Ex 1. Recall that for A-B-bimodule M and left A-module X and left B-module Y, we have a left B-module given by $\mathrm{Hom}_A(M, X)$ and a left A-module $M \otimes_A Y$. This question is to complete the missing pieces in Section 25 of the lecture notes; the setup remains the same throughout all parts.

(1) For a left KG-module V and a subgroup $H \leq G$, show that $\mathrm{Res}^G_H(V) \cong \mathrm{Hom}_{KG}(KG, V)$ as KH-module.

(2) Show that $\mathrm{Res}^G_H(V) \cong KG \otimes_{KG} V$ as KH-module.

(3) Describe the G-action on $\mathrm{Hom}_{KH}(KG, V)$.

(4) For a KH-module U, consider the map $\alpha : KG \otimes U \to \mathrm{Hom}_{KH}(KG, U)$ given by

$$g \otimes u \mapsto \left(x \mapsto \begin{cases} (xg)u & \text{if } xg \in H \\ 0 & \text{otherwise.} \end{cases} \right)$$

Show that this defines a KG-module homomorphism.

(5) Show that the map $\beta : \mathrm{Hom}_{KH}(KG, U) \to KG \otimes_{KH} U$ given by

$$f \mapsto \sum_{i=1}^{k} t_i \otimes f(t_i^{-1}),$$

Show that this defines a KG-module homomorphism.

(6) Show that $\alpha\beta$ and $\beta\alpha$ are both the identity map.

Ex 2. Consider two 2-part partitions $\lambda = (n-\ell, \ell)$ and $\mu = (n-k, k)$ with $0 \leq \ell \leq k \leq n/2$.

(1) Show that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(M^\lambda, M^\mu) = \ell + 1$.

(2) Show that $M^{(n-k,k)} \cong \bigoplus_{i=0}^{k} S^{(n-i,i)}$ for $r \leq n/2$ when n is even and for $r \leq (n-1)/2$ when n is odd.

Hint:
(a) Both (1) and (2) can be done with character theory (of course, tableaux combinatorics is also possible).
(b) $M^\lambda \cong K\Omega_\ell$ where Ω_ℓ is the set of ℓ-subsets of $[n]$.
(c) For (1), see Proposition 20.3 of lecture notes.
(d) For (2), starts with $k = 1$, then try $k = 2$, etc.

Ex 3. Let $\lambda = (3, 2) \vdash 5$. Write down the standard λ-polytabloids, and show that S^λ any other λ-polytabloids is in the span of the standard ones.

Ex 4. Let $\mathfrak{t}$ be any λ-tableau for $\lambda \vdash n$.

Let λ' be the partition obtained from reflecting λ along the diagonal $y = -x$ (with origin being the top-left corner of Young daigram; e.g. $\lambda = (4,3) \Rightarrow \lambda' = (2^3, 1)$).

Let $\mathfrak{t}'$ be the λ'-tableau given by reflecting $\mathfrak{t}$ along the same diagonal. Suppose that $S^{(1^n)} = Ku$.

$$\rho_{\mathfrak{t}'} := \sum_{\sigma \in R_{\mathfrak{t}'}} \sigma \in K\,\mathfrak{S}_n$$

Consider a map $\theta : M^{\lambda'} \to S^{\lambda} \otimes S^{(1^n)}$ given by

$$\{\mathfrak{t}'\} \mapsto \rho_{\mathfrak{t}'}(\{\mathfrak{t}\} \otimes u).$$

(1) Explain why $\rho_{\mathfrak{t}'}(\{\mathfrak{t}\} \otimes u) = (\kappa_{\mathfrak{t}}\{\mathfrak{t}\}) \otimes u$.

(2) Show that $\theta(\{\sigma\mathfrak{t}'\}) = \operatorname{sgn}(\sigma)\kappa_{\sigma\mathfrak{t}}\{\sigma\mathfrak{t}\} \otimes u$ for any $\sigma \in \mathfrak{S}_n$.

(3) Show that θ is a surjective $K\,\mathfrak{S}_n$-module homomorphism.

(4) Show that the coefficient of $\{\mathfrak{t}\}$ in $\rho_{\mathfrak{t}}\kappa_{\mathfrak{t}}\{\mathfrak{t}\}$ is $\#R_{\mathfrak{t}}$.
Hint: Coefficient of $\{\mathfrak{t}\}$ in a vector v is the same as the coefficient of $\{\sigma^{-1}\mathfrak{t}\}$ in $\sigma^{-1}v$.

(5) Deduce that $S^{\lambda'} \not\subseteq \ker\theta$.

(6) Recall that for any simple KG-module V and a 1-dimensional KG-module S, the tensor product $V \otimes_K S$ is also simple. Suppose that $\operatorname{char} K \nmid |\,\mathfrak{S}_n\,|$, show that $S^{\lambda} \otimes_K S^{(1^n)} \cong S^{\lambda'}$.

Deadline: 19th January, 2024
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