

**Ex 1.** Suppose  $\rho : G \rightarrow \text{GL}(V)$  is a representation. Show that  $\det \rho$  is also a representation.

**Ex 2.** Let  $V$  be the 1-dimensional subspace spanned by  $\sum_{g \in G} g \in KG$ . Show that  $V$  is a  $KG$ -module and that  $\text{triv}_G \cong V$ .

**Ex 3.** Fix any  $n \geq 2$ .

- (i) Find a generator  $v$  such that  $\text{sgn} = Kv$ . (Hint: Modify the generator  $\sum_{g \in G} g$  of the trivial representation.)
- (ii) Show that  $\text{Hom}_{\mathfrak{S}_n}(\text{triv}, \text{sgn}) = 0 = \text{Hom}_{\mathfrak{S}_n}(\text{sgn}, \text{triv})$  when  $\text{char } K = 2$ , otherwise,  $\text{triv} \cong \text{sgn}$ .

**Ex 4.** Classify (with reason) all simple  $\mathbb{C}\mathfrak{S}_3$ -modules (up to isomorphism). (Hint: there are only three of them.)

**Ex 5.** Recall that the center of  $GL_n(K)$  is just  $K^\times \text{Id} = \{\lambda \text{Id} \mid \lambda \in K^\times = K \setminus \{0\}\}$ .

- (i) Classify (with reason) all simple  $\mathbb{C}C_n$ -modules (up to isomorphism) of the cyclic group  $C_n$  of order  $n \geq 2$ . (Hint: To find, notice that  $\mathbb{C}$  has all roots of 1. To show you have all, consider dimensionality and Artin-Wedderburn.)
- (ii) Classify all simple  $\mathbb{C}G$ -modules (up to isom.) of the abelian group  $G = C_m \times C_n$  for  $m, n \geq 2$ .
- (iii) Classify all simple  $\mathbb{C}G$ -modules (up to isom.) of a finite abelian group  $G$  (recall that a finite abelian group is a finite direct product of finite cyclic groups, i.e.  $G = C_{n_1} \times \cdots \times C_{n_r}$ ).

**Ex 6.** Let  $A$  be the algebra of upper triangular  $n \times n$ -matrices:

$$A := \begin{pmatrix} K & K & \cdots & K \\ 0 & K & \cdots & K \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & K \end{pmatrix} = \left\{ (a_{i,j})_{1 \leq i,j \leq n} \mid \begin{array}{l} a_{i,j} \in K \ \forall i,j \\ a_{i,j} = 0 \ \forall i > j \end{array} \right\}$$

For  $1 \leq i \leq j \leq n$ , let  $M_{i,j} \subset K^{\oplus n}$  be the vector space given by column vectors  $v = (v_k)_{1 \leq k \leq n}$  where  $v_k = 0$  for  $k \notin \{i, i+1, \dots, j\}$ .

- (i) Determine which  $M_{i,j}$ 's are simple.
- (ii) Describe the composition series of  $M_{i,j}$ .

Deadline: 4th November, 2022