Topics in Mathematical Science VIII
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Introduction to quiver representations and homological algebras

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Convention

Throughout the course, $\Bbbk$ will always be a field. All rings are unital and associative. We only really work with artinian rings (but sometimes noetherian is also OK). We always compose maps from right to left.

1 Reminder on some basics of rings and modules

Definition 1.1. *Let R be a ring. A right R-module M is an abelian group $(M, +)$ equipped with a (linear) R-action on the right of M $\cdot : M \times R \to M$, meaning that for all $r, s \in R$ and $m, n \in M$, we have*

- $m \cdot 1 = m$,
- $(m + n) \cdot r = m \cdot r + n \cdot r$,
- $m \cdot (r + s) = m \cdot r + m \cdot s$,
- $m(sr) = (ms)r$.

Dually, a left R-module is one where R acts on the left of M (details of definition left as exercise). Sometimes, for clarity, we write M_A for right A-module and ${}_AM$ for left A-module.

Note that, for a commutative ring, the class of left modules coincides with that of right modules.

Example 1.2. *R is naturally a left, and a right, R-module. Both are free R-module of rank 1. Sometimes this is also called regular modules but it clashes with terminology used in quiver representation and so we will avoid it.*

In general, a free R-module F is one where there is a basis $\{x_i\}_{i\in I}$ such that for all $x \in F$, $x = \sum_{i\in I} x_i r_i$ with $r_i \in R$. We only really work with free modules of finite rank, i.e. when the indexing set I is finite. In such a case, we write R^n.

Convention. *All modules are right modules unless otherwise specified.*

Definition 1.3. *Suppose R is a* commutative *ring. A ring A is called an R-algebra if there is a (unital) ring homomorphism $\theta : R \to A$ with image $\theta(R)$ being in the center $Z(A) := \{z \in A \mid za = az\ \forall a \in A\}$ of A. In such a case, A is an R-module and so we simply write ar for $a \in A, r \in R$ instead of $a\theta(r)$.*

An (unital) R-algebra homomorphism $f : A \to A'$ is a (unital) ring homomorphism f that intertwines R-action, i.e. $f(ar) = f(a)r$.

The dimension of a $\Bbbk$-algebra A is the dimension of A as a $\Bbbk$-vector space; we say that A is finite-dimensional if $\dim_\Bbbk A < \infty$.

Note that commutative ring theorists usually use dimension to mean Krull dimension, which has a completely different meaning.

Example 1.4. *Every ring is a $\mathbb{Z}$-algebra.*

The matrix ring $M_n(R)$ given by n-by-n matrices with entries in R is an R-algebra.

We will only really work with $\Bbbk$-algebras, where $\Bbbk$ is a field. Most of the time, we will also assume $\Bbbk$ is algebraically closed for simplicity. But it worth reminding there are many interesting R-algebras for different R, such as group algebra. Recall that the *characteristic* of R, denoted by $\operatorname{char} R$, is 0 if the additive order of the identity 1 is infinite, or else the additive order itself.

Example 1.5. *Let G be a finite (semi)group and R a commutative ring. Let $A := R[G]$ be the free R-module with basis G, i.e. every $a \in A$ can be written as the formal R-linear combination $\sum_{g \in G} \lambda_g g$ with $\lambda_g \in R$. Then group multiplication extends (R-linearly) to a ring multiplication on $R[G]$, making A an R-algebra.*

Example 1.6. *Recall that the direct product of two rings A, B is the ring $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with unit $1_{A\times B} = (1_A, 1_B)$. It is straightforward to check that if A, B are R-algebras, then $A \times B$ is also an R-algebra.*

Example 1.7. *Suppose that A is a $\Bbbk$-algebra and B is a $\Bbbk$-subspace of A containing 1_A and closed under multiplication. Then B is also a $\Bbbk$-algebra. We call such a B a subalgebra of A. For a concrete example, the space of diagonal matrices forms a subalgebra of $M_n(\Bbbk)$.*

Definition 1.8. *A map $f : M \to N$ between right R-modules M, N is a homomorphism if it is a homomorphism of abelian groups (i.e. $f(m + n) = f(m) + f(n)$ for all $m, n \in M$) that intertwines R-action (i.e. $f(mr) = f(m)r$ for all $m \in M$ and $r \in R$). Denote by $\operatorname{Hom}_R(M, N)$ the set of all R-module homomorphisms from M to N. We also write $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$.*

Lemma 1.9. *$\operatorname{Hom}_R(M, N)$ is an abelian group with $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \operatorname{Hom}_R(M, N)$ and all $m \in M$. If R is commutative, then $\operatorname{Hom}_R(M, N)$ is an R-module, namely, for a homomorphism $f : M \to N$ and $r \in R$, the homomorphism fr is given by $m \mapsto f(mr)$.*

Definition 1.10. *$\operatorname{End}_R(M)$ is an associative ring where multiplication is given by composition and identity element being id_M. We call this the endomorphism ring of M.*

Lemma 1.11. *If A is an R-algebra over a commutative ring R, then any right A-module is also an R-module, and $\operatorname{Hom}_A(M, N)$ is also an R-module (hence, $\operatorname{End}_R(M)$ is an R-algebra).*

Example 1.12. *$A \cong \operatorname{End}_A(A)$ given by $a \mapsto (1_A \mapsto a)$ is an isomorphism of rings (or of R-algebras if A is an R-algebra). Note that if we work with left modules, then $A \cong \operatorname{End}_A({}_AA)^{\mathrm{op}}$, where $(-)^{\mathrm{op}}$ denotes the opposite ring given by the same underlying set with reverse direction of multiplication, i.e. $a \cdot_{\mathrm{op}} b := b \cdot a$.*

Recall that an R-module M is *finitely generated* if there exists as surjective homomorphism $R^n \twoheadrightarrow M$, or equivalently, there is a finite set $X \subset M$ such that for any $m \in M$, we have $m = \sum_{x \in X} x r_x$ for some $r_x \in R$.

Notation. *We write $\mathsf{mod}\, A$ for the collection of all finitely generated right A-modules.*

2 Indecomposable modules and Krull-Schmidt property

We recall two types of building blocks of modules. The first one is indecomposability.

Definition 2.1. *Let M be a R-module and $N_1, \ldots, N_r$ be submodules. We say that M is the direct sum $N_1 \oplus \cdots \oplus N_r$ of the N_i's if $M = N_1 + \cdots + N_r$ and $N_j \cap (N_1 + \cdots + N_{\hat{j}} + \cdots N_r) = 0$. Equivalently, every $m \in M$ can be written uniquely as $n_1 + n_2 + \cdots + n_r$ with $n_i \in N_i$ for all i. In such a case, we write $M \cong N_1 \oplus \cdots \oplus N_r$. Each N_i is called a direct summand of M.*

M is called indecomposable if $M \cong N_1 \oplus N_2$ implies $N_1 = 0$ or $N_2 = 0$.

We say that $M = \bigoplus_{i=1}^m M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable.

Convention. *We write $(n_1, \ldots, n_r)$ instead of $n_1 + \cdots + n_r$ with $n_i \in N_i$ for a direct sum $N_1 \oplus \cdots \oplus N_r$.*

We will only work with direct sum with finitely many indecomposable direct summands.

Example 2.2. *Suppose that R_R is indecomposable as an R-module. If F is a free R-module of rank n, then $R^{\oplus n} := R \oplus R \oplus \cdots \oplus R$ (with n copies of R) is a decomposition of F.*

Example 2.3. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Let V be the 'row space', i.e. $V = \{(v_j)_{1 \le j \le n} \mid v_j \in \Bbbk\}$ where $X \in \mathrm{Mat}_n(\Bbbk)$ acts on $v \in V$ by $v \mapsto vX$ (matrix multiplication from the right). Since for any pair $u, v \in V$, there always exist X so that $v = uX$, we see that there is no other A-submodule of V other than 0 or V itself. Hence, V is an indecomposable A-module. In particular, the n different ways of embedding a row into an n-by-n-matrix yields an A-module isomorphism between $V^{\oplus n} \cong A_A$, which is the decomposition of the free A-module A_A.*

The above example shows indecomposability by showing that V is a *simple* A-module, which is a stronger condition that we will come back later. Let us give an example of a different type of indecomposable (but non-simple) modules.

Example 2.4. *Let $A = \Bbbk[x]/(x^k)$ the truncated polynomial ring for some $k \ge 2$. This is an algebra generated by (1_A and) x, and an A-module is just a $\Bbbk$-vector space V equipped with a linear transformation $\rho_x \in \mathrm{End}_{\Bbbk}(V)$ (representing the action of x) such that $\rho_x^k = 0$.*

Consider a 2-dimensional space $V = \Bbbk\{v_1, v_2\}$ and a linear transformation

$$\rho_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By definition $(av_1 + bv_2)x = (a + b)v_2$, and so any submodules must contains $\Bbbk v_2$, i.e. v_2 spans a unique non-zero submodules. If, on the contrary, V is not indecomposable, then we have $V = U_1 \oplus U_2$ for (at least) two non-zero submodules U_1, U_2. But v_2 must be contained in any submodule of V, hence, we have $v_2 \in U_1 \cap U_2$, i.e. $U_1 \cap U_2 \neq 0$ – a contradiction not decomposability.

Proposition 2.5. *There is a canonical R-module isomorphism*

$$\begin{array}{ccc} \mathrm{Hom}_A(\bigoplus_{j=1}^m M_j, \bigoplus_{i=1}^n N_i) & \xrightarrow{\cong} & \bigoplus_{i,j} \mathrm{Hom}_A(M_j, N_i) \\ f & \longmapsto & (\pi_i f \iota_j)_{i,j} \end{array}$$

where $\iota_j : N_j \to \bigoplus_j N_j$ is the canonical inclusion for all j and $\pi_i : \bigoplus_i M_i \to M_i$ is the canonical projection for all i.

One can think of the right-hand space above as the space of m-by-n matrix with entries in each corresponding Hom-space.

Recall that an *idempotent* $e \in R$ is an element with $e^2 = e$. For example, the identity map $\mathrm{id}_M \in \mathrm{End}_A(M)$ (the unit element of the endomorphism ring) is an idempotent. From the previous proposition, we see that for a decomposition $M = N_1 \oplus N_2$, we have idempotents

$$e_i : M \xrightarrow{\pi_i} N_i \xrightarrow{\iota_i} M$$

for both $i = 1, 2$. Hence, being decomposable implies existence of multiple idempotents; this turns out characterise indecomposability completely.

Proposition 2.6. *Let A be a finite-dimensional algebra and M be a finite-dimensional non-zero A-module. Then the following hold.*

(1) *(Fitting's lemma) For any $f \in \mathrm{End}_A(M)$, there exists $n \geq 1$ such that $M \cong \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n)$.*

(2) *The following are equivalent.*
- *M is indecomposable.*
- *The endomorphism algebra $\mathrm{End}_A(M)$ does not contain any idempotents except 0 and id_M.*
- *Every homomorphism $f \in \mathrm{End}_A(M)$ is either an isomorphism or is nilpotent.*
- *$\mathrm{End}_A(M)$ is local (see below).*

Remark 2.7. It is known that if M is only artinian or only noetherian, then Fitting's lemma (and hence part (2)) fails. Nevertheless, in general, the proposition still hold for M that is both artinian and noetherian.

Let us briefly recall various characterisation of local rings.

Definition 2.8. *A ring R is local if it has a unique maximal right (equivalently, left; equivalently, two-sided) ideal.*

Remark 2.9. When R is non-commutative, the 'non-invertible elements' are the ones that do not admit (right) inverses.

Lemma 2.10. *The following are equivalent for a finite-dimensional algebra A.*

- *A is local (i.e. has a unique maximal right ideal).*
- *Non-invertible elements of A form a two-sided ideal.*
- *For any $a \in A$, one of a or $1 - a$ is invertible.*
- *0 and 1_A are the only idempotents of A.*
- *$A/J(A) \cong \Bbbk$ as rings, where $J(A)$ is the two-sided ideal of A given by the intersection of all maximal right (equivalently, left) ideals.*

Example 2.11. *Consider the upper triangular 2-by-2 matrix ring*

$$A = \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq 2} \,\middle|\, \begin{array}{l} a_{i,j} \in \Bbbk \; \forall i \leq j \\ a_{i,j} = 0 \; \forall i > j \end{array} \right\}.$$

Let $M = \{(x, y) \in \Bbbk^2\}$ be the 2-dimensional space where A acts as matrix multiplication (on the right). Suppose $f \in \mathrm{End}_A(M)$, say, $f(x, y) = (ax + by, cx + dy)$ for some $a, b, c, d \in \Bbbk$. Then being an A-module homomorphisms means that

$$(ax + by, cx + dy) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = f\left((x, y) \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \right) = (aux + bvx + wy, cux + dvx + dwy)$$

for all $u, v, w, x, y \in \Bbbk$. This means that

$$\begin{cases} buy = bvx + bwy \\ avx + bvy + cxw = cux + dvx \end{cases}.$$

The first line yields $b = 0$, and the second line yields $c = 0 = b$ and $a = d$. In other words, $\mathrm{End}_A(M) \cong \Bbbk$ which is clearly a local algebra. Hence, M is indecomposable.

A natural question is to ask when is a decomposition of modules, if it exists, unique up to permuting the direct summands.

Definition 2.12. *We say that an indecomposable decomposition $M = \bigoplus_{i=1}^{m} M_i$ is unique if any other indecomposable decomposition $M = \bigoplus_{j=1}^{n} N_j$ implies that $m = n$ and there is a permutation σ such that $M_i \cong N_{\sigma(i)}$ for all $1 \leq i \leq m$. $\mathsf{mod}\, A$ is said to be Krull-Schmidt if every (finitely generated) A-module M admits a unique indecomposable decomposition.*

Theorem 2.13. *For a finite-dimensional algebra A, $\mathsf{mod}\, A$ is Krull-Schmidt.*

Remark 2.14. This is a special case of the Krull-Schmidt theorem - whose proof we will omit to save time.

Theorem 2.15 (Krull-Schmidt). *Suppose $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \leq i \leq m$, then the decomposition of M is unique.*

Remark 2.16. Some people refer to this result as Krull-Remak-Schmidt theorem.

3 Simple modules, Schur's lemma

Definition 3.1. *Let M be an R-module.*

(1) *M is simple if $M \neq 0$, and for any submodule $L \subset M$, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 3.2. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Remark 3.3. Note that a module is semisimple if and only if every submodule is a direct summand.

Example 3.4. *Consider the matrix ring $A := \mathrm{Mat}_n(\Bbbk)$ over a field $\Bbbk$. Then the row-space representation V is an n-dimensional simple module. Since $A_A \cong V^{\oplus n}$, we have that A_A is a semisimple module.*

Example 3.5. *The ring of dual numbers is $A := \Bbbk[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as $(x) = AxA$ is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by $\Bbbk v$ where $v = 1+(x)$. If A is semisimple, then the 1-dimensional space $\Bbbk v$ is isomorphic to a submodule of A. Such a submodule must be generated by $a + bx$ (over A) for some $a, b \in \Bbbk$. If $a \neq 0$, then $(a + bx)A = A$. So $a = 0$, and $\Bbbk v \cong (x)$, a contradiction.*

Lemma 3.6. *S is a simple A-module if and only if for any non-zero $m \in S$, we have $mA := \{ma \mid a \in A\} = S$. In particular, simple modules are cyclic (i.e. generated by one element).*

Let us see how one can find a simple module.

Definition 3.7. *Let M be an A-module and take any $m \in M$. The annihilator of m (in A) is the set $\mathrm{Ann}_A(m) := \{a \in A \mid ma = 0\}$.*

Note that $\mathrm{Ann}_A(m)$ is a right ideal of A - hence, a right A-module.

Lemma 3.8. *For a simple A-module S and any non-zero $m \in S$, we have $S \cong A/\mathrm{Ann}_A(m)$ as A-module. In particular, if A is finite-dimensional, then every simple A-module is also finite-dimensional.*

Suppose I is a two-sided ideal of A. Then we have a quotient algebra $B := A/I$. For any B-module M, we have a canonical A-module structure on M given by $ma := m(a + I)$. This is (somewhat confusingly) the *restriction of M along the algebra homomorphism $A \twoheadrightarrow A/I$*.

Lemma 3.9. *Suppose $B := A/I$ is a quotient algebra of A by a strict two-sided ideal $I \neq A$. If $S \in \mathsf{mod}\, B$ is simple, then S is also simple as A-module*

Proof This follows from the easy observation that any a B-submodule of S_B is also a A-submodule of S_A under restriction. $\square$

The following easy, yet fundamental, lemma describes the relation between simple modules. Recall that a division ring is one where every non-zero element admits an inverse (but the ring is not necessarily commutative).

Lemma 3.10 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S,T) = \begin{cases} \text{a division ring}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.11. Note that if A is an R-algebra, then the division ring appearing is also an R-algebra (since it is the endomorphism ring of an A-module). In particular, if R is an algebraically closed field $\Bbbk = \overline{\Bbbk}$, then any division $\Bbbk$-algebra is just $\Bbbk$ itself.

Proof The claim is equivalent to saying that any $f \in \mathrm{Hom}_A(S,T)$ is either zero or an isomorphism. Since $\mathrm{Im}(f)$ is a submodule of T, simplicity of T says that $\mathrm{Im}(f) = 0$, i.e. $f = 0$, or $\mathrm{Im}(f) \cong T$. In the latter case, we can consider $\mathrm{Ker}(f)$, which is a submodule of S, so by simplicity of S it is either 0 or S itself. But this cannot be S as this means $f = 0$, hence, $\mathrm{Im}(f) \cong T$ implies that $\mathrm{Ker}(f) = 0$, i.e. f is an isomorphism. □

Example 3.12. *In Example 2.11, we showed that the upper triangular 2-by-2 matrix ring A has a 2-dimensional indecomposable module $P_1 = \{(x,y) \mid x, y \in \Bbbk^2\}$ given by 'row vectors'. It is straightforward to check that there is a 1-dimensional (hence, simple) submodule given by $S_2 := \{(0,y) \mid y \in \Bbbk^2\}$.*

Consider the module $S_1 := P_1/S_2$. This is a 1-dimensional (simple) module spanned by, say, w with A-action given by

$$w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := wa.$$

Consider a homomorphism $f \in \mathrm{Hom}_A(S_1, S_2)$. This will be of the form $w \mapsto (0,y)$ for some $y \in \Bbbk$ and has to satisfy

$$(0, ya) = (0,y)a = f(wa) = f(w \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = f(w) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (0,y)c = (0, yc)$$

for any $a, b, c \in \Bbbk$. Hence, we must have $y = 0$, which means that $f = 0$. In particular, by Schur's lemma $S_1 \not\cong S_2$.

Lemma 3.13. *Suppose that S is a simple A-module. Consider a semisimple A-module $M = S_1 \oplus \cdots \oplus S_n$ with $S_i \cong S$ for all i. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_n(D)$, where $D := \mathrm{End}_A(S)$.*

Proof We have canonical inclusion $\iota_j : S_j \hookrightarrow M$ and projection $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this is an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \le i,j \le r},$$

which is clearly injective. Conversely, for $(a_{i,j})_{1 \le i,j \le r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \overset{a_{i,j}}{\to} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. □

Example 3.14. *For a tautological example, take $A = \Bbbk$ to be just a field. Then we have a 1-dimensional simple A-module $S = \Bbbk$ with $\mathrm{End}_A(S^{\oplus n}) = \mathrm{Mat}_n(\mathrm{End}_A(\Bbbk)) = \mathrm{Mat}_n(\Bbbk)$. Note that now we have an n-dimensional simple $\mathrm{Mat}_n(\Bbbk)$-module (given by the row vectors).*

4 Quiver and path algebra

Definition 4.1. *A (finite) quiver is a datum $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ for finite sets Q_0, Q_1. The elements of Q_0 are called vertices and those of Q_1 are called arrows. The source (resp. target)of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).*

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 4.2. *Let Q be a quiver.*

- *A trivial path on Q is a "stationary walk at i", denoted by e_i for some $i \in Q_0$.*
- *A path of Q is either a trivial path or a word $\alpha_1\alpha_2 \cdots \alpha_\ell$ of arrows with $s(\alpha_i) = t(\alpha_{i+1})$.*

The source and target functions extend naturally to paths, with $s(e_i) = i = t(e_i)$. Two paths p, q can be concatenated to a new one pq if $t(p) = s(q)$; note that our convention is to read *from left to right*.

Definition 4.3. *The path algebra $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra whose underlying vector space is given by $\bigoplus_{p:\text{paths of } Q} \Bbbk p$, with multiplication given by path concatenation. That is $x \in \Bbbk Q$ is a formal linear combinations of paths on Q.*

Note that $e_i e_j = \delta_{i,j} e_i$, where $\delta_{i,j} = 1$ if $i = j$ else 0. In other words, e_i is an *idempotent* of the path algebra $\Bbbk Q$. Moreover, we have an idempotent decomposition

$$1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$$

of the unit element of $\Bbbk Q$.

Example 4.4. *Consider the one-looped quiver, a.k.a. Jordan quiver,*

$$Q = \left(\begin{tikzcd} \bullet \arrow[loop above, "\alpha"] \end{tikzcd} \right)$$

Then $\Bbbk Q$ has basis $\{\alpha^k \mid k \geq 0\}$ (note that the trivial path at the unique vertex is the identity element). Then $\Bbbk Q \cong \Bbbk[x]$.

An *oriented cycle* is a path of the form $v_1 \to v_2 \to \cdots v_r \to v_1$, i.e. starts and ends at the same vertex. If Q does not contain any oriented cycle, we say that it is *acyclic*.

Proposition 4.5. *$\Bbbk Q$ is finite-dimensional if, and only if, Q is finite acyclic.*

Proof If there is an oriented cycle c, then $c^k \in \Bbbk Q$ for all $k \geq 0$, and so $\Bbbk Q$ is infinite-dimensional. Otherwise, there are only finitely many paths on Q. □

Example 4.6. *Consider the linearly oriented $\vec{\mathbb{A}}_n$-quiver*

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then the path algebra $\Bbbk Q$ has basis $\{e_i, \alpha_{j,k} \mid 1 \leq i \leq n, 1 \leq j \leq k \leq n\}$, where $\alpha_{j,k} := \alpha_j \alpha_{j+1} \cdots \alpha_k$.

Consider the upper triangular n-by-n matrix ring

$$\begin{pmatrix} \Bbbk & \Bbbk & \cdots & \Bbbk \\ 0 & \Bbbk & \cdots & \Bbbk \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \Bbbk \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i \leq j \leq n} \middle| \begin{array}{l} a_{i,j} \in \Bbbk \ \forall i \leq j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}.$$

Denote by $E_{i,j}$ the elementary matrix whose entries are all zero except at (i,j) where it is one. This ring is isomorphic to $\Bbbk Q$ via $E_{i,i} \mapsto e_i$ and $E_{i,j} \mapsto \alpha_{i,j-1}$ for $1 \le j < k \le n$.

From now on, we will focus in the following setting.

Assumption 4.7. (1) *Quivers are finite (i.e. finitely many vertices and arrows).*

(2) *Representations (equivalently, modules) are finite-dimensional.*

5 Duality

For a quiver Q, the opposite quiver Q^{op} has the same set of vertices with the reverse direction of arrows, i.e. $Q_0^{\mathrm{op}} = Q_0, Q_1^{\mathrm{op}} = Q_1, s_{Q^{\mathrm{op}}} = t_Q$, and $t_{Q^{\mathrm{op}}} = s_Q$.

Exercise 5.1. *Show that there is a canonical isomorphism $(\Bbbk Q)^{\mathrm{op}} \cong \Bbbk(Q^{\mathrm{op}})$.*

Let M be a finite-dimensional A-module. Then we have a dual space

$$D(M) := M^* := \mathrm{Hom}_{\Bbbk}(M, \Bbbk),$$

which has a natural A^{op}-module structure, namely, $(a \cdot f)(m) := f(ma)$ for any $a \in A, f \in M^*, m \in M$. Moreover, for an A-module homomorphism $\theta : M \to N$, we have also an A^{op}-module homomorphism $\theta^* : N^* \to M^*$ with $\theta^*(f)(m) = f(\theta(m))$.

Lemma 5.2. *There is a $\Bbbk$-vector space isomorphism $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$.*

Proof Just a straightforward check that $(\theta^*)^* = \theta$. □

We note as a fact that D preserves indecomposability of (finite-dimensional) modules. This can be seen using the fact that $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(DN, DM)$ and can be upgraded to an algebra isomorphism for the case when $N = M$; then uses characterisation of indecomposable module by local endomorphism ring.

Example 5.3. *The* left *A-module ${}_AA$ yields a right A-module structure on $D(A)$. More generally, suppose we have a left ideal Ae of A for some element $e \in A$, then $D(Ae)$ is a right ideal of A.*

Remark 5.4. There is another natural duality, which we will not used, between $\mathsf{mod}\, A$ and $\mathsf{mod}\, A^{\mathrm{op}}$ given by sending M to $\mathrm{Hom}_A(M, A)$. In general, this duality is different from the $\Bbbk$-linear dual unless A is a so-called *symmetric algebra*, meaning that $A \cong DA$ as bimodule; in which case, $\mathrm{Hom}_A(-, A)$ dual is naturally isomorphic to D (as functors).

6 Representations of quiver

Definition 6.1. *A $\Bbbk$-linear* representation *of Q is a datum $(\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$ where M_i is a $\Bbbk$-vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is $\Bbbk$-linear map for each $\alpha \in Q_1$.*

Such a representation is finite-dimensional *if $\dim_{\Bbbk} M_i < \infty$ for all $i \in Q_0$.*

Notation. *For a representation M of Q, we take $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$ for a path $p = \alpha_1 \cdots \alpha_\ell$.*

It is easy to notice that every representation of Q is equivalent to a $\Bbbk Q$-module, namely,

$$\text{representation } (\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1}) \leftrightarrow \begin{array}{c}\Bbbk Q\text{-module } \prod_{i\in Q_0} M_i \\ \text{s.t. } \sum_{p:\text{path}} \lambda_p p \text{ acts as } \sum_p \lambda_p M_p.\end{array}$$

Example 6.2 (Simple). *For $x \in Q_0$, denote by S_x (or $S(x)$) the representation given by putting a 1-dimensional space on x, zero on all other vertices, and zero on all arrows. This corresponds to a 1-dimensional $\Bbbk Q$-module and so we call it the simple at x.*

Note: at this stage, it is not clear if these are all the simple $\Bbbk Q$-modules (up to isomorphism) yet.

Example 6.3 (Projective). *For $x \in Q_0$, denote by P_x (or $P(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p\text{:path with}\\ s(p)=x,\\ t(p)=y}} \Bbbk p, \qquad \textit{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p\alpha=q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\mathrm{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the projective at x. This corresponds to the right ideal $e_x \Bbbk Q$ of $\Bbbk Q$.

Example 6.4 (Injective). *Dual to the projective module construction, for $x \in Q_0$, denote by I_x (or $I(x)$) the representation given by $(\{M_y\}_{y\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$, where*

$$M_y := \bigoplus_{\substack{p\text{:path with}\\ s(p)=y,\\ t(p)=x}} \Bbbk p, \qquad \textit{and} \qquad (M_\alpha : M_y \to M_z) := \sum_{p=\alpha q} (M_y \twoheadrightarrow \Bbbk p \xrightarrow{\mathrm{id}} \Bbbk q \hookrightarrow M_z).$$

This is called the injective at x. This corresponds to the dual of the left ideal generated by e_x, i.e. $D(\Bbbk Q e_x)$.

Example 6.5. *The representation of $Q = \vec{\mathbb{A}}_n$ given by*

$$U_{i,j} := 0 \to \cdots 0 \to \Bbbk \xrightarrow{\mathrm{id}}\to \cdots \xrightarrow{\mathrm{id}} \Bbbk \to 0 \to \cdots \to 0$$

with a copy of $\Bbbk$ on vertices $i, i+1, \ldots, j$ is the uniserial $\Bbbk Q$-module corresponding to the column space (under the isomorphism of $\Bbbk Q$ with the lower triangular matrix ring) with non-zero entries in the k-th row for $i \le k \le j$.

Example 6.6. *Let Q be the Jordan quiver with unique arrow α. Then a representation of Q is nothing but an n-dimensional vector space equipped with a linear endomorphism, equivalently, an n-by-n matrix.*

Definition 6.7. *A homomorphism $f : M \to N$ of ($\Bbbk$-linear) quiver representations $M = (M_i, M_\alpha)_{i,\alpha}$ and $N = (N_i, N_\alpha)_{i,\alpha}$ is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' actions, i.e. we have a commutative diagram*

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[d, "M_\alpha"'] & N_i \arrow[d, "N_\alpha"] \\ M_j \arrow[r, "f_j"'] & N_j \end{tikzcd}$$

for all arrows $\alpha : i \to j$ in Q.

A homomorphism $f = (f_i)_{i\in Q_0} : M \to N$ of quiver representations is injective, resp. surjective, resp. an isomorphism, if every f_i is injective, resp. surjective, resp. an isomorphism, for all $i \in Q_0$.

Example 6.8. *Let Q be the Jordan quiver. Recall that a representation of Q is equivalent to a choice of n-by-n matrix M_α. By definition, the isomorphism class of such a representation is given by the conjugacy classes of M_α. If we assume $\Bbbk$ is algebraically closed, then a representative of the isomorphism class of M_α is given by the Jordan normal form of M_α. That is, M_α can be block-diagonalise into Jordan blocks $J_{m_1}(\lambda_1), \ldots, J_{m_l}(\lambda_l)$, where $J_m(\lambda)$ is the m-by-m Jordan block with eigenvalue $\lambda \in \Bbbk$.*

Proposition 6.9. *There is an isomorphism between the category of representations of Q and* $\mathsf{mod}\,\Bbbk Q$, *where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \prod_{i \in Q_0} M_i$ with $\Bbbk Q$-action given by (linear combinations of compositions of) M_α's, and isomorphism classes of Q-representations correspond to isomorphism classes of $\Bbbk Q$-modules.*

7 Idempotents

Recall that an *idempotent* of an algebra A is an element x with $x^2 = x$.

The right A-modules of the form eA and $D(Ae)$ for an idempotent $e \in A$ are of central importance in representation theory and in homological algebra.

Lemma 7.1. *The the following hold for any idempotent $e \in A$.*

(1) (Yoneda's lemma) $\mathrm{Hom}_A(eA, M) \cong Me$ *as a $\Bbbk$-vector space for all $M \in \mathsf{mod}\,A$.*

(2) *There is an isomorphism of rings* $\mathrm{End}_A(eA) \cong eAe$.

Proof For (1), check that $\mathrm{Hom}_A(eA, M) \ni f \mapsto f(e) = f(1)e \in Me$ defines a $\Bbbk$-linear map with inverse $me \mapsto (ea \mapsto mea)$. (2) follows from (1) by putting $M = eA$ with straightforward check of correspondence of multiplication on both sides. $\square$

Remark 7.2. Under the isomorphism $A \cong \mathrm{End}_A(A)$, an idempotent e of A corresponds to the 'project to direct summand $P = eA$ endomorphism', i.e. $A \twoheadrightarrow P \hookrightarrow A$. This is compatible with Yoneda lemma (think about this!) which says that there is a vector space isomorphism $fAe \cong \mathrm{Hom}_A(eA, fA)$ for any idempotents e, f.

Lemma 7.3. *For idempotents $e, f \in A$, we have $eA \cong fA$ as right A-module if and only if $f = ueu^{-1}$ for some unit $u \in A^\times$.*

Given an idempotent $e = e^2 \in A$ in an algebra A, then eA and $(1-e)A$ are both right ideal of A. Since $e(1-e) = 0 = (1-e)e$, we have $eA \cap (1-e)A = 0$, which means that $A \cong eA \oplus (1-e)A$ as right A-module. In particular, in the setting of the above lemma, we have that $eA \cong fA$ and $(1-e)A \cong (1-f)A$ by Krull-Schmidt property.

Definition 7.4. *Two idempotents e, f are orthogonal if $ef = 0 = fe$. An idempotent e is primitive if $e \neq f + f'$ for some orthogonal (pair of) idempotents f, f'.*

It follows from the definition of primitivity that

$$eA \text{ and } D(Ae) \text{ are indecomposable } A\text{-modules for a primitive idempotent } e.$$

Example 7.5. *The trivial paths e_x for $x \in Q_0$ is (by design) a primitive idempotent of the path algebra $\Bbbk Q$, and $1 = \sum_{x \in Q_0} e_x$ is an orthgonal decomposition of primitive idempotents. Hence, we have a decomposition*

$$\Bbbk Q \cong \bigoplus_{x \in Q_0} e_x \Bbbk Q = \bigoplus_{x \in Q_0} P_x \text{ and } D(\Bbbk Q) \cong \bigoplus_{x \in Q_0} D(\Bbbk Q e_x) \cong \bigoplus_{x \in Q_0} I_x.$$

8 Composition series, Jordan-Hölder Theorem

Definition 8.1. *Let A be a $\Bbbk$-algebra and $M \in A\,\mathsf{mod}$. A composition series of M is a finite chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that M_i/M_{i-1} is simple for all $1 \le i \le \ell$. The number ℓ here is the length of the composition series. The module M_i/M_{i-1} for each $1 \le i \le \ell$ are called the composition factors of the series.

Theorem 8.2 (Jordan-Hölder Theorem). *Any two composition series have the same length and the multi-sets of their composition factors (up to isomorphisms) are the same.*

We omit the proof. The strategy is basically by induction on the length of series.

Remark 8.3. Jordan-Hölder theorem holds as long as a module, regardless of what kind of algebra, has a (finite) composition series; this condition is actually equivalent to saying that it is noetherian and artinian.

Remark 8.4. The Jordan-Hölder theorem may not hold if one relaxes the form of composition factors from simple modules to something else. There are a few active research themes, including one related to quasi-hereditary algebras, that are stemmed from this.

Lemma 8.5. *Let M be a finite-dimensional right A-module. Then M has a composition series.*

Proof Induction on $\dim_\Bbbk M$, at each step choose a maximal submodule (i.e. a submodule whose quotient is simple). □

Example 8.6. *Let $A = \Bbbk\vec{\mathbb{A}}_n$. Then the module $U_{i,j}$ has a composition series*

$$0 \subset U_{j,j} \subset U_{j-1,j} \subset \cdots \subset U_{i+1,j} \subset U_{i,j}$$

with composition factors $S_k = U_{k,j}/U_{k+1,j}$ for $i \le k \le j$. Note that this composition series is unique - such kind of modules are called uniserial.

Lemma 8.7. *If $M \in \mathsf{mod}\,A$ and $N \subset M$ is a submodule, then there is a composition series $(M_i)_{0 \le i \le \ell}$ so that $N = M_k$ for some $0 \le k \le \ell$.*

Proof N has a composition series, say, of length k, so we take that as the first k terms of the required composition series of M. On the other hand, M/N also has a composition series, and since every submodule of M/N is of the form L/N (for a submodule U of M/N, take $L := \{m \in M \mid m+N \in U\}$; it is routine to check that this is an inverse operation as quotienting N on the submodules of M that contains N), a composition series of M/N is of the form $(L_i/N)_{0 \le i \le r}$. Now take $M_{k+i} = L_i$. □

Proposition 8.8. *Suppose A is a $\Bbbk$-algebra such that A_A has a composition series. Then there are only finitely many simple A-modules up to isomorphisms, and they all appear in the form A/I for some A-submodule I of A.*

Note that while this does not require A to be finite-dimensional, it requires A_A to be of finite length (equivalently, noetherian and artinian).

Proof The final clause of the claim is just restating Lemma 3.8: any simple S is given by $A/\operatorname{Ann}_A(m)$ for any non-zero $m \in S$. Now fix such an S and $I := \operatorname{Ann}_A(m)$. Since A has a composition series, I also have one by Lemma 8.7 so that the series ends with $I \subset A$. Since this is possible for any simple S, it follows from Jordan-Hölder theorem that all simple modules other than S must appear as composition factors of I.

Since composition series is a finite chain, there must be finitely many composition factors - hence, the simple modules of A must be finite. □

9 Semisimplicity and Artin-Wedderburn theorem

In order to obtain all (isomorphism classes of) simple A-modules - or equivalently maximal right A ideal (i.e. maximal submodules of A_A) - for a finite-dimensional $\Bbbk$-algebra A, we will use the following.

Definition 9.1. *Let A be a $\Bbbk$-algebra and $M \in \mathsf{mod}\, A$.*

(1) *The (Jacobson) radical $\mathrm{rad}(A)$ (sometimes also written as $J(A)$) of A is the intersection of all maximal right ideals (i.e. maximal A-submodules) of A.*

(2) *A is semisimple if $\mathrm{rad}(A) = 0$.*

Example 9.2. *For $A = \Bbbk Q$ of a finite quiver Q and $x \in Q_0$. The projective P_x at x contains a submodule spanned by all paths starting from x with length at least 1. This is a maximal submodule of P_x since the cokernel of the natural embedding to P_x is a one-dimensional module spanned by the coset of e_x – in particular, this simple module is isomorphic to S_x. Thus, we have $\mathrm{rad}(A) = \Bbbk Q_{\geq 1}$ the submodule of A_A spanned by all paths of length at least 1.*

Proposition 9.3. *Suppose A_A has a composition series. Then the following holds for the Jacobson radical $\mathrm{rad}(A)$.*

- *$\mathrm{rad}(A)$ is the intersection of finitely many maximal right ideals.*
- *$\mathrm{rad}(A)$ is the intersection of all two-sided ideals $\mathrm{Ann}_A(S) := \{a \in A \mid ma = 0 \forall m \in S\}$, in other words*
$$\mathrm{rad}(A) = \{a \in A \mid Sa = 0 \text{ for all simple } S\}.$$
- *$\mathrm{rad}(A)$ is a two-sided ideal of A.*
- *$\mathrm{rad}(A)^\ell = 0$ for ℓ at most the length of A_A.*
- *$(A/\mathrm{rad}(A))_{A/\mathrm{rad}(A)}$ is a semisimple (as a module).*
- *A_A is a semisimple (as a module) if, and only if, $\mathrm{rad}(A) = 0$ (i.e. A semisimple as an algebra).*

Proof omitted. We note that all of these claims do make use of the Jordan-Hölder theorem.

Example 9.4. (1) *Direct product of two semisimple algebras is semisimple.*

(2) *$A = \mathrm{Mat}_n(D)$ with D a division $\Bbbk$-algebra is a semisimple $\Bbbk$-algebra. We have decomposition $A_A \cong V^{\oplus n}$ into n copies of n-dimensional simple module*
$$V = \{(v_i)_{1 \leq i \leq n} \mid v_i \in D \; \forall i\}.$$

(3) *$A := \Bbbk[x]/(x^n)$ is not semisimple for any $n \geq 2$ as it has a non-trivial (unique) maximal ideal $\mathrm{rad}(A) = (x)$.*

Theorem 9.5 (Artin-Wedderburn theorem). *Let A be a finite-dimensional $\Bbbk$-algebra and let r be the number of isoclasses of simple A-modules, say, with representatives $S_1, \ldots, S_r$. Let $D_i := \mathrm{End}_A(S_i)$ be the division $\Bbbk$-algebra given by endomorphism of the simple module S_i. Then there is an isomorphism of $\Bbbk$-algebras*
$$A/\mathrm{rad}(A) \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r).$$

As before, if we work over algebraically closed field $\Bbbk = \overline{\Bbbk}$, then all the D_i's are just $\Bbbk$.

Proof Let $B := A/\mathrm{rad}(A)$. By definition of $\mathrm{rad}(A)$, the A-module $A/\mathrm{rad}(A)$ is semisimple, and any A-submodule M of $A/\mathrm{rad}(A)$ satisfies $M\,\mathrm{rad}(A) = 0$. Hence, $M = M/M\,\mathrm{rad}(A)$ is naturally a B-module and $\mathrm{End}_B(M) \cong \mathrm{End}_A(M)$ (even as algebras!).

By Lemma 7.1, we have $B \cong \mathrm{End}_B(B)$. Since B is semisimple, the B_B is a semisimple B-module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B-modules. Hence, it follows from Schur's lemma and its consequence (Lemma 3.10 and Lemma 3.13) that

$$B \cong \mathrm{End}_B(B) \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r),$$

where $D_i := \mathrm{End}_B(S_i)$ for all $1 \leq i \leq r$. This completes the proof. $\square$

Corollary 9.6. *For any finite-dimensional $\Bbbk$-algebra A, let $\mathrm{Sim}(A)$ be the set of isomorphism-class representatives of simple A-modules. Then there is a one-to-one correspondence*

$$\begin{aligned} \mathrm{Sim}(A) &\xleftrightarrow{1:1} \mathrm{Sim}(A/\operatorname{rad}(A)) \\ S &\longmapsto \overline{S} := S/S\operatorname{rad}(A) \\ &\qquad\quad (= S \text{ as underlying vector space}) \\ \mathrm{res}T &\longleftarrow\!\shortmid\; T \end{aligned}$$

where $\mathrm{res}T$ is the restriction of T along $A \twoheadrightarrow A/\operatorname{rad}(A)$.

Example 9.7. *Suppose that Q is finite acyclic, i.e. $\Bbbk Q$ is finite-dimensional. Since $\operatorname{rad}(\Bbbk Q)$ is spanned by all non-trivial paths, $\Bbbk Q/\operatorname{rad}(\Bbbk Q)$ is just the semisimple $\Bbbk Q$-module $\bigoplus_{i \in Q_0} S_i$. In particular, the Artin-Wedderburn decomposition reads*

$$\Bbbk Q \cong \Bbbk \times \cdots \times \Bbbk$$

with one copy of $\Bbbk$ for each $i \in Q_0$ on the right-hand side. Moreover, every simple $\Bbbk Q$-module is isomorphic to one of S_i for $i \in Q_0$.

Exercise 9.8. *Show that when Q is the Jordan quiver, then $\Bbbk Q$ has infinitely many simple modules and that $\operatorname{rad}(\Bbbk Q) = 0$.*

10 Radical and socle

Definition 10.1. *The radical of an A-module M is $\operatorname{rad}(M) := M\operatorname{rad}(A)$. In general, take $\operatorname{rad}^0(M) := M$ and denote by $\operatorname{rad}^{k+1}(M) := \operatorname{rad}(\operatorname{rad}^k(M)) = \operatorname{rad}^k(M)\operatorname{rad}(A)$ for all $k \geq 0$.*

Successively taking the radical yields a series:

$$0 \subset \operatorname{rad}^\ell(M) \subset \cdots \subset \operatorname{rad}(M) \subset M$$

This is called the radical series. The quotient $M/\operatorname{rad}(M)$ is called the top of M, and is denoted by $\operatorname{top}(M)$.

Proposition 10.2. *The following hold for $M \in \mathsf{mod}\,A$.*

(1) *$\operatorname{rad}(M)$ is the intersection of all maximal submodules of M.*

(2) *$\operatorname{top}(M) := M/\operatorname{rad}(M)$ is the maximal semisimple quotient of M.*

(3) *$\operatorname{rad}(M \oplus N) = \operatorname{rad}(M) \oplus \operatorname{rad}(N)$.*

(4) *If $f : M \to N$ is a surjective A-module homomorphism, then $f(\operatorname{rad} M) = \operatorname{rad} N$.*

(5) *(Nakayama's Lemma, special case) For a submodule $N \subset M$, $(N + \operatorname{rad}(M) = M) \Rightarrow N = M$.*

Proof omitted; this follows the same kind of arguments as in the case for $\operatorname{rad}(A)$.

Example 10.3. *Let A be a finite-dimensional algebra. Suppose that e is a primitive idempotent, i.e. $P := eA$ is an indecomposable A-module. Since $A = P \oplus Q$ (by taking $Q := (1-e)A$), we have*

$$\operatorname{rad}(P) \oplus \operatorname{rad}(Q) = \operatorname{rad}(P \oplus Q) = \operatorname{rad}(A).$$

Since P and Q has no common (non-trivial) submodule, we get that

$$A/\operatorname{rad}(A) = \frac{P \oplus Q}{\operatorname{rad}(P \oplus Q)} = P/\operatorname{rad}(P) \oplus \frac{Q}{\operatorname{rad}(Q)}.$$

Thus, it follows from Corollary 9.6 that $P/\operatorname{rad}(P)$ is a simple module and that every simple A-module arises this way. In other words, let $\operatorname{PIM}(A)$ be the set of isoclass (=isomorphism class) representatives of indecomposable direct summands of A, then we have a correspondence

$$\begin{array}{ccc} \operatorname{PIM}(A) & \xleftrightarrow{1:1} & \operatorname{Sim}(A) \\ P & \longmapsto & P/\operatorname{rad}(P) \end{array} \tag{10.1}$$

For a simple A-module S, denote by P_S the corresponding direct summand P of A under the correspondence (10.1).

There is a construction dual to $\operatorname{rad}(M)$.

Definition 10.4. *The socle of an A-module M is $\operatorname{soc}(M)$, which is defined as the maximal semisimple submodule of M. More generally, take $\operatorname{soc}^0(M) = 0$ and for $k \geq 0$, let $\operatorname{soc}^{k+1}(M)$ to be the submodule of M generated by the lift of $\operatorname{soc}(M/\operatorname{soc}^k(M)) \subset M/\operatorname{soc}^k(M)$. This yields a series*

$$0 \subset \operatorname{soc}(M) \subset \operatorname{soc}^2(M) \subset \cdots \subset \operatorname{soc}^\ell(M) = M$$

called the socle series of M.

Example 10.5. *Consider a path algebra $\Bbbk Q$ of a finite acyclic (for simplicity) quiver Q, and $x \in Q_0$. The indecomposable injective $I_x = D(\Bbbk Q e_x)$ has a simple socle isomorphic to S_x. Essentially this can be seen by a dual argument in showing $\operatorname{top}(P_x) \cong S_x$. More generally, analogous to Example 10.3, for a finite-dimensional algebra A, every simple A-module appears as $\operatorname{soc}(I)$ for an indecomposable direct summand of $D(A)$.*

Lemma 10.6. *For $M \in \mathsf{mod}\, A$, the socle series and radical series has the same length, and this length is called the Loewy length of M, and is denoted by* $\mathrm{LL}(M)$.

Proof Let r_M (resp. s_M) denotes the length of the radical (resp. socle) series of M. First, we show that $s_M \leq r_M$ by induction on s_M. This is clearly fine if $s_M = 0$.

Suppose that $s_M > 0$. By definition we have $\mathrm{rad}^{r-1}(M)$ a semisimple submodule of M, and so $\mathrm{rad}^{r-1}(M) \subset \mathrm{soc}(M)$. This means that there is a surjective homomorphism $M/\mathrm{rad}^{r-1}(M) \twoheadrightarrow M/\mathrm{soc}(M)$, and so $r_{M/\mathrm{rad}^{r-1}(M)} \geq r_{M/\mathrm{soc}\, M}$ (EXERCISE!). In particular, we have

$$r_M = r_{M/\mathrm{rad}^{r-1}(M)} + 1 \geq r_{M/\mathrm{soc}\, M} + 1.$$

Since $s_{M/\mathrm{soc}\, M} = s_M - 1$, it follows from the induction hypothesis that $s_{M/\mathrm{soc}\, M} \leq r_{M/\mathrm{soc}\, M}$, and hence

$$s_M = s_{M/\mathrm{soc}\, M} + 1 \leq r_{M/\mathrm{soc}\, M} + 1 \leq r_M,$$

as required.

One can show that $r_M \leq s_M$ dually. $\square$

Note that the semisimple subquotients in (between the layers of) the socle series and the radical series of a module may not coincide.

Example 10.7. *Let Q be the quiver $1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and consider the projective P_2 which has the form*

$$\Bbbk \xleftarrow{1} \Bbbk \xrightarrow{1} \Bbbk \xrightarrow{1} \Bbbk$$

Then we have radical series

$$0 \subset S_4 = \Bbbk\beta\gamma \overset{S_1 \oplus S_3}{\subset} \mathrm{rad}(P_2) = \Bbbk\alpha + \Bbbk\beta + \Bbbk\beta\gamma \overset{S_2}{\subset} P_2$$

and socle series

$$0 \subset S_2 \oplus S_4 = \Bbbk\alpha + \Bbbk\beta\gamma \overset{S_3}{\subset} \mathrm{rad}(P_2) \subset P_2.$$

11 Example: Topological data analysis

Topological data analysis concerns the "rough shape of data". Here, we regard data as just a finite discrete set X in $\mathbb{R}^d$ (with usual Euclidean metric if you like). X itself is not particular interesting space (in terms of geometry or topology) for further analysis; yet, we can often see "pattern" – whether they look more or less randomly distributed, whether they are distributed in the space in a way that avoid certain areas, etc.

A more well-known mathematical approach to addressing this issue is statistics, where we try to see if the pattern tells us correlation between different parameters. For topological data analysis (TDA) we want to just tell if the data form some 'shapes with holes' (this is where 'topology' comes in). The idea is to replace each data point $x \in X$ by a ball $B_t(x)$ of very small radius t, slowly increase the radius and observe how the topology (e.g. by looking at topological invariant such as the 'genus') of the space $X_t := \bigcup_{x\in X} B_t(x)$ changes.

Note that if $s \leq t$, then we have a subspace $X_s \subset X_t$. Moreover, in practice, it makes sense to sample t to a finite sequence $t_1 < t_2 < \cdots < t_n$ and take $X_i := X_{t_i}$. Since we only concern topology of X_t, we can replace X_t by a simplicial complex $\triangle_t$ where 0-cells (points) are x, and $\{x_1, \ldots, x_r\}$ form an r-cells if $B_t(x_1) \cap \cdots \cap B_t(x_r) \neq \emptyset$. Having a simplicial complex means that we can take (e.g. the p-th) homology group $H_p(X_t) = H_p(\triangle_t)$. The fact that we have $X_{t_i} \subset X_{t_{i+1}}$ means that we have a chain

$$H_p(X_1) \to H_p(X_2) \to \cdots \to H_p(X_n).$$

If we linearise these abelian groups to $\Bbbk$-vector spaces, then we get a chain of vector spaces and linear transformations – this is nothing but a representation of the $\vec{\mathbb{A}}_n$-quiver

$$1 \to 2 \to \cdots \to n.$$

In TDA, such a chain is called *persistence module* (of 1 parameter / finite linear poset). Understanding the indecomposable decomposition of a persistence module is an important aspect in TDA, this can even be used to characterise the nature of the data set (e.g. one may record some data from various metals, and the topological information can be used to distinguish each metal just from the data set).

An *interval module* $M_{[a,b]}$ for $1 \leq a \leq b \leq n$ is the $\vec{\mathbb{A}}_n$-quiver representation given by

$$0 \to \cdots \to 0 \to \Bbbk \xrightarrow{\mathrm{id}} \Bbbk \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \Bbbk \to 0 \to \cdots 0$$

where the non-zero space starts at a and ends at b. This is clearly an indecomposable representation. In fact, forms *all* indecomposable representation – known by Gabriel in the 70s (this is one special case of the Gabriel's theorem). The following is then just a consequence of Krull-Schmidt theorem, but turns out to be fundamental in TDA.

Proposition 11.1. *Every persistence module can be decomposed uniquely to a direct sum of interval module.*

The above is what people call 'single parameter', or (finite) 'linear poset', case. There are other possible forms:

(1) Multi-parameter case: the quiver $\vec{\mathbb{A}}_n$ is replaced by the 'commutative cube', i.e. the bound quiver $\vec{\mathbb{A}}_{n_1} \times \cdots \times \vec{\mathbb{A}}_{n_r}$ with relation $\alpha\beta - \beta\alpha$ for arrows α, β going in different directions. In other words, a persistence module in this case is the same as an A-module, where $A = \bigotimes_{i=1}^{r} \Bbbk\vec{\mathbb{A}}$.

(2) Poset case: the quiver $\vec{\mathbb{A}}_n$ is replace by the bound quiver (Q, I) whose underlying quiver Q is the Hasse quiver of the poset P, and I includes all commutation

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z - x \xrightarrow{\alpha'} w \xrightarrow{\beta'} z$$

whenever $x \geq y \geq z$ and $x \geq w \geq z$. In other words, a persistence module in this case isthe same as a module over the *incidence algebra* of the poset P.

In these general cases, one can still define interval modules, but it is no longer true that every A-module can be decomposed into interval modules. Much of the recent algebraic and homological aspect of TDA concerns how to overcome such a problem.

For other aspects and more in-depth study of applying quiver representation to TDA, see, for exmample, book of Steve Oudot.

12 Example: Linear matrix pencil

A *linear matrix pencil* is a matrix $A+\lambda B$ with $A, B \in M_{m\times n}(\Bbbk)$ and λ being an indeterminant, i.e. $A+\lambda B \in M_{m\times n}(\Bbbk[k])$. For simplicity, we just say 'matrix pencil' and drop the adjective 'linear'. Matrix pencil is used in the study of the so-called *generalised eigenvalue problem*, and has applications to various applied mathematics like control theory, differential algebraic equations, numerical linear algebra, etc.

Two matrix pencils $A+\lambda B$ and $A'+\lambda B'$ are *strictly equivalent* if there are invertible matrices $P \in M_m(\Bbbk), Q \in M_n(\Bbbk)$ such that $A'+\lambda B' = P(A+\lambda B)Q$. This is equivalent to $A' = PAQ$ and $B' = PBQ$.

For simplicity, let us specialise $\Bbbk$ to an algebraically closed field; this means that we can use Jordan canonical form $J_m(\alpha) \in M_m(\Bbbk)$.

Let $\overline{H}_m$ be the $m\times(m+1)$-matrix given by removing the last row of $J_m(0)$, and $(I_m|0)$ be the $m\times(m+1)$-matrix given by adding a column of zero to the identity matrix I_m. Define

$$L_m := \lambda(I_m|0) + \overline{H}_m = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & 1 \end{pmatrix} \in M_{m\times(m+1)}(\Bbbk[\lambda]).$$

Similar to the Smith/Jordan canonical form, each matrix pencil of is equivalent to one in *Kronecker canonical form*.

Theorem 12.1. *Every matrix pencil is strictly equivalent to a block-diagonal matrix, where each block is of one of the following form:*

(1) L_m *or* L_m^{tr} *for some* $m \geq 1$.

(2) $I_m + \lambda J_m(0)$ *or* $J_m(\alpha) + \lambda I_m$, *for some* $m \geq 1$ *and* $\alpha \in \Bbbk$.

One way to prove this theorem is to observe the following.

Proposition 12.2. *For any* $m, n \in \mathbb{Z}_{\geq 0}$, *there is a one-to-one correspondence between the strictly equivalent classes of* $m\times n$*-matrix pencil and the isomorphism classes of representations over the Kronecker quiver* $K_2 := (1 \rightrightarrows 2)$ *with dimension vector* (n, m).

Proof Exercise. □

Under this correspondence, the Kronecker canonical form (the blocks appearing in the block-diagonal form) corresponds to indecomposable representations of the Kronecker quiver. In quiver representation theory, such classification can be done using a Kac's theorem.

Corollary 12.3. *Every indecomposable* $\Bbbk K_2$*-modules is isomorphic to one of the following.*

(1) *Preinjective modules, which correspond to* L_m *for* $m \geq 1$.

(2) *Preprojective modules, which correspond to* L_m^{tr} *for* $m \geq 1$.

(3) *Regular modules* $R_m(x)$ *for* $x \in \mathbb{P}^1_{\Bbbk}$ *and* $m \geq 1$, *where*

$$\begin{cases} R_m(\alpha) \text{ corresponds to } J_m(\alpha) + \lambda I_m & \text{if } \alpha = [x:1]; \\ R_m(\infty) \text{ corresponds to } I_m + \lambda J_m(0) & \text{if } \alpha = [1:0] = \infty. \end{cases}$$

With this, various problems about linear matrix pencil can be transformed to problems about representations of the Kronecker quiver. There are also 'higher variation' of matrix pencils that correspond to the n-Kronecker quiver where there are $n \geq 2$ arrows between the 2 vertices (instead of just $n = 2$). Examples problem includes the "matrix subpencil" problem, which are studied by Claus Ringel, Han Yang, Ştefan Şuteu-Szöllősi.

Exercise 12.4. *Write down the indecomposable* $\Bbbk K_2$*-modules as representations.*

13 Bounded path algebra

For general quiver, we loses finite-dimensionality, and so many nice things we explained do not hold any more. To retain finite-dimensionality, we need to consider nice quotients of path algebras.

Definition 13.1. *An ideal $I \lhd \Bbbk Q$ is admissible if $(\Bbbk Q_1)^k \subset I \subset (\Bbbk Q_1)^2$ for some $k \geq 2$, i.e. I is generated by linear combinations of paths of finite length at least 2. The pair (Q, I) is sometimes called bounded quiver. A bounded path algebra or quiver algebra (with relations) is an algebra of the form $\Bbbk Q/I$ for some quiver Q and admissible ideal I.*

Remark 13.2. Admissiblity ensures there is no redundant arrows (which appears if there is a relation like, for example, $\alpha - \beta\gamma \in I$ for some $\alpha \neq \beta, \gamma \in Q_1$) and there is enough vertices (trivial paths may not be primitive if there is a loop x at a vertex with relation $x^2 - x \in I$).

Lemma 13.3. *A bounded path algebra is finite-dimensional.*

Proof There exists a surjective algebra homomorphism $\Bbbk Q/(\Bbbk Q_1)^k \twoheadrightarrow \Bbbk Q/I$; the former is finite-dimensional. $\square$

Example 13.4. *Let Q be the Jordan quiver with unique arrow α. Let I be the ideal of $\Bbbk Q$ generated by α^k for some $k \geq 2$. Then I is an admissible ideal and $\Bbbk Q/I \cong \Bbbk[x]/(x^k)$ is a truncated polynomial ring.*

Definition 13.5. *A representation M of a bounded quiver (Q, I) is a representation $M = (M_i, M_\alpha)_{i,\alpha}$ of Q such that $M_a = 0$ for all $a \in I$; here $M_a := \sum_p \lambda_p M_p$ for $a = \sum_p \lambda_p p$ written as a linear combinations of paths p.*

A homomorphism $f : M \to N$ of representations of (Q, I) is a collection of linear maps $f_i : M_i \to N_i$ that intertwines arrows' action.

As before, representations are really just synonyms of modules.

Lemma 13.6. *A representation of a bounded quiver (Q, I) is equivalent to a $\Bbbk Q/I$-module, and homomorphisms between representations are equivalent to those between $\Bbbk Q/I$-modules.*