Ex 1. Let A be a K-algebra. We call a K-algebra B a direct factor of A if $A \cong B \times B'$ for some K-algebra B'.

(1) Show that $Z(A) \cong \mathrm{End}_{A\otimes_K A^{\mathrm{op}}}(A)$ as rings.

(2) Show that B is a direct factor of A if, and only if, B is a direct summand of A as $A \otimes_K A^{\mathrm{op}}$-module. (Hint: Idempotent e of a K-algebra Λ gives rise to decomposition $\Lambda e \oplus \Lambda(1-e)$ of left Λ-modules.)

(3) Show that B is a direct factor of KG if, and only if, B is a direct summand of $K(G \times G)$.

Ex 2. For $V, W \in KG\,\mathsf{mod}$, show that the following are KG-modules isomorphisms

$$(1)\ \ (V \otimes_K W)^* \cong V^* \otimes_K W^*, \qquad (2)\ \ V^* \otimes_K W \cong \mathrm{Hom}_K(V, W).$$

Ex 3. Recall that $\mathrm{End}_K(V)$ is a ring where multiplication is composition.

(1) Find the multiplication on $V^* \otimes_K V$ that makes the canonical isomorphism $\mathrm{End}_K(V) \cong V^* \otimes_K V$ as an isomorphism of rings.

(2) Suppose $V \in KG\,\mathsf{mod}$. Show that triv_G is a direct summand of $V^* \otimes_K V$ when $\dim_K M$ is invertible in K. (Hint: fix a basis of V and consider also the dual basis.)

Ex 4. Suppose $H \leq G$ is a subgroup. For any $g \in G$, let ${}^gH := gHg^{-1} = \{{}^gh := ghg^{-1} \mid h \in H\}$, and define for each KH-module W a $K({}^gH)$-module ${}^gW := \{{}^gw \mid w \in W\}$ (${}^gw$ is just a formal symbol) with ${}^gh \cdot {}^gw := {}^g(hw)$. Show that

(1) $\mathrm{Ind}_{{}^gH}^G({}^gW) \cong \mathrm{Ind}_H^G(W)$ as KG-module;

(2) ${}^gV \cong V$ as KG-module for all $V \in KG\,\mathsf{mod}$.

Ex 5. Let $H \leq G$, $V \in KG\,\mathsf{mod}$ and $W \in KH\,\mathsf{mod}$. Show that

(1) $V \otimes_K \mathrm{Ind}_H^G(W) \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes_K W)$;

(2) $\mathrm{Ind}_H^G(W^*) \cong (\mathrm{Ind}_H^G(W))^*$;

(3) use (2) to give an alternative proof of permutation module being self-dual.

Ex 6. Let Ω_r be the set of r-subsets (=subsets of size r) of $\{1, 2, \ldots, n\}$ for an integer $n \geq 1$.

(1) Find a subgroup $H \leq \mathfrak{S}_n$ (and the reason) such that $K\Omega_r \cong \mathrm{Ind}_H^{\mathfrak{S}_n} \mathrm{triv}_H$.

(2) Let π_r be the character of $K\Omega_r$. Find $\pi_r(1)$ for any r, and find $\pi_2(g)$ for all $g \in \mathfrak{S}_4$.

Deadline: 22nd November, 2022