Ex 1. Let A be the ring

$$\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \in \mathbb{R}, b, c \in \mathbb{C} \right\}.$$

1. Show that A is an $\mathbb{R}$-algebra and find the dimension of A over $\mathbb{R}$.
2. There are two simple A-modules. Describe them.
3. Write down a composition series of the free A-module A_A.

Ex 2. Let $A = A_1 \times A_2$ be a direct product of two $\Bbbk$-algebras A_1, A_2. Note that there are algebra homomorphisms $\pi_i : A \to A_i$. Consider the idempotents $e_1 := (1_{A_1}, 0), e_2 := (0, 1_{A_2})$ of A.

1. For $M \in \mod A$, show that $M = M_1 \oplus M_2$ with $M_i = Me_i$ for both $i \in \{1, 2\}$, i.e. $M = Me_1 + Me_2$ with $Me_1 \cap Me_2 = 0$.
2. Show that π_i induces a simple A-module structure on the simple A_i-modules for both $i \in \{1, 2\}$.
3. Show that there exists a natural A_i-action on the direct summand M_i of M (in the notation of (1)). In particular, classify the simple A-modules.

Ex 3. Consider the formal power series ring $\Bbbk[[x]] := \{\sum_{i=0}^{\infty} \lambda_i x^i \mid \lambda_i \in \Bbbk\}$ with

- addtition $(\sum_i \lambda_i x^i) + (\sum_i \mu_i x^i) = \sum_i (\lambda_i + \mu_i) x^i$,
- scalar multiplication $\lambda(\sum_i \lambda_i x^i) = \sum_i (\lambda \lambda_i) x^i$,
- multiplication $(\sum_i \lambda_i x^i)(\sum_j \mu_j x^j) = \sum_k (\sum_{i+j=k} \lambda_i \mu_j) x^k$.

1. Show that $\Bbbk[[x]]$ is a commutative $\Bbbk$-algebra.
2. Determine the invertible elements in $\Bbbk[[x]]$.
3. Show that $\Bbbk[[x]]$ is a local algebra, i.e. there exists a unique maximal left (equivalently, right) ideal.
4. Classify the simple modules of $\Bbbk[[x]]$.
5. Classify the simple modules of $\Bbbk[x]$. *Hint*: (a) Maximal two-sided ideals of a ring are determined by the irreducible elements. (b) $\Bbbk[x]$ is commutative.

Ex 4. Let A be the following $\Bbbk$-algebra:

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, x, y \in \Bbbk \right\}$$

Note that for every matrix in A, the (1,1)-entry and the (3,3)-entry must be the same. Let e_1 be the idempotent of A given by the matrix with 1 in the (1,1)- and (3,3)-entry and 0 everywhere else; and $e_2 = 1_B - e_1$.

1. Show that both e_1A and e_2A have a unique simple submodules and they are isomorphic.
2. Let S_1 be the simple module in (1). Show that $S_2 := e_2A/S_1 \not\cong S_1$.
3. Find the composition series of e_1A and e_2A.

Ex 5. Consider the truncated polynomial ring $B = \Bbbk[x]/(x^2)$ and let S be its unique simple module $S = \Bbbk y$.

1. Find a basis for the Hom-spaces $\mathrm{Hom}_B(X, Y)$ for $X, Y \in \{B, S\}$. *Note*: One of these spaces have dimension 2, and all other have dimension 1.
2. Show that $\mathrm{End}_B(S \oplus B) \cong A$ where A is the algebra in Exercise 3 above.
3. Find the bouned path algebra presentation of B, i.e. a $\Bbbk$-algebra isomorphism $B \cong \Bbbk Q/I$.

Deadline: 3rd November, 2022

Submission: E-mail to (replace at by @) aaron.chan at math.nagoya-u.ac.jp