Ex 1. Recall that for a G-set X, π_X denotes the permutation representation associated to X with underlying KG-module being KX.

(i) Let X, Y be two finite G-sets. Show that $\pi_{X \sqcup Y} \cong \pi_X \oplus \pi_Y$.

(ii) Suppose that X is a finite G-set with G-orbit decomposition $X = O_1 \sqcup \cdots \sqcup O_m$. Show that we have $\pi_X = \pi_{O_1} \oplus \cdots \oplus \pi_{O_m}$.

(iii) Recall that there is a 2-dimensional irreducible representation $V = K\{u, v\}$ of $G = D_6 = \langle a, b \mid a^3 = 1 = b^2, abab = 1 \rangle$ with action matrices

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find $x, y \in KG$ so that the $K\{x, y\}$ is the subrepresentation of KG that is isomorphic to V.

Ex 2.

(i) For any finite-dimensional KG-modules U, V, W, show that

(a) $\mathrm{Hom}_{KG}(U \oplus V, W) \cong \mathrm{Hom}_{KG}(U, W) \oplus \mathrm{Hom}_{KG}(V, W)$.

(b) $\mathrm{Hom}_{KG}(U, V \oplus W) \cong \mathrm{Hom}_{KG}(U, V) \oplus \mathrm{Hom}_{KG}(U, W)$.

(c) Suppose U is a simple and K is algebraically closed. Show that there is a ring isomorphism $\mathrm{End}_{KG}(S^{\oplus m})^{\mathrm{op}} \cong \mathrm{Mat}_m(K)$.

(d) Show that, if $\mathrm{Hom}_{KG}(U, V) = 0 = \mathrm{Hom}_{KG}(V, U)$, then there is a ($K$-linear) ring isomorphism $\mathrm{End}_{KG}(U \oplus V) \cong \mathrm{End}_{KG}(U) \times \mathrm{End}_{KG}(V)$.

(ii) Consider $G = C_3 = \langle g \mid g^3 = 1 \rangle$ and K be a field with $\operatorname{char} K = 0$. Define a matrix G-representation $R : G \to \mathrm{GL}_2(K)$ given by

$$R_g := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

(a) Show that when $K = \mathbb{R}$, R is an irreducible $\mathbb{R}$-linear C_3-representation.

(b) For $K = \mathbb{C}$, find $i, j \in \{1, 2, 3\}$ so that $R \cong R^{(i)} \oplus R^{(j)}$ where $R^{(a)}$ is the representation given by $R_g^{(a)} := \omega^a$.

Ex 3.

(i) Show that $\mathrm{triv}_G \otimes_K V \cong V \cong V \otimes_K \mathrm{triv}_G$ for all KG-module V.

(ii) For finite-dimensional KG-modules U, V, W, show that $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$ as KG-modules.

(iii) Recall that for a cyclic group C_n, the irreducible $\mathbb{C}$-linear representations of $C_n = \langle g \mid g^n = 1 \rangle$ (up to isomorphism) are given by S_i for $i \in \{1, \ldots, n\}$ where g-action is given by ξ^i for $\xi := \exp(2\pi i/n)$ the n-root of 1. Calculate the $\mathbb{C}G$-module $S_i \otimes S_j$ for all $i, j \in \{1, \ldots, n\}$ and express the result in terms of direct sums of the S_k's.

(iv) Show that for finite groups G, H, $KG \otimes_K KH$ has a canonical ring structure so that $KG \otimes_K KH \cong K(G \times H)$ as rings.

Ex 4. Let U, V, W be KG-modules.

(i) Find a KG-module structure on the space $\mathrm{Hom}_K(V, W)$.

(ii) Show that there are the following isomorphisms of KG-modules

(a) $(V \otimes_K W)^* \cong V^* \otimes_K W^*$.

(b) $V^* \otimes_K W \cong \mathrm{Hom}_K(V, W)$.

(iii) Suppose X is a finite G-set or a KG-module. Define the G-invariant subset (subspace) as

$$X^G := \{x \in X \mid gx = x \, \forall g \in G\}.$$

(a) Show that $(V^* \otimes_K V)^G \cong \mathrm{End}_{KG}(V)$.

(b) Show that $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W)$

Deadline: 1st December, 2024

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