Topics in Mathematical Science VI
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Group Representations and Character Theory

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Contents

Lecture 1

Throughout, 'group' means 'finite group', unless otherwise stated. K will always be a field.

Definition 1.1. *A finite-dimensional (resp. n-dimensional) K-linear representation of a group G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V), \qquad g \mapsto \rho_g,$$

for some finite-dimensional (resp. n-dimensional) K-vector space V. The linear transformation ρ_g here is called the action of g on V.

Often, the symbol ρ is suppressed and we write $G \curvearrowright V$ instead, and say 'G acts on V'. In particular, instead of $\rho_g(v)$ for $v \in V$, we write $g(v)$ instead.

Example 1.2. *(1) The trivial representation of G is the one-dimensional representation*

$$\mathrm{triv}_G : G \to \mathrm{GL}(K), \qquad g \mapsto \mathrm{id}\,.$$

(2) $G = \mathfrak{S}_n$ the symmetric group of rank n. The sign representation of $\mathfrak{S}_n$ is the one-dimensional representation

$$\mathrm{sgn} : G \to \mathrm{GL}(K), \qquad \sigma \mapsto \mathrm{sgn}(\sigma),$$

where $\operatorname{sgn}(\sigma) \in \{\pm 1\}$ *is the parity (or sign) of the permutation* σ.

Exercise 1.3. *Suppose* $\rho : G \to \mathrm{GL}(V)$ *is a representation. Show that* $\det \rho$ *is also a representation.*

Definition 1.4. *Let* KG *be the* K*-vector space with basis* G*, i.e.* $x \in KG \Leftrightarrow x = \sum_{g \in G} \lambda_g g$ *with* $\lambda_g \in K$ *for all* $g \in G$.

Define a map

$$KG \times KG \to KG, \qquad (\sum_{g \in G} \lambda_g g, \sum_{h \in G} \mu_h h) \mapsto \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

It is routine to check that this defines a ring structure on KG *with identity given by that of* G*. We call this ring the group algebra of* G *over* K.

Clearly, $G \curvearrowright KG$ naturally; this is called the *regular representation*.

Exercise. *Show that there is an injective ring homomorphism* $K \to Z(KG) := \{x \in KG \mid xy = yx \ \forall y \in KG\}$*. In other words, the group algebra* KG *is a* K*-algebra.*

Lemma 1.5. $\rho : G \to \mathrm{GL}(V)$ *is a (finite-dimensional)* K*-linear representation of* G *if, and only if,* V *has the structure of a (finite-dimensional) left* KG*-module.*

Proof $\underline{\Rightarrow:}$ For $x = \sum_g \lambda_g g \in KG, v \in V$. It is routine to check that $x \cdot v := \sum_g \lambda_g \rho_g(v)$ defines a left KG-module structure.

$\underline{\Leftarrow:}$ Define a map $\rho_g : V \to V$ by $v \mapsto gv$. Since $g^{-1}g(v) = v$, we have $\rho_{g^{-1}}\rho_g = \mathrm{id}$, and so $\rho_g \in \mathrm{GL}(V)$. It is routine to check that $g \mapsto \rho_g$ is a group homomorphism. $\square$

Remark 1.6. One may find in older textbooks that use terminologies like 'the KG-module V is afforded by ρ' in the setting of this lemma.

Definition 1.7. $V = (V, \rho), W = (W, \theta)$ *be* K*-linear representations of* G*. A homomorphism from* V *to* W *is a* K*-linear transformation such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\theta_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$*, i.e.* $f\rho_g = \theta_g f$ *for all* $g \in G$.

An isomorphism from V *to* W *is a homomorphism that is invertible, i.e.* $\exists g$ *s.t.* $gf = \mathrm{id}_V$ *and* $fg = \mathrm{id}_W$*. In this case,* V *and* W *are equivalent representations, and write* $V \cong W$.

Write $\mathrm{Hom}_G(V, W)$ *to be the (*K*-vector) space of all homomorphisms from* V *to* W.

Lemma 1.8. $f : V \to W$ *is a homomorphism of* K*-linear* G*-representations if, and only if, it is a homomorphism of left* KG*-modules; in other words,* $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{KG}(V, W)$*. Consequently,* $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$, $W/\mathrm{Im}(f)$ *are naturally* K*-linear* G*-representations.*

Proof This first part is clear (if not, think through it).

For the second part, just recall that the kernel, image, and quotient of image of any homomorphism of modules are also modules. $\square$

Remark. In the language of category theory, Lemma 1.5 and 1.8 together says that the category of finite-dimensional K-linear G-representations (where morphisms are homomorphisms) and the category of finitely generated left KG-modules are isomorphic (note that this is stronger than just equivalence of categories).

Exercise 1.9. *Let V be the 1-dimensional subspace spanned by $\sum_{g \in G} g \in KG$. Show that V is a KG-module and that $\mathrm{triv}_G \cong V$.*

Recall that for a ring R with identity 1, either 1 has infinite order (under addition) or has prime, say p, order. The *characteristic* of R, denoted by $\operatorname{char} R$, is 0 in the former case, p in the latter.

Exercise. *Fix any $n \geq 2$.*

(i) Find a generator v such that $\mathrm{sgn} = Kv$. (Hint: Modify the generator $\sum_{g \in G} g$ of the trivial representation.)

(ii) Show that $\mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{triv}, \mathrm{sgn}) = 0 = \mathrm{Hom}_{\mathfrak{S}_n}(\mathrm{sgn}, \mathrm{triv})$ when $\operatorname{char} K = 2$, otherwise, $\mathrm{triv} \cong \mathrm{sgn}$.

Two classes of group representations. In the literature, by *ordinary representations* we mean K-linear representations with $\operatorname{char} K = 0$; by *modular representations* we mean K-linear representations with $\operatorname{char} K \mid |G|$.

The *Maschke's theorem* (and its consequence) justifies that ordinary representation theory is (significantly) easier to understand than modular ones - this will be our next goal. The material we will use is based on a more ring theoretic approach (from Benson's book Chapter 1) to the subject, which has the advantage of shedding some light on what happen on the modular side too. The proof of Maschke's theorem will follow the exposition of James and Liebecks.

Interlude on terminology and notation. For a field K, recall that a *K-algebra* is a ring R equipped with a ring homomorphism $K \to Z(R) := \{x \in R \mid xy = yx \ \forall y \in R\}$. This is equivalent to saying that R is a K-vector space equipped with a ring structure.

For a K-algebra A, let $A\,\mathsf{mod}$ be the category of finitely generated left A-modules. So by $M \in A\,\mathsf{mod}$ we mean that M is an A-module, and by $(f : M \to N) \in A\,\mathsf{mod}$ we mean that f is an A-module homomorphism. We will use 0 to denote either the zero homomorphism, or the zero element in a vector space, or the vector space with only the zero element; this should be clear from context.

Like numbers, we like to break down modules into simpler 'components'. The first candidate is via the notion of direct sum. Recall that an A-module M is a *direct sum*, say $M = M_1 \oplus M_2$, if $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$. We will come back to this next lecture. In this lecture, we consider a more refined way to break down M into smaller modules.

Definition 1.10. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$.*

(1) *M is simple if for any submodule L of M, we have $L = 0$ or $L = M$.*

(2) *M is semisimple if it is a direct sum of simples.*

Remark 1.11. In the language of representations, simple modules are called *irreducible* representations, and semisimple modules are called *completely reducible* representations.

Example 1.12. (1) *Trivial module and sign module are both simple. In general, any 1-dimensional representation of a group G will be simple for dimension reason.*

(2) *Consider the matrix ring $A := \mathrm{Mat}_n(K) := \{n \times n$ matrices with entries in $K\}$. Let V be the 'column space', i.e. $V = \{(v_j)_{1 \leq j \leq n} \mid v_j \in K\}$ where $X \in \mathrm{Mat}_n(K)$ acts on $v \in V$ by $v \mapsto Xv$ (matrix multiplication from the left). Then V is an n-dimensional simple module. The regular representation A is semisimple as it is isomorphic to the direct sum of n column spaces (corresponding to the n choices of column we can cut matrix into V).*

(3) *The ring of dual numbers is $A := K[x]/(x^2)$. The module (x) is simple. The regular representation A is non-simple (as (x) is a non-trivial submodule). It is also not semisimple. Indeed, (x) is a submodule of A, and the quotient module can be described by Kv where $v = 1 + (x)$. If A is semisimple, then Kv is isomorphic to a submodule of A. Such a submodule must be generated*

by $a + bx$ (over A) for some $a, b \in K$. If $a \neq 0$, then $A(a + bx) = A$. So $a = 0$, and $Kv \cong (x)$, a contradiction.

The following easy yet fundamental lemma describes the relation between simple modules.

Lemma 1.13 (Schur's lemma). *Suppose S, T are simple A-modules, then*

$$\mathrm{Hom}_A(S, T) = \begin{cases} \text{a division } K\text{-algebra}, & \text{if } S \cong T; \\ 0, & \text{otherwise.} \end{cases}$$

Proof For $f \in \mathrm{Hom}_A(S, T)$, $\mathrm{Im}(f)$ is a submodule of T, and so f is either zero or a K-vector space isomorphism, and the latter case only happens when $S \cong T$. $\square$

Remark 1.14. If K is algebraically closed, then any division K-algebra is just K itself. The complication with the divison K-algebra appearing is the reason why most literature consider only the case when K is algebraically closed. In particular, for ordinary representation one usually just consider $K = \mathbb{C}$. In this course, this will also often be the case - perhaps the only exception is when we consider general K-algebra instead of group algebra.

Lemma 1.15. *Consider $M = S_1 \oplus \cdots S_r$ with simples $S_i \cong S_j$ for all i, j. Then $\mathrm{End}_A(M) \cong \mathrm{Mat}_r(D)$ as K-algebras, where $D := \mathrm{End}_A(S_i)$.*

Note that $\mathrm{End}_A(M)$ is a ring where multiplication is given by composition. Since A is a K-algebra, $\mathrm{End}_A(M)$ is also a K-algebra as K acts by scalar multiplications and commutes with homomorphisms, i.e. $(\lambda \cdot f)(m) := \lambda f(m) = f(\lambda m) = (f \cdot \lambda)(m)$ for all $(f : M \to M) \in A\,\mathsf{mod}$ and $m \in M$.

Proof We have canonical homomorphisms $\iota_j : S_j \hookrightarrow M$ and $\pi_i : M \twoheadrightarrow S_i$. So for $f \in \mathrm{End}_A(M)$, we have a homomorphism $\pi_i f \iota_j : S_j \to S_i$, and by Schur's lemma, this can be identified with an element of D. Now we have a ring homomorphism

$$\mathrm{End}_A(M) \to \mathrm{Mat}_r(D), \quad f \mapsto (\pi_i f \iota_j)_{1 \leq i,j \leq r},$$

which is clearly injective. Conversely, for $(\lambda_{i,j})_{1 \leq i,j \leq r} \in \mathrm{Mat}_r(D)$, we have an endomorphism $M \overset{\pi_j}{\twoheadrightarrow} S_j \overset{\lambda_{i,j}}{\to} S_i \overset{\iota_i}{\hookrightarrow} M$, which yields the required surjection. $\square$

Lecture 2

Definition 2.1. *Let A be a K-algebra and $M \in A\,\mathsf{mod}$. A composition series of M is a finite chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$$

such that M_i/M_{i-1} is simple for all $1 \leq i \leq \ell$. The number ℓ here is the length of the composition series. The module M_i/M_{i-1} for each $1 \leq i \leq \ell$ are called the composition factors of the series.

Composition series allows us to understand the structure of a module by simple modules. It is desirable to have a rigidity result - that composition factors do not change.

Lemma 2.2. *Let M be a finite-dimensional left A-module. Then composition series of M exists.*

Proof This is by induction on $\dim_K M$. For $\dim_K M = 0$ this is trivial. For $\dim_K M > 0$, if M is simple, then we are done. Otherwise, M proper non-zero submodule, and we pick N such a submodule so that M/N is simple. Clearly $\dim_K N < \dim_K M$ and so we can apply induction hypothesis. □

Theorem 2.3 (Jordan-Hölder Theorem). *Any two composition series have the same length and their composition factors are the same up to permutations.*

Proof Suppose we have two composition series

$$\begin{array}{ccccccc} 0 = M_0 & \subset & M_1 & \subset & \cdots & \subset & M_\ell = M, \\ 0 = N_0 & \subset & N_1 & \subset & \cdots & \subset & N_n = M. \end{array}$$

Without loss of generality, we can assume $n > \ell$. We claim that $N_\ell = M$. Indeed, we can do this by induction on ℓ. If $\ell = 0$, then clearly $M_0 = 0 = N_0$ and we are done; likewise, when $\ell = 1$, then M is simple and we have $M_1 = M = N_1$. For $\ell > 1$, we have

$$0 = M_1 \cap N_0 \subset M_1 \cap N_1 \subset \cdots \subset M_1 \cap N_n = M_1 \cap M = M_1.$$

So as M_1 simple, there is some n_0 such that $N_{n_0} \cap M_1 = M_1$ and $N_i \cap M_1 = 0$ for all $i < n_0$.

We now consider two new chains

$$\begin{array}{ccccccccccccccccc} 0 & \subset & \frac{M_2}{M_1} & \subset & \cdots & \cdots & & \cdots & & \cdots & \cdots & & \cdots & \cdots & \cdots & \subset & \frac{M_\ell}{M_1} = \frac{M}{M_1}, \\ 0 & \subset & N_1 & \subset & \cdots & \subset & N_{n_0-1} & \subset & \frac{N_{n_0+1}}{M_1} & \subset & \frac{N_{n_0+2}}{M_1} & \subset & \cdots & & & \subset & \frac{N_n}{M_1} = \frac{M}{M_1}, \end{array}$$

which are both composition series of M/M_1. By induction hypothesis, we thus have $n-1 = \ell-1$ and the composition factors of these two series coincide up to permutation. □

Remark. This (simpler) version of proof relies on M having composition series of finite length. One can expect similar more careful argument applies for modules that are both noetherian and artinian. In fact, for general K-algebra, M admits a composition series of finite length if and only if it is noetherian and artinian. In this case, Jordan-Hölder theorem also holds.

Exercise 2.4. *Let A be the algebra of upper triangular $n \times n$-matrices:*

$$A := \begin{pmatrix} K & K & \cdots & K \\ 0 & K & \cdots & K \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & K \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i,j \leq n} \,\middle|\, \begin{array}{l} a_{i,j} \in K \ \forall i,j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}$$

For $1 \leq i \leq j \leq n$, let $M_{i,j} \subset K^{\oplus n}$ be the vector space given by column vectors $v = (v_k)_{1 \leq k \leq n}$ where $v_k = 0$ for $k \notin \{i, i+1, \ldots, j\}$.

(i) *Determine which $M_{i,j}$'s are simple.*

(ii) *Describe the composition series of* $M_{i,j}$.

Jordan-Hölder theorem effectively says that the notion of length and composition factor of a module is well-defined without any reference to a chosen composition series.

Now that we no longer worries about building blocks (composition factors) of a module is non-well-defined, we can move on to understand the simplest form of algebra - where every module is semisimple.

Definition 2.5. *Let* A *be a* K*-algebra and* $M \in A\,\mathsf{mod}$.

(1) *The (Jacobson) radical of* A *is the (two-sided) ideal*

$$J(A) := \{a \in A \mid aM = 0 \; \forall \textit{simple } M\}.$$

This is equivalent to saying that $J(A)$ *is the intersection of all maximal left ideals of* A*, as well as the intersection of all maximal right ideals of* A.

(2) A *is semisimple if* $J(A) = 0$*. This is equivalent to saying that left (equivalently, right) regular* A*-module* ${}_AA$ *is semisimple.*

Example 2.6. (1) *A field* K *on its own is a semisimple* K*-algebra.*

(2) *Suppose* D *is a division* K*-algebra, then* $\mathrm{Mat}_n(D) := \{n \times n \textit{ matrices with entries in } D\}$ *is a semisimple* K*-algebra.*

(3) *A finite product of semisimple algebras is semisimple.*

(4) *The ring of dual numbers* $A := K[x]/(x^2)$ *is not semisimple since it has a non-trivial maximal ideal* $J(A) = (x)$*. More generally, the truncated polynomial ring* $K[x]/(x^n)$ *for any* $n \geq 2$ *is also non-semisimple.*

Theorem 2.7. (see [Benson, Lemma 1.2.4] or [Erdmann-Holm, Theorem 4.11, 4.23]) *The following are equivalent for a* K*-algebra* A.

(i) A *is a semisimple algebra.*

(ii) The regular representation ${}_AA$ *is a semisimple module.*

(iii) Every A*-module is semisimple.*

A natural question is whether all semisimple is always a product of matrix rings over division rings. To answer this question, we need some elementary (but fundamental) properties of simple modules first.

Lemma 2.8. *Let* $e \in A$ *be an idempotent, i.e.* $e = e^2 \in A$*. Then the following hold.*

(1) (Yoneda's lemma) $\mathrm{Hom}_A(Ae, M) \cong eM$ *as a* K*-vector space for all* $M \in A\,\mathsf{mod}$.

(2) *There is an isomorphism of rings* $\mathrm{End}_A(Ae)^{\mathrm{op}} \cong eAe$.

Proof (1): Check $\mathrm{Hom}_A(Ae, M) \ni f \mapsto f(e) \in eM$ defines a K-linear map with inverse $em \mapsto (ae \mapsto aem)$.

(2): Take $M = Ae$ in (1) and notice that order of multiplication in reverse that of homomorphism composition. □

Exercise. *Recall (or check any reference book) the notion of free module and the rank of it. Check that for an idempotent* $e \in A$*,* Ae *is a direct summand of* A*. In ring/module theory terms, (by definition)* Ae *is thus a projective module since it is a direct summand of a free module.*

Theorem 2.9 (Artin-Wedderburn's theorem). *Let A be a finite-dimensional K-algebra and let r be the number of isoclasses of simple A-modules, say, with representatives $S_1, \dots, S_r$. Let $D_i := \mathrm{End}_A(S_i)^{\mathrm{op}}$ be the division K-algebra given by endomorphism of the simple module S_i. Then there is an isomorphism of K-algebras*

$$A/J(A) \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_r}(D_r).$$

Proof Let $B := A/J(A)$. By definition of $J(A)$, the A-module $A/J(A)$ is semisimple, and any A-submodule M of $A/J(A)$ satisfies $J(A)M = 0$. Hence, $M = M/J(A)M$ is naturally a B-module and $\mathrm{End}_B(M) \cong \mathrm{End}_A(M)$ (even as rings!).

By Lemma 2.8, we have $B \cong \mathrm{End}_B(B)^{\mathrm{op}}$. Since B is semisimple, the regular representation B is a semisimple B-module, say, $B \cong S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$ where S_i are the (representatives of the) isomorphism classes of simple B-modules. Hence, it follows from Lemma 1.13 and Lemma 1.15 that

$$B \cong \mathrm{End}_B(B)^{\mathrm{op}} \cong \big(\mathrm{Mat}_{n_1}(E_1) \times \cdots \times \mathrm{Mat}_{n_r}(E_r)\big)^{\mathrm{op}} \cong \mathrm{Mat}_{n_1}(E_1^{\mathrm{op}}) \times \cdots \times \mathrm{Mat}_{n_r}(E_r^{\mathrm{op}}),$$

where $E_i := \mathrm{End}_B(S_i)$ for all $1 \le i \le r$. This completes the proof. □

Theorem 2.10 (Maschke's theorem). *If $\mathrm{char}\, K \nmid |G|$, then for any KG-module V and submodule $U \subset V$, there is a KG-module W such that $V = U \oplus W$.*

Proof Let W_0 be any K-vector space complement of U in V, and $\pi : V \to U$ be the K-linear projection map. If π is a homomorphism, then W_0 is a KG-module and we are done by Lemma 1.8 – unfortunately this is not true in general. So our goal is to modify π into an idempotent homomorphism. The clever trick is to consider

$$p : V \to V, \quad v \mapsto \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(v).$$

Let us now show that $p \in \mathrm{End}_{KG}(V)$. Indeed, for any $g \in G$, we have

$$p(gv) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h(gv) = \frac{1}{|G|} \sum_{h \in G} g(g^{-1}h^{-1}) \pi (hg) v = g \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi h v = gp(v).$$

Now we check that $p^2 = p$. It is easy to see that, as $\mathrm{Im}(\pi) = U$, we have $\mathrm{Im}(p) \subset U$. Hence, it remains to show that $p(u) = u$ for all $u \in U$. Indeed, we have

$$p(u) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \pi \underbrace{h(u)}_{\in U} = \frac{1}{|G|} \sum_{h \in G} h^{-1} h(u) = \frac{1}{|G|} \sum_{h \in G} u = u.$$

This completes the proof. □

Corollary 2.11. *KG is semisimple if, and only if, $\mathrm{char}\, K \nmid |G|$.*

Proof $\underline{\Leftarrow:}$ Consequence of iteratively applying Maschke's theorem (Theorem 2.10) starting with $V = KG$.

$\underline{\Rightarrow:}$ Suppose on the contrary that KG is semisimple. Let $a := \sum_g g \in KG$. Recall that $\mathrm{triv}_G \cong V := Ka \subset KG$. So we must have $KG \cong V \oplus W$ for some left ideal W of KG.

Consider $w = \sum_h \lambda_h h \in KG$. Since W is a left ideal of KG, we have $aw \in W$. On the other hand, we also have

$$aw = (\sum_g g)(\sum_h \lambda_h h) = \sum_h \lambda_h (\sum_g gh) = \sum_h \lambda_h a,$$

which means that $aw \in V$. But $V \cap W = 0$ and so we must have $\sum_h \lambda_h = 0$, which means that

$$W \subset W' := \left\{ \sum_g \mu_g g \in KG \,\middle|\, \sum_g \mu_g = 0 \right\}.$$

The space W' can be rewritten as the kernel of the map (a.k.a. the augmentation map)

$$\epsilon : KG \to K \text{ given by } \sum_g \mu_g g \mapsto \sum_g \mu_g.$$

Thus, $\dim_K W' = |G| - 1 = \dim_K W$ which means that $W = W'$.

However, we can also see that $\epsilon(a) = 0$, and so $V \subset W$, a contradiction. □

Remark. Note that the proof of this result (both directions) relies neither on Jordan-Hölder nor Artin-Wedderburn. From ring theory perspective, it makes more sense to first talk about unicity of composition factors and structure theory for semisimple algebras, so that we know semisimple modules (and algebras) can be completely understood once we know their composition factors.

Lecture 3

We have seen Jordan-Hölder theorem, which tells us that the decomposition of a module into composition factors (‘irreducible constituents’ in the language of classical representation theory) does not ‘change’. One could have also considered the decomposition of a module into direct sum of smaller ones, and ask whether such a decomposition is unique (up to permutation of the direct summands).

Definition 3.1. *Let A be a K-algebra and M be an A-module.*

(1) *M is indecomposable if $M = L \oplus N$ implies that either L or N is zero.*

(2) *We say that $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition (or just decomposition for short if context is clear) of M if each M_i is indecomposable. Such a decomposition is said to be unique if for any other decomposition $M = \bigoplus_{j=1}^{n} N_j$, we have $n = m$ and the N_j's are permutation of the M_i's.*

(3) *$A\,\mathsf{mod}$ is said to be Krull-Schmidt if every finitely generated A-module M admits a unique indecomposable decomposition.*

(4) *A ring R is local if it has a unique maximal left (equivalently, right) ideal.*

Theorem 3.2. *Suppose $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition of M. If $\mathrm{End}_A(M_i)$ is local for all $1 \le i \le m$, then the decomposition of M is unique.*

Proof We proceed by induction on m. This is clear if $m = 0, 1$. Suppose that $m > 1$ and we have another decomposition $M = \bigoplus_{j=1}^{n} N_j$. Consider the homomorphisms given by composing canonical inclusions and projections

$$\begin{tikzcd} & M \arrow[dr, two heads] & \\ N_j \arrow[ur, hook] \arrow[rr, "\alpha_j"] & & M_1, \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & M \arrow[dr, two heads] & \\ M_1 \arrow[ur, hook] \arrow[rr, "\beta_j"] & & N_j. \end{tikzcd}$$

Then we have $\sum_j \alpha_j\beta_j = \mathrm{id}_{M_1}$. Since $\mathrm{End}_A(M_1)$ is local and each $\alpha_j\beta_j \in \mathrm{End}_A(M_1)$, there is some j such that $\alpha_j\beta_j$ is a unit. Without loss of generality, we can take $j = 1$, and so $M_1 \cong N_1$.

In order to apply induction hypothesis, we need isomorphism $f : \bigoplus_{i=2}^{m} M_i = M/M_1 \to M/N_1 = \bigoplus_{j=2}^{n} N_j$. This amounts to finding an isomorphism $\hat{f} : M \to M$ such that $\hat{f}(M_1) = N_1$. Let $\hat{f} := \mathrm{id}_M - p + qp \in \mathrm{End}_A(M)$, where p and q are given by

$$\begin{tikzcd} & M_1 \arrow[dr, hook] & \\ M \arrow[ur, two heads] \arrow[rr, "p"] & & M, \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & N_1 \arrow[dr, hook] & \\ M \arrow[ur, two heads] \arrow[rr, "q"] & & M \end{tikzcd}$$

respectively.

We first show that $\hat{f}$ is an isomorphism; it suffices to show that this is injective by dimension of the domain and the range. Indeed, if $\hat{f}(m) = 0$, then as $p^2 = p$, we have

$$0 = (p\hat{f})(m) = p(m) - p^2(m) + pqp(m) = pqp(m)$$

Observe from the definition of pqp that we have the following commutative diagram:

$$\begin{tikzcd} & M_1 \arrow[rr, "\beta_1"] \arrow[dr, hook] & & N_1 \arrow[rr, "\alpha_1"] \arrow[dr, hook] & & M_1 \arrow[dr, hook, "\iota_1"] & \\ M \arrow[ur, two heads, "\pi_1"] \arrow[rr, "p"] & & M \arrow[ur, two heads] \arrow[rr, "q"] & & M \arrow[ur, two heads] \arrow[rr, "p"] & & M \end{tikzcd}$$

Since $\alpha_1\beta_1$ is a unit and ι_1 is an injection, $\iota_1\alpha_1\beta_1$ is injective. Hence, $pqp(m) = \iota_1\alpha_1\beta_1(\pi_1(m)) = 0$ implies that $\pi_1(m) = 0$. But $p = \iota_1\pi_1$, and so $p(m) = \iota_1(\pi_1(m)) = 0$, which then implies that $\hat{f}(m) = m - p(m) + qp(m) = m$. Hence, $\hat{f}(m) = 0$ implies that $m = 0$ as required.

Let us now consider $\hat{f}(M_1)$. Since $qp = \iota_1\alpha_1\pi_1$ and we have shown that α_1 is an isomorphism, $\hat{f}(m_1) = m_1 - m_1 + \iota(\alpha_1(m_1)) = \iota(\alpha_1(m_1))$ for all $m_1 \in M_1$. Hence, $\hat{f}(M_1) = N_1$ as required. $\square$

Tensor and dual

Let us now come back to the setting of group algebra (group representation) and look at various natural way to construct new representations from old.

Definition 3.3. *Let V, W be finite-dimensional K-vector space with bases, say, $\mathcal{B}, \mathcal{C}$ respectively. Then the tensor product $V \otimes_K W$ (or simplifies to $V \otimes W$ if context is clear) is the finite-dimension K-vector space with bases given by*

$$\{v \otimes w \mid v \in \mathcal{B}, w \in \mathcal{C}\}.$$

Notation. For $V \in K\,\mathsf{mod}$, $V^* := \mathrm{Hom}_K(V, K)$ denotes the dual vector space.

The following innocent looking isomorphisms are arguably the most used isomorphisms in homological algebra.

Lemma 3.4. *For any finite-dimensional K-vector spaces U, V, W, the following hold.*

(1) $V^* \otimes_K W \cong \mathrm{Hom}_K(V, W)$.

(2) $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W))$.

Proof (1) Let $\mathcal{B} = \{v_1, \ldots, v_m\}, \mathcal{C} = \{w_1, \ldots, w_n\}$ be bases of V, W respectively. Let $\mathcal{B}^* = \{f_1, \ldots, f_m\}$ be the canonical dual basis, i.e. $f_i(v_j) = \delta_{i,j}$ for all $1 \leq i, j \leq m$.

Define $\theta(f_i \otimes w_j)$ to be the K-linear map that extends $v_k \mapsto f_i(v_k)w_j \in W$ and check that θ is K-linear.

Conversely, for $\alpha \in \mathrm{Hom}_K(V, W)$, let $\phi(\alpha) := \sum_i f_i \otimes \alpha(v_i)$. Check that ϕ and θ are inverse to each other.

(2) Define

$$\theta : \mathrm{Hom}_K(U \otimes V, W) \to \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)), \quad f \mapsto \theta_f,$$

where $\theta_f(u) : V \to W$ is the map that sends $v \in V$ to $f(u \otimes v) \in W$.

Define also

$$\phi : \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)) \to \mathrm{Hom}_K(U \otimes V, W), \quad f \mapsto \phi_f,$$

where $\phi_f(u \otimes v) := (f(u))(v)$. Check that ϕ and θ are inverse to each other. $\square$

Remark 3.5. The isomorphism (1) absolutely require finite-dimensionality. The isomorphism (2) is called 'currying' in computer science, coined from Curry-Howard correspondence. This isomorphism is actually natural, and yields an adjoint pair $(- \otimes_K V, \mathrm{Hom}_K(V, -))$ of functors.

Proposition 3.6. *Let A, B be K-algebras. Then $A \otimes_K B$ is also a K-algebra with multiplication given by extending $(a \otimes b)(a' \otimes b') \mapsto aa' \otimes bb'$ linearly. For $M \in A\,\mathsf{mod}$ and $N \in B\,\mathsf{mod}$, we have $M \otimes_K N \in A \otimes_K B\,\mathsf{mod}$.*

Proof Routine checking. $\square$

Example 3.7. *Consider $A = (a_{i,j})_{1 \leq i,j \leq m} \in \mathrm{Mat}_m(K)$ and $B \in \mathrm{Mat}_n(K)$ and defines (what is* *sometimes called Kronecker product of matrices)*

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & \ddots & & a_{2,m}B \\ \vdots & & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,m}B \end{pmatrix}.$$

Then we have an isomorphism of algebras

$$\mathrm{Mat}_m(K) \otimes_K \mathrm{Mat}_n(K) \to \mathrm{Mat}_{mn}(K), \quad (A, B) \mapsto A \otimes B.$$

From this, we can see that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, *if (and only if) both* A, B *are invertible. Thus, the isomorphism restricts to a group isomorphism* $\mathrm{GL}(K^{\oplus m}) \otimes_K \mathrm{GL}(K^{\oplus n}) \cong \mathrm{GL}(K^{\oplus mn})$.

Exercise 3.8. *Show that the tensor product algebra* $KG \otimes_K (KG)^{\mathrm{op}}$ *is isomorphic to the group algebra* $K(G \times G)$ *of the direct product* $G \times G$ *as* K*-algebras.*

One thing that makes group algebras special is that we can always 'tensor within the category of G-representations':

Proposition 3.9. *For any* $V, W \in KG\,\mathsf{mod}$, *we have* $V \otimes_K W \in KG\,\mathsf{mod}$ *where the action of* g *is given by* $v \otimes w \mapsto gv \otimes gw$.

Proof Let $\mathcal{B}, \mathcal{C}$ be the K-linear bases of V, W respectively and consider their respective representations $\rho : G \to \mathrm{GL}(V)$ and $\phi : G \to \mathrm{GL}(W)$. Then ρ and ϕ extends to a group homomorphism $G \to \mathrm{Mat}_r(K)$ for $r = m := |\mathcal{B}|$ and $r = n := |\mathcal{C}|$ respectively. Define

$$\rho \otimes \phi : G \to \mathrm{Mat}_{mn}(K) = \mathrm{Mat}_m(K) \otimes \mathrm{Mat}_n(K), \quad g \mapsto \rho_g \otimes \phi_g,$$

where ρ_g, ϕ_g are regarded as matrices. By the discussion in Example 3.7, this map factors through $\mathrm{GL}(V \otimes W)$. Hence, $\rho \otimes \phi$ is a representation of G, and it is clear by construction $\rho_g \otimes \phi_g$ corresponds to the given action of g on the vector space $V \otimes W$. □

Remark 3.10. Proposition 3.9 holds for any *Hopf algebra* in place of KG. Otherwise, for $M \in A\,\mathsf{mod}$ and $N \in B\,\mathsf{mod}$ with A, B algebras, then $M \otimes_K N$ is only a $A \otimes_K B$-module. In the case when $B = A$, we need a ring homomorphism $A \to A \otimes_K A$ in order to induce an A-module structure on $M \otimes_K N$; when A is a Hopf algebra, then such a ring homomorphism is given by the comultiplication map.

Exercise 3.11. *Let* A *be the ring of upper* 2×2*-triangular matrices. Let* V_1 *be the column space* $\binom{K}{0}$ *and* V_2 *be the column space* $\binom{K}{K}$; *i.e. the modules* $M_{1,1}$ *and* $M_{1,2}$, *respectively, in the notation of Example 2.4. Consider the identity element* $1_A = e_1 + e_2$ *where* e_i *is the matrix with* (i, i)*-entry 1 and zero everywhere else. Use this decomposition of* 1_A *to show that* $V_1 \otimes_K V_2$ *cannot be an* A*-module if we define a candidate* A*-action by* $v_1 \otimes v_2 \mapsto av_1 \otimes av_2$ *for all* $a \in A$.

Exercise 3.12. *Show that* $\mathrm{triv}_G \otimes_K V \cong V$ *for all* $V \in KG\,\mathsf{mod}$.

Detour: Even in good characteristics, tensor products of group (or Hopf algebra in general) representations is still active theme of researches that falls under the realm of *categorification* - the more precise problem is: For $V, W \in KG\,\mathsf{mod}$, describes the indecomposable direct summands of $V \otimes_K W$.

For example, in the representation theory of symmetric groups (its generalisations such as the Hecke algebra), the Mullineux problem asks for the description of $V \otimes_K \mathrm{sgn}$ for each irreducible V. Another example is McKay correspondence (which has deep implications in algebraic geometry) which comes from looking at tensor product representation of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Let us move on to the next construction.

Definition 3.13. *Let* $V, W \in KG\,\mathsf{mod}$ *and* $g \in G$.

(1) *For any K-linear map f in the (K-linear) dual space $V^* := \mathrm{Hom}_K(V, K)$, define $g \cdot f$ to be the K-linear map $v \mapsto f(g^{-1}v)$ for all $v \in V$.*

(2) *For any K-linear map $f \in \mathrm{Hom}_K(V, W)$, define $g \cdot f$ to be the K-linear map $v \mapsto gf(g^{-1}v)$ for all $v \in V$.*

Exercise. *Check that the two maps in the above definition yield two representations of G.*

Remark 3.14. Let ρ be the representation corresponding to $V \in KG\,\mathsf{mod}$, and ρ^* be the representation corresponding to V^*. Then $(\rho^*)_g = (\rho_{g^{-1}})^\top$ (the transpose of $\rho_{g^{-1}}$).

Although $V^* \cong V$ for any (finite-dimensional) K-vector space, this generally does not lift to an isomorphism of KG-modules.

Definition 3.15. *$V \in KG\,\mathsf{mod}$ is self-dual if $V^* \cong V$ as KG-modules.*

Exercise. *Trivial representation is clearly self-dual. Check that $\mathrm{sgn} \in K\mathfrak{S}_n\,\mathsf{mod}$ is self-dual.*

Proposition 3.16. *The regular representation is self-dual.*

Proof KG has K-linear basis G. The canonical (dual) basis of $(KG)^*$ is given by $\{f_g \mid g \in G\}$ where $f_g(h) := \delta_{g,h}$, i.e. $f_g(g) = 1$ and $f_g(h) = 0$ for all $h \in G \setminus \{g\}$.

Consider the K-linear map $\alpha : KG \to (KG)^*$ given by linearly extending $g \mapsto f_g$. This is clearly a K-vector space isomorphism. So we only need to show that $\alpha \in KG\,\mathsf{mod}$. For any $g, h, k \in G$, we have

$$(h\alpha(g))(k) = (h \cdot f_g)(k) = f_g(h^{-1}k) = \delta_{g,h^{-1}k} = \delta_{hg,k} = f_{hg}(k) = (\alpha(gh))(k).$$

This shows the claim. □

Remark. In ring theory, this is the same as saying that KG is self-injective (and in fact, Frobenius and symmetric).

In general, finding self-dual representations amounts to finding a 'G-invariant bilinear form'.

Proposition 3.17. *Suppose $\langle -, - \rangle : U \times V \to K$ is a G-invariant non-degenerate bilinear pairing of $U, V \in KG\,\mathsf{mod}$, i.e. $\langle gu, gv \rangle = \langle u, v \rangle$ for all $g \in G$ and all $u \in U, v \in V$. Then $U \cong V^*$ as KG-module.*

Proof Recall that for finite-dimensional K-vector spaces U, V, a non-degenerate bilinear pairing $\langle -, - \rangle : U \otimes V \to K$ yields an isomorphism $U \cong V^*$ via $u \mapsto \langle u, - \rangle$. One just needs to check that when $\langle -, - \rangle$ is G-invariant, then this K-vector space isomorphism lifts to a KG-module homomorphism. Indeed, if we write $f_u := \langle u, - \rangle$, then we have

$$f_{gu}(v) = \langle gu, v \rangle = \langle gu, g(g^{-1}(v)) \rangle = \langle u, g^{-1}(v) \rangle = f_u(g^{-1}v) = (g \cdot f_u)(v).$$

This shows the claim. □

Exercise 3.18. *For $V, W \in KG\,\mathsf{mod}$, show that there are the following isomorphisms.*

(1) *$(V \otimes_K W)^* \cong V^* \otimes_K W^*$ as KG-modules.*

(2) *$V^* \otimes_K W \cong \mathrm{Hom}_K(V, W)$ as KG-modules.*

Exercise 3.19. *Suppose X is a G-set (i.e. G acts by permuting elements of X) or a KG-module, denote by X^G the invariant subspace $\{x \in X \mid gx = x \,\forall g \in G\}$ of X. Let $U, V, W \in KG\,\mathsf{mod}$.*

(1) *Show that $(V^* \otimes_K V)^G \cong \mathrm{End}_{KG}(V)$.*

(2) *Show that $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W)$*

Lecture 4

To understand operation on a representation, it is natural to start looking at its effect on the simples. Naively, one may guess that being simple is preserved under taking the dual representation. This is our next aim. To this end, we want to construct submodule of the dual representation from a submodule of the original. Since duality swaps injective map with surjective map, simply taking the dual of a submodule will not gives us the submodule of the dual. But we may consider its complement in the following sense.

Definition 4.1. *Let $V \in K\,\mathsf{mod}$. For a K-linear subspace $U \subset V$, define a K-vector subspace*

$$U^\circ := \{f \in V^* \mid f(u) = 0, \forall u \in U\} \subset V^*.$$

For a K-linear subspace $L \subset V^*$, define a K-vector subspace

$$L^\perp := \{v \in V \mid f(v) = 0\, \forall f \in L\} \subset V.$$

Lemma 4.2. *Consider $V \in K\,\mathsf{mod}$, $U \subset V$ and $L \subset V^*$ are K-linear subspaces.*

(1) $\dim_K L^\perp = \dim_K V - \dim_K L$

(2) $\dim_K U^\circ = \dim_K V - \dim_K U$.

Proof We show the first one; the other one is analogous. Pick a basis $\{f_1, \dots, f_m\}$ of L and extends it to a basis $\{f_1, \dots, f_n\}$ of V^*. Let $\{e_1, \dots, e_n\}$ be the dual basis, i.e. $f_i(e_j) = \delta_{i,j}$. Then by definition $e_j \in L^\perp$ if and only if $m < j \leq n$. $\square$

Lemma 4.3. *Consider $V \in KG\,\mathsf{mod}$.*

(1) *If $L \subset V^*$ a KG-submodule, then $L^\perp$ is a KG-submodule of V.*

(2) *If $U \subset V$ is a KG-submodule, then U° is a KG-submodule of V^*.*

Proof (1) For any $g \in G$ and any $w \in L^\perp$, since $(g^{-1} \cdot f)(w) = f(g(w))$ and $g^{-1} \cdot f \in L$, we have $f(g(w)) = 0$, and so $L^\perp$ is closed under G-action.

(2) For any $g \in G$ and any $f \in {}^\perp U$, since $(g \cdot f)(u) = f(g^{-1}(u))$ and $g^{-1}(u) \in U$, we have $(g \cdot f)(u) = 0$, and so ${}^\perp U$ is closed under G-action. $\square$

Proposition 4.4. *For $V \in KG\,\mathsf{mod}$, V is simple if and only if so is V^*.*

Proof Consequence of Lemma 4.2 and Lemma 4.3. $\square$

In general, simple KG-module is not always self-dual, not even when $K = \mathbb{C}$, but ordinary character theory provides a simple way to check whether a simple $\mathbb{C}G$-module is self-dual.

Definition 4.5. *Let ρ be a representation of G over $\mathbb{C}$, and V be its corresponding $\mathbb{C}G$-module. Then the (ordinary) character of ρ (or of V) is the map*

$$\chi_\rho = \chi_V : G \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}(\rho(g)),$$

where Tr is the trace function (i.e. sum of all eigenvalues).

We will explore more on characters later in the course. For now, we just note that character is a representation-invariant, i.e. $V \cong W$ as $\mathbb{C}G\,\mathsf{mod}$ implies that $\chi_V = \chi_W$.

Lemma 4.6. *For any $g \in G$, $\chi_{V^*}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$, where $\overline{z}$ denotes the conjugate of $z \in \mathbb{C}$. In particular, if V is self-dual, then its character χ_V is real-valued.*

Proof Recall that $\rho^*(g) = (\rho(g)^{-1})^\top$. Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues (counted with multiplicity, i.e. $n = \dim_{\mathbb{C}} V$) of $\rho(g)$. Since G is finite, $\rho(g)$ has finite order, and so every eigenvalue is a root of unity. So we have

$$\chi_{V^*}(g) = \mathrm{Tr}(\rho_V(g^{-1})^\top) = \mathrm{Tr}(\rho_V(g^{-1})) = \chi_V(g^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\chi_V(g)}$$

for all $g \in G$. □

Remark 4.7. This requires G being finite.

Induction and Restriction.

Definition 4.8. *Let A be a K-algebra, M be a right A-module, and N be a left A-module. Then the tensor product $M \otimes_A N$ of M and N over A is the quotient K-vector space $M \otimes_K N/R$, where*

$$R = \{ma \otimes n - m \otimes an \mid m \in M, a \in A, n \in N\}.$$

WARNING: $M \otimes_A N$ is generally not an A-module.

Definition 4.9. *Let A, B be K-algebras. An K-vector space M is an A-B-bimodule if it is a left A-module and right B-module with commuting A- and B-action, i.e. $r(ms) = (rm)s$ for all $r \in R, m \in M, s \in S$. In other words, it is a left module over $A \times B^{\mathrm{op}}$ (equivalently, right module over $B \times A^{\mathrm{op}}$).*

Lemma 4.10. *Consider rings A, B, C. Let M be an A-B-bimodule, N be an B-C-bimodule, and L be an A-C-bimodule.*

(1) *$M \otimes_B N$ is a A-C-bimodule given by $a \cdot (m \otimes n) := (am) \otimes n$ and $(m \otimes n) \cdot c := m \otimes (nc)$.*

(2) *$\mathrm{Hom}_A(L, M)$ is a C-B-bimodule given by $(c \cdot f)(l) := (f(lc))$ and $(f \cdot b)(l) := f(l)b$.*

Proof Exercise. □

The above lemma tells us that tensor and Hom can be used to transfer modules (in fact, even homomorphisms) between different rings. Another consequence of Lemma 4.10 is that, if R is a commutative ring, then R-modules are the same as R-R-bimodules, and so $M \otimes_R N$ are automatically R-modules for R-modules M and N. Similarly, as left (resp. right) modules over a K-algebra, say A, are really A-K-bimodules (resp. K-A-bimodules), and so $M \otimes_A N$ is automatically a K-vector space.

Example 4.11. (1) *$A \otimes_A M \cong M$ as left A-module for all $M \in A\,\mathsf{mod}$.*

(2) *Suppose $\phi \in \mathrm{Aut}_K(A)$ is a K-linear (ring) automorphism of A. For $M \in A\,\mathsf{mod}$, let ${}_\phi M$ be the left A-module where the left A-action is twisted by ϕ, i.e. am on ${}_\phi M$ is given by $\phi(a)m$ on the original M. Consider A as an A-A-bimodule (action being multiplication), and write ${}_\phi A_1$ the A-A-bimodule with left action twisted by ϕ. Then ${}_\phi A \otimes_A M \cong {}_\phi M$.*

Recall the 'useful isomorphism' in Lemma 3.4; it has the following enhanced version.

Lemma 4.12. *Suppose A, B are K-algebras, X is an A-B-bimodule. Then for any $M \in B\,\mathsf{mod}, N \in A\,\mathsf{mod}$, there is a K-vector space isomorphism $\mathrm{Hom}_A(X \otimes_B M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(X, N))$.*

Proof Verbatim to the proof of Lemma 3.4. □

Definition 4.13. *Suppose $H \leq G$.*

(1) *For $V \in KG\,\mathsf{mod}$, its restriction to H, denoted by $\mathrm{Res}_H^G(V)$ or $V\downarrow_H^G$, is KH-module given by the same K-vector space where H-action is inherited from G-action.*

(2) *For $U \in KH\,\mathsf{mod}$, its induction to G (a.k.a. induced representation, induced module), denoted by $\operatorname{Ind}_H^G(U)$ or $U\uparrow_H^G$, is the KG-module given by $KG \otimes_{KH} U$.*

Remark 4.14. G-action on $\operatorname{Ind}_H^G(U)$ can be described as follows. Take coset representatives $g_1, \dots, g_n$, i.e. $G/H = \{g_1H, \dots, g_nH\}$. It is customary to just write $g_i \in G/H$ instead of $g_iH \in G/H$. For $g \in G$, we have $gg_iH = g_jH$ for some j, i.e. $gg_i = g_jh$ for some $h \in H$. This yields, for any $m \in M$, the following g-action on $g_i \otimes m \in \operatorname{Ind}_H^G(U)$:

$$g(g_i \otimes u) = (gg_i) \otimes u = g_jh \otimes u = g_j \otimes hu.$$

Remark 4.15. $KG \otimes_{KH} -$ is functorial (i.e. it can be applied to homomorphisms in a way that preserves axioms regarding compositions). Restriction can be made functorial by noticing that

$$\operatorname{Res}_H^G(V) = \operatorname{Hom}_{KG}({}_{KG}KG_{KH}, V)$$

where KG in the domain here is regarded as a KG-KH-bimodule.

Lemma 4.16. *Consider subgroup $H \leq G$ with coset representatives $g_1, \dots, g_n$.*

(1) *The right KH-module KG is free of rank n, namely, $(KG)_{KH} \cong (KH)^{\oplus n}$ in $\mathsf{mod}\,KH$.*

(2) *If $U \in KH\,\mathsf{mod}$ has K-basis $\mathcal{B}$, then $\operatorname{Ind}_H^G(U)$ has K-basis $\{g_i \otimes b \mid b \in \mathcal{B}, 1 \leq i \leq n\}$, i.e. $\dim_K \operatorname{Ind}_H^G(U) = |G/H| \dim_K(U)$.*

Proof (1) Clearly, as K-vector space we have decomposition $KG = \bigoplus_{i=1}^n K(g_iH)$. Since $g_ihh' \in g_iH$ for all $h, h' \in H$, each $K(g_iH)$ is isomorphic to KH as a right H-module.

(2) Now we have K-vector space isomorphisms:

$$\operatorname{Ind}_H^G(U) = KG \otimes_{KH} U \cong (\bigoplus_{i=1}^n g_i \cdot KH) \otimes_{KH} U \cong \bigoplus_{i=1}^n g_i \cdot U,$$

and the claim follows. □

Example 4.17. *Suppose $H \leq G$ is a subgroup. Consider the K-vector space $M_H := K(G/H)$ whose basis is given by the set of left G-cosets G/H. Then M_H is a KG-module. It follows from Lemma 4.16 (1) that $M_H \cong \operatorname{Ind}_H^G(\mathrm{triv}_H)$.*

Lemma 4.18. *Suppose we have subgroups $L \leq H \leq G$. Then $\operatorname{Ind}_H^G \operatorname{Ind}_L^H(U) = \operatorname{Ind}_L^G(U)$ for all $U \in KL\,\mathsf{mod}$.*

Proof This follows from the fact that $M \otimes_A (N \otimes_B L) \cong (M \otimes_A N) \otimes_B L$ as bimodules (check yourself). Namely, $KG \otimes_{KH} (KH \otimes_{KL} U) \cong (KG \otimes_{KH} KH) \otimes_{KL} U = KG \otimes_{KL} U$. □

Exercise 4.19. *Let $H \leq G$, $V \in KG\,\mathsf{mod}$ and $W \in KH\,\mathsf{mod}$. Show that*

(1) $\operatorname{Ind}_H^G(W^*) \cong (\operatorname{Ind}_H^G(W))^*$.

(2) $V \otimes_K \operatorname{Ind}_H^G(W) \cong \operatorname{Ind}_H^G(\operatorname{Res}_H^G(V) \otimes_K W)$.

Lemma 4.20 (Eckmann-Shapiro lemma). *There are K-vector space isomorphisms:*

(1) (Frobenius reciprocity) $\operatorname{Hom}_{KG}(\operatorname{Ind}_H^G U, V) \cong \operatorname{Hom}_{KH}(U, \operatorname{Res}_H^G V)$.

(2) $\operatorname{Hom}_{KG}(V, \operatorname{Ind}_H^G U) \cong \operatorname{Hom}_{KH}(\operatorname{Res}_H^G V, U)$.

Proof (1) Special case of Lemma 4.12.

(2) For $f : \mathrm{Res}^G_H V \to U$, define $\theta_f : V \to \mathrm{Ind}^G_H(U)$ to be the map $v \mapsto \sum_{g_i \in G/H} g_i \otimes f(g_i^{-1}v)$. It is routine to check that θ_f is a KG-module homomorphism and so we have a map $\theta : \mathrm{Hom}_{KH}(\mathrm{Res}^G_H V, U) \to \mathrm{Hom}_{KG}(V, \mathrm{Ind}^G_H U)$. It is clear that θ is K-linear and injective.

To show surjective, take homomorphism $f : V \to \mathrm{Ind}^G_H(U)$ and write $f(v) = \sum_{g_i \in G/H} g_i \otimes f_i(v)$. We have $h(f(v)) = h \sum_i g_i \otimes f_i(v) = \sum_i hg_i \otimes f_i(v)$, and $f(hv) = \sum_i g_i \otimes f_i(hv)$ for all $h \in H$. Since f is a KG-module homomorphism, we have $\sum_i hg_i \otimes f_i(v) = \sum_i g_i \otimes f_i(hv)$. Note that we can take g_1 to be the identity element of G, and so using $h \otimes f_1(v) = g_1 h \otimes f_1(v) = g_1 \otimes hf_1(v)$ we have

$$g_1 \otimes hf_1(v) + \sum_{i \neq 1} hg_i \otimes f_i(v) = g_1 \otimes f_1(v) + \sum_{i \neq 1} g_i \otimes f_i(hv).$$

This means that $v \mapsto f_1(v)$ is a KH-module homomorphism.

On the other hand, if we consider $g_j^{-1} f(v) = f(g_j^{-1} v)$, then we have

$$\sum_i g_j^{-1} g_i \otimes f_i(v) = \sum_i g_i \otimes f_i(g_j^{-1} v),$$

which yields

$$g_1 \otimes f_j(v) + \sum_{i \neq j} g_j^{-1} g_i \otimes f_i(v) = g_1 \otimes f_1(g_j^{-1} v) + \sum_{i \neq j} g_i \otimes f_i(g_j^{-1} v),$$

meaning that $f_1(g_j^{-1} v) = f_j(v)$. Hence, we have the map θ_{f_1} is given by

$$\sum_i g_i \otimes f_1(g_i^{-1} v) = \sum_i g_i \otimes f_i(v) = f(v).$$

This proves the required surjection. □

Remark 4.21. Both of these isomorphisms are (bi-)natural. In particular, this means that Ind^G_H and Res^G_H are biadjoint functors.

For time constraint, we omit the proof of the following theorem.

Theorem 4.22 (Mackey decomposition theorem). *For $H, L \leq G$. Let $U \in KL\,\mathsf{mod}$. Then there is the following KH-module isomorphism*

$$U \uparrow^G_L \downarrow^G_H \cong \bigoplus_{t \in H \backslash G / L} ({}^tU) \downarrow^L_{H \cap {}^tL} \uparrow^H_{H \cap {}^tL},$$

where $H\backslash G/L$ denotes the set of double cosets $\{HgL \mid g \in G\}$, and ${}^tL := \{t\ell t^{-1} \mid \ell \in L\}$ and ${}^tU \in K{}^tL\,\mathsf{mod}$ is given by $x \cdot u := txt^{-1}u$ for all $x \in L$ and $u \in U$.

Exercise 4.23. *Suppose $N \lhd G$ is a normal subgroup of G and $W \in KN\,\mathsf{mod}$. Show that*

$$\mathrm{Res}^G_N \mathrm{Ind}^G_N W \cong \bigoplus_{x \in G/N} {}^xW.$$

Lecture 5

Recall that a *G-set* or *G-acted set* is a set Ω equipped with a G-action map, i.e. a group homomorphism $G \to \mathrm{Sym}(\Omega)$, where $\mathrm{Sym}(\Omega) \cong \mathfrak{S}_{|\Omega|}$ is the group of symmetries on Ω.

Definition 5.1. *A permutation module of G over K is the KG-module given by $K\Omega$ (the K-vector space with basis Ω) for a (finite) G-set Ω, with the obvious G-action.*

Remark 5.2. For the representation ρ corresponding to a permutation module, the matrix $\rho(g)$ for every $g \in G$ with respective to the basis Ω is a permutation matrix (i.e. every row and column has exactly one non-zero entry and such an entry is equal to 1).

Example 5.3. *The regular representation is a permutation representation associated to the G-set G itself.*

Lemma 5.4. *Permutation representations are self-dual.*

Proof Define $\langle -, - \rangle : K\Omega \times K\Omega \to K$ by bilinearly extending $\langle \omega, \omega' \rangle = \delta_{\omega,\omega'}$. This is clearly non-degenerate. It is G-invariant as $g\omega = g\omega' \Leftrightarrow \omega = \omega'$. Now apply Proposition 3.17. $\square$

Recall that a G-action on a set Ω is *transitive* if for all $x, y \in \Omega$ there exists $g \in G$ with $gx = y$. Recall also that the stabiliser $\mathrm{Stab}_G(x)$ of $x \in \Omega$ is the subgroup $\{g \in G \mid gx = x\}$.

Lemma 5.5. *If G acts transitively on Ω and $x \in \Omega$, then the map*

$$\Omega \to G/\operatorname{Stab}_G(x), \quad gx \mapsto g\operatorname{Stab}_G(x),$$

is a bijection that commutes with G-action, i.e. $\Omega \cong G/\operatorname{Stab}_G(x)$ are isomorphic as G-set. In particular, $K\Omega \cong K(G/\mathrm{Stab}_G(x))$ is isomorphic as $KG\,\mathsf{mod}$.

Proof Since $gx = hx \Leftrightarrow x = g^{-1}hx \Leftrightarrow g^{-1}h \in \operatorname{Stab}_G(x) \Leftrightarrow g\operatorname{Stab}_G(x) = h\operatorname{Stab}_G(x)$, the map is well-defined and injective. Surjective follows from orbit-stabiliser theorem and transitivity $|G/\operatorname{Stab}_G(x)| = |Gx| = |\Omega|$.

Finally, commutation with G-action follows from the assumption that Ω as $g(hx) = (gh)x$ for all $x \in \Omega$ and all $g, h \in G$. $\square$

Proposition 5.6. *Every permutation KG-module is a direct sum of induced modules of the form $\mathrm{Ind}_H^G(\mathrm{triv}_H)$.*

Proof Let $K\Omega$ be a permutation module. Decompose Ω into G-orbits $\Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_r$. Then each G-acts on each Ω_i transitively and so by Lemma 5.5 says that Ω_i is isomorphic to G/H_i for some subgroup $H_i \leq G$ as G-set for all $i = 1, \ldots, r$. Hence, we have a chain of isomorphisms

$$\begin{aligned} K\Omega &\cong K(\Omega_1 \sqcup \cdots \Omega_r) \cong K\Omega_1 \oplus \cdots \oplus K\Omega_r \\ &\cong K(G/H_1) \oplus \cdots K(G/H_r) \cong \mathrm{Ind}_{H_1}^G(\mathrm{triv}_{H_1}) \oplus \cdots \oplus \mathrm{Ind}_{H_r}^G(\mathrm{triv}_{H_r}) \end{aligned}$$

of KG-modules. Note that last isomorphism is from Example 4.17. $\square$

Exercise 5.7. *Recall that $\mathrm{Ind}_H^G(W^*) \cong \mathrm{Ind}_H^G(W)^*$. Use this to give an alternative proof of self-duality of permutation modules.*

Exercise 5.8. *Consider an integer $n \geq 1$ and an integer $r \leq n/2$. Let Ω_r be the set of r-subsets (=subsets of size r) of $\{1, 2, \ldots, n\}$. Find (and prove) a subgroup $H \leq \mathfrak{S}_n$ such that $K\Omega_r \cong \mathrm{Ind}_H^{\mathfrak{S}_n} \mathrm{triv}_H$.*

Exercise 5.9. *Show that triv_G is a direct summand of $\mathbb{C}\Omega$ (or a submodule of $K\Omega$ for arbitrary field K) for any G-set Ω. (Hint: We have done a similar proof on the case $\Omega = G$.)*

Artin-Wedderburn decomposition of $\mathbb{C}G$.

Definition 5.10. *Let C be a conjugacy class in G. The class sum is the element $\overline{C} := \sum_{g\in C} g \in KG$.*

Recall that the *center* $Z(A) := \{a \in A \mid ab = ba \forall b \in A\}$ of an algebra A is a commutative ring.

Proposition 5.11. *Suppose $C_1, \dots, C_r$ are all conjugacy classes of G. Then $\{\overline{C}_1, \dots, \overline{C}_r\}$ is a K-basis of $Z(KG)$.*

Proof Let us first show $\overline{C}_i \in Z(KG)$ for all i. By definition, $g\overline{C}_i g^{-1} = \overline{C}_i$ for any $g \in G$, so we have $g\overline{C}_i = \overline{C}_i g$ which implies by linearity $\overline{C}_i \in Z(KG)$.

Since each $g \in G$ lies in precisely one conjugacy class, it follows that $\{\overline{C}_i\}_{i=1,\dots,r}$ is a linear independent set.

Finally, suppose that $v = \sum_g \lambda_g g \in Z(KG)$. Then for all $h \in G$ we have

$$v = hvh^{-1} = \sum_g \lambda_g hgh^{-1} = \sum_{k\in G} \lambda_{h^{-1}kh} k.$$

Hence, as G is the basis of KG, comparing coefficients yields $\lambda_g = \lambda_{hgh^{-1}}$ for all $g, h \in G$. In other words, λ_g is constant over the conjugacy class containing g. This means that v is in the span of $\{\overline{C}_i\}_{i=1,\dots,r}$. □

Theorem 5.12. *Let $\mathbb{C}G \cong \mathrm{Mat}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{n_r}(\mathbb{C})$ be the Artin-Wedderburn decomposition of $\mathbb{C}G$. Then the number r (i.e. the number of isoclasses of simple $\mathbb{C}G$=modules) is the same as the number of conjugacy classes of G*

Proof Since $Z(\mathbb{C}G)$ is direct product of $Z(\mathrm{Mat}_{n_i}(\mathbb{C}))$, each of which is a 1-dimensional $\mathbb{C}$-algebra (namely, given by $\lambda\,\mathrm{id}$ for $\lambda \in \mathbb{C}$ where id is the identity matrix), so $r = \dim_{\mathbb{C}} Z(\mathbb{C}G)$, which is the same as the number of conjugacy classes in G by Proposition 5.11. □

Remark 5.13. For K algebraically closed with $\operatorname{char} K = p > 0$, the number of isoclasses of simple KG-modules coincides with the p'-conjugacy classes, i.e. conjugacy class C such that p does not divides $|C|$. The proof is much more involved and require closer comparison bewteen $KG/\operatorname{rad} KG$ and $Z(KG)$.

Exercise 5.14. *Let A be a semisimple K-algebra such that the endomorphism ring of every simple is isomorphic to K. Show that $\dim_K(Z(A))$ coincide with the number of isoclasses of simple A-modules.*

Ordinary character theory.

From now on until further notice, we take $K = \mathbb{C}$.

Recall from Definition 4.5 that the character χ_ρ associated to a $\mathbb{C}$-linear representation ρ is the assign to each group element the trace of its representing linear transformation. This is clearly a representation-invariant (i.e. isomorphic representations yield the same character), and provides a very helpful way to understand representations.

Definition 5.15. *Let $V \in \mathbb{C}G\,\mathsf{mod}$. We call χ_V an irreducible character if V is a simple $\mathbb{C}G$-module. In the special case of $V = \mathrm{triv}_G$, write $\mathbf{1}_G$ and call it the trivial character. We call χ_V a permutation character if $V = K\Omega$ for some G-set Ω; in this case, it is conventional to write π_Ω for χ_V.*

Lemma 5.16. *Let $\chi = \chi_V$ be the character associated to $V \in \mathbb{C}G\,\mathsf{mod}$.*

(1) *χ_V is constant on each conjugacy class of G.*

(2) *$\chi(g)$ is a sum of m-th roots of unity if $g \in G$ is of order m.*

(3) *The degree of* χ *is* $\deg\chi := \chi(1) = \dim_{\mathbb{C}} V$.

(4) $\chi(g^{-1}) = \overline{\chi(g)}$ *for any* $g \in G$.

(5) $\chi(g) \in \mathbb{R}$ *if* g *and* g^{-1} *is in the same conjugacy class.*

(6) $\pi_\Omega(g) = \#\Omega^g$, *where* $\Omega^g := \{\omega \in \Omega \mid g\omega = \omega\}$, *for all* $g \in G$ *and any* G*-set* Ω.

Proof (1) Since $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$ for any linear transformations f, g. We have $\mathrm{Tr}(\rho_{hgh^{-1}}) = \mathrm{Tr}(\rho_h\rho_g\rho_h^{-1}) = \mathrm{Tr}(\rho_h\rho_h^{-1}\rho_g) = \mathrm{Tr}(\rho_g)$.

(2) See proof of Lemma 4.6.

(3) Clear since $\chi(1) = \mathrm{Tr}(\mathrm{id}_V)$.

(4) This is Lemma 4.6.

(5) Consequence of (1) and (4).

(6) Consider the matrix corresponding to $\rho(g)$ with respect to the basis Ω. Then a diagonal entry, say, corresponding to $\omega \in \Omega$ is non-zero if, and only if, $g\omega = \omega$. Moreover, in such a case, the entry is exactly 1. □

Exercise 5.17. *Show that for a character* $\chi = \chi_V$, $\mathrm{Ker}\,\chi := \{g \in G \mid \chi(g) = \chi(1)\}$ *is a normal subgroup of* G.

Recall that we can take direct sum and tensor products of representations, which behaves like $+$ and $\times$ respectively. Indeed, this is the case for K-vector spaces, namely, that $\dim K\,\mathsf{mod} \to \mathbb{Z}$ 'sends' $\oplus$ to $+$ and $\otimes$ to $\times$. Note that $\mathbb{C} = \mathbb{C}1$ is the group algebra of the trivial group, and so character of $\mathbb{C}1$ is nothing but just the degree of the character, i.e. $\dim_{\mathbb{C}}$ by Lemma 5.16 (3). Hence, it makes sense to view characters as a generalisation of $\dim_{\mathbb{C}}$. Let us see how well this philosophy works.

Definition 5.18. *A class function on* G *is a* $\mathbb{C}$*-valued function* $\psi : G \to \mathbb{C}$ *that is constant over each conjugacy class, i.e.* $\psi(g) = \psi(h)$ *whenever* g *and* h *are in the same conjugacy class. Denote by* $\mathcal{C}(G)$ *the set of all class functions on* G.

For $\psi, \phi \in \mathcal{C}(G)$ *and* $\lambda \in \mathbb{C}$, *define:*

(1) $\lambda\phi$ *the class function given by* $(\lambda\phi)(g) := \lambda(\phi(g))$;

(2) $\psi + \phi$ *the class function given by pointwise addition (i.e.* $(\psi + \phi)(g) := \psi(g) + \phi(g)$*);*

(3) $\psi\phi$ *the class function given by pointwise multiplication (i.e.* $(\psi\phi)(g) := \psi(g)\phi(g)$*).*

In particular, $\mathcal{C}(G)$ *is a* $\mathbb{C}$*-vector space (and a* $\mathbb{C}$*-algebra).*

From now on, unless otherwise specified, unadorned $\otimes$ means $\otimes_{\mathbb{C}}$.

Lemma 5.19. *For any* $V \in \mathbb{C}G\,\mathsf{mod}$, χ_V *is a class function on* G. *Moreover, we have* $\chi_{V\oplus W} = \chi_V + \chi_W$ *and* $\chi_{V\otimes W} = \chi_V\chi_W$.

Proof First point follows immediately from Lemma 5.16.

Addition corresponds to direct sum follows from the fact that (we can choose a basis so that) the matrix corresponding to $\rho_{V\oplus W}(g)$ is given by the block diagonal matrix with entries $\rho_V(g)$ and $\rho_W(g)$.

Multiplication corresponds to tensor product follows from the fact that the matrix corresponding to $\rho_{V\otimes W}(g)$ is the Kronecker product (Example 3.7) of $\rho_V(g)$ and $\rho_W(g)$. □

Exercise 5.20. *Write $\overline{\chi_V}$ the function $g \mapsto \overline{\chi_V(g)}$. Show that $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)} = \overline{\chi_V}\chi_W$.*

Exercise 5.21. *Suppose $\mathbb{C}G$ has r conjugacy classes. Prove that $\pi_G = \sum_{i=1}^r \deg(\chi_i)\chi_i$, where $\chi_i = \chi_{S_i}$ is the character of a simple $\mathbb{C}G$-module such that $S_i \not\cong S_j$ for all $i \neq j$. Moreover, determine the value $\chi_V(g)$ for all $g \in G$.*

Exercise 5.22. *Let Ω be a G-set.*

(1) *Show that $\nu(g) := \#\Omega^g - 1$ is a character of (some representation of) G.*

(2) *In the case of $G = \mathfrak{S}_n$ and $\Omega = \{1, 2, \ldots, n\}$. Let V be the representation with $\chi_V = \nu$ as in (1). Show that $\mathrm{sgn} \otimes V \cong V$ if and only if $n = 3$.*

Inner product

Recall that an inner product on a $\mathbb{C}$-vector space X is a non-degenerate Hermitian form $\langle -, - \rangle : X \times X \to \mathbb{C}$, i.e.

(1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$;

(2) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, y \rangle + \mu \langle x, y \rangle$ for all $\lambda, \mu \in \mathbb{C}$ and all $x, y, z \in X$;

(3) $\langle x, x \rangle \in \mathbb{R}_{>0}$ for all non-zero $x \in X$.

Note that (1) and (2) combines to $\langle x, \lambda y + \mu z \rangle = \overline{\lambda}\langle x, y \rangle + \overline{\mu}\langle x, z \rangle$.

Exercise 5.23. *Show that $\langle \pi_X, \mathbf{1}_G \rangle$ is the number of G-orbits on the G-set X.*

Definition 5.24. *For $\chi, \psi \in \mathcal{C}(G)$, define*

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}$$

It is easy to check that this defines an inner product on $\mathcal{C}(G)$.

Recall that for $g \in G$, its centraliser subgroup is $C_G(g) := \{h \in G \mid hgh^{-1} = g\}$, i.e. the stabiliser subgroup of $g \in G$ under conjugation (=adjoint) action of G on G itself.

Proposition 5.25. *Let $\chi, \psi \in \mathcal{C}(G)$.*

(1) *If χ, ψ are characters, then $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle \in \mathbb{R}$.*

(2) *If $g_1, \ldots, g_r$ are representatives of the conjugacy classes of G, then $\langle \chi, \psi \rangle = \displaystyle\sum_{i=1}^r \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$.*

Proof (1) Since $\overline{\psi(g)} = \psi(g^{-1})$, we have

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}) = \frac{1}{|G|} \sum_{h \in G} \chi(h^{-1})\psi(h) = \langle \psi, \chi \rangle.$$

But $\langle \chi, \psi \rangle = \overline{\langle \psi, \chi \rangle}$ as $\langle -, - \rangle$ is an inner product, so $\langle \chi, \psi \rangle \in \mathbb{R}$.

(2) Let C_i be the conjugacy class whose representative is g_i. Since characters are class functions, we have $\sum_{g \in C_i} \chi(g)\overline{\psi(g)} = |C_i|\chi(g_i)\overline{\psi(g_i)}$. Orbit-stabiliser theorem implies that $|C_i| = |G|/|C_G(g_i)|$ and that $G = \sqcup_{i=1}^r C_i$. Hence, we have

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{i=1}^r \frac{|G|}{|C_G(g_i)|} \chi(g_i)\overline{\psi(g_i)} = \sum_{i=1}^r \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$$

as required. $\square$

Lecture 6

The first aim of this lecture is to show the following theorem:

Theorem 6.1. *For $V, W \in \mathbb{C}G\,\mathsf{mod}$, we have*

$$\langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V, W).$$

In particular, any inner product of characters is always integer-valued.

Lemma 6.2. $\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)^G := \{f \mid g \cdot f = f\}$.

Proof For $f \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, we have

$$f \in \mathrm{Hom}_{\mathbb{C}G}(V, W) \Leftrightarrow g(f(v)) = f(gv) \forall g, v \Leftrightarrow (g \cdot f)(v) = gf(g^{-1}v) = g(g^{-1}f(v)) = f(v) \forall v.$$

The claim now follows. □

Lemma 6.3. *For $V \in \mathbb{C}G\,\mathsf{mod}$, we have*

(1) *a vector space isomorphism* $\mathrm{Hom}_{\mathbb{C}G}(\mathrm{triv}_G, V) \cong V^G$ *given by* $f \mapsto f(1)$*;*

(2) $\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$.

Proof (1) By definition, V^G is the maximal submodule of V that is isomorphic to a sum of triv_G. Since $\mathbb{C}G$ is semisimple, V^G is the maximal direct summand of V given by direct sum of triv_G, i.e. $V^G = eV$ for e the idempotent in $\mathbb{C}G$ such that $\mathrm{triv}_G = \mathbb{C}Ge$. Now the claim follows from Yoneda lemma: $\mathrm{Hom}_{\mathbb{C}G}(\mathrm{triv}_G, V) = \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}Ge, V) \cong eV = V^G$.

(2) Recall that $\mathrm{triv}_G = \mathbb{C}v$ where $v = \sum_{g \in G} g \in \mathbb{C}G$. Hence, we have $v^2 = \sum_{g \in G} gv = |G|v$. In particular, if we take $e := \frac{1}{|G|}v$, then $e^2 = e$ is an idempotent in $\mathbb{C}G$ with image triv_G.

By (1), we have $eV^G = e(eV) = eV$, and so e acts as identity on V^G. Therefore,

$$\dim_{\mathbb{C}} V^G = \mathrm{Tr}\left(\sum_{g \in G} \frac{1}{|G|} \rho(g) \right) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}\, \rho(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

as required. □

Proof of Theorem 6.1 Using Lemma 6.2 first, and then Lemma 6.3 (with V in the statement replaced by $\mathrm{Hom}_{\mathbb{C}}(V, W)$ in the setting of the claim), we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V, W) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(V, W)^G = \frac{1}{|G|} \sum_{g \in G} \chi(g),$$

where χ is the character of $\mathrm{Hom}_{\mathbb{C}}(V, W)$. Since $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$ as $\mathbb{C}G$-modules, we have

$$\chi(g) = \chi_{V^* \otimes W}(g) = \chi_{V^*}(g)\chi_W(g) = \overline{\chi_V(g)}\chi_W(g) = \chi_V(g^{-1})\overline{\chi_W(g^{-1})}.$$

Substitute this back into the previous formula yields the claim. □

Corollary 6.4. *Suppose $\mathbb{C}G$ has r simple modules $S_1, \ldots, S_r$ with characters $\chi_1, \ldots, \chi_r$ respectively. Then the following hold.*

(1) $\langle \chi_i, \chi_j \rangle = \delta_{i,j}$ *and* $\langle \chi_V, \chi_W \rangle \in \mathbb{Z}$ *for all* $V, W \in \mathbb{C}G\,\mathsf{mod}$.

(2) $\{\chi_i\}_{1 \le i \le r}$ *is an orthonormal (with respect to $\langle -, - \rangle$) basis of* $\mathcal{C}(G)$.

(3) $V \cong \bigoplus_{i=1}^r S_i^{\oplus \langle \chi_i, \chi_V\rangle}$ *and* $\chi_V = \sum_{i=1}^r \langle \chi_i, \chi_V\rangle \chi_i$ *for all* $V \in \mathbb{C}G\,\mathsf{mod}$.

(4) *We have*

$$\langle \chi_V, \chi_V\rangle = \sum_{i=1}^r \langle \chi_i, \chi_V\rangle^2$$

for all $V \in \mathbb{C}G\,\mathsf{mod}$.

(5) *If* $H \leq G$ *is a subgroup, then* $\langle \mathrm{Ind}_H^G \chi_W, \chi_V\rangle_{\mathcal{C}(G)} = \langle \chi_W, \mathrm{Res}_H^G \chi_V\rangle_{\mathcal{C}(H)}$ *and* $\langle \mathrm{Res}_H^G \chi_V, \chi_W\rangle_{\mathcal{C}(H)} = \langle \chi_V, \mathrm{Ind}_H^G \chi_W\rangle_{\mathcal{C}(G)}$ *for all* $W \in \mathbb{C}H\,\mathsf{mod}$ *and all* $V \in \mathbb{C}G\,\mathsf{mod}$.

Proof (1) Combine Theorem 6.1 with Schur's lemma.

(2) By (1), we have $\{\chi_i\}_{1\leq i\leq r}$ is an orthonormal set of vectors in $\mathcal{C}(G)$. In particular, it is linear independent.

Recall that r is the same as the number of conjugacy classes of G. Let $C_1, \ldots, C_r$ be the conjugacy classes of G. Observe that $\mathcal{C}(G)$ has a 'canonical basis' given by $\{\delta_i\}_{1\leq i\leq r}$ with

$$\delta_i(g) := \begin{cases} 1, & \text{if } g \in C_i; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we have $\dim_{\mathbb{C}} \mathcal{C}(G) = r$, which then implies that $\{\chi_i\}_{1\leq i\leq r}$ is a maximal linear independent set. Now the claim follows.

(3) By Jordan-Hölder theorem and Maschke's theorem, we have $V \cong \bigoplus_{i=1}^r S_i^{\oplus \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(S_i, V)}$, then apply Theorem 6.1. The statement for the characters then follow by considering the characters on both sides.

(4) Combines (2) and (3).

(5) Follows from Eckmann-Shapiro Lemma 4.20: $\mathrm{Hom}_{\mathbb{C}G}(\mathrm{Ind}_H^G(W), V) \cong \mathrm{Hom}_{\mathbb{C}H}(W, \mathrm{Res}_H^G(V))$ and $\mathrm{Hom}_{\mathbb{C}G}(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_{\mathbb{C}H}(\mathrm{Res}_H^G(V), W)$. □

Remark 6.5. We note that there is another orthonormal basis given by $\{\sqrt{|G|/|C_i|}\delta_i = \sqrt{|C_G(g_i)|}\delta_i\}_{1\leq i\leq r}$, where $C_1, \ldots, C_r$ are the conjugacy classes of G with representatives $g_1, \ldots, g_r$ respectively.

The following result which tells us that characters not only are representation-invariant, but can also tell apart non-isomorphic representations!, i.e. a *complete invariant* of representations.

Theorem 6.6. *For any* $V, W \in \mathbb{C}G\,\mathsf{mod}$, $V \cong W$ *as* $\mathbb{C}G$*-module if and only if* $\chi_V = \chi_W$.

Proof ⇒: Clear as every g acts in the 'same' way.

⇐: Let $S_1, \ldots, S_r$ be the complete set of (isoclass representatives of) simple $\mathbb{C}G$-modules with characters $\chi_1, \ldots, \chi_r$ respectively. From Corollary 6.4 (3), we can write

$$V = \bigoplus_{i=1}^r S_i^{\oplus \langle \chi_i, \chi_V\rangle}, \quad \text{and} \quad W = \bigoplus_{i=1}^r S_i^{\oplus \langle \chi_i, \chi_W\rangle}.$$

$\chi_V = \chi_W$ implies that composition factors of both V and W are exactly the same, and so they are isomorphic. □

Exercise 6.7. *Show that, for any subgroup* $H \leq G$*, any simple* $\mathbb{C}G$*-module is isomorphic to a direct summand of some module induced from* H.

We can now strengthen a previous lemma.

Corollary 6.8. *$V \in \mathbb{C}G\,\mathsf{mod}$ is self-dual if, and only if χ_V is real-valued.*

Proof We have already shown $\Rightarrow$ direction before.

$\Leftarrow$: χ_V is real-valued implies that $\chi_{V^*} = \chi_V$; now apply Theorem 6.6. $\square$

Character table.

Definition 6.9. *Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G, and $g_1, \ldots, g_r$ be the representative of the conjugacy classes of G. Then the character table of G is the matrix $(\chi_i(g_j))_{1 \le i,j \le r}$.*

We will fix the notation for χ_i and g_i as in the definition until further notice. It is customary to take $\chi_1 = \mathbf{1}_G$ the trivial character and $g_1 = 1$ the identity element of G.

Example 6.10 (Character table of C_3). *Suppose $G = C_3 = \{1, g, g^2\}$. Let $\omega := \exp(2\pi i/3)$. Then $\rho_k : g \mapsto \omega^{k-1}$ for $k = 1, 2, 3$ defines a 1-dimensional (hence, simple) representation of G. So $\chi_k = \rho_k$ and the character table is:*

	1	g	g^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

It is easy to generalise this example to any cyclic group C_n by replacing ω by $\zeta := \exp(2\pi i/n)$.

Example 6.11 (Character table of $D_6 \cong \mathfrak{S}_3$). *Suppose $G = \mathfrak{S}_3 \cong D_6 = \langle a, b \mid a^3 = 1 = b^2, b^{-1}ab = a^{-1}\rangle$. In terms of $\mathfrak{S}_3$, we can choose the isomorphism where a is identified with (123) and b is identified with (12). There are three conjugacy classes $C_1 := \{1\}, C_a := \{a, a^2\}, C_b := \{b, ab, a^2b\}$.*

Take

$$v_k := 1 + \omega^k a + \omega^{2k} a^2 \quad \text{for} \quad k = 0, 1, 2 \text{ with } \omega := \exp(2\pi i/3).$$

We have (see Homework 1, or [JL, Example 10.8])

(1) *trivial module* $\mathrm{triv} = K(1 + a + a^2 + b + ab + a^2b) = K(v_0 + bv_0)$,

(2) *sign module* $\mathrm{sgn} = K(1 + a + a^2 - b - ab - a^2b) = K(v_0 - bv_0)$, *and*

(3) *two isomorphic 2-dimensional simple modules* $V := K\{v_1, bv_2\} \cong \mathrm{Ind}_{\langle a\rangle}^{\mathfrak{S}_3} S_2 \cong V' := K\{v_2, bv_1\} \cong \mathrm{Ind}_{\langle a\rangle}^{\mathfrak{S}_3} S_3$, *where* $\rho_{S_k} = \rho_k$ *from Example 6.10,*

so that $\mathbb{C}G = \mathrm{triv} \oplus \mathrm{sgn} \oplus V \oplus V'$.

Let ρ_1, ρ_2, ρ_3 be the three simple representations corresponding to $\mathrm{triv}, \mathrm{sgn}, V$ respectively. Note that

$$\rho_3(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \text{ and } \rho_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we can compute the corresponding χ_i directly, which gives the character table:

	C_1	C_a	C_b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

In particular, we see that every simple modules over $\mathfrak{S}_3$ is self-dual.

As a side remark, if you are symmetric group representation person, then you may prefer to write χ_1 as the partition (3) of 3, χ_2 as the partition (1^3) of 3, and χ_3 as the partition $(2,1)$ of 3.

We can refine this more.

Lemma 6.12. *The matrix $U := (u_{i,j})_{1\leq i,j\leq r}$ given by*

$$u_{i,j} := \frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}$$

is a unitary matrix*, i.e. invertible with $U^{-1} = \overline{U^\top}$. In particular, the character table of G is invertible.*

Proof By Proposition 5.25 (2) and Corollary 6.4 (1), we have

$$\delta_{i,j} = \langle \chi_i, \chi_j \rangle = \sum_{k=1}^{r} \frac{\chi_i(g_k)\overline{\chi_j(g_k)}}{|C_G(g_k)|} = \sum_{k=1}^{r} u_{i,k}\overline{u_{j,k}}.$$

This means that the identity matrix $I = (\delta_{i,j})_{1\leq i,j\leq r}$ is given by $U\overline{U^\top}$; the claim now follows. □

Theorem 6.13. *The following hold.*

(1) *Row orthogonality:* $\displaystyle\sum_{i=1}^{r} \frac{\chi_s(g_i)\overline{\chi_t(g_i)}}{|C_G(g_i)|} = \delta_{s,t}$ *for any $1 \leq s,t \leq r$.*

(2) *Column orthogonality:* $\displaystyle\sum_{k=1}^{r} \chi_k(g_s)\overline{\chi_k(g_t)} = \delta_{s,t}|C_G(g_t)|$ *for any $1 \leq s,t \leq r$.*

Proof (1) Apply Proposition 5.25 (2) to Corollary 6.4 (1).

(2) Lemma 6.12 says that $\overline{U^\top}U = I$, which is equivalent to

$$\delta_{s,t} = \sum_{k=1}^{r} \overline{u_{k,s}}u_{k,t} = \sum_{k=1}^{r} \frac{\overline{\chi_k(g_s)}\chi_k(g_t)}{|C_G(g_k)|},$$

as required. □

We can also refine Corollary 6.4 (3).

Proposition 6.14. *For any class function $\psi \in \mathcal{C}(G)$, we have $\psi = \sum_{i=1}^{r}\langle \psi, \chi_i\rangle \chi_i$.*

Proof Consider the character table matrix $X := (\chi_i(g_j))_{1\leq i,j\leq r}$. This is the change of basis matrix from $\{\chi_i\}_i$ to $\{\delta_j\}_j$. By Lemma 6.12, the inverse of X is given by $M := (m_{i,j})_{1\leq i,j\leq r}$ where

$$m_{i,j} := \langle \delta_j, \chi_i \rangle = \frac{\overline{\chi_i(g_j)}}{|C_G(g_j)|}.$$

Hence, M is the change of basis matrix from $\{\delta_j\}_j$ to $\{\chi_i\}_i$.

Since $\psi = \sum_{j=1}^{r} \psi(g_j)\delta_j$, applying M yields

$$\psi = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} \frac{\overline{\chi_i(g_j)}}{|C_G(g_j)|}\psi(g_j) \right) \chi_i$$

which yields $\sum_{i=1}^{r}\langle \psi, \chi_i\rangle \chi_i$ by Lemma 5.25 (2). □

Lecture 7

Induced character

Considering Corollary 6.4 (5) and Example 6.11, it should be helpful to clarify values of characters for induced modules. Let us start with the obvious formulae first.

Lemma 7.1. *Suppose we have* $V \in \mathbb{C}G\,\mathsf{mod}$ *and* $W \in \mathbb{C}H\,\mathsf{mod}$ *for* $H \leq G$. *Then* $\chi_W \uparrow^G (g) = \sum_{t \in G/H} \hat{\chi}_W(tgt^{-1}) = \frac{1}{|H|}\sum_{x \in G} \hat{\chi}_W(xgx^{-1})$, *where*

$$\hat{\chi}_W(g) := \begin{cases} \chi_W(g), & \text{if } g \in H, \\ 0, & \text{if } g \notin H. \end{cases}$$

Proof We give two different proofs. First one uses only structure of induced module and definition of characters; second one uses only character theory but require Theorem 6.1 and Corollary 6.4.

Module theoretic proof:

Fix representatives $t_1, \ldots, t_c$ for the left cosets of H in G. Recall that if W has basis $\{w_i\}_{1 \leq i \leq n}$, then $\mathrm{Ind}_H^G(W)$ has basis $\{t_a \otimes w_i \mid 1 \leq a \leq c, 1 \leq i \leq n\}$.

For $g \in G$, and basis element $t_a \otimes v_j \in \mathbb{C}G \otimes_{\mathbb{C}H} W = \mathrm{Ind}_H^G(W)$. Write $gt_a = t_b h$ for $h \in H$, then we have

$$g(t_a \otimes v_i) = (gt_a) \otimes v_i = t_b h \otimes v_i = t_b \otimes hv_i.$$

By definition, $\chi_W \uparrow^G (g)$ is given by the sum of the coefficient of $t_a \otimes v_i$ in $g(t_a \otimes v_i)$. If $a = b$, i.e. $t_a^{-1} g t_a \in H$, then this coefficient is given by that of v_i in $hv_i = (t_a^{-1} g t_a) v_i$; otherwise, this coefficient is zero. This gives the first equality. Then we have

$$\sum_{t \in G/H} \hat{\chi}_W(tgt^{-1}) = \sum_{t \in G/H} \frac{1}{|H|} \sum_{h \in H} \hat{\chi}_W(h^{-1}t^{-1}gth) = \frac{1}{|H|}\sum_{x \in G} \hat{\chi}_W(xgx^{-1}).$$

Character theoretic proof:

Let us define $\psi : G \to \mathbb{C}$ to be $\frac{1}{|H|}\sum_{x \in G} \hat{\chi}_W(xgx^{-1})$. The summation over all $x \in G$ implies that ψ is constant over each conjugacy class of G and so is in $\mathbb{C}(G)$.

For simplicity, write $\hat{\chi} := \hat{\chi}_W$. Since irreducible characters $\{\chi_i\}_i$ is a(n orthonormal) basis of $\mathbb{C}(G)$, and it is enough to show that $\langle \chi_W \uparrow^G, \chi_i \rangle = \langle \psi, \chi_i \rangle$ for all i. Let us compute the right-hand side:

$$\begin{aligned}
\langle \psi, \chi_i \rangle = \frac{1}{|G|}\sum_{g \in G} \psi(g)\overline{\chi_i(g)} \quad &= \quad \frac{1}{|G|}\frac{1}{|H|}\sum_{g \in G}\sum_{x \in G} \hat{\chi}(xgx^{-1})\overline{\chi_i(g)} \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{y \in G}\sum_{x \in G} \hat{\chi}(y)\overline{\chi_i(x^{-1}yx)} \quad (\text{by taking } y := xgx^{-1}) \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{y \in G} |G|\hat{\chi}(y)\overline{\chi_i(y)} \quad (\text{as } \overline{\chi_i} \in \mathcal{C}(G)) \\
&= \frac{1}{|H|}\sum_{y \in H} \chi(y)\overline{\chi_i(y)} \quad (\text{by definition of } \hat{\chi}) \\
&= \langle \chi, \chi_i \downarrow_H \rangle_H \quad = \quad \langle \chi \uparrow^G, \chi_i \rangle_G \quad (\text{by Corollary 6.4 (5)}).
\end{aligned}$$

This completes the proof. $\square$

Proposition 7.2. *Let $H \leq G$ be a subgroup and $\chi := \chi_W$ be the character for some $W \in \mathbb{C}H\,\mathsf{mod}$. Suppose that $h_1, \ldots, h_m$ are H-conjugacy classes representatives such that h_i are G-conjugate to $g \in G$ for all $1 \leq i \leq m$. Then*

$$\chi_W \uparrow^G (g) = |C_G(g)| \sum_{i=1}^{m} \frac{\chi(h_i)}{|C_H(h_i)|}.$$

Proof Let $C_1, \ldots, C_m$ be the H-conjugacy classes containing $h_1, \ldots, h_m$ respectively. Then we have $\{xgx \mid x \in G\} \cap H = C_1 \sqcup \cdots \sqcup C_m$.

Let us write $g' \sim_G g$ if $g' = xgx^{-1}$ for some $x \in G$. Starting with Lemma 7.1, we have

$$\begin{aligned}\chi \uparrow^G (g) &= \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) = \frac{|C_G(g)|}{|H|} \sum_{g' \sim_G g} \hat{\chi}(g') = \frac{|C_G(g)|}{|H|} \sum_{i=1}^{m} \sum_{h \sim_H h_i} \chi(h) \\ &= \frac{|C_G(g)|}{|H|} \sum_{i=1}^{m} |C_i| \chi(h_i) = |C_G(g)| \sum_{i=1}^{m} \frac{\chi(h_i)}{|C_H(h_i)|},\end{aligned}$$

where the last equality follows from orbit-stabiliser theorem that $|H|/|C_i| = |C_H(h_i)|$. □

Restricted character

It actually can happen that it is easier to calculate the characters on a larger group (e.g. $\mathfrak{S}_n$ versus its alternating subgroup $\mathfrak{A}_n$). So let us have a look at some results on restricted characters too.

First, by definition, it is clear that

$$\chi_V \downarrow_H (h) = \chi_V(h) \qquad \forall h \in H \leq G.$$

Normal subgroups are often of particular interest; the theory around it (including the positive characteristic case) is called *Clifford theory.*

Theorem 7.3 (Clifford's theorem). *Suppose $H \lhd G$ is a normal subgroup and $\chi = \chi_V$ is an irreducible character for some simple $\mathbb{C}G$-module V. Let $\mathrm{Res}^G_H(V) = W_1 \oplus \cdots \oplus W_k$ be the decomposition of the restricted $\mathbb{C}H$-module. Then the following hold.*

(1) *For $W \in \mathbb{C}H\,\mathsf{mod}$, let*

$$T(W) := \{t \in G \mid {}^tW \cong W\} = \{t \in G \mid {}^t\chi_W = \chi_W\} \leq G$$

be the inertial group of W. Then $W_i = {}^{t_i}W$.

(2) $\deg \psi$ *is constant for all irreducible $\psi = \chi_{W_i}$. In other words, the direct summand W_i has equal dimensions.*

(3) *If $\psi_1, \ldots, \psi_k$ are the corresponding characters of H, then there is some positive integer e such that $\chi \downarrow_H = e \sum_{i=1}^{k} \psi_i$.*

More examples of character tables

Example 7.4 (Character table of D_{2n} for n odd). *This is mostly similar to Example 6.11. Recall that*

$$D_{2n} = \langle a, b \mid a^n = 1 = b, bab = a^{-1} \rangle.$$

When n odd, we have $(n+3)/2$ conjugacy classes:

$$C_1 = \{1\}, C_{a^k} = \{a^k, a^{-k}\} \text{ for } 1 \leq k \leq (n-1)/2, C_b = \{a^i b \mid 1 \leq i \leq n\}.$$

Now we have data

g_i	1	a^r	b
$\lvert C_G(g_i)\rvert$	$2n$	n	2
χ_1	1	1	1

We need $(n+1)/2$ more irreducible characters. Consider the irreducible character ϕ_j of $C_n = \langle a \rangle \leq D_{2n}$ associated to the 1-dimensional representation W_j where a acts by ξ^j for $\xi := \exp(2\pi i/n)$ and $0 \leq j \leq n-1$. Then using the formula for induced character we have

g_i	1	a^r	b
$\phi_j \uparrow$	2	$\xi^{rj} + \xi^{-rj}$	0

In particular, we have $\phi_j \uparrow = \phi_{n-j} \uparrow$. One then shows that $\psi_j := \phi_j \uparrow$ is an irreducible character for each $1 \leq j \leq (n-1)/2$; one way to do this is to use the same argument as in Example 6.11 (i.e. consider a 1-dimensional subspace and show it cannot be closed under D_{2n}-action). There is an alternative, but not really practical way, namely, using row orthogonality – this yields a sum with terms involving $\cos(k\theta)$ so one need superior knowledge on trigonometry to solve this; on the other hand, showing ψ_j module-theoretically allows us to deduce such daunting trigonometry formula!

Now we need one more irreducible character. We can consider D_{2n} as a subgroup of $\mathfrak{S}_n$ with $a = (12\cdots n)$ and $b = (12)$. Then $\mathrm{Res}(\mathrm{sgn})$ yields a 1-dimensional module where b acts as -1. Hence, this is simple; let χ_2 be the corresponding irreducible character. We have the full character table.

g_i	1	a^r	b
$\lvert C_G(g_i)\rvert$	$2n$	n	2
χ_1	1	1	1
χ_2	1	1	-1
$\phi_j \uparrow$	2	$\xi^{rj} + \xi^{-rj}$	0

Lemma 7.5. *Let Ω be a G-set and π be the associated permutation character. Then $\langle \pi, \mathbf{1} \rangle$ is the number of G-orbits on Ω. In particular, the trivial $\mathbb{C}G$-module is always a direct summand of $\mathbb{C}\Omega$.*

Proof Consider first the case when when G acts transitively on Ω. Now by Lemma 5.16 (6) and exchange of summation we have

$$\begin{aligned}
\langle \pi, \mathbf{1} \rangle = \frac{1}{|G|}\sum_g \pi(g) = \frac{1}{|G|}\sum_g \#\Omega^g = \frac{1}{|G|}\sum_g \#\{\omega \in \Omega \mid g\omega = \omega\} \\
= \frac{1}{|G|}\#\{(g,\omega) \in G \times \Omega \mid g\omega = \omega\} \\
= \frac{1}{|G|}\sum_{\omega\in\Omega} |\mathrm{Stab}_G(\omega)|
\end{aligned}$$

By orbit-stabiliser theorem we have

$$\langle \pi, \mathbf{1} \rangle = \frac{1}{|G|}\sum_{\omega\in\Omega}\frac{|G|}{|\Omega|} = \frac{1}{|G|}\cdot|\Omega|\cdot\frac{|G|}{|\Omega|} = 1.$$

This proves the claim when G acts transitively. In general, partitioning Ω into orbits $\Omega_1 \sqcup \cdots \sqcup \Omega_m$ yields $\mathbb{C}\Omega = \mathbb{C}\Omega_1 \oplus \cdots \oplus \mathbb{C}\Omega_m$, and so the claim follows immediately. □

Example 7.6 (Character table of $\mathfrak{S}_4$). *Recall that conjugacy classes correspond to cycle types. So for $\mathfrak{S}_4$ we have conjugacy class representatives $1, (12), (12)(34), (123), (1234)$. Writing down trivial and sign characters we have*

g_i	1	(12)(34)	(123)	(1234)	(12)
$\lvert C_G(g_i)\rvert$	24	8	3	4	4
χ_1	1	1	1	1	1
χ_{sgn}	1	1	1	-1	-1

Let $\Omega = \{1,2,3,4\}$ and so $\mathfrak{S}_4$ acts on it by permutation, and we have the permutation module $\mathbb{C}\Omega$. Clearly, $\mathfrak{S}_4$ acts transitively on $\mathbb{C}\Omega$, and so we have $\mathbb{C}\Omega = \text{triv} \oplus V$ for some V (and triv is not a direct summand of V). The character χ_V is then given by $\pi_\Omega - \text{triv}$, i.e.

g_i	1	(12)(34)	(123)	(1234)	(12)
π	4	0	1	0	2
χ_V	3	-1	0	-1	1

Check that $\langle \chi_V, \chi_V \rangle = 3^2/24 + 1/8 + 1/4 + 1/4 = 1$ and we now know that V is irreducible. Now $\text{sgn} \otimes V$ *yields a new simple module, and so we have*

g_i	1	(12)(34)	(123)	(1234)	(12)
$\lvert C_G(g_i)\rvert$	24	8	3	4	4
χ_1	1	1	1	1	1
χ_{sgn}	1	1	1	-1	-1
χ_V	3	-1	0	-1	1
$\chi_{\text{sgn}}\chi_V$	3	-1	0	1	-1

One last irreducible character χ_U remains, and we can use column orthogonality on each column to deduce entries; alternatively, one can use column orthogonality on the first column, which yields $\chi_U(1) = 2$. Then by Artin-Wedderburn we have $\chi_{\mathbb{C}G} = \chi_1 + \chi_{\text{sgn}} + 3\chi_V + 3\chi_{\text{sgn}}\chi_V + 2\chi_U$, and we can get the remaining entries.

g_i	1	(12)(34)	(123)	(1234)	(12)
$\lvert C_G(g_i)\rvert$	24	8	3	4	4
χ_1	1	1	1	1	1
χ_{sgn}	1	1	1	-1	-1
χ_V	3	-1	0	-1	1
$\chi_{\text{sgn}}\chi_V$	3	-1	0	1	-1
χ_U	2	2	-1	0	0

The fact that $V = \mathbb{C}\{1,2,3,4\}/\text{triv}$ is simple is not just fluke.

Lemma 7.7. *Let X, Y be G-sets. Then we have a G-set $X \times Y$ given by diagonal action $g(x,y) := (gx, gy)$, with $\langle \pi_X, \pi_Y \rangle$ being the number of G-orbits on $X \times Y$.*

Proof Permutation character are $\mathbb{R}$-valued and so we have

$$\begin{aligned}
\langle \pi_X, \pi_Y \rangle &= \frac{1}{|G|} \sum_g \pi_X(g)\overline{\pi_Y(g)} \\
&= \frac{1}{|G|} \sum_g \pi_X(g)\pi_Y(g)\overline{\mathbf{1}(g)} \\
&= \langle \pi_X\pi_Y, \mathbf{1} \rangle = \langle \pi_{X\times Y}, \mathbf{1} \rangle
\end{aligned}$$

and the claim follows from Lemma 7.5. $\square$

Definition 7.8. *Let Ω be a G-set. We say that G-action on Ω is 2-transitive if the diagonal action $g(x,y) := (gx, gy)$ on $\Omega \times \Omega$ has precisely 2 orbits, namely, $\{(x,x) \mid x \in \Omega\}$ and $\{(x,y) \mid x \neq y \in \Omega\}$.*

Example 7.9. *For $G = \mathfrak{S}_n$ with $n > 1$. The permutation G-action on $\Omega = \{1, 2, \dots, n\}$ is 2-transitive.*

Lemma 7.10. *Let G acts on Ω with $|\Omega| > 2$. Then $\pi_\Omega - \mathbf{1}$ is irreducible if and only if G-action on Ω is 2-transitive.*

Proof Since $\mathcal{C}(G)$ is spanned by irreducible charactesr, we can decompose $\pi := \pi_\Omega$ into

$$\pi = m_1\mathbf{1} + m_2\chi_2 + \cdots + m_r\chi_r$$

with $m_i \in \mathbb{Z}_{\geq 0}$. Moreover, we have $\langle \pi, \pi \rangle = \sum_{i=1}^r m_i^2$ by Corollary 6.4 (4).

By Lemma 7.7 this is the number of G-orbits in $X \times X$. So 2-transitivity is equivalent to $r = 2$ and $m_1 = m_i = 1$ for a unique $i \in \{2, \dots, r\}$, which is the same as saying that $\pi - \mathbf{1}$ is irreducible. $\square$

Example 7.11. *For $\mathfrak{S}_n$ with $n > 1$, we have an $(n-1)$-dimensional simple $\mathbb{C}G$-module whose character is $\pi_\Omega - \mathbf{1}$ where $\Omega = \{1, 2, \dots, n\}$.*

Example 7.12 (Character table of $\mathfrak{A}_4$). *Let $G = \mathfrak{A}_4$ the alternating group of rank 4. This has 4 conjugacy classes with representatives $1, (12)(34), (123), (132)$. So we have*

g_i	1	$(12)(34)$	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1

The restriction $\chi_4 := \mathrm{Res}^{\mathfrak{S}_4}(\chi_V)$ of the character χ_V of $\mathfrak{S}_4$ (see Example 7.6) evaluates on the conjugacy class representatives as $3, -1, 0, 0$ respectively. Then one can check from $\langle \chi_4, \chi_4 \rangle$ that it is indeed irreducible. So we have character table:

g_i	1	$(12)(34)$	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	d_2	a	b	c
χ_3	d_3	x	y	z
χ_4	3	-1	0	0

d_2, d_3 are positive integers as they are dimensions of the respective simple modules. By column orthogonality (or Artin-Wedderburn), we have $1 + d_2^2 + d_3^2 + 9 = 12$, and so $d_2 = 1 = d_3$.

Now, $(12)(34)$ is clearly conjugate to its inverse so $\chi((12)(14)) \in \mathbb{R}$ and so $a, x \in \mathbb{R}$. By column orthogonality of the second column with itself yields $a^2 + x^2 = 2$, whereas that of the second column with the first column yields $a + x = 2$. Consider $(a+x)^2$ and compare with the previous equation we get that $2ax = 2$ and so $x = a^{-1}$. Put this back into $a^2 + x^2 = 2$ we get that $\frac{a^2+1}{a} = 2$ and so $a^2 - 2a + 1 = 0$, i.e. $a = 1 = c$.

Now $\chi((123)) = \overline{\chi((132))}$ since (123) and (132) are in different conjugacy classes. Hence, $c = \overline{b}$ and $z = \overline{y}$. Using column orthogonality of the (123) column with $(12)(34)$, (123), (132), we get that

$$\begin{aligned}
(123) \text{ vs } (123):&\quad 1 + x\overline{x} + y\overline{y} &= 3,\\
(123) \text{ vs } (132):&\quad 1 + x^2 + y^2 &= 0,\\
(123) \text{ vs } (12)(34):&\quad 1 + x + y &= 0.
\end{aligned}$$

From the last line we have $y = -1 - x$. Put this into the second line we get that $x^2 + (x+1)^2 = -1$, and so $x^2 + x + 1 = 0$. Hence $x = \omega = \exp(2\pi i/3)$ is the third root of unity (or its conjugate). Now we have the full character table

g_i	1	(12)(34)	(123)	(132)
$\lvert C_G(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

Lifted characters

Exercise 7.13. *Fix a normal subgroup $N \lhd G$ and let $\pi : G \to G/N$ be the canonical projection $g \mapsto gN$. For $W \in \mathbb{C}N\,\mathsf{mod}$, let $\mathrm{Inf}(W)$ be the $\mathbb{C}G$-module whose corresponding representation is given by $\rho \circ \pi$ for ρ the representation corresponding to W (equivalently, the pullback of W via algebra homomorphism $\mathbb{C}G \to \mathbb{C}(G/N)$).*

(1) *Show that $\chi_{\mathrm{Inf}(W)}(g) = \chi_W(gN)$ for all $g \in G$.*

(2) *Show that $\mathrm{Inf}(-)$ preserves simple modules.*

(3) *Show that $\mathrm{Inf}(-)$ induces a bijection between the set of characters (resp. irreducible characters) of G/N and the set of characters (resp. irreducible characters) ψ of G such that $N \leq \operatorname{Ker}\psi := \{g \in G \mid \psi(g) = \psi(1)\}$.*

(4) *Show that any normal subgroup $L \lhd G$ can be written as $\bigcup_\psi \operatorname{Ker}(\psi)$, where ψ varies over all irreducible characters of G that satisfies $N \leq \operatorname{Ker}\psi$.*

(5) *Show that G is simple (i.e. normal subgroups of G are trivial) if and only if $\chi(g) \neq \chi(1)$ for all non-identity $g \in G$ and all non-trivial irreducible character χ.*

Lecture 8

We will look into some (relatively) easy classes of algebras appearing in modular representation theory of finite groups. As before, K will denote a field of any possible characteristic. All algebras are assumed to be finite-dimensional over K.

We use $D(-) := \mathrm{Hom}_K(-, K)$ to denote the K-linear duality. Note that for a left A-module M, DM is a right A-module given by $(fa)(b) := f(ab)$. Likewise, for a right A-module N, DN is a left A-module. Most of the time, DA will be understood as the left A-module given by the right regular representation A_A (depending on context DA could be understood as an A-A-bimodule).

Lemma 8.1. *The following are equivalent for an algebra A.*

(1) *$\exists$ linear map $\lambda : A \to K$ such that $\mathrm{Ker}\,\lambda$ does not contain a non-zero left ideal. (i.e. $I \lhd A$ left ideal with $\phi(I) = 0$ implies $I = 0$.)*

(2) *$\exists$ linear map $\rho : A \to K$ such that $\mathrm{Ker}\,\lambda$ does not contain a non-zero right ideal.*

(3) *$\exists$ non-degenerate bilinear form $\langle -, - \rangle : A \times A \to K$ that is associative, i.e. $\langle ab, c\rangle = \langle a, bc\rangle$.*

(4) *$\exists$ left A-module isomorphism $f_\lambda : A \to DA$.*

(5) *$\exists$ right A-module isomorphism $f_\rho : A \to DA$.*

In such a case, we say that A is Frobenius.

Proof

(1) $\Rightarrow$ (3): Take $\langle a, b\rangle := \lambda(ab)$. Associativity comes from associativity of A. If $\langle -, a\rangle = 0$, then $\pi(Aa) = 0$, meaning that a generates a left ideal, and so the assumption says that $a = 0$.

(3) $\Rightarrow$ (1): Take $\lambda(x) := \langle x, 1\rangle$. Suppose $I \lhd A$ a left ideal with $\lambda(I) = 0$ and $a \in I$. Then $\langle A, a\rangle = \lambda(Aa) = 0$ as $Aa \subset I$. Hence, $a = 0$ by non-degeneracy of $\langle -, -\rangle$. Thus, $I = 0$.

(3) $\Rightarrow$ (4): Define $f_\lambda(a) := \langle -, a\rangle$. Then non-degeneracy says that f_λ is an isomorphism. Associativity implies that f_λ is a left A-module homomorphism.

(4) $\Rightarrow$ (3): Define $\langle a, b\rangle := (f_\lambda(b))(a)$. Then f_λ being isomorphism is equivalent to non-degeneracy. Note that $a\langle -, b\rangle = (x \mapsto \langle xa, b\rangle)$, and so $f_\lambda(ab) = a(f_\lambda(b))$ implies that $\langle -, -\rangle$ is associative.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5): Same as above, but use $\langle a, -\rangle$ instead of $\langle -, a\rangle$. $\square$

Definition 8.2. *For an A-B-bimodule M, and $\phi \in \mathrm{Aut}_K(A)$, $\psi \in \mathrm{Aut}_K(B)$ are K-algebra automorphisms (i.e. K-linear ring automorphism), we can twist actions and get a new A-B-module ${}_\phi M_\psi$ where*

$$a \cdot m := \phi(a)m \text{ and } m \cdot b := m\psi(b).$$

It is customary to write 1 for the identity map when twisting.

Definition 8.3. *Suppose A is a Frobenius algebra with $\langle -, -\rangle$ as in Lemma 8.1. In such a case, associativity of the bilinear forms implies that the following formula*

$$\langle b, a\rangle = \langle a, \nu_A(b)\rangle$$

defines a K-linear automorphism $\nu = \nu_A \in \mathrm{Aut}_K(A)$. This is unique up to conjugation by a unit (by Exercise 8.7), and we call any such automorphism a Nakayama automorphism. In this case we have a bimodule isomorphism $f : {}_1A_\nu \to DA$ given by $x \mapsto \langle -, x\rangle$.

Remark 8.4. (1) Note that f here is exactly f_λ in Lemma 8.1, and so when working with right modules, one should instead use '$\langle b, a\rangle = \langle \nu(a), b\rangle$' as the defining property of ν and the bimodule isomorphism is replaced by ${}_\nu A_1 \to DA \cong$ given by $x \mapsto \langle x, -\rangle$.

(2) *Inner automorphisms* are the automorphisms given by conjugation by a unit element and they form a normal subgroup $\mathrm{Inn}_K(A) \lhd \mathrm{Aut}_K(A)$. K-algebra automorphisms that are not inner are called *outer*. The quotient group $\mathrm{Out}_K(A) := \mathrm{Aut}(A)/\mathrm{Inn}_K(A)$ is called the group of (K-linear) outer automorphisms – even though the elements are not really automorphisms. Thus, the Nakayama automorphisms form a unique class in $\mathrm{Out}_K(A)$. In the special case when A is basic, i.e. $A/\operatorname{rad} A$ is a direct product of fields, then the only inner automorphism is the identity map.

Lemma 8.5. *Suppose A is a Frobenius algebra with $\lambda, \rho, \langle -,-\rangle, f_\lambda, f_\rho$ (resp. $\rho, \langle -,-\rangle, f_\rho$) as in Lemma 8.1. Then the following are equivalent.*

(1) $\lambda(ab) = \lambda(ba)$.

(2) $\rho(ab) = \rho(ba)$

(3) $\langle a, b\rangle = \langle b, a\rangle$.

(4) $\nu_A \in \mathrm{Inn}(A)$.

(5) $A \cong DA$ *as A-A-bimodule.*

Proof (1) ⇔ (3) ⇔ (2): Follows from the relation between $\lambda, \rho, \langle -,-\rangle$; see the proof of Lemma 8.1.

(4) ⇔ (5): Follows from the definition of ν_A and f being the same as f_λ in the left module setting or f_ρ in the right module setting. □

Example 8.6. (1) *$A = KG$ with λ the augmentation map, i.e. $\lambda(\sum_g c_g g) = c_1$, is a symmetric algebra. The defining bilinear form is given by $\langle g, h\rangle = \delta_{g,h^{-1}}$ for all $g, h \in G$.*

(2) *$A = \mathrm{Mat}_n(K)$ with $\lambda = \mathrm{Tr}$ (i.e. $\langle X, Y\rangle = \mathrm{Tr}(XY)$) is a symmetric algebra.*

(3) *$A = \Lambda \ltimes D\Lambda$ the trivial extension algebra of a finite-dimensional algebra Λ, which is the vector space $\Lambda \oplus D\Lambda$ with multiplication $(a, f)(b, g) := (ab, ag + fb)$. We have a bilinear form*

$$\langle (a, f), (b, g)\rangle := f(b) + g(a).$$

This is clearly a symmetric form. For non-degeneracy, suppose that $\langle (a, f), -\rangle = 0$, so $0 = \langle (a, f), (b, 0)\rangle = f(b)$ says that $f = 0$; likewise $0 = \langle (a, f), (0, g)\rangle = g(a)$ says that $a = 0$. For associativity:

$$\begin{aligned}
\langle (a, f)(b, g), (c, h)\rangle &= \langle (ab, ag + fb), (c, h)\rangle \\
&= h(ab) + (ag + fb)(c) = h(ab) + (ag)(c) + (fb)(c) \\
&= h(ab) + g(ca) + f(bc) = f(bc) + (bh)(a) + (gc)(a) \\
&= f(bc) + (bh + gc)(a) = \langle (a, f), (bc, bh + gc)\rangle \\
&= \langle (a, f), (b, g)(c, h)\rangle.
\end{aligned}$$

Exercise 8.7. *Suppose $\langle -,-\rangle$ is the defining symmetrising form of a symmetric algebra A. By considering $\mathrm{End}_{A\otimes_K A^{\mathrm{op}}}(A) \cong Z(A)$, show that any other non-degenerate associative symmetrising form $(-,-)$ on A is of the form $(a, b) = \langle uau^{-1}, b\rangle$ for some unit $u \in A^\times$.*

Exercise 8.8. *Use $DA \cong A$ (as bimodule) and tensor-hom adjunction to show that $\mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_K(M, K) = DM$ for all $M \in A\,\mathsf{mod}$.*

Definition 8.9. *An A-module P is projective if any given surjective homomorphism $\mu : M \twoheadrightarrow M'$ and any homomorphism $\lambda : P \to M$, we have λ factors through μ, i.e. $\exists \nu : P \to M'$ s.t. there is the* *following commutative diagram*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\nu"'] \arrow[d, "\lambda"] \\ M' \arrow[r, two heads, "\mu"] & M. \end{tikzcd}$$

In other words, $\mu_* = \mathrm{Hom}_A(P, \mu) : \mathrm{Hom}_A(P, M') \to \mathrm{Hom}_A(P, M)$ *given by* $\nu \mapsto \mu\nu$ *is surjective.*

Dually, an A*-module* I *is* *injective* *if for any given injective homomorphism* $\mu : M' \hookrightarrow M$ *and any homomorphism* $\lambda : M \to I$*,* λ *factors through* μ*. This is equivalent to saying that* $\mu^* := \mathrm{Hom}_A(\mu, I) : \mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(M', I)$ *given by* $\nu \mapsto \nu\mu$ *is surjective.*

Write $\mathsf{proj}\, A$ *to be the 'collection' (category) of all finitely generated projective* A*-module.*

Remark 8.10. Since we use finite-dimensional A, finitely generated is the same as finite-dimensional.

Remark 8.11. Note that if f is an injective homomorphism, then both $\mathrm{Hom}_A(N, f)$ and $\mathrm{Hom}_A(f, N)$ are injective for any $N \in A\,\mathsf{mod}$. So P being projective (resp. I being injective) means that if one have a *short exact sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

(meaning that f is injective, g is surjective, and $gf = 0$) in $A\,\mathsf{mod}$, then we have short exact sequences

$$0 \to \mathrm{Hom}_A(P, L) \xrightarrow{f_*} \mathrm{Hom}_A(P, M) \xrightarrow{g_*} \mathrm{Hom}_A(P, N) \to 0,$$
$$0 \to \mathrm{Hom}_A(N, I) \xrightarrow{g^*} \mathrm{Hom}_A(M, I) \xrightarrow{f^*} \mathrm{Hom}_A(N, I) \to 0,$$

in $K\,\mathsf{mod}$.

Lemma 8.12. *The following are equivalent of an* A*-module* P*.*

(1) P *is projective.*

(2) *Every surjective map* $f : M \to P$ *splits, i.e.* $M = \mathrm{Ker}(f) \oplus P$*.*

(3) P *is a direct summand of a free module.*

Proof See, for example, Rotman's homological algebra book Prop 3.3, Thm 3.5. □

Lemma 8.13. *For idempotents* $e, f \in A$*, we have* $Ae \cong Af$ *as left* A*-module if and only if* $f = ueu^{-1}$ *for some unit* $u \in A^\times$*.*

Proof $\Leftarrow$: Since $A \cong Ae \oplus A(1-e)$ and $A \cong Af \oplus A(1-f)$, we have $A(1-e) \cong A(1-f)$ by Krull-Schmidt property. By Yoneda lemma, an isomorphism $\phi \in \mathrm{Hom}_A(Ae, Af)$ corresponds to an element in $x \in eAf \subset A$; likewise an isomorphism $\psi \in \mathrm{Hom}_A(A(1-e), A(1-e))$ corresponds to $y \in (1-e)A(1-f) \subset A$. Let $x' \in fAe$ and $y' \in (1-f)A(1-e)$ be the elements corresponding to ϕ^{-1} and ψ^{-1} respectively. Since $\phi^{-1}\phi = \mathrm{id}_{Ae}$ corresponds to $e \in eAe$, we have

$$x'x = f, xx' = e, y'y = 1-f, yy' = 1-e.$$

Take $u := x + y$ and $v := x' + y'$. Then we have $vu = f + (1-f) = 1$ and $uv = e + (1-e) = 1$. Therefore, u, v are units such that $uf = x = eu$, i.e. $e = ufu^{-1}$ as required.

$\Rightarrow$: The required isomorphism $Af \to Ae$ is given by $af \mapsto aue$. □

Given an idempotent $e = e^2 \in A$ in an algebra A, then Ae and $A(1-e)$ are both left ideal of A. Since $e(1-e) = 0 = (1-e)e$, we have $Ae \cap A(1-e) = 0$, which means that $A \cong Ae \oplus A(1-e)$ as left A-module. By Lemma 8.12 both Ae and $A(1-e)$ are then projective A-modules. This leads to the following characterisation of idempotent that yields *indecomposable* projective modules.

Definition 8.14. *Two idempotents e, f are orthogonal if $ef = 0 = fe$. An idempotent e is primitive if $e \neq f + f'$ for some orthogonal (pair of) idempotents f, f'.*

Lemma 8.15. *$P \in \mathsf{proj}\, A$ is indecomposable if and only if $P = Ae$ for some primitive idempotent e.*

Proof Follows from definition of primitive. □

Indecomposable projective modules - as they are direct summands of A - can be regarded as the 'largest unbreakable building block' (not in the sense of dimension, but from the Jordan-Hölder filtration perspective) of A-modules, whereas a simple A-modules are the smallest unbreakable building block. The following part details their relation.

Theorem 8.16. *(Idempotent lifting) If I is a nilpotent ideal of A and $\overline{e} = \overline{e}^2 \in A/I$, then there is a lift $e = e^2 \in A$ of $\overline{e}$, i.e. $\overline{e} = e + I$.*

Proof Since I is nilpotent, we have a chain of quotient algebras Let $e_1 := \overline{e} \in A/I$. We are going to inductively an idempotent $e_m \in A/I^m$ for $1 \leq m \leq n$ so that $e_{m-1} = e_m + I^{m-1}$. Since $A/I^m \to A/I^{m-1}$ is surjective, we have some $a \in A/I^m$ with $a + I^{m-1} = e_{m-1}$. Since $(a + I^{m-1})^2 = a + I^{m-1}$, we have $a^2 - a \in I^{m-1}/I^m$, and so $(a^2 - a)^2 \in I^{2(m-1)}/I^m = 0$ (last equality comes from $m > 1$).

Define

$$e_m := \begin{cases} a^p, & \text{if } \operatorname{char} K = p > 0; \\ 3a^2 - 2a^3, & \text{if } \operatorname{char} K = 0. \end{cases}$$

For the positive characteristic case, we have $e_m^2 - e_m = a^{2p} - a^p = (a^2 - a)^p = 0$. For the characteristic zero case, we have

$$e_m^2 - e_m = e_m(e_m - 1) = (3a^2 - 2a^3)(3a^2 - 2a^3 - 1) = -(3 - 2a)(1 + 2a)(a^2 - a)^2 = 0$$

as required. □

Corollary 8.17. *Let I be an nilpotent ideal in A. Let*

$$1 = f_1 + \cdots + f_n \qquad \text{with } f_i \text{ primitive orthogonal idempotents}$$

Then we can write

$$1 = e_1 + \cdots e_n \qquad \text{with } e_i \text{ primitive orthogonal idempotents with } \overline{e_i} = f_i$$

By abuse of terminology, we refer this correspondence between e_i's and f_i's (hence, between indecomposable projective and simple modules) as idempotent

Proof Define idempotents e_i' inductively.

Set $e_1' = 1$. For each $i > 1$, take e_i' as any lift of $f_i + \cdots + f_n$ in the ring $e_{i-1}'Ae_{i-1}'$. Then for any $j \geq i$, e_j' is an idempotent in the ring , and so $e_i'e_j' = e_j' = e_j'e_i'$.

Define $e_i := e_i' - e_{i+1}'$ and so we have $e_i + I = f_i$. Now we need to check orthogonality. If $j > i$, then by using $e_{i+1}'e_j' = e_j'$ and $e_{i+1}'e_{j+1}' = e_{j+1}'$ we have

$$e_{i+1}'e_j = e_{i+1}'(e_j' - e_{j+1}') = e_{i+1}'e_j' - e_{i+1}'e_{j+1}' = e_j' - e_{j+1}' = e_j,$$

and so

$$e_ie_j = (e_i' - e_{i+1}')e_{i+1}'e_j = e_{i+1}'e_j - e_{i+1}'e_j = 0.$$

By a dual argument we have $e_je_i = 0$. □

Now we apply the above corollary to $I = J(A)$. We usually use the following convention of notation:

$$A/J(A) = S_1 \oplus \cdots S_t$$

for the decomposition corresponding to idempotent decomposition 1 in the semisimple algebra $A/J(A)$. Note that different S_i can be isomorphic here. Then by idempotent lifting we have idempotent decomposition $1 = e_1 + \cdots + e_t$ and indecomposable projective $P_i := Ae_i$.

Lemma 8.18. *We have* $\mathrm{Hom}_A(P_i, S_j) \cong \begin{cases} \mathrm{End}_A(S_i), \textit{ if } S_i \cong S_j; \\ 0, \textit{ otherwise.} \end{cases}$

Proof If non-zero homomorphism $\theta : P_i \to S_j$ then $P_i/\ker\theta$ is a non-trivial submodule of S_j and so by simplicity of S_j we have $P_i/\ker\theta \cong S_j$ itself. By Corollary 8.17, we have $P_i/J(A)P_i \cong S_i$. As $P_i/\ker\theta$ surjects onto $P_i/J(A)P_i = \cong S_i$, we have $S_i \cong S_j$ and θ lifts to an endomorphism of S_i. □

Lemma 8.19. *Suppose K is algebraically closed. For any $M \in \mathsf{mod}\, A$, we have* $\dim_K \mathrm{Hom}_A(P_i, M) = [M : S_i] :=$ *number of composition factors of M that is isomorphic to S_i.*

Proof Consider a Jordan-Hölder filtration $M \supset M_1 \supset \cdots M_\ell \supset 0$. P_i being projective implies that $\mathrm{Hom}_A(P_i, M_j/M_{j+1}) \cong \mathrm{Hom}_A(P_i, M_j)/\mathrm{Hom}_A(P_i, M_{j+1})$ by Remark 8.11.

Note that M_j/M_{j+1} is simple, and algebraically closed implies that $\mathrm{End}_A(S_i) \cong K$, so inductively applying the previous lemma yields the claim. □

We will rearrange the indices into $P_1, \ldots, P_n, P_{n+1}, \ldots, P_t$ so that $P_1, \ldots, P_n$ are the isoclass representatives of indecomposable projective A-modules.

Definition 8.20. *Suppose K is algebraically closed. Define*

$$c_{i,j} := \dim_K \mathrm{Hom}_A(P_i, P_j) = \dim_K e_i A e_j = [P_j : S_i]$$

where first equality is from Yoneda's lemma. The Cartan matrix of A is $C_A = (c_{i,j})_{1 \le i,j \le n}$.

Note that if S is a simple A-module, then $S \cong A/I$ for some maximal left ideal of A. Indeed, fix any non-zero element $x \in S$, then Ax is a non-zero A-submodule of S and so is S itself. The map $f : A \to S$ given by $a \mapsto ax$ thus defines a surjective homomorphism, meaning that $S \cong A/\mathrm{Ker}(f)$ with $\mathrm{Ker}(f)$ a left ideal (=left submodule) of A. Since submodule of $A/\mathrm{Ker}(f)$ lifts to left ideal of A containing $\mathrm{Ker}(f)$, simplicity implies that $\mathrm{Ker}(f)$ is maximal.

By definition of Jacobson radical, every indecomposable left $A/\mathrm{rad}(A)$-module is a simple $A/\mathrm{rad}(A)$-module and can be regarded naturally as a simple A-module. Consequently, our choice of indexing is equivalent to that $S_1, \ldots, S_n$ are the isoclass representatives of simple A-modules.

Lecture 9

Let us now specialise to the case when the working field is of characteristic $p > 0$.

Convention: To adopt notation closer to group representation theorists' convention[1], we use the notation $\Bbbk$ for the underlying field. For simplicity, we will assume $\Bbbk$ is algebraically closed (most convenient consequence being $\mathrm{End}_{\Bbbk G}(S) \cong \Bbbk$ for all simple S). We will also assume $|G| = p^a r$ where the p-exponent a is maximal possible (i.e. $p \nmid r$); it is also customary to write $|G|_p = p^a$ and $|G|_{p'} = r$.

Definition 9.1. *An element $g \in G$ is p-regular if $p \nmid \mathrm{ord}(g)$ (not divisible by p, or equivalently, coprime to p). It is sometimes abbreviated as p'-element.*

A conjugacy class C of G is p-regular if any (hence, all) of its elements are p-regular. It is sometimes abbreviated as p'-conjugacy class.

Recall from Remark 5.13 that we have the following characteristic $p > 0$ version of Theorem 5.12.

Theorem 9.2. *The number of (isoclasses of) simple $\Bbbk G$-modules is the number of p-regular conjugacy classes of G.*

Proof See end of Chapter 1 in Alperin's book. □

Recall that a *p-group* is a group whose non-identity elements are always of order divisible by p.

Corollary 9.3. *If G is a p-group, then there is only one simple $\Bbbk G$-module which is $\mathrm{triv}_G \cong \Bbbk G / \mathrm{rad}\, \Bbbk G$.*

Proof Identity is the only p'-elements of G and so there is only one simple $\Bbbk G$-module, and so it is given by $\Bbbk G / \mathrm{rad}\, \Bbbk G$. But triv_G is always a simple $\Bbbk G$-module and so the claim follows. □

Next we look at cyclic group; the material in this almost all come from Alperin's book. We need a fact from Galois theory.

Lemma 9.4. *If $p = \mathrm{char}(\Bbbk) > 0$ does not divide m, then $x^m - 1 \in \Bbbk[x]$ is separable (i.e. all roots have multiplicity 1). In particular, all solutions are given by elements of the group of m-th roots of unity $\mu_m(\Bbbk) := \{\zeta_i \in \Bbbk \mid 0 \le i < m\}$, where ζ is the primitive m-th root.*

Proof A polynomial $f \in \Bbbk[x]$ is separable (meaning all roots have multiplicity 1) if and only if $\gcd(f, df) = 1$ where $df \in \Bbbk[x]$ denotes the formal derivative of f. Taking $f(x) = x^m - 1$, then $df(x) = mx^{m-1}$ only has roots at 0 and so f is separable. □

Lemma 9.5. *Suppose $n = p^a r$ with $p \nmid r$. If $\lambda \in \Bbbk^\times$ satisfies $\lambda^n = 1$, then $\lambda^r = 1$ and $\lambda \in \mu_r(\Bbbk)$.*

Proof Over a field of characteristic $p > 0$, $x \mapsto x^p$ is an automorphism (called the Frobenius automorphism) of $\Bbbk$, so $\lambda^n = (\lambda^r)^{p^a} = 1$ implies that $\lambda^r = 1$. By Lemma 9.4, we have $\lambda \in \mu_r(\Bbbk)$. □

Proposition 9.6. *Let $G = \langle g \rangle$ be a cyclic group of order $p^a r$. For $\lambda \in \mu_r(\Bbbk)$, let S_λ be a 1-dimensional vector space and define, for every $v \in S_\lambda$, $gv := \lambda v$. Then S_λ becomes a simple $\Bbbk G$-module and all simple $\Bbbk G$-module is of this form.*

Proof $\mu_r(\Bbbk)$ is well-defined by Lemma 9.4. It is clear that $\lambda \in \mu_r(\Bbbk)$ satisfies $\lambda^r = 1$, hence $\lambda^n = 1$, and thus $S_\lambda \in \Bbbk G\,\mathsf{mod}$. $\dim_\Bbbk S_\lambda$ implies that S_λ is a simple module. It is clear that $S_\lambda \not\cong S_{\lambda'}$ for $\lambda \neq \lambda'$ distinct elements in $\mu_r(\Bbbk)$.

[1] The usual convention is k or $\mathbb{F}$. K is used to denote a field of characteristic zero given by the field of fractions of a discrete valuation ring $\mathcal{O}$ whose residue field is k; in this setting $(K, \mathcal{O}, k)$ is called a p-modular system. This gives a way relates representations across characteristic 0, integral, and modular settings.

Since G is cyclic, there are exactly $r = |\mu_r(\Bbbk)|$ elements of order not divisible by p – namely, $g^{p^a k}$ for $0 \le k < r$. Hence, we have r distinct simple $\Bbbk G$-modules. The proof finishes if we invoke Theorem 9.2 now as the conjugacy class of any element of G is of size 1. $\square$

Remark 9.7. It is possible to avoid using Theorem 9.2. One first shows that every $\Bbbk G$-simple is of $\Bbbk$-dimension 1; we show this in a more general setting in the lemma below. With this fact in hand, as it suffices to look at action of the generator g and $g^{|G|} = 1$, g must acts by multiplying some $\lambda \in \Bbbk^\times$ such that $\lambda^n = 1$, so it follows from Lemma 9.5 that $\lambda \in \mu_r(\Bbbk)$.

Lemma 9.8. *If A is a (finite-dimensional) commutative K-algebra over some algebraically closed field K (such as $A = \Bbbk G$ for G abelian), then $\dim_\Bbbk S = 1$ for every simple A-module S.*

Proof By Artin-Wedderburn we have $A/\operatorname{rad}(A)$ a product of matrix rings over division $\Bbbk$-algebras. $\Bbbk$ being algebraically closed implies that all matrix ring is over $\Bbbk$. A being commutative means that so is $A/\operatorname{rad}(A)$, and so it must be a product of $\Bbbk$ (as $\operatorname{Mat}_n(\Bbbk)$ is non-commutative for $n > 1$). $\square$

For cyclic $G = \langle g \rangle$, we have now known all the simple $\Bbbk G$-modules and that they are 1-dimensional. Next we look at the projective modules. First tool is the following general result.

Proposition 9.9. *Let G be a finite group and $H \le G$ a subgroup. If $P \in \Bbbk G\,\mathsf{mod}$ is projective, then $\operatorname{Res}^G_H P$ is a projective $\Bbbk H$-module.*

Proof Partition $G = Hx_1 \sqcup Hx_2 \sqcup \cdots \sqcup Hx_r$ into right H-cosets. Then for each i, we have $\Bbbk C_i \cong \Bbbk H$ as $\Bbbk H$-modules (via $hx_i \mapsto h$). Hence, $\operatorname{Res}^G_H \Bbbk G \cong \Bbbk H^{\oplus r}$. Since restriction preserves direct sum and direct summand, so any decomposition $\Bbbk G^{\oplus n} = P \oplus Q$ of $\Bbbk G$-module yields an isomorphism

$$\operatorname{Res}^G_H P \oplus \operatorname{Res}^G_H Q \cong \operatorname{Res}^G_H P \oplus Q \cong \operatorname{Res}^G_H \Bbbk G^{\oplus n} \cong \Bbbk H^{\oplus rn},$$

and so $\operatorname{Res}^G_H P$ is projective by Lemma 8.12. $\square$

Remark 9.10. One may prefer a group-theorectic-independent homological explanation: the right (resp. left) adjoint (e.g. restriction Res^G_H) of a left (resp. right) exact functor (e.g. $\operatorname{Ind}^G_H = \Bbbk G \otimes_{\Bbbk H} -$) on abelian categories preserves injective (resp. projective) objects. Note that, since group algebras are symmetric algebras, injectives and projectives are the same.

As an application, we can obtain some numerical information about projective $\Bbbk G$-module (for arbitrary G).

Recall that a Sylow p-subgroup of G is a p-subgroup (a subgroup that is a p-group) of maximal order, i.e. a subgroup $P \le G$ with $|P| = |G|_p$.

Lemma 9.11. *If P is a Sylow p-subgroup of a finite group G, then $p^a := |P|$ divides $\dim_\Bbbk Q$ for any projective $\Bbbk G$-module Q.*

Proof From Corollary 9.3 we have $\Bbbk P/\operatorname{rad}\Bbbk P \cong \operatorname{triv}_P$ and projective $\Bbbk P$-modules are always free. Hence, $p^a = \dim_\Bbbk \Bbbk P$ divides $\dim_\Bbbk R$ for any projective $\Bbbk P$-module R. By Proposition 9.9, if Q is a projective $\Bbbk G$-module, then so is $\operatorname{Res}^G_H Q$, but this has the same dimension as Q. $\square$

Proposition 9.12. *Suppose $G = \langle g \rangle$ is a cyclic group of order $n = p^a r$. Let V be an indecomposable $\Bbbk G$-module with $\dim_\Bbbk V = d$. Then V has a unique (semi)simple submodule isomorphic to S_λ for some $\lambda \in \mu_r(\Bbbk)$. In particular, d, λ uniquely determines V has a unique Jordan-Hölder filtration with exactly d composition factors each isomorphic to S_λ.*

$$V = V_1 \overset{S_\lambda}{\supset} V_2 \overset{S_\lambda}{\supset} \cdots V_{d-1} \overset{S_\lambda}{\supset} V_d \overset{S_\lambda}{\supset} 0.$$

Denote such a V by $V_d(\lambda)$.

Proof Then g acts on V as a linear transformation, say, T of order n, and so every eigenvalue of T is an n-th root of unity. We can then pick a basis of V so that T is block-diagonalised into Jordan blocks $T_1, \ldots, T_k$. Thus, every g^i acts as a block-diagonal matrix with blocks $T_1^i, \ldots, T_k^i$, and so yields a decomposition $V = V_1 \oplus \cdots \oplus V_k$ into indecomposable modules. By indecomposability we have $k = 1$ and thus T is just a single Jordan block $J_d(\lambda)$ with eigenvalue, say, $\lambda \in \mu_n$ of size d.

Now we can see that there is a unique 1-dimensional submodule of V where g acts by multiplying λ (this submodule corresponds to the corner entry of $J_d(\lambda)$). It then follows from Lemma 9.5 and Proposition 9.6 that this is isomorphic to S_λ. The Jordan-Hölder filtration in the claim can then be obtained by repeating this procedure. Clearly, if g acts by a different Jordan block $J_d(\lambda')$ then we have a distinct (non-isomorphic) module. This completes the proof. $\square$

Proposition 9.13. *Suppose $G = \langle g \rangle$ is a cyclic group of order $n = p^a r$. Then there are exactly n isomorphism classes of indecomposable $\Bbbk G$-modules with representative $V_d(\lambda)$ for $1 \leq d \leq p^a$ and $\lambda \in \mu_r(\Bbbk)$. In particular, $\Bbbk G$ is isomorphic (as an algebra) to the direct product of r copies of $\Bbbk[x]/(x^{p^a})$.*

Proof Let $R = V_d(\lambda)$ be an indecomposable projective $\Bbbk G$-module of dimension d. Consider $\langle g^r \rangle \leq G$, this is a Sylow p-subgroup of order p^a, and so by Lemma 9.11, we have $p^a \mid d$, i.e. $d = p^a s$ for some non-zero s.

We also knew from Proposition 9.6 there are r distinct simple modules, hence r distinct indecomposable projective $\Bbbk G$-modules $\{P_\lambda \mid \lambda \in \mu_r(\Bbbk)\}$. It then follows by idempotent lifting and $\dim_\Bbbk S_\lambda = 1$ that $\Bbbk G = \bigoplus_{\lambda \in \mu_r(\Bbbk)} P_\lambda$.

Let $s_\lambda \geq 1$ be such that $\dim_\Bbbk P_\lambda = p^a s_\lambda$. Then we have

$$p^a r = \dim_\Bbbk \Bbbk G = \sum_{\lambda \in \mu_r(\Bbbk)} \dim_\Bbbk P_\lambda = \sum p^a s_\lambda = p^a \sum_{\lambda \in \mu_r(\Bbbk)}^{r} s_\lambda.$$

As $|\mu_r(\Bbbk)| = r$, each s_i is necessary 1. Considering the submodules of $P_\lambda = V_{p^a}(\lambda)$, then we have n isoclasses of indecomposable $\Bbbk G$-modules.

It remains to argue that $V_d(\lambda) \in \Bbbk G\,\mathsf{mod}$ implies that $d \leq p^a$. By Yoneda's lemma we have $\dim_\Bbbk \mathrm{Hom}_A(P_\lambda, V_d(\lambda))$ the same as the number of composition multiplicity $[V_d(\lambda) : S_\lambda]$ of S_λ in $V_{p^a}(\lambda)$, which is precisely d by Proposition 9.12. Hence, there is a homomorphism that maps the idempotent $e_\lambda \in \Bbbk G e_\lambda = P_\lambda$ (which lies in the top composition factor) to the top composition factor $V_d(\lambda)$. Since image of a homomorphism is necessary a submodule of the range, this is a surjection from P_λ to $V_d(\lambda)$, and so $d \leq p^a$. $\square$

Remark 9.14. In the last part where we show $d \leq p^a$, we used module-theoretic argument. One can use more basic linear algebra (which is Alperin's approach) as follows. Consider again the action of g on $V_d(\lambda)$, which is given by Jordan block $T := J_d(\lambda)$. It satisfies

$$x^n - 1 = (x^r - 1)^{p^a} = (x - \lambda) \prod_{\lambda \neq \omega \in \mu_r(\Bbbk)} (x - \omega).$$

Let $S := \prod_{\lambda \neq \omega \in \mu_r(\Bbbk)} (T - \omega I_d)$. Since $(T - \lambda I_d)(T - \omega I_d) = (T - \omega I_d)(T - \lambda I_d)$ for all $\omega \in \mu_r(\Bbbk)$, we have

$$0 = T^n - I_d = (T^r - I_d)^{p^a} = (T - \lambda I_d)^{p^a} S^{p^a}.$$

Each $\omega \in \mu_r(\Bbbk) \setminus \{\lambda\}$ is not an eigenvalue of the invertible matrix T, so $(T - \omega I_d)^k \neq 0$ for all $k \geq 1$, and hence $S^{p^a} \neq 0$. Consequently, we have $(T - \lambda I_d)^{p^a} = 0$. But $T = J_d(\lambda)$ and so d is the smallest positive integer k such that $(T - \lambda I_d)^k = 0$. Thus $p^a \geq d$ as required.

Corollary 9.15. *For cyclic group G of order $p^a r$. The Cartan matrix of $\Bbbk G$ is a $p^a I_r$.*

Lecture 10

In the language of artin algebra this type of algebra where every module has a unique Jordan-Hölder filtration has a special name.

Definition 10.1. *A non-zero A-module $M \in A\,\mathsf{mod}$ is uniserial if it has a unique Jordan-Hölder filtration (equivalently, composition series).*

An algebra A is if uniserial if every indecomposable projective A-module in $A\,\mathsf{mod}$ and $A^{\mathrm{op}}\,\mathsf{mod} = \mathsf{mod}\,A$ is uniserial.

In such a case, it is convenient to display the composition series of the modules as follows.

Example 10.2. $\Bbbk C_6 \cong \Bbbk[x]/(x^3) \times \Bbbk[y]/(y^3)$ *for* $\operatorname{char}\Bbbk = 3$*, denote by S_1 the simple corresponding to $P_1 := \Bbbk[x]/(x^3)$ and S_2 the simple corresponding $P_2 := \Bbbk[y]/(y^3)$. Then we can display the left regular representation as follows:*

$$\Bbbk C_6 = P_1 \oplus P_2 = \begin{smallmatrix} S_1 \\ S_1 \\ S_1 \end{smallmatrix} \oplus \begin{smallmatrix} S_2 \\ S_2 \\ S_2 \end{smallmatrix} = \begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}.$$

Note that on the far-right we further simplified the notation; this is also rather common in practice.

Notice that when $r > 1$ (recall that $n = p^a r$). Then we can see that each indecomposable projective involves only a single simple; or equivalently $\operatorname{Hom}_{\Bbbk G}(P_i, P_j) = 0$ whenever $i \neq j$. The Cartan matrix also becomes (block-)diagonal. There is 'no interaction' between each indecomposable projective.

Definition 10.3. *Suppose $A = B_0 \oplus B_1 \oplus \cdots B_r$ is the decomposition of A as A-A-bimodule. Then each B_i is called a block of A. If B is a block of A, then by abuse of terminology we also call the central idempotent $e^2 = e \in Z(A)$ of A such that $Ae = eA = AeA = B_i$ (for some i) a block of A.*

The block $e \in Z(\Bbbk G)$ (or $\Bbbk Ge$) such that $e\,\mathrm{triv}_G \neq 0$ is called the principal block of $\Bbbk G$; often more conveniently denoted by $B_0(\Bbbk G)$ or even B_0.

Remark 10.4. A block idempotent of A is the same as a primitive central idempotent of A, i.e. primitive idempotent in $Z(A)$.

Remark 10.5. One who is serious about categorical rigour will be annoyed with $\oplus$ since the context usually infers that we are thinking about ring decomposition $A = B_1 \times B_2 \times \cdots \times B_r$; the use here justified by the fact that we are looking at A-A-bimodule. See the relevant discussion at StackExchange here (https://math.stackexchange.com/questions/345501/is-a-times-b-the-same-as-a-oplus-b/346140).

In the rest, we will use block to refer to block of group algebra, unless otherwise stated.

Example 10.6. *For $G = \langle g \rangle$ of order $n = p^a r$, the group algebra $\Bbbk G$ has r blocks.*

The terminology 'block' is used in group representation theory and also in algebraic Lie theory (e.g. blocks of the BGG category); ring theorists will just say direct factor. Block decomposition of $\Bbbk G$ induces module decomposition: $A = B_0 \oplus \cdots \oplus B_r$ corresponding to primitive central idempotent decomposition $1 = e_0 + \cdots e_r$ yields $M = M_1 \oplus \cdots \oplus M_r \in A\,\mathsf{mod}$ with $M_i = Me_i \in B_i\,\mathsf{mod}$.

We say that an indecomposable module M *belongs to* block $B = Ae$ if $Me = M$. By iteratively quotienting out a simple submodule, we get the following.

Proposition 10.7. *If M is an indecomposable $\Bbbk G$-module, then all composition factors of M belong to the same block.*

Remark 10.8. Categorically, we have $\Bbbk G\,\mathsf{mod} = B_0\,\mathsf{mod} \oplus \cdots \oplus B_r\,\mathsf{mod}$.

Another property satisfied by $\Bbbk G$ for cyclic G is the following.

Definition 10.9. *An algebra A is representation-finite if there is only finitely many isoclasses of indecomposable A-modules. In this case, we may also that that A is of finite representation-type, or sometimes of finite-type for short. Otherwise, we say that A is representation-infinite, or of infinite representation-type, or of infinite-type.*

Remark 10.10. We will see in a later lectures that that even for an abelian group as small as $C_2 \times C_2$, the group algebra $\Bbbk G$ can be of infinite-type.

Remark 10.11. For $\Bbbk G$, it is possible that one block is of finite-type while another is of infinite-type. Finite-type can be detected by a block-invariant called *defect group*; it is the minimal subgroup $D \leq G$ such that the canonical map $\operatorname{Ind}_D^G B = B \otimes_{\Bbbk D} \Bbbk G \to B$ given by $b \otimes g \mapsto b$ splits. For technicality we will not explain the details behind.

It turns out that representation-finite blocks can be described by a family of algebras called the *Brauer tree algebras.*

Theorem 10.12. *Suppose B is a representation-finite block algebra. Then B is a Brauer tree algebra.*

We will not give a proof of this result in this course. The history is a bit more complicated; we refer to Craven's book (Representation Theory of Finite Groups – a Guidebook; Springer 2019) for detailed account; for simplicity, one usually attribute this to Dade.

Dade's statement does not concern representation-finiteness: If B has cyclic defect, then B is a Brauer tree algebra. The representation-finite part is purely a result about the family of Brauer tree algebras itself, and comes from works of Gabriel and Riedtmann (and possibly independent from the eastern European school of the 80's under the name of the so-called 'matrix problems').

Dade used character theory in his proof. A purely module-theoretic proof can be found in Alperin's book. A slightly more streamlined version can be found in my notes for Sejong Park's course on Derived Equivalence of Blocks of Group Rings on my webpage.

Full statement of Dade's result also contains how some information of B can be obtained direct from the cyclic defect group; as we demonstrate in the next proposition.

Proposition 10.13. *If $G = C_{p^n} \rtimes C_e$ where C_k denotes the cyclic group of order k, then $\Bbbk G$ is (its own block and) a uniseral Brauer tree algebra with exceptional multiplicity $(p^n - 1)/e$.*

We will omit proofs of this.

Combinatorics of Brauer trees

In the rest of this lecture, we explain the composition series of the indecomposable projective module over a Brauer tree algebra.

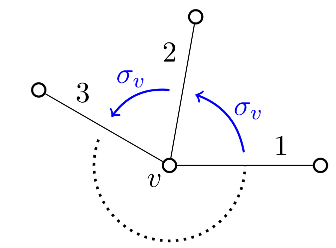
Definition 10.14. *A Brauer tree is a datum (T, σ, v_0, m_0) where*

- *$T = (T_0, T_1)$ is a (graphical) tree,*
- *$\sigma = (\sigma_v)_{v \in T_0}$ records the cyclic ordering $\sigma_v \in \operatorname{Sym}(T_1|_v)$ of edges $T_1|_v$ around each vertex v,*
- *an exceptional vertex v_0,*
- *an exceptional multiplicity $m_0 \in \mathbb{Z}_+$ attached to v_0.*

Every non-exceptional vertex v is regarded to have trivial multiplicity $m_v = 1$. In the case when $m_0 = 1$, we say that the Brauer tree is multiplicity-free.

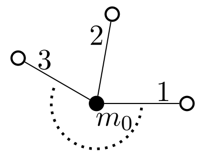
Remark 10.15. (T, σ) is equivalent to specifying a planar embedding of T, i.e. embedding T on the $\mathbb{R}^2$-plane (or equivalent a disk) in a way where edges do not cross each other.

Our convention is to display the cyclic ordering in the counter-clockwsie direction, and ordinary vertices in white hollow circle, the exceptional vertex in black with the exceptional multiplicity written near to it. We will suppress the notation σ from (T, σ, v_0, m_0). We always assume the underlying tree is connected.



$$\sigma_v : 1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto 1$$

Example 10.16. *One extreme cases are given by the Brauer star below*

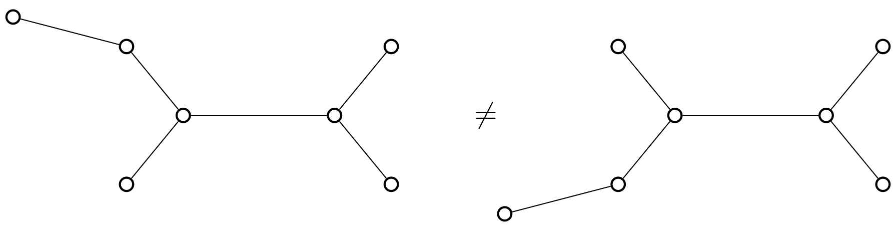


Brauer star with exceptional multiplicity m

where the exceptional vertex is required to be the central vertex.

Another extreme case is the multiplicity-free Brauer line algebra, where the underlying tree is a line (so valency of vertex is at most 2 for all); this often appear in Lie-theoretic setting.

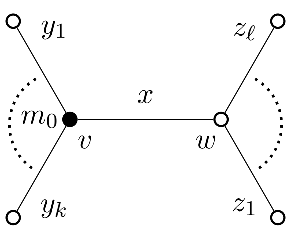
Example 10.17. *Note that the cyclic ordering makes a difference; the following two tree are the same as graph since we can move around the edges, but they are not the same as Brauer graph (or planar graph) as the cyclic ordering forbid us from moving the branches.*



Reading indecomposable projective from Brauer trees

Let us explain how to read the composition series of an indecomposable projective A-module for A a Brauer tree algebra associated to (T, σ, v_0, m_0) - before even giving the construction of these algebras.

First, the (isoclasses of) indecomposable projective and simple A-modules are enumerated by the edges of T. Let P_x be the indecomposable projective A-module corresponding to $x \in T_1$. On the Brauer tree, we only need to consider the edges around the edge x, which can be displayed as follows.



Here, the two endpoints of x are labelled v and w. To shorten exposition, we take $v_0 = v$ here but the general case is just the same as taking $m_0 = 1$. The edges around v is labelled by $y_1, y_2, \ldots, y_k$ under the cyclic ordering; likewise, around w we have $z_1, \ldots, z_\ell$.

P_x has a unique simple quotient and unique simple submodule both isomorphic to the simple S_x. Note that A is a symmetric algebra and the the indecomposable injective module I_x is isomorphic to P_x. Removing the simple quotient and submodule from P_x, we have $\operatorname{rad} P_x/S_x = J(A)e_x/S_x$ being isomorphic to a direct sum of two uniserial modules U_v, U_w, where $v, w \in T_0$ are the two ends of the edge $x \in T_1$. The composition series of U_v is given by

$$S_{y_1}, S_{y_2}, \ldots, S_{y_k}, \left(S_x, S_{y_1}, S_{y_2}, \ldots, S_{y_k}\right)^{m-1}.$$

Here, $(S_x, \ldots, S_{y_k})^{m-1}$ means that this subseries repeats itself $m-1$ times. Likewise, U_w has composition series $S_{z_1}, \ldots, S_{z_\ell}$ (so just like U_v with $m_0 = 1$). Note that in the case when one of v, w has valency 1 with trivial multiplicity, then the corresponding uniserial module is zero, and so P_x in this case is uniserial. Summarising, we can display P_x in diagrammatic form as follows.

$$\begin{array}{r} x \text{ appears } m-1 \text{ times,} \\ y_j \text{ appears } m \text{ times} \end{array} \left\{ \begin{array}{ccc} & x & \\ y_1 & & \\ \vdots & & z_1 \\ x & & \vdots \\ \vdots & & z_\ell \\ y_k & & \\ & x & \end{array} \right.$$

Example 10.18. *Suppose (T, v_0, m_0) is a Brauer star with n edges as in Example 10.16, and B be the associated Brauer star algebra. Then the indecomposable projective B-module associated to an edge $x \in \{1, \ldots, n\}$ has a unique composition series*

$$\underbrace{S_x, S_{x+1}, \ldots, S_{x-1}}_{\textit{repeat } m_0 \textit{ times}} S_x.$$

In particular, B is a uniserial algebra. The Cartan matrix of B has the form

$$\begin{pmatrix} m_0+1 & m_0 & \cdots & m_0 \\ m_0 & m_0+1 & & \vdots \\ \vdots & & \ddots & m_0 \\ m_0 & \cdots & m_0 & m_0+1 \end{pmatrix}$$

Indecomposable modules over Brauer tree algebras

Indecomposable modules over a Brauer tree algebras can be completely classified and described by certain combinatorics on the *quiver* (and relation) of the algebra. More recent advances tells us that these combinatorics can be reflected by considering certain type of curves on the disk where the Brauer tree embeds into. Nevertheless, the rigourous mathematics behind these relies on using the quiver and relation defining the Brauer tree algebra. This will be the focus of the next lecture.

Lecture 11

Quiver algebras and Brauer tree algebras

Definition 11.1. *A (finite) quiver is a datum $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ for finite sets Q_0, Q_1. The elements of Q_0 are called vertices and those of Q_1 are called arrows. The source (resp. target)of an arrow $\alpha \in Q_1$ is the vertex $s(\alpha)$ (resp. $t(\alpha)$).*

This is equivalent to specifying an oriented graph (possibly with multi-edges and loops); Gabriel coined the term quiver as a way to emphasise the context is not really about the graph itself.

Definition 11.2. *Let Q be a quiver.*

- *A trivial path at $i \in Q_0$ is a walk on Q stationary at i. Denote such a path by e_i.*
- *A path of Q is either a trivial path or a word $\alpha_1\alpha_2\cdots\alpha_\ell$ where $s(\alpha_i) = t(\alpha_{i+1})$. The source and target functions extend naturally to paths.*
- *The path algebra KQ of a quiver Q is the K-algebra whose underlying space is given by $\bigoplus_{p:\text{paths of } Q} Kp$, with multiplication given by path concatenation:*

$$p \cdot q := \begin{cases} pq, & \text{if } s(p) = t(q); \\ 0, & \text{otherwise.} \end{cases}$$

  That is, we compose arrow from right to left (in the same direction as we compose maps): "$\overleftarrow{pq} = \overleftarrow{p} \cdot \overleftarrow{q}$"

Note that the trivial paths e_i are primitive idempotents of KQ, and the radical $\operatorname{rad} KQ$ of KQ is the ideal generated by all arrows.

Example 11.3. *Consider the linear $\mathbb{A}_n$-quiver*

$$Q = \vec{\mathbb{A}}_n = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

Then KQ is isomorphic to the lower triangular n-by-n matrix ring where the diagonal elementary matrix $E_{i,i}$ corresponds to e_i and (i,j)-th elementary matrix $E_{i,j}$ for $i > j$ corresponds to the path $\alpha_{i-1}\cdots\alpha_{j+1}\alpha_j$.

Definition 11.4. *An ideal $I \lhd KQ$ is admissible if $I \subset (KQ)^2$, i.e. generated by polynomials in paths of length at least 2. A bounded path algebra or quiver algebra (with relations) is an algebra of the form KQ/I for some quiver Q and admissible ideal I.*

Remark 11.5. Admissiblity ensures there is no redundant arrows (which appears if there is a relation like, for example, $\alpha - \beta\gamma \in I$ for some $\alpha \neq \beta, \gamma \in Q_1$) and there is enough vertices (trivial paths may not be primitive if there is a loop x at a vertex with relation $x^2 - x \in I$).

Definition 11.6. *Suppose $(T = (T, \sigma), v_0, m_0)$ is a Brauer tree. We define a quiver Q_T as follows.*

- *The vertices of Q_T are given by the edges of T.*
- *There is an arrow $y \xleftarrow{(y|x)_v} x$ if x, y have a common endpoint v with $y = \sigma_v(x)$.*

Suppose $x_1, \ldots, x_\ell$ are edges of T all sharing a common vertex v of valency k with $x_{i+1} = \sigma_v(x_i)$ and $\ell \leq k+1$, then we write

$$(x_\ell|x_1)_v := (x_\ell|x_{\ell-1})_v \cdots (x_3|x_2)_v(x_2|x_1)_v \in KQ_T.$$

Let I_{T,v_0,m_0} be the ideal of KQ_T generated by the following.

- (Bouncing relation) $(z|y)_v(y|x)_u$ *if* $v \neq u$*;*
- (Brauer commutation) $(x|x)_v^{m_v} - (x|x)_u^{m_u}$ *for each edge x with endpoints u, v.*

The basic Brauer tree algebra $B(T, v_0, m_0)$ associated to the Brauer tree (T, v_0, m_0) is the bounded path algebra $KQ_T/I_{T,v_0,m_0}$.

In general, a Brauer tree algebra is one that is Morita equivalent to a basic one, that is an algebra A with

$$A \cong \mathrm{End}_B(\bigoplus_{x \in T_1} Be_x^{\oplus r_x})^{\mathrm{op}}$$

for some basic Brauer tree algebra $B = B(T, v_0, m_0)$ and integers $d_x \geq 1$ over all $x \in T_1$.

Example 11.7. *Suppose T is a Brauer star with 1 edge and exceptional multiplicity $m = m_0$. Then $B(T, v_0, m_0) \cong K[x]/(x^{m+1})$ given by $(e|e)_{v_0} \mapsto x$ where e is the unique edge of T.*

Note that I_{T,v_0,m_0} is not an admissible ideal. If one insists on using admissible ideal, then we need to tweak as follows.

(1) Replace Q_T by Q_{T,v_0,m_0} the quiver, which is obtained from Q_T by removing every arrow $(x|x)_v$ from Q_T for each non-exceptional vertex v of valency 1.

(2) Remove any generating relation of I_{T,v_0,m_0} that involves the removed arrows (i.e. we replace I_{T,v_0,m_0} by $I_{T,v_0,m_0} \cap KQ_{T,v_0,m_0}$).

(3) Add new generating relations:
   - (Length relation) $(\sigma_v(x)|x)_v(x|x)_v^{m_v}$ and $(x|x)_v^{m_v}(x|\sigma_v^{-1}(x))_v$ for all x, v.

Representations of bounded quivers

Definition 11.8. *A K-linear representation of Q is a datum $(\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1})$ where M_i is a K-vector space for each $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is K-linear map for each $\alpha \in Q_1$.*

Proposition 11.9. *There is an isomorphism between the category of representations of Q and KQ* mod*, where $(M_i, M_\alpha)_{i,\alpha}$ corresponds to $M = \bigoplus_i M_i$ with KQ-action given by (linear combinations of compositions) M_α's.*

Example 11.10. *The representation of $Q = \vec{\mathbb{A}}_n$ given by*

$$U_{i,j} := 0 \to \cdots 0 \to K \xrightarrow{\mathrm{id}} \to \cdots \xrightarrow{\mathrm{id}} K \to 0 \to \cdots \to 0$$

with a copy of K on vertices $i, i+1, \ldots, j$ is the uniserial KQ-module corresponding to the column space (under the isomorphism of KQ with the lower triangular matrix ring) with non-zero entries in the k-th row for $i \leq k \leq j$.

Suppose $M = (M_i, M_\alpha)_{i,\alpha}$ is a representation of Q, and I is an admissible ideal of KQ. For a path $p = \alpha_1 \cdots \alpha_\ell$, let $M_p := M_{\alpha_1} \cdots M_{\alpha_\ell}$; similarly, for $a = \sum_{p\text{: path}} \lambda_p p \in KQ$ (with $\lambda_p \in K$), let $M_a := \sum_p \lambda_p M_p$. Then we write $M \in \mathrm{rep}(Q, I)$ if $M_a = 0$ for all $a \in I$.

Proposition 11.11. *Suppose $A = KQ/I$ is a bounded path algebra. Then A* mod *is isomorphic to the full subcategory $\mathrm{rep}(Q, I)$ of K-linear representations of Q.*

String combinatorics

The indecomposable modules over a Brauer tree algebra A can be described by the so-called *string combinatorics*.

Let Q be a quiver. Consider the set

$$Q_1^{-1} = \{\alpha^- \mid \alpha \in Q_1\}$$

of formal inverses of arrows in Q. For notational convenience sometimes we write α^{-1} for α^-.

A *walk* on Q is either a *stationary* walk (=trivial path) e_x for some $x \in Q_0$ or a word $w = w_\ell \cdots w_2 w_1$ in $Q_1 \sqcup Q_1^{-1}$ such that $t(w_i) = s(w_{i+1})$ for all $1 \leq i < \ell$. A non-stationary walk w is *directed* if w_i's are all arrows; likewise, w is *inverse* if w_i's are all inverses. The *reflection* of a walk $w = w_\ell \cdots w_1$ is the walk $w^- := w_1^- \cdots w_\ell^-$. This defines an equivalence relations on the set of walks on Q.

Suppose R is a set of monomials in KQ (think: the bouncing relation and length relation). A *walk* on (Q, R) is a walk on Q such that there is no directed subwalk w with $w \in R$, and there is no inverse subwalk w with $w^{-1} \in R$. A *string* of (Q, R) is a reflection equivalence class of walks on (Q, R). In practice, we always choose a representative walk to work with whenever we say 'a string'.

Given a non-stationary walk $w = w_\ell \cdots w_1$ on (Q, R), we can assign to it an $\mathbb{A}_{\ell+1}$-quiver, i.e. a quiver whose underlying undirected graph is just a line with $\ell+1$ vertices. We enumerate the vertex from right to left by $0, 1, \ldots, \ell$. The arrow connecting $i-1$ and i points to the left if w_i is an arrow; otherwise, to the right. Denote by $\mathbb{A}_w$ this quiver. Then there is a morphism of quivers (in the obvious sense) from $c : \mathbb{A}_w \to Q$ where $i \in \{0, 1, \ldots, \ell\} = (\mathbb{A}_w)_0$ is mapped to $s(w_i)$ and the arrow connecting $i-1$ and i is mapped to w_i if it is an arrow; or else to w_i^-. This induces naturally an algebra homomorphism $f : KQ \to K\mathbb{A}_w$.

Consider the K-linear representation M of the $\mathbb{A}_w$-quiver where every vertex is assigned a 1-dimensional K-space and every arrow is the identity map. Then we have a pullback representation $M(w) \in KQ\,\mathsf{mod}$ where $a \in KQ$ acts by $f(a) \in K\mathbb{A}_w$. The subpath condition on w ensures that $M(w)$ is indeed a module over the bounded path algebra $KQ/(R)$. In fact, this is an indecomposable A-module. More generally, if $A \twoheadrightarrow KQ/(R)$ with nilpotent kernel[2], then we can regard $M(w)$ as an A-module. We call this $M(w)$ the *string module* associated to w.

For stationary walk $w = e_x$, the associated string module $M(w)$ is just the simple module where the vertex x is assigned a 1-dimensional K-space, and all other vertices are assigned the zero space.

Theorem 11.12. *Let $Q = Q_{T,v_0,m_0}$ and R be the following set of monomials*

$$R := \{(\sigma_v(x)|x)_v(x|\sigma_u^{-1}(x))_u \mid x \in T_1 \text{ with endpoints } u \neq v\} \sqcup \{(x|x)_v^{m_v} \mid x \in T_1\}.$$

Then we have a one-to-one correspondence

$$\{\text{strings on } (Q,R)\} \overset{1:1}{\leftrightarrow} \{\text{indecomposable non-projective } B(T, v_0, m_0)\text{-modules}\}$$
$$w \mapsto M(w).$$

Moreover, the number of strings on (Q, R) is $m_0|T|$.

[2] $KQ/(R)$ is given by $A/\mathrm{soc}(A)$ where $\mathrm{soc}(A)$ denotes the maximal semisimple submodule of A

Lecture 12

Example 12.1. *Let $(T, v_0, m = m_0)$ be the Brauer star with one edge x and multiplicity m. Then the only possibly walks on the induced (Q, R) are given by e_x and $(x|x)^k$ for $1 \leq k \leq m$. This yields all indecomposable $K[x]/(x^{m+1})$-module of length at most m, i.e. all indecomposable non-projective modules.*

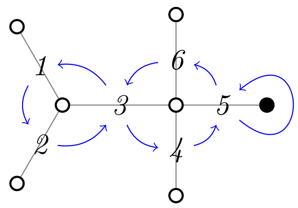
More generally, we have the following.

Example 12.2. *Uniserial modules of a Brauer tree algebras are given by $M(w)$ for a string $w = (x|y)_v$ whose underlying walk is directed (up to reflection).*

Example 12.3. *Let B be the Brauer star algebra associated to the Brauer star with n edges and exceptional multiplicity $m = m_0$. Since there is no vertex with two incoming arrows on the quiver of B, the strings are all given by paths $(y|x)(x|x)^k = (y|y)^k(y|x)$ with $0 \leq k < m$. In particular, all indecomposable modules are uniserial.*

The following example shows that one should be careful with the direction of the letters in the walk.

Example 12.4. *Consider the following Brauer tree with exceptional multiplicity $m_0 > 1$.*



For simplicity, we use $(i|j)$ instead of $(i|j)_u$ whenever with u is non-exceptional, and use $v := v_0$ (and so $(5|5)_v$ is an arrow). We have a walk $(3|6)(6|5)$. Let us try to extend this on the right, that is, to try to attach $(5|5)_v^{\pm}$ or $(4|5)^{\pm}$ at the end. One needs to be careful when attaching $(5|5)_v$ as

- *$(3|6)(6|5)(5|5)_v$ is not a string because $(6|5)(5|5) \in R$; whereas*
- *$(3|6)(6|5)(5|5)_v^{-1}$ is a valid string.*

Module diagram

It is convenient to display the structure of a module using module diagram.[3] Recall that to a (non-trivial) string $w = w_1 \cdots w_\ell$, we can associate to it an $\mathbb{A}_{\ell+1}$-quiver that we denoted by $\mathbb{A}_w$ along with a quiver morphism $c : \mathbb{A}_w \to Q$. The module diagram is obtained by

(1) Draw the quiver $\mathbb{A}_w$ with vertex $v \in (\mathbb{A}_w)_0$ labelled (coloured) by $c(v) \in Q_0$ and arrow α labelled (coloured) by $c(\alpha) \in Q_1$.

(2) Arrows are drawn diagonally with peaks (i.e. sources, i.e. vertices with only outgoing arrow) on top and trough (i.e. sinks, i.e. vertices with only incoming arrow) in the bottom.

For trivial string e_x (for $x \in Q_0$), then the corresponding module diagram is just x (i.e., the $\mathbb{A}_1$-quiver where the only vertex is coloured by x).

In the case of Brauer tree algebras, it is not even necessary to draw the arrows and their labelling as there will not be any ambiguity.

It is easier to explain with example.

[3]There is no widely agreed name to these diagrams; for convenience, we just call them 'module diagram' in this notes.

Example 12.5. *Let us consider again the Brauer tree in Example 12.4 with exceptional multiplicity $m_0 > 1$. Consider the string $w = (2|1)^{-1}(3|2)^{-1}(3|6)(6|5)(5|5)_v^{-1}$. Then we have module diagram*

$$M(w) = \begin{tikzcd}[column sep=small] 1 \arrow[dr, "(2|1)"] & & & & 5 \arrow[dl, "(6|5)"'] \arrow[dr, "(5|5)_v"] & \\ & 2 \arrow[dr, "(3|2)"] & & 6 \arrow[dl, "(3|6)"'] & & 5 \\ & & 3 & & & \end{tikzcd}$$

or more compactly (which is only possible in the case of Brauer tree algebras)

$$M(w) = \begin{smallmatrix} 1 & & & & 5 & \\ & 2 & & 6 & & 5 \\ & & 3 & & & \end{smallmatrix}.$$

Walking along w we pass through the vertices $5, 5, 6, 3, 2, 1$ in order. We then have a canonical basis *$\{v_0, v_1, \ldots, v_5\}$ of $M(w)$ so that*

$$(5|5)_v \cdot v_1 = v_0, \quad (6|5) \cdot v_1 = v_2, \quad (3|6) \cdot v_2 = v_3, \quad (3|2) \cdot v_4 = v_3, \quad (2|1)v_5 = v4,$$
$$\text{and } e_5 \cdot v_0 = v_0, \quad e_5 \cdot v_1 = v_1, \quad e_6 \cdot v_2 = v_2, \quad e_3 \cdot v_3 = v_3, \quad e_2 \cdot v_4 = v_4, \quad e_1 \cdot v_5 = v_5,$$

and $\alpha v_i = 0$ for all other combinations of i and $\alpha \in Q_1 \cup \{e_x\}_{x \in Q_0}$. In other words, the canonical basis matches the vertices of the module diagram so that the colouring of the arrows in the diagram act as the identity map between the corresponding subspace.

We have only explained the module diagrams from string modules, but it works in the same way for indecomposable projective module; those are exactly the pictures we drew in Lecture 10.

Module diagram allows us to see certain submodules and quotients easily. Notice that, as an element in a quiver algebra, the arrows (including the trivial ones) are the generators of the algebra. This means that, if we have a subdiagram of a module diagram such that for each coloured arrow $x \xrightarrow{a} y$, any out-going arrow $y \xrightarrow{b} z$ attached to this particular y is also in the subdiagram, then the subdiagram specifies a submodule. Dually, subdiagram where coloured arrow $w \xrightarrow{c} x$ are also included yields a quotient module.

Moreover, if we have subdiagram this consist of several connected components, then each of these components represents an indecomposable direct summand.

Example 12.6. *Consider $M(w)$ in Example 12.5. Then we have a chain of submodules:*

$$M(w) \overset{1}{\supset} \begin{smallmatrix} & & & 5 & \\ 2 & & 6 & & 5 \\ & 3 & & & \end{smallmatrix} \overset{2}{\supset} \begin{smallmatrix} & & 5 & \\ & 6 & & 5 \\ 3 & & & \end{smallmatrix}.$$

The number above $\supset$ denotes the quotient of the larger one by the succeeding submodule. Now for the last module is the above chain, if we remove the 5 in the peak, then we have a diagram with two disconnected components $\begin{smallmatrix} 6 \\ 3 \end{smallmatrix}$ and 5, which means that $\begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \oplus 5$ is a submodule. Continuing the chain like so we obtain:

$$M(w) \overset{1}{\supset} \begin{smallmatrix} & & & 5 & \\ 2 & & 6 & & 5 \\ & 3 & & & \end{smallmatrix} \overset{2}{\supset} \begin{smallmatrix} & & 5 & \\ & 6 & & 5 \\ 3 & & & \end{smallmatrix} \overset{5}{\supset} \left(\begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \oplus 5\right) \overset{6}{\supset} (3 \oplus 5) \overset{5}{\supset} 3 \supset 0.$$

Note that in this chain, we remove precisely one vertex at each step, that means that the subquotient of this chain are all simple modules. Hence, this is a Jordan-Hölder filtration of $M(w)$ with composition series $(1, 2, 5, 6, 5, 3)$. There are many ways to obtain different Jordan-Hölder filtrations of $M(w)$, but all of these can be found simply using module diagram as demonstrated. Let us just show another example:

$$M(w) \overset{5}{\supset} \left(\begin{smallmatrix} 1 & & \\ & 2 & & 6 \\ & & 3 & \end{smallmatrix} \oplus 5\right) \overset{5}{\supset} \begin{smallmatrix} 1 & & & \\ & 2 & & 6 \\ & & 3 & \end{smallmatrix} \overset{1}{\supset} \begin{smallmatrix} 2 & & 6 \\ & 3 & \end{smallmatrix} \overset{6}{\supset} \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \overset{2}{\supset} 3 \supset 0.$$

This has composition series $(5, 5, 1, 6, 2, 3)$.

Another convenience brought by module diagram is homological calculation. We highlight one here. Recall that a module is *finitely generated* if there is some $n > 0$ such that $A^{\oplus} \twoheadrightarrow M$. Note that the kernel of such a map is usually quite large – many of the direct summands of A will appear. One is often interests in a more optimised version.

Definition 12.7. *A surjective homomorphism $\pi_M : P_M \twoheadrightarrow M$ from a projective A-module P_M is called the projective cover of M if no indecomposable direct summand of P_M is in the kernel of π_M. By abuse of terminology, we often say 'P_M is the projective cover of M' to mean the existence of π_M.*

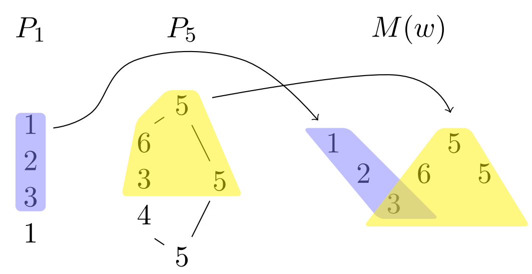
The syzygy of M, denoted by $\Omega(M)$, is the kernel of the projective cover of M.

Finding projective cover is extremely easy with module diagram, namely, the projective module covering $M(w)$ is given by $\bigoplus_x P_x$ where x varies over all peaks of the module diagram of $M(w)$ (counted *with* multiplicity), and the map is given by mapping e_x to the corresponding element in the canonical basis of $M(w)$.

Example 12.8. *We continue with the previous Example 12.5. The module $M(w)$ have projective cover $P_M = P_1 \oplus P_5$ with $e_1 \mapsto v_6$ and $e_5 \mapsto v_1$; or flexing out the full details:*

$$\left\{\begin{array}{rcl} P_1 & \to & M(w) \\ e_1 & \mapsto & v_6 \\ (2|1) & \mapsto & v_5 \\ (4|1) & \mapsto & v_4 \\ (3|1) & \mapsto & v_3; \\ \textit{everything else} & \mapsto & 0, \end{array}\right. \quad \textit{and} \quad \left\{\begin{array}{rcl} P_5 & \to & M(w) \\ (3|6) & \mapsto & v_3 \\ (5|6) & \mapsto & v_2 \\ e_5 & \mapsto & v_1 \\ (5|5)_v & \mapsto & v_0 \\ \textit{everything else} & \mapsto & 0, \end{array}\right.$$

We can visualise this map by highlighting the involved parts in the module diagram.



Then it is easy to see what the kernel is – the parts that are not highlighted, and the part with overlapping highlight, which is the (space spanned by) $(3|1) - (3|6)$ in this example. Now, the module diagram of $\Omega(M(w))$ can be obtained by gluing the unhighlighted parts in P_1, P_5 along the overlapping highlight parts. So in this example we have

$$\Omega(M(w)) = \begin{smallmatrix} & 3 & & \\ 1 & & 4 & \\ & & & 5 \end{smallmatrix}.$$

An interesting example is what is called the *Green's walk around Brauer tree* discovered by J. A. Green in the finite group representation context. In modern setting, this is a special Ω-orbit. We demonstrate an example of this in the following.

Example 12.9. *We continue with the Brauer tree used in Example 12.4. Let us start with the simple S_1 (=module diagram 1). The projective cover is $P_1 = \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \end{smallmatrix}$, with kernel $\operatorname{rad} P_1 = \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}$.*

This diagram has peak 2, so it has projective cover $P_2 = \begin{smallmatrix} 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix}$ with kernel S_2. The next syzygy is then $\operatorname{rad} P_2 = \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$*, and the next is $\begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix}$, etc. The full Ω-orbit is then*

$$1,\ \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix},\ 2,\ \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix},\ \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix},\ 4,\ \begin{smallmatrix} 5 \\ 6 \\ 3 \\ 4 \end{smallmatrix},\ \begin{smallmatrix} 5 \\ 5 \end{smallmatrix},\ \begin{smallmatrix} 6 \\ 3 \\ 4 \\ 5 \end{smallmatrix},\ 6,\ \begin{smallmatrix} 3 \\ 4 \\ 5 \\ 6 \end{smallmatrix},\ \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, 1,\ \ldots$$

Example from group representation theory

Example 12.10. *$A = \Bbbk\mathfrak{S}_3$ over $\Bbbk$ of characteristic 3. As before, we take $\alpha := (1,2,3), \beta := (1,2)$. Let $e_1 := -1+\beta$ and $e_2 := -1-\beta$. Then one can check that $e_i^2 = e_i$ for both $i \in \{1,2\}$ and $e_1e_2 = 0 = e_2e_1$ and $e_1+e_2 = 1$ (using $-2 = 1$ in $\Bbbk$).*

Consider the left ideal (hence, projective module) generated by e_i. This has basis $\{e_i, \alpha e_i, \alpha^2 e_i\}$, for which we will transform to $\{e_i, e_j\alpha e_i, (1+\alpha+\alpha^2)e_i\}$ with $\{i,j\} = \{1,2\}$.

Let $(j|i) := e_j\alpha e_i$, then one can check that

(1) $(j|i)(i|j) = (1+\alpha+\alpha^2)e_j$, *and*

(2) $(j|i)(i|j)(j|i) = 0$.

Hence, let A be the basic Brauer tree algebra associated to the Brauer line with 2 edges and trivial multiplicity, we have an algebra isomorphism

$$\Bbbk\mathfrak{S}_3 \xrightarrow{\sim} A \text{ given by } \begin{cases} e_i & \mapsto & e_i, \\ (j|i) = e_j\alpha e_i & \mapsto & (j|i), \\ (1+\alpha+\alpha^2)e_i & \mapsto & (i|j)(j|i). \end{cases}$$

More generally, we can obtain Brauer line algebra (i.e. Brauer tree algebra associated to a line graph with trivial multiplicity) from the principal block of $\Bbbk\mathfrak{S}_p$ over $\operatorname{char}\Bbbk = p > 0$. For reader with knowledge in symmetric group representation, we know that the simple modules (equivalently, conjugacy classes) are labelled by partitions of n. The edges of the Brauer line corresponds to p-regular *hook type partition* $(p-k, 1^k)$ for $0 \leq k < p$, with the edge labelled by $(p-k, 1^k)$ connected to $(p-(k\pm 1), 1^{k\pm 1})$ (whenever the notation makes sense).

Note that, as we have mentioned before, in general, it is rare to have a block of group algebra to be isomorphic to a (basic) Brauer tree algebra on the spot. In fact, this is already the case for $B_0(\Bbbk\mathfrak{S}_5)$, where the simple module corresponding to $(p-2, 1^2)$ is not of dimension 1. As a final (slightly unrelated) remark, dimension of simple module over a symmetric group algebra is still an open problem in general!