Ext-algebra of Standard Modules of Rhombal Algebras

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Quasi-hereditary algebras with duality

Let (A, χ) be quasi-hereditary (qh) algebra

Definition (Irving/C.Xi)

We say A is qh with duality (or BGG-algbera) if there is a contravariant functor $\delta : A\text{-mod} \to \text{mod-}A$ which fixes simples.

Equivalently, $\delta P(x) = I(x)$
Equivalently, $\delta \Delta(x) = \nabla(x)$

Example: δ induced by an involutary anti-automorphism of the algebra.

Standard Koszul algebras

Definition (Ágoston-Dlab-Lukács)

A qh algebra is called standard Koszul if all standard module $\Delta(x)$ admits linear projective resolution $\widetilde{\Delta}(x)$:

$$\cdots \to \widetilde{\Delta}^i(x)\langle i\rangle \to \cdots \widetilde{\Delta}^1(x)\langle 1\rangle \to \widetilde{\Delta}^0(x) = P(x) \twoheadrightarrow \Delta(x)$$

and all costandard module $\nabla(x)$ admits linear injective coresolution.

1. If A has duality, only need to check condition on Δ
2. Standard Koszul $\Rightarrow$ Koszul (i.e. there is a grading on A with degree 0 part of A admiting linear projective resolution)

Ext-algebras

A qh $\Rightarrow$ 6 families of special modules whose indecomposables are indexed by χ:

- projectives P
- injectives I
- simples L
- standard Δ
- costandard ∇
- (characteristic) tilting modules T

Let $X \in \{L, P, I, \Delta, \nabla, T\}$
Question: How does A relate to Ext-algebra of X?

Ext-algebras

From now on, assume A standard Koszul
For convenience: $A^X := \mathrm{Ext}_A(X,X)^{\mathrm{op}}$
Recall: any A is Morita to a basic algebra given by quivers and relations kQ/I.

- $A^P = \mathrm{End}_A(P)^{\mathrm{op}} = kQ/I = \mathrm{End}_A(I)^{\mathrm{op}} = A^I$
- $A^T = \mathrm{End}_A(T)^{\mathrm{op}} =$ Ringel dual of A
- $A^L = A^!$ Koszul dual of A

In these cases, A^X is also qh with respect to $(\chi, \leq^{\mathrm{op}})$
Quiver of A^X is the opposite quiver of A
Moreover, A^X is derived equivalent to A

Ext-algebra of standards

In the case of $A^{\Delta} := \mathrm{Ext}_A(\Delta, \Delta)^{\mathrm{op}}$, the algebra and homological structure is not so clear in general.

Works appeared so far:

1. (general theory)
   - Y.Drozd-V.Mazorchuk (on Koszulity)
   - D.Madsen (on derived equivalence)
   - L.P.Li (generalising to...)
2. (particular examples)
   - V.Miemietz-W.Turner [MT] ($GL_2(\overline{\mathbb{F}_p})$)
   - A.Klamt-C.Stroppel [KS] (some generalised Khonvanov arc algebras)

The zigzag algebra

There is one (family of) algebra A_n which satisfies (H).

(Some) Concrete version of "the algebra A_n":

1. $A_p =$ weight 1 block of Schur algebra.
2. first case in [KS]: $A_n =$ basic algebra K_1^n of principal block of parabolic category $\mathcal{O}$ of $\mathfrak{gl}_{n+1}$ with Levi $\mathfrak{gl}_n \oplus \mathfrak{gl}_1$

Second case of [KS]: $\mathfrak{gl}_{n+2}$ with Levi $\mathfrak{gl}_2 \oplus \mathfrak{gl}_n$. Call this algebra K_2^n. So "K_2^n generalises A_n".

Rhombal also generalises zigzag!

Let B be a weight 2 block of symmetric group and $\overline{B}$ be weight 2 block of Schur algebras, then there is a rhombal algebra U_χ such that

$$\begin{aligned} eU_\chi e &\cong e'Be' \\ f(U_\chi/U_\chi gU_\chi)f &\cong f'(\overline{B}/\overline{B}g'\overline{B})f' \end{aligned}$$

where the unexplained symbols are some sums of primitive idempotents.

Some (unrelated) remarks

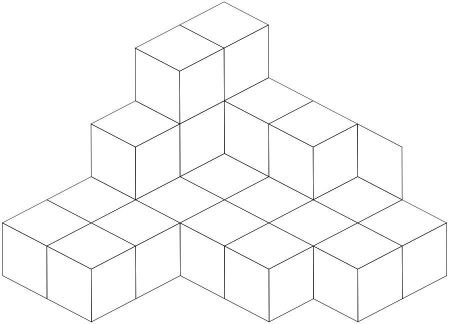
1. Conjectured relation of higher weight blocks and cubist algebra. (True for the Rouquier blocks.)
2. Rhombal algebra can relates to K_2^n via truncation, but this truncation satisfy (H).
3. The truncations of principal and RoCK block of weight w, satisfy (H); but not for other weight 2 blocks in general.

Construction in brief

1. You start off living in $\mathbb{R}^3$ ($r = 3$).
2. You pick a subset of $\chi \subset \mathbb{Z}^3$ such that $\chi \cap \{x + (1,1,1) | x \in \chi\} = \emptyset$.
3. Connect vertices with distance 1 by an edge.
4. This gives a rhombic tiling for the wallpaper (2-dimensional space) in your living room via suitable projection.
5. Construct quiver: with vertices χ, and place a pair of arrows, one in each direction, on each edge.
6. And then you impose some relations...

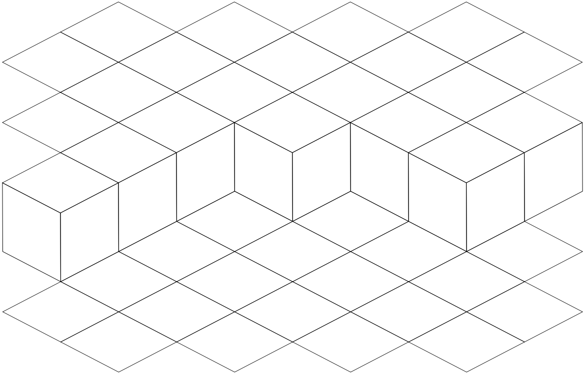
Example

(Part of) An example:



Weight 2 block example

Weight 2 block example



Idea

The combinatorics of cubist (rhombal) algebras give the following:

Theorem (Chuang-Turner)

The infinite dimensional algebra U_χ is symmetric, standard Koszul, with duality.

To calculate A^Δ:

$\rightsquigarrow$ calculate its basis: $\mathrm{Ext}_A(\Delta(x), \Delta(y)) = e_x A^\Delta e_y$

$\rightsquigarrow$ look at $\mathrm{Hom}_U(\widetilde{\Delta}(x), \Delta(y))$

$\rightsquigarrow$ this can boils down to combinatorics of cubists set (the rhombic tiling)

Homological structure

The partial order on $\mathbb{Z}^3 \supset \chi \quad \rightsquigarrow \quad$ bijection $\lambda : \chi \to \{\text{rhombi}\}$

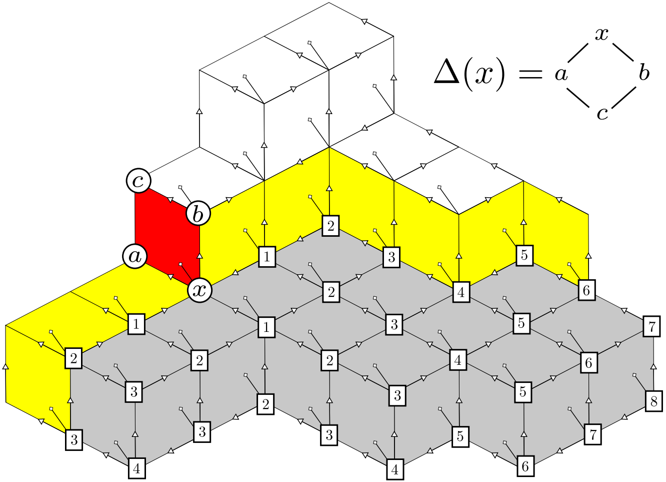
This gives the standard modules of U_χ:

$$[\Delta(x) : L(y)] = \begin{cases} q^{d(x,y)} & y \leq d(x,y) \leq w = 2 \\ 0 & \text{otherwise} \end{cases}$$

There is also a map $\mu : \chi \to \mathbb{R}^3$ representing $\widetilde{\Delta}(x)$ of $\Delta(x)$ i.e.

$$P(y)\langle i \rangle \text{ a summand of } \widetilde{\Delta}^j(x) \quad \Leftrightarrow \quad i = j = d(x,y) \text{ and } y \in \mu x$$

Example of visualising $\lambda(x)$ and $\mu(x)$



Combinatorial non-vanishing condition

Theorem

For rhombal algebra U, and $x < y \in \chi$
$\mathrm{Ext}_A(\Delta(x), \Delta(y)) \neq 0$ *precisely when* $\lambda y \cap \mu x \neq \emptyset$ *and for all* $z \in \lambda y \cap \mu x$, $d(x, y) = d(x, z) + d(z, y)$.

Remark: Also true in cubist algebra, $r = w + 1$, when $\lambda y \subset \mu x$.

The proof of the theorem is given by looking at the decomposition of the Ext-group into the graded ext-groups

Graded decomposition

If $\mathrm{Ext}^*_U(\Delta(x), \Delta(y))$ non-zero, then it has the following decomposition:

$$\bigoplus_{i=i_0}^{i_0+s} \mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle i - \ (d-i)\ \rangle)$$

The basis of each graded ext-group is indexed by $z \in \lambda y \cap \mu x$ which are of distance i from x.

Restating the graded decomposition

Another way to look at the graded decomposition:

$$\mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle j\rangle) \neq 0 \;\; \Rightarrow \;\; 2i - j = d(x, y)$$

Recall that for Koszul algebras:

$$\mathrm{ext}^i_A(L(x), L(y)\langle j\rangle) \neq 0 \;\; \Rightarrow \;\; i - j = 0$$

Quiver description

An application of the non-vanishing condition and the description of the graded decomposition is:

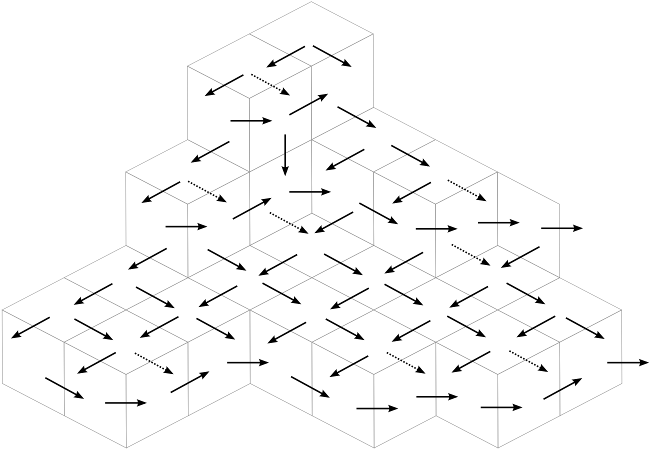
Theorem

U be a rhombal algebra. Then there is a combinatorial description of the quiver of U^{Δ}.

For rhombal algebra which relates with block of symmetric group/Schur algebra, we also calculated all the relations.

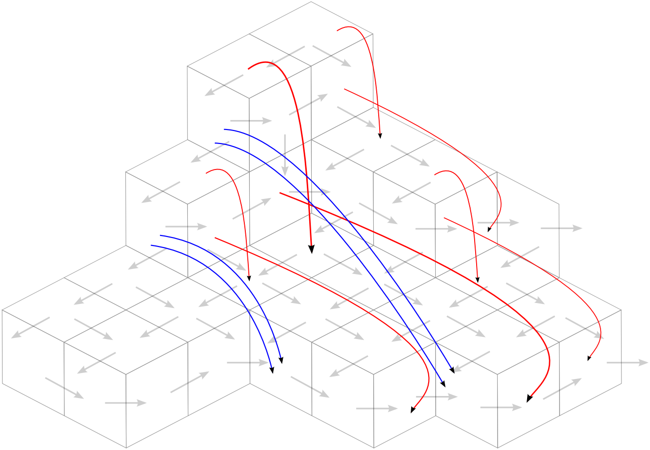
Example

Example continued:



Example

Example continued:



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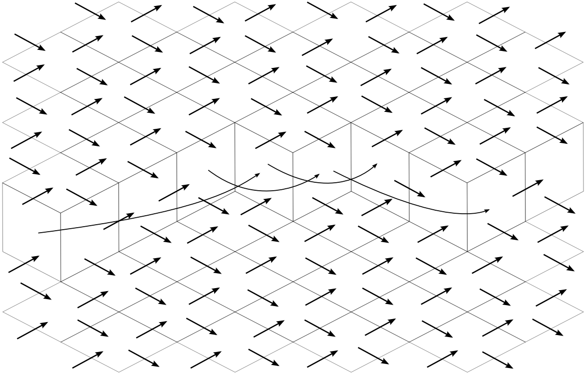

What's next?

Really not much insight to understanding A^{Δ} in general:

- Structure of A^{Δ} in general: Quiver of A^{Δ}?
- Homological properties: Formality and derived equivalence?
- How much does this help to calculate B^{Δ} for B a weight 2 block of symmetric group/Schur algebra?