Quivers, their representations and applications

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Outline

First Definitions

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4. A morphism between representations of Q is a collection of linear maps $\{f_i\}_{i \in Q_0}$ defined over the vector spaces, and commute with the linear maps representing the arrows.
5. Denote the category of k-representations of Q as $\mathrm{Rep}_k(Q)$

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The following facts are immediate from definitions:

- If there is no oriented cycle and Q finite, then kQ is f.d. k-algebra
- If Q finite, the kQ is f.g. k-algebra

Question: What is a kQ-module M?

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Answer This is just the same thing as a representation of Q given by $(\bigoplus V_i, \{\theta_\alpha\}_{\alpha\in Q_1})$:

$M = \bigoplus V_i$ (set/v.s.-wise), then $\alpha \in Q_1$ acts on M by applying θ_α to $\bigoplus V_i$. More specifically:

$$\bigoplus V_i \twoheadrightarrow V_{s(\alpha)} \xrightarrow{\theta_\alpha} V_{t(\alpha)} \hookrightarrow \bigoplus V_i$$

In another words, we have $\mathrm{Mod}(kQ) \cong \mathrm{Rep}_k(Q)$

Examples

One vertex with a loop:

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One vertex with a loop:

One vertex with two loops:

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Kronecker algebra:

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Answer: Reverse process is taking Ext^1-quiver. But there are also other way to construct (useful) quivers out of A (e.g. Auslander-Reiten quiver, see later). The answer for second questions is no for any A (but almost) and yes for any Q.

Ext^1-quiver

Definition

The Ext^1-quiver of an algebra A has vertices $\leftrightarrow$ isoclasses of simples ($\leftrightarrow$ isoclasses of projective indecomposables). The number of arrows from vertex representing simple S_i to the vertex representing simple S_j is equal to
$\dim_k \mathrm{Ext}^1(S_i, S_j) = \dim_k \mathrm{Hom}_A(P_j, J(P_i)/J^2(P_i))$

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$$\mathrm{Ext}^1(S_i, S_j) \leftrightarrow \{\text{equiv class of } 0 \to S_j \to X \to S_i \to 0\}$$

Note that equiv. ses $\Rightarrow X \cong X'$, but not the converse. So we count dimension instead of the size of $\mathrm{Ext}^1(S_i, S_j)$

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Note: X has composition factors S_j (at bottom) and S_i (on top)

Hereditary algebra

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- gl.dim(A)=1
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For Q finite without oriented cycle, this is always the case: Let $\{\alpha_1, \ldots, \alpha_t\}$ be arrows starting from i. Then

$$J(kQe_i) = J(kQ)e_i = \bigoplus_{k=1}^{t} kQe_{t(\alpha_k)}\alpha_k \cong \bigoplus kQe_{t(\alpha_k)}$$

Gabriel's Theorem 1

Lemma

Let A be finite dimensional hereditary basic k-algebra with k algebraically closed. Then $A \cong kQ$ where Q is Ext^1-quiver of A.

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Theorem (Gabriel)

A is a finite dimensional k-algebra with k algebraically closed. Then $A \cong kQ/I$ where Q is Ext^1-quiver of A and I generated by some paths of length ≥ 2

Examples

- Nakayama algebra, Uniserial algebra (such as $k[X]/(X^n)$, Brauer star algebra)
- (char k=2) Group algebra of V_4.
  $kV_4 = \langle X, Y \rangle / \langle X^2, Y^2, XY - YX \rangle$
  This somehow links to Kronecker algebra [Benson vol.1], and thus somehow link to coherent sheaves of projective space (see later)

Computation of Lowey structure using quivers

Easy computation of top/soc/rad of representation of Q:

Lemma

$M = (\bigoplus V_i, \phi_\alpha)$ *representation of* kQ/I

- M *is semisimple* $\Leftrightarrow \phi_\alpha = 0\ \forall\alpha$
- *Let* $W_i = V_i$ *for* i *sink, otherwise* $W_i = \bigcap_{\alpha:i\to j} \ker(\phi_\alpha)$
  soc$(M) = (\bigoplus_i W_i, 0)$
- *Let* $J_i = \sum_{\alpha:j\to i}$ *Im*(ϕ_α). *rad*$(M) = (\bigoplus_i J_i, \phi_\alpha|_{J_{s(\alpha)}})$
- *Let* $L_i = V_i$ *for* i *source, otherwise* $L_i = \sum_{\alpha:j\to i}$ *coker*(ϕ_α).
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- *Let $L_i = V_i$ for i source, otherwise $L_i = \sum_{\alpha:j\to i}$ coker(ϕ_α). top$(M) = (\bigoplus_i J_i, 0)$*

(Exercise!?) Find similar easy computation for proj and inj.
Examples will be given if time allows

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More is true: [Gabriel, Dlab-Ringel] tame representation type, if and only if, Euclidean diagrams

Auslander-Reiten quiver

Ext^1-quiver is only one way of getting information about the algebra A using quiver. The information that it obtains is usually about extensions and composition series/factors and such. Another useful/popular quiver obtained from A is the Auslander-Reiten quiver (AR-quiver), which we will denote as Γ_A.

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Irreducible morphism

Definition

Let M, N be indecomposable A-module. $f : M \to N$ is an irreducible morphism if

- f is not an isomorphism
- if f factors through, say $f = f_2 f_1$, then EITHER f_1 has right inverse OR f_2 has left inverse

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AR-quiver (cont.)

Definition

The Auslander-Reiten quiver Γ_A of k-algebra A is quiver with vertex correspond to isoclass of indecomposables.
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The AR-quiver should actually be defined using a type of ses called almost split sequences, which are defined in terms of (left/right) almost split morphisms. These definitions are highly non-trivial but Auslander and Reiten showed that almost split sequences ALWAYS exists for non-proj modules and non-inj modules, and are uniquely determined by starting/ending term of the ses. They also showed irreducible morphisms are left or right (or both) almost split morphisms.

An almost split sequence ending at M can be written as:

$$0 \to \mathsf{DTr}(M) \xrightarrow{(f_1,\dots,f_n)^T} X_1 \oplus \cdots \oplus X_n \xrightarrow{(g_1,\dots,g_n)} M \to 0$$

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For symmetric algebra (e.g. group algebra), DTr $\cong \Omega^2$ (square of syzygy/Heller fucntor) and induces stable self-equivalence.
In general, calculating DTr(M) for arbitrary M can be done (provided you know the Lowey structure of the projective indecomposables and M) but might take some time...

There are yet two more ways to realise Γ_A from A. One is via functor category, which we will skip; another is using Auslander algebra of A, denote as $\Xi_A := \mathrm{End}_A(\bigoplus_{\mathrm{ind}} M)$.

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Theorem

Let A be f.d. basic k-algebra with k algebraically closed. Then the Ext-quiver of Auslander algebra Ξ_A, is the same as Γ_A

Path algebra of $Q = A_4$

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The convention is to put DTr(M) on the left of M, DTr$^2(M)$ on the left of DTr(M) and so on. So given algebra A, we have a rather "rigid" graph Γ_A, the reverse process is to start with a graph called stable translation quiver, this is studied by Gabriel, Riedtmann, Ringel etc.

For naive example like this, the following facts are usually used for computing AR-translates:

- $f : \text{rad}(P) \to P$ (resp. $f : I \to I/\text{soc}(I)$) is irreducible for proj. ind. P (resp. inj. ind I)
- S simple proj. non-inj. (resp. inj. non-proj.) If $f : S \to M$ (resp. $f : M \to S$) irreducible, then M projective (resp. injective)
- P non-simple ind. proj-inj. then the following is almost split seq.:

$$0 \to \text{rad}(P) \to \text{rad}(P)/\text{soc}(P) \oplus P \to P/\text{soc}(P) \to 0$$

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Sometimes (but I wonder how often), none of this are needed to compute Γ_A, see later example on Brauer tree algebra.

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For infinite representation type algebra A, its AR-quiver Γ_A are often splits into connected components (and usually each of them are also of infinite size). The components are classified as preprojective, regular, preinjective. For Kronecker algebra, this looks like this:...

Kronecker algebra (cont.)

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Happel showed that the derived category of hereditary category $\mathcal{C}$ is $\bigvee_{i\in\mathbb{Z}} \mathcal{C}[i]$.Further, we can define almost split sequences on derived categories. With some more technologies...we can draw out the AR-quiver of $D^b(\mathcal{C})$.

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Using extensive knowledge about $\mathrm{Coh}(\mathbb{P}^1)$. We can see that $D^b(A\text{-mod}) \cong D^b(\mathrm{Coh}(\mathbb{P}^1))$.We will skip about this business of representation type and derived equivalence of hereditary categories.

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We now find the (stable) AR-quiver of Brauer tree algebra.

Finally...

Theorem (Gabriel-Riedtmann, Rickard)

Two Brauer tree algebras are stably equivalent if and only if they have the same number of edges and same multiplicity on the exceptional vertex. In particular, the stable AR-quiver of Brauer tree algebra with e edges and multiplicity m is the stable tube $(\mathbb{Z}/e)A_{me}$

One of the application of almost split sequences (and Auslander-Reiten theory) is it provides a much shortened proof for "block with cyclic defect = Brauer tree algebra" [Benson vol.1].

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One of the application of almost split sequences (and Auslander-Reiten theory) is it provides a much shortened proof for "block with cyclic defect = Brauer tree algebra" [Benson vol.1].It also inspired Erdmann a way to study the tame blocks of group algebras.