Abstract

Modular representation theory (i.e. representation of finite groups of field of characteristic p) despite of its failure to produce theorem as nice as Maschke's, is a great tool in the study of group. One of the reason why is that we are able to discover some nice relation between the representation (kG-modules) of the group and the representation of its p-(local) subgroup. The two important ones are Green's correspondence and Brauer's correspondence.

In this talk, we will introduce this famous correspondence discovered by Richard Brauer in the 1950s about the correspondence between blocks of a group and blocks of some of its subgroups.

Prerequisites includes block of group algebra, vertex of an indecomposable kG-module and defect group associated to a block and Green's correspondence. These will be covered in the talk by Gregor.

1 Introduction

Gregor has let us seen what nice relation we have used on studying the indecomposable kG-module. Now we are interested in another object, the blocks of the group algebra.

Setup:
k is the (residue) field of characteristic p which also is the splitting field for G ($\Rightarrow$ vertex of indecomposable modules are p-subgroups)
Fix a block B of the group algebra kG, recall that B is just an indecomposable $k(G\times G)$-summand of kG, i.e. B is one of the B_i in the following $k(G \times G)$-decomposition of kG

$$kG = B_0 \oplus B_1 \oplus \cdots \oplus B_k$$

via the action:

$$(g, h)a = gah^{-1} \quad \forall a \in kG$$

Under this action,

$$kG \cong k_{\Delta(G)} \uparrow^{G\times G}$$

So the indecomposable summands are $\Delta(G)$-projective, and has vertex of form $\Delta(D)$, where D is a p-subgroup of G, or order p^d. We call this D the defect group of B, and say B has defect d.

In particular, a defect zero block has all its modules projective (and injective)

To better understand the structure of kG, in modular representation theory, we usually do this by investigating the structure of kH for some H. Usually this H is a p-local subgroup, i.e. $H = N_G(P)$ for some P a p-subgroup of G, as this contains most information about the relation between G and P.

From the Green's correspondence, and the above view of block as indecomposable module, we can ask ourselves, is there a correspondence between the blocks of G and blocks of H, with the same vertex (i.e. defect group D)? The answer is certain and this is the Brauer correspondence. On a historical notes, Brauer correspondence is obtained before

Green's one, but Green's correspondence is a lot easier to understand and can use it to get the Brauer's correspondence in the case $H \geq N_G(P)$ with minimal effort (see later). This essentially sees the shift of the study of character and block theory to the study of module theory for the representation theorists in the 1970s.

2 The Correspondence And The First Main Theorem

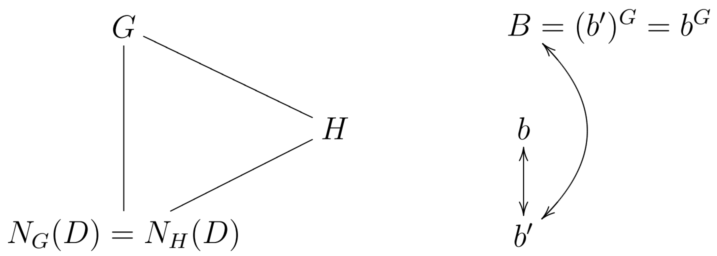
There are three main theorem related to this correspondence, termed as the Brauer First, Second, Third Main Theorem. I will briefly talk about each of them here and their application, no proof will be given, the audience can refer to [Bn],[Al] and [Na] for more detailed description and for proofs. Another point to mention is that Brauer correspondence works on the k-representation (modules) but not necessary on $\mathcal{O}$-representation (modules).

Theorem 2.1 (Brauer's First Main Theorem)
Let D be a p-subgroup of G, define the *Brauer map* (or *Brauer homomorphism*) as the well-defined k-algebra homomorphism

$$\begin{aligned} \mathrm{Br}_D : Z(kG) &\to Z(kC_G(D)) \\ \sum_{g \in G} a_g g &\mapsto \sum_{x \in C_G(D)} a_x x \end{aligned}$$

This map sets up a one-to-one correspondence between the idempotents associated to kG-block with defect group D and idempotents associated to $kN_G(D)$-block with defect group D.

Remark. This correspondence extends to $H \geq N_G(D)$, since $N_H(D) = H \cap N_G(D) = N_G(D)$. So given an H-block, there is a unique $N_H(D)$-block, which is a $N_G(D)$-block, and this gives a unique G-block



Let b be a kH-block, we denote b^G to be the corresponding kG-block under the Brauer correspondence.

When H goes below $N_G(D)$, there is no unique correspondence. However, we still want to generalise our result as much as possible, in the sense that we are happy even if the Brauer map is just a surjection. In fact, this is the case:

Theorem 2.2
Let H be a subgroup of G containing $DC_G(D)$, then the Brauer map defines a surjection from the set of kG-block with defect groups containing D to the set of kH-blocks with

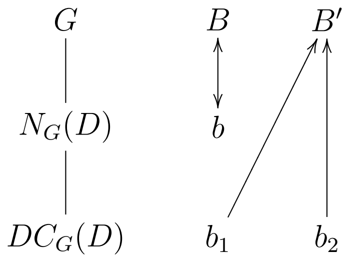
defect group containing D
Moreover, if b_1, b_2 are the kH-blocks in the former set, then $b_1^G = b_2^G$, if and only if, $b_1 =_G b_2$

Given this, now the notation b^G makes sense for b a block of $H \geq DC_G(D)$, we say b^G is defined in this case, and somehow literature will also call this imperfect relation of b and b^G as Brauer correspondence of each other. It should also be noted that the correct way of introducing the Brauer correspondence is to first show that b^G is defined for $DC_G(D) \leq H \leq N_G(D)$, then prove the First Main Theorem.

The above is just one way of setting up the correspondence. As we have seen before, from the application of Green's correspondence, we can introduce this correspondence without using the Brauer map. This way of introducing Brauer's correspondence is done in detail in [Al]. Here we only list this results of Alperin's.

Lemma 2.3 (Alperin)
For $DC_G(D) \leq H \leq G$ and b an H-block with defect group D. Then b^G is defined and is a unique block B of kG such that $b | B \downarrow_{H \times H}$ regarded as $k(H \times H)$-module

To summarise, we have



3 The Third Main Theorem

This general form tells us that correspondence exists, but given a block, we do not exactly know what the corresponding block is. Now the Brauer's Third Main Theorem comes to rescue:

Theorem 3.1 (Brauer Third Main Theorem)
Let H be a subgroup of G containing $DC_G(D)$, and $B_0(G)$ denote the *principal block* of kG, i.e. the block which the trivial module k lies. Then $b = B_0(H)$ (principal kH-block), if and only if, $b^G = B_0(G)$

Principal block is usually the block with the most complex structure in the group algebra (which means it contains more information), because it has the largest (full) defect group (a Sylow p-subgroup), which effectively means there may be modules which are 'furtherest' away from being projective, in the sense of relative projectivity.

On the other hand, The Third Main Theorem says that principal block is actually the easiest to work with because, we can study the principal block of kH, rather than the more complicated kG.

4 Second Main Theorem

The next thing we are interested in is, what criteria will be sufficient to help us determine whether two blocks corresponds under the Brauer map. This is what the Second Main Theorem tells us. Instead of the original version by Brauer, which uses generalised decomposition number, we give the modular version of it, originated from Nagao.

Theorem 4.1 (Second Main Theorem, Nagao's modular version)
Let D be a p-subgroup of G. Let M be an indecomposable kG-module lying in B, block of kG.
Let N be an indecomposable kH-module lying in b, block of kH, with H containing $C_G(D)$ and vertex of N is D.
If N is a direct summand of $M \downarrow_H$, then $b^G = B$

The Second Main Theorem gives another connection of Brauer's and Green's correspondence as follows:

Corollary 4.2
Let M be indecomposable kG-module lying in kG-block B with vertex D
Consider the map f as Green's correspondence depends on G, $H = N_G(D)$, $P = D$ (c.f. Green's correspondence).
If $f(M)$ lies in kH-block b, then $b^G = B$

The following is also a corollary of the Second Main Theorem, which is an interesting result about indecomposable modules lying in B with defect group D

Corollary 4.3
If B is a kG-block with defect group D, then there is an indecomposable kG-block in B with vertex being D

The interested reader should note that the Theorem on blocks of defect zero is an application of the Brauer's three main theorems.

We also give this relation between the Green's correspondence and the Brauer's correspondence:

Proposition 4.4 (Alperin)
Let H be a subgroup of G containing $N_G(D)$; M be indecomposable kG-module and N be indecomposable kH-module.
Let B be a kG-block with defect D and b be kH-block with defect D such that B is the Brauer correspondent of b. Then

$$\begin{aligned} M \text{ lies in } B \quad &\Leftrightarrow \quad fM \text{ lies in } b \\ gN \text{ lies in } B \quad &\Leftrightarrow \quad N \text{ lies in } b \end{aligned}$$

where f, g are the Green correspondence depending on G, H, D

5 Example

5.1 Correspondence of A_5

$|A_5| = 2^3 \cdot 3 \cdot 5$
Character table of A_5:

ccl	1	(12)(34)	(123)	(12345)	(13452)
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	-1
χ_5	5	-1	1	0	0

To see the blocks of kG, we need the central character, we use the formula

$$\omega_i(\mathcal{C}) = \frac{|\mathcal{C}|\chi_i(g)}{\deg \chi_i}$$

(where g is the representative of $\mathcal{C}$) and the following fact:

Lemma 5.1
Central character ω_i, ω_j lies in the same p-block $\Leftrightarrow$ $\omega_i(\mathcal{C}) \equiv \omega_j(\mathcal{C}) \mod p \;\; \forall \mathcal{C}$

And we have the following table for central character:

ccl	1	(12)(34)	(123)	(12345)	(13452)	mod 5
ω_1	1	15	20	12	12	B_0
ω_2	1	-5	0	4α	4β	B_0
ω_3	1	-5	0	4β	4α	B_0
ω_4	1	0	5	-3	-3	B_0
ω_5	1	-3	4	0	0	B_1

We can see the defect zero block using this criteria:

Lemma 5.2
$|G| = p^a b$ with $(p, b) = 1$, every central character of degree divisible by p^a is contained in a block of defect zero.
If central character has degree divides by a power of p, $p^b \neq p^a$, then it is contained in a block of defect $a - b$

Therefore, B_0 (in the above table) is the principal block of A_5, and B_1 is a block of defect zero. So, by the Brauer correspondence and the Third Main Theorem, we have an uninteresting correspondence between the principal block B_0 and the principal block of $N_{A_5}(C_5)$. (Note C_5 is the only (Sylow) p-subgroup of A_5) We can quickly verify this. $N_{A_5}(C_5) = D_{10}$. :

Character table of D_{10}

ccl	1	(25)(34)	(12345)	(13542)
ψ_1	1	1	1	1
ψ_2	1	-1	1	1
ψ_3	2	0	α	β
ψ_4	2	0	β	α

Central character table:

ccl	1	(25)(34)	(12345)	(13542)	mod 5
ω_1'	1	5	2	2	b_0
ω_2'	1	-5	2	2	b_0
ω_3'	1	0	α	β	b_1
ω_4'	1	0	β	α	b_1

Now we look at less trivial example: S_7 mod 3

Character table of S_7 (using 𝔸𝕋𝕃𝔸𝕊 [Con] notation)
(up to multiplication by χ_ϵ)

ccl	1A	2A	3A	3B	4A	5A	6A	7A	2B	2C	4B	6B	6C	10A	12A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_ϵ	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
χ_6	6	2	3	0	0	1	-1	-1	4	0	2	1	0	-1	-1
χ_{10+10}	20	-4	2	2	0	0	2	-1	0	0	0	0	0	0	0
χ_{14a}	14	2	2	-1	0	-1	2	0	6	2	0	0	-1	1	0
χ_{14b}	14	2	-1	2	0	-1	-1	0	4	0	-2	1	0	-1	1
χ_{15}	15	-1	3	0	-1	0	-1	1	5	-3	1	-1	0	0	-1
χ_{21}	21	1	-3	0	-1	1	1	0	1	-3	-1	1	0	1	-1
χ_{35}	35	-1	-1	-1	1	0	-1	0	5	1	-1	-1	1	0	-1

(Reduced) Central character table mod 3:

Block																Contain
B_0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	$\left\{\begin{matrix} 1, 20, 14a, \\ 14b, 35 \end{matrix}\right\} \otimes \{1, \epsilon\}$
B_1	1	2	2	0	0	0	1	0	2	0	1	1	0	0	2	$\{6, 15 \otimes \epsilon, 21 \otimes \epsilon\}$
B_2	1	2	2	0	0	0	1	0	1	0	2	2	0	0	1	$\{6 \otimes \epsilon, 15, 21\}$

Principal block B_0 has the full defect group $C_3 \times C_3$
B_1, B_2 have defect group C_3 (since they have degree divisible by 3)

So by Third Main Theorem, we know immediately that B_0 correspond to the principal block of $N_{S_7}(C_3 \times C_3)$, and they are the only block with full defect in both groups.

By First Main Theorem, as $N_{S_7}(C_3) \geq N_{S_7}(C_3 \times C_3)$, we know $N_{S_7}(C_3)$ will have 3 blocks:

$N_{S_7}(C_3)$-block	S_7-block	Defect Group
Principal b_0	Principal B_0	$C_3 \times C_3$
b_1	B_1	C_3
b_2	B_2	C_3

Let the smaller defect group $C_3 = \langle(123)\rangle$, then its normaliser permutes $\{1,2,3\}$ on one side and permutes $\{4,5,6,7\}$ on other side, so $N_{S_7}(C_3) = S_3 \times S_4$. Now we can tensor product the character table of S_3 and S_4 to get the character table of $S_3 \times S_7$. We only quote the blocks we obtained via method similar to above:

	1A	2B1	2A1	3A1	4B	2B2	2A2	2C	6B1	4A	3A2	6B2	6A	3B	12A
b_0	1	0	0	2	0	0	0	0	0	0	2	0	0	1	0
b_x	1	2	2	0	1	0	0	0	0	0	2	1	1	0	2
b_y	1	1	2	0	2	0	0	0	0	0	2	2	1	0	1

We do not know yet whether b_x (or b_y) is b_1 or b_2. So rearrange the ccl to compatible with previous form:

	1A	2A1	2A2	3A1	3A2	3B	4A	6A	2B1	2B2	2C	4B	6B1	6B2	12A
b_0	1	0	0	2	2	1	0	0	0	0	0	0	0	0	0
b_a	1	2	0	0	2	0	0	1	2	0	0	1	0	1	2
b_b	1	2	0	0	2	0	0	1	1	0	0	2	0	2	1

Now by comparing with previous block table of S_7, we know b_x is b_1 and b_y is b_2.

The reader should note that these example are worked out mainly using character theory. So we see the advantage of modular approach, which helps us to prove many more advanced results, but the down side of it is that it usually does not suffice for us to actually compute the blocks and the modules.

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