Ex 1.

(i) Show that for a character $\chi = \chi_V$, $\mathrm{Ker}\chi := \{g \in G \mid \chi(g) = \chi(1)\}$ is a normal subgroup of G.

(ii) Show that $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)} = \overline{\chi_V}\chi_W$.

(iii) For a $\mathbb{C}H$-module U and $g \in G$, consider the set

$$^gU := \{gu \mid u \in U\}.$$

Show that

$$h(gu) := g((g^{-1}hg) \cdot u)$$

defines an H-action on ^gU. Show also that isomorphism of KH-modules $^{g_1}U \cong {}^{g_2}U$ implies that $^{gg_1}U \cong {}^{gg_2}U$ for all $g \in G$.

(iv) (Exercise 23.11 of lecture notes) Consider a left transversal $t_1, t_2, \ldots, t_k$ of $H \leq G$ and a $\mathbb{C}H$-module U. For $g \in G$ and $1 \leq i \leq k$, write $gt_i = t_j h$ with $h \in H$. Show that, for $u \in U$, the following

$$g(t_i \otimes u) := t_j \otimes (t_j^{-1} g t_i) u$$

defines a left G-action on $\mathrm{Ind}_H^G(U) := \mathbb{C}G \otimes_{\mathbb{C}H} U$.

(v) Suppose that $H \leq G$ is a subgroup. Show that for $\mathbb{C}H$-module W, we have $\mathrm{Ind}_H^G(W^*) \cong (\mathrm{Ind}_H^G(W))^*$ as $\mathbb{C}G$-module.

Ex 2. Let $V := \mathbb{C}\{v_1, \ldots, v_n\}$ be an n-dimensional $\mathbb{C}$-vector space with basis $\{v_1, \ldots, v_n\}$. Consider the tensor product $V^{\otimes 2} := V \otimes_{\mathbb{C}} V$ and the linear map $\tau : V^{\otimes 2} \to V^{\otimes 2}$ given by linearly extending

$$\tau : v_i \otimes v_j \mapsto v_j \otimes v_i$$

for all $1 \leq i, j \leq n$. Let

$$S^2(V) := \{v \in V^{\otimes 2} \mid \tau(v) = v\} \text{ and } \Lambda^2(V) := \{v \in V^{\otimes 2} \mid \tau(v) = -v\}.$$

(a) Fix a pair $1 \leq i, j \leq n$ and let $v_\pm := \frac{1}{2}(v_i \otimes v_j \pm v_j \otimes v_i)$. Show that $\tau(v_\pm) = \pm v_\pm$.

(b) Show that $\{v_i v_j := \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i) \mid 1 \leq i \leq j \leq n\}$ form a basis of $S^2(V)$ and compute $\dim_{\mathbb{C}} S^2(V)$.

(c) Show that $\{v_i \wedge v_j := \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i) \mid 1 \leq i < j \leq n\}$ form a basis of $\Lambda^2(V)$ and compute $\dim_{\mathbb{C}} \Lambda^2(V)$.

(d) Suppose now that V is a $\mathbb{C}G$-module. Show that $\tau(x) = x$ implies that $\tau(gx) = gx$. Likewse, show that $\tau(x) = -x$ implies that $\tau(gx) = -gx$.

(e) Show that $S^2(V) \oplus \Lambda^2(V) \cong V^{\otimes 2}$ as $\mathbb{C}G$-modules. (At least show they are isomorphic as $\mathbb{C}$-vector spaces if you cannot do it on the $\mathbb{C}G$-module level.)

Ex 3.

(i) A certain group G has two columns of its character table as follows:

g_i	g_1	g_2
$\lvert C_G(g_i)\rvert$	21	7
χ_1	1	1
χ_2	1	1
χ_3	1	1
χ_4	3	ζ
χ_5	3	$\overline{\zeta}$

where $g_1 = 1$ and $\zeta \in \mathbb{C}$.

(a) Find ζ.

(b) Find one other column of the character table.
Hint: (1) Recall that if g, g^{-1} are in the same conjugacy class, then $\chi(g) \in \mathbb{R}$.
(2) Recall that if χ_i irreducible, then so is $\overline{\chi_i}$.
(3) $\lvert C_G(g)\rvert = \lvert C_G(g^{-1})\rvert$
(4) You can use $\zeta \notin \mathbb{R}$ if you cannot complete (a).
(5) Part (b) is not about using orthogonality relation.

(ii) A group of order 720 has 11 conjugacy classes. Two representations of the group are known and have corresponding characters α and β with values shown in the table below. Prove that the group has an 16-dimensional irreducible representation and calculate its character.

$\lvert C_G(g_i)\rvert$	720	48	18	8	16	6	5	6	8	48	18
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Ex 4.

(i) Consider the alternating group $\mathfrak{A}_4$.

(a) Write down all conjugacy classes of $\mathfrak{A}_4$.

(b) By using permutation character associated to $\{1, 2, 3, 4\}$ or using restriction from $\mathfrak{S}_4$, show that there is a degree 3 irreducible character of $\mathfrak{A}_4$.

(c) Compute the character table of $\mathfrak{A}_4$. Explain your reasoning.

(ii) Consider the dihedral group $D_{2n} = \langle a, b \mid a^n = 1 = b^2, bab = a^{-1}\rangle$ of order $2n$ for $n = 2m$ even.

(a) Write down all conjugacy classes of D_{2n}.
Hint: There are $m + 3$ of them, of which $m - 1$ (resp. $m + 1 = 3$) of them has size 2 when $n > 4$ (resp. $n = 4$).

(b) Find the derived subgroup D'_{2n} of D_{2n} and compute its quotient.

(c) Show that there are precisely 4 (up to isomorphism) one-dimensional representations of D_{2n}.

(d) Compute the induced characters of the irreducible characters of $\langle a \rangle \leq D_{2n}$.

(e) Compute the character table of D_{2n}. Explain your reasoning.

Deadline: 5th January, 2024

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