrm{rad}^2(A)} e_y \end{aligned}$$

where the last isomorphism uses Yoneda's lemma. □

Remark 20.7. Note that when $A = \Bbbk Q/I$ a bounded path algebra, then arrows from x to y in Q correspond bijectively to basis elements of $\mathrm{Ext}^1_A(S_x, S_y)$.

21 Induced long exact sequence

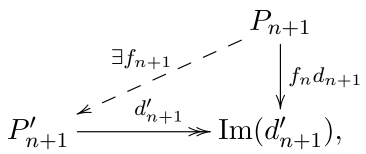
Definition 21.1. *Suppose $C_\bullet = (C_k, d_k)_k, C'_\bullet = (C'_k, d'_k)_k$ are (chain) complexes of A-modules. A chain map is $f_\bullet : C_\bullet \to C'_\bullet$ given by A-module homomorphisms $f_k : C_k \to C'_k$ over all $k \in \mathbb{Z}$ such that $d'_k f_k = f_{k+1} d_k$.*

Two chain maps $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ are homotopic (or "the same up to homotopy") if there exists a sequence of homomorphisms $(h_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ such that $f_n - g_n = d'_{n-1} h_n + h_{n+1} d_n$ for all $n \in \mathbb{Z}$. A chain map homotopic to 0 is said to be null-homotopic, or just call it a null-homotopy.

Theorem 21.2 (Comparison theorem). *An A-module homomorphism $f : M \to N$ extends to a chain map on their projective resolutions, as well as a chain map on their injective coresolutions. Moreover, changing the choice of (co)resolution results in a homotopic chain map.*

Proof Suppose $P_\bullet, P'_\bullet$ are projective resolutions of M and N respectively. Define the desired chain map $f_\bullet : (P_\bullet \to M \to 0) \to (P'_\bullet \to N \to 0)$ starting from $f_{-1} = f : M \to N$ inductively as follows. We take $P_{-1} = M$ and $P'_{-1} = N$.

Given $f_n : P_n \to P'_n$ defined, using the fact that P_n is projective we can lift $f_n d_{n+1}$, which yields a commutative diagram



with the desired chain map property $d'_{n+1} f_{n+1} = f_n d_{n+1}$.

Independence up to homotopy: EXERCISE.

The claim for injective coresolution can be shown analogously. □

Notation. *For a complex* $C_\bullet = (\cdots C_k \xrightarrow{d_k} C_{k+1} \to \cdots)$, *and* $z_k \in \mathrm{Ker}(d_k)$, *denote by* $[z_k] := z_k + \mathrm{Im}(d_{k-1})$.

Lemma 21.3. *Suppose* $C_\bullet, C'_\bullet$ *are complexes of* A*-modules and* $f_\bullet : C_\bullet \to C'_\bullet$ *is a chain map. Then for each* $k \in \mathbb{Z}$, *we have an induced* A*-module homomorphism* $H^k(f_\bullet) : H^k(C_\bullet) \to H^k(C'_\bullet)$ *given by* $[z_k] \mapsto [f_k(z_k)]$ *for any* $z_k \in \mathrm{Ker}(d_k : C_k \to C_{k+1})$. *Moreover,* H^k *preserves identity map and additive, as well as intertwines with composition, i.e.* H^k *is a functor from the category of complexes of* A*-modules to the category of* A*-modules.*

Proof Since $d_k(z_k) = 0$, we have

$$d'_k(f_k(z_k)) = f_{k+1} d_k(z_k) = f_{k+1}(0) = 0,$$

i.e. f_k restricts to a map $\mathrm{Ker}(d_k) \to \mathrm{Ker}(d'_k)$.

Suppose now that $z_k \in \mathrm{Im}(d_{k-1})$, say, $z_k = d_{k-1}(x_{k-1})$. Then we have

$$f_k(z_k) = f_k d_{k-1}(x_{k-1}) = d'_{k-1} f_{k-1}(x_{k-1}),$$

i.e. $\mathrm{Im}(f_k|_{\mathrm{Im}(d_{k-1})}) \subset \mathrm{Im}(d'_{k-1})$. Hence, $H^k(f_\bullet) : H^k(C_\bullet) \to H^k(C'_\bullet)$ is well-defined.

We leave the rest as exercise. □

The following is a powerful tool in computing Ext-groups.

Theorem 21.4 (Induced long exact sequence). *Suppose* $0 \to X \to Y \to Z \to 0$ *is a short exact sequence of* A*-modules. For any* A*-module* M, *there is the following long exact sequence:*

$$\begin{aligned} 0 \to \mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, Z) \to \\ \mathrm{Ext}^1_A(M, X) \to \mathrm{Ext}^1_A(M, Y) \to \mathrm{Ext}^1_A(M, Z) \to \\ \cdots \to \mathrm{Ext}^k_A(M, X) \to \mathrm{Ext}^k_A(M, Y) \to \mathrm{Ext}^k(M, Z) \to \cdots \end{aligned}$$

Proof omitted.

21.1 Other homological lemmata

Lemma 21.5 (Horseshoe lemma). *Suppose* $0 \to L \to M \to N \to 0$ *is a short exact sequence. Then a projective resolution* $P_\bullet$ *of* L *and a projective resolution* $Q_\bullet$ *induces a projective resolution of* M *given by with degree* $k \geq 0$ *term given by* $P_k \oplus Q_k$.

In pictorial form:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & P_1^L & & P_1^L \oplus P_1^N & & P_1^N & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & P_0^L & & P_0^L \oplus P_0^N & & P_0^N & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Lemma 21.6. (1) *Suppose there are exact rows and homomorphisms w, u such that the left-hand square commutes:*

$$\begin{array}{ccccccc}
L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
\downarrow{\scriptstyle w} & \circlearrowleft & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \exists v} & & \\
L' & \xrightarrow[f']{} & M' & \xrightarrow[g']{} & N' & \longrightarrow & 0
\end{array}$$

Then there exists a unique homomorphism $v : N \to N'$ such that the right-hand square commutes. Moreover, if w, u are isomorphisms, then so is v.

(2) *Suppose there are exact rows and homomorphisms u, v such that the right-hand square commutes:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \\
 & & \downarrow{\scriptstyle \exists w} & & \downarrow{\scriptstyle u} & \circlearrowleft & \downarrow{\scriptstyle v} \\
0 & \longrightarrow & L' & \xrightarrow[f']{} & M' & \xrightarrow[g']{} & N'
\end{array}$$

Then there exists a unique homomorphism $w : L \to L'$ such that the left-hand square commutes. Moreover, if u, v are isomorphisms, then so is w.

Proof Diagram chasing. □

Lemma 21.7 (Short 5-lemma). *Suppose there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle w} & \circlearrowleft & \downarrow{\scriptstyle u} & \circlearrowleft & \downarrow{\scriptstyle v} & & \\
0 & \longrightarrow & L' & \xrightarrow[f']{} & M' & \xrightarrow[g']{} & N' & \longrightarrow & 0
\end{array}$$

with exact rows. Then the following hold.

- *If w, v are both injective, then so is u.*
- *If w, v are both surjective, then so is v.*

Proof Diagram chasing. □

21.2 Ext-group versus Extensions

The previous proposition has a better intuition using another manifestation of the Ext-groups.

Definition 21.8. *Consider two extensions of N by L given by short exact sequences $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ and $0 \to L \xrightarrow{f'} M' \xrightarrow{g'} N \to 0$ are* equivalent *if there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle u} & & \| & & \\
0 & \longrightarrow & L & \xrightarrow[f']{} & M' & \xrightarrow[g']{} & N & \longrightarrow & 0
\end{array}$$

Remark 21.9. The map u is necessarily an isomorphism (as a consequence of 5-lemma (Lemma 21.7) or snake lemma).

Theorem 21.10. *There is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{extensions of } N \textit{ by } L \end{array} \right\} \leftrightarrow \operatorname{Ext}^1_A(N, L)$$

such that the split exact sequence corresponds to $0 \in \operatorname{Ext}^1_A(N, L)$.

Proof Let $\mathcal{E}(N, L)$ be the left-hand set. Let us first define a map $\phi : \mathcal{E}(N, L) \to \operatorname{Ext}^1_A(N, L)$. Consider an extension $\xi : 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$. Suppose that $P_\bullet$ is a projective resolution of N. Then, by the same yoga as in the proof of Comparison Theorem (Theorem 21.2), we can lift $\mathrm{id}_N : N \to N$ to a chain map:

$$\begin{array}{cccccccccc}
 & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & N & \longrightarrow & 0 \\
 & \vdots{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \| & & \\
\xi : & 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0
\end{array}$$

In particular, we have $d_2^*(\alpha) = \alpha_1 d_2 = 0$, so we have an element $[\alpha_1] \in \operatorname{Ext}^1(N, L) := \operatorname{Ker}(d_2^*)/\operatorname{Im}(d_1^*)$.

We claim that $[\xi] \mapsto [\alpha_1]$ is a well-defined map.

- Changing projective resolution: By the Comparison Theorem, we get a new chain map $(\alpha'_n)_n$ that is homotopic to $(\alpha_n)_n$. Spelling this out means that $\alpha'_1 - \alpha_1 = 0 \cdot h_1 + h_0 d_1 = h_0 d_1 = d_1^*(h_0) \in \operatorname{Im}(d_1^*)$. Hence, $[\alpha'_1] = [\alpha_1]$.
- Changing equivalence extension: By the definition of equivalence betwen extensions, we have a commutative diagram

$$\begin{array}{cccccccccc}
 & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & N & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \| & & \\
\xi : & 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \beta} & & \| & & \\
\xi' : & 0 & \longrightarrow & L & \xrightarrow{f} & M' & \xrightarrow{g} & N & \longrightarrow & 0
\end{array}$$

  This means that ξ' defines the same element $[\alpha_1] \in \operatorname{Ext}^1_A(N, L)$.

Now we show that if ξ is a split extension, then $\phi([\xi]) = 0$. Indeed, since ϕ is well-defined map to equivalence of extensions, we can just take the canonical split extension $L \oplus N$, then the canonical inclusion $\iota : N \to L \oplus N$ yields the following commutative diagram:

$$\begin{array}{ccccccccccc}
 & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & N & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \iota d_0} & & \| & & \\
\xi : & 0 & \longrightarrow & L & \xrightarrow{(1,0)^t} & L \oplus N & \xrightarrow{(0,1)} & N & \longrightarrow & 0
\end{array}$$

Hence, $\phi([\xi]) = 0$ as required.

Now we need to construct the inverse of ϕ.

- Construction of assignment $[\alpha] \mapsto [\xi]$: Consider the short exact sequence $0 \to \operatorname{Ker} d_0 \xrightarrow{i} P_0 \xrightarrow{d_0} N \to 0$ Applying $\operatorname{Hom}_A(-, L)$ and using $\operatorname{Ext}^1_A(P, -) = 0$ for any projective module P, we get an exact sequence

  $$\operatorname{Hom}_A(P, L) \to \operatorname{Hom}_A(\operatorname{Ker} d_0, L) \xrightarrow{\partial} \operatorname{Ext}^1_A(N, L) \to 0$$

  from the induced long exact sequence. Hence, for each $[\alpha] \in \operatorname{Ext}^1_A(N, L)$, we have some $\overline{\alpha} \in \operatorname{Hom}_A(\operatorname{Ker} d_0, L)$ such that $\partial(\overline{\alpha}) = \alpha$ [4] We form the pushout M of i and f, i.e.

  $$M := \operatorname{Cok}\big((i, -\overline{\alpha}) : \operatorname{Ker} d_0 \to P_0 \oplus L\big) = \frac{P_0 \oplus L}{S}, \text{ where } S := \{(i(x), -f(x)) \in P_0 \oplus L \mid x \in \Omega(N)\},$$

  then we get a commutative diagram

  $$\begin{array}{ccccccccccc}
   & 0 & \longrightarrow & \operatorname{Ker} d_0 & \xrightarrow{i} & P_0 & \xrightarrow{d_0} & N & \longrightarrow & 0 \\
   & & & \downarrow{\scriptstyle \overline{\alpha}} & & \downarrow{\scriptstyle \alpha_0} & & \| & & \\
  \xi : & 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0
  \end{array}$$

  where $\alpha_0(p) := (0, p) + S$, $f(l) := (l, 0) + S$, and $g([p, l]) := j(p)$. Note that the bottom row is also exact (Exercise: check this!). We now have an assignment $[\alpha] \mapsto [\xi]$.

- Well-definedness: Suppose we choose another $\alpha' \in [\alpha]$ and associated $\overline{\alpha}' : \operatorname{Ker} d_0 \to L$. Then following the same procedure we have another extension $0 \to L \xrightarrow{f'} M' \xrightarrow{g'} N \to 0$ and homomorphism $\alpha'_0 : P_0 \to M'$. Define $\theta : M \to M'$ by $(l, p) + S \mapsto f'(l) + \alpha'_0(p) \in M'$ yields a commutative diagram:

  $$\begin{array}{ccccccccccc}
  \xi : & 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{d_0} & N & \longrightarrow & 0 \\
   & & & \| & & \downarrow{\scriptstyle \theta} & & \| & & \\
  \xi' : & 0 & \longrightarrow & L & \xrightarrow{f'} & M' & \xrightarrow{g'} & N & \longrightarrow & 0
  \end{array}$$

  Hence, we have $[\xi] = [\xi']$.

- Inverse to ϕ: This is straightforward to check that $\phi\theta = \mathrm{id}$ by construction; for $\theta\phi = \mathrm{id}$, note that α_1 induces naturally $\overline{\alpha_1} : \operatorname{Ker}(d_0) \to L$ which allows us to check it is compatible with θ.

□

$\operatorname{Ext}^1_A(N, L)$ is an abelian group, so there is a binary operation. There is indeed a corresponding operation on short exact sequences (which we omit in this text).

Theorem 21.11. *The set of equivalence classes of short exact sequence with first term L and last term N form an abelian group under Baer sum, and this abelian group is isomorphic to $\operatorname{Ext}^1_A(N, L)$, with the zero element corresponding to the equivalence class of split short exact sequences.*

[4] This is, in fact, the homomorphism induced by some $P_1 \to L$ by looking at the definition of $\operatorname{Ext}^1_A(N, L)$ as a homology group of the Hom-complex.

There exists similar description for $\mathrm{Ext}^n_A(N, L)$ but the notion of splitness is not as nice as in the case of ses. In any case, for us, we only need to keep in mind that $\mathrm{Ext}^1_A(N, L)$ contains information about short exact sequence of the form $0 \to L \to M \to N \to 0$; c.f. Proposition 20.6 and relation with arrows of quiver. Having said that, we should warn that equivalence classes of ses is not the same as isomorphism classes of the middle term, i.e. there exists non-equivalent ses with the same middle term.