Summary on τ-tilting theory

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A f.d. algebra over a algebraically closed field k
$X \in$ mod-A
$P \in$ **proj**-A

- X is <u>τ-rigid</u>: $\mathrm{Hom}_A(X, \tau X) = 0$
- (X, P) is <u>τ-rigid pair</u>: (a) X is τ-rigid, (b) $\mathrm{Hom}_A(P, X) = 0$
- X is <u>τ-tilting</u>: (a) X is τ-rigid, (b) $|X| = |A|$
- X is <u>support τ-tilting</u>: $\exists e = e^2 \in A$ s.t. X is τ-tilting A/AeA-module
- (X, P) is <u>support τ-tilting pair</u>: (a) (X, P) is τ-rigid pair, (b) $|X| + |P| = |A|$
- (X, P) support τ-tilting pair $\Leftrightarrow$ X τ-tilting A/AeA-module with $P = eA$.

Theorem[Adachi-Iyama-Reiten]:

$$\begin{array}{rcccc} \text{f-tors}(A) & \leftrightarrow & \text{s}\tau\text{-tilt}(A) & \leftrightarrow & \text{2-silt}(A) \\ \mathrm{Fac}(X) = {}^{\perp}(\tau X) \cap P^{\perp} & \leftarrow\!\shortmid & (X, P) & \mapsto & P_X \oplus P[1] \\ \mathcal{C} & \mapsto & P(\mathcal{C}) & & \end{array} \tag{0.1}$$

- P_X is projective presentation of X, concentrated in degree $-1, 0$, say $P^{-1} \to P^0$
- $\mathrm{Fac}(X)$ smallest full subcat of mod-A containing X, closed under quotients and extensions
- $P(\mathcal{C})$ = (basic) direct sum of isoclass representative Ext-projective objects
- f-tors(A) collection of functorially finite torsion classes of A

Note: $X \in \mathcal{C}$ is Ext-projective object in $\mathcal{C}$ if $\mathrm{Ext}^1_A(X, \mathcal{C}) = 0$

Theorem[Adachi-Iyama-Reiten]: Above bijection restricts to:

$$\begin{array}{rcl} {}^{\perp}(\tau X) \cap P^{\perp} & \leftrightarrow & (X, P) \\ \text{f-tors}(A) & \leftrightarrow & \text{s}\tau\text{-tilt}(A) \\ \cup & \leftrightarrow & \cup \\ \text{(s=sincere)}\ \text{sf-tors}(A) & \leftrightarrow & \tau\text{-tilt}(A) \\ \cup & \leftrightarrow & \cup \\ \text{(f=faithful)}\ \text{ff-tors}(A) & \leftrightarrow & \text{tilt}(A) \\ {}^{\perp}(\tau X) & \leftrightarrow & (X, 0) \end{array} \tag{0.2}$$