Highest weight categories and quasi-hereditary algebras

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February 21, 2012

1 Introduction

Throughout the seminars, k will denote an algebraically closed field unless otherwise specified. All finite dimensional k-algebra will contain a unit 1. Modules are assumed to be left modules. A-mod denote the category of finitely generated modules of A, which is the same as the category of finite dimensional modules of A if A is finite dimensional. A-Mod denote the category of all A-module. The derived category of an abelian category $\mathcal{A}$ is denoted as $\mathbf{D}(\mathcal{A})$.

1.1 Equivalence notion

First of all, we should underline that highest weight categories (HWC) and quasi-hereditary (qh) algebras are (almost) the same thing. If your favourite highest weight category $\mathcal{C}$ has only finite many isoclass and has enough projectives (resp. injectives), indexed the simple isoclasses by $\lambda \in \Lambda$ say, then take a representative $P(\lambda)$ (resp. $I(\lambda)$) from each isoclass of projective (resp. injective) indecomposable objects; then $A = \mathrm{End}_{\mathcal{C}}(\bigoplus_{\lambda\in\Lambda} P(\lambda))$ (resp. $\mathrm{End}_{\mathcal{C}}(\bigoplus_{\lambda\in\Lambda} I(\lambda))$) is a quasi-hereditary algebra, and there is an equivalence of categories $\mathcal{C} \cong A$-mod. Alternatively, we can take the endomorphism ring of injective hulls of the simples, especially if $\mathcal{C}$ does not have enough projective but has enough injectives. Conversely, the module category of a finite dimensional qh algebra is automatically a HWC, which we will see later.

1.2 Motivation by examples

Many interesting algebras and categories appears to possess what we call the "highest weight theory", which should motivate us to study HWC and qh algebras.

(1) (The *regular blocks* of) BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ of complex semisimple Lie algebra $\mathfrak{g}$.

(2) Parabolic analogue of category $\mathcal{O}$, often denote by $\mathcal{O}^{\mathfrak{p}}$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$; or $\mathcal{O}^{\mu}$ where μ is a composition of n. The quantum analogue of this category if the quantum parameter is a root of unity, is also a HWC. [Lusztig(?)]

(3) Rational Cherednik algebra (DAHA) analogue of category $\mathcal{O}$. Often denoted by $\mathcal{O}_p(W)$ where p is an r-tuple of complex numbers and W is a Weyl group. [Dunkl-Opdam, Berest-Etingolf-Ginzburg]

(4) (some blocks of) analogue of category $\mathcal{O}$ for Lie superalgebra $\mathfrak{gl}_{m+n}(\mathbb{C})$. [Brundan-Stroppel] This category is denoted by $\mathcal{O}(m,n)$. Note this is not true for all complex (semi)simple Lie superalgebra, there are some blocks of category $\mathcal{O}$ of the queer (or strange) superalgebra $\mathfrak{q}(n)$ which is not HWC (but standardly stratified).

(5) Schur algebra $S(n,r)$ and its many generalisations, include the quantum q analogue, the cyclotomic analogue and symplectic analogue, etc.

(6) $\text{Rep}^r(GL_n(k))$: the category of polynomial representation of $GL_n(k)$ of degree r. Note that this is in fact equivalence to $S(n,r)$-mod [Green] via Schur-Weyl duality.

(7) More generally, G-mod for G a linear algebraic group (e.g. $GL_n(k)$), the category of rational representation of G. Note this category usually does not have enough projectives.

(8) G-mod for $G = GL(m|n)$ = general linear supergroup. Note this is the easiest example of an algebraic supergroup; and this is only worked out very recently. [Brundan-Stroppel]

(9) (1-faithful) Quasi-hereditary cover of many "well-behaved" algebras (e.g. cellular algebras). e.g. Brundan-Stroppel's generalised Khovanov arc algebras, which are used to study categorification. Schur algebra analogue of (cellular) diagram algebras invented by Henke-Hartmann-Koenig-Paget.

(10) (f.d.) Algebra of global dimension 2. In particular, Auslander algebra of a finite-representation type algebra. [Dlab-Ringel]

(11) qh algebra arise from fusion system.

(12) qh algebras arise from the category algebra of regular monoid.

There are also more example of algebras which can be said to be quasi-hereditary, but NOT in the sense of CPS. The original definition for quasi-hereditary algebras require the algebra to be finite dimensional, or equivalently with finite number of (isoclass of) irreducibles. Nevertheless, we can weaken the definition of qh algebras using its homological properties and say that certain locally unital infinite dimensional algebras are qh; or equivalently, saying A is qh if A-mod is HWC. For example, the "full" Brundan-Stroppel's generalised Khovanov arc algebras, and

The abstract definitions of HWC and qh algebras grew out of the study of (co)homology of BGG category $\mathcal{O}$, category of rational representations of algebraic groups and the study of Schur algebra. The definitions, as we will see later, rely on the existence of an indexing set Λ of the simple modules (objects) which has a partial order structure, together with existence of a special family of modules (objects). This resembles (or generalise) the categorical and homological behaviour of a class of algebras, called the hereditary algebra. This class of algebra is arguably the best understood class of algebras. They can be thought as algebras which can be built inductively, by attaching a layer of k-mod on top each time. Hence, the cohomology should also be biuilt inducitvely in a similar manner. HWC/qh is exactly the category which satisfty this condition.

On the other hand, if one interest lies in the structure of modules, it can be shown that decomposition matrix is unitriangular, hence one can find an inverse of the matrix and compute the dimension of simple modules. In another words, computing the dimension of simple modules has become a problem of linear algebra (although it could still be very hard). In fact, this has also become one of the most powerful tool when people study modules of cellular algebras.

1.3 Subclasses and generalisations

In the realm of qh algebras, there are certain subclasses of algebras which exhibit nice property that one would like to study on their own. The BGG algebras are qh algebras satisfying some duality property; the naming comes from the fact that all BGG category $\mathcal{O}$ is equivalent to module category of BGG algebra. Agoston-Dlab-Lukacs has been studying many subclasses of qh algebras since its birth, the most well-known one is the standard Koszul algebras, which are qh algebras that are also Koszul. Again, blocks of $\mathcal{O}$ are equivalent to module category of standard Koszul algebras.

A big theme of representation theory is to study the cohomology of modules over algebras. HWC/qh algebras are arguably the next best (easiest) class of algebras, after hereditary algebras, for which we can simplify the calculation of cohomology through what we called "stratification" or "recollement of derived categories". This says that the derived category can be built by layer of derived categories, each of them equivalent to the derived category of k-mod. This can also be viewed as the categorical analgoue of a short exact sequence, as well as the generalistion of the phenomenon, for any algebra A and any idempotents of $e \in A$, there are triplets of functors (which appears naturally by adjointness):

$$eAe\text{-mod} \overset{\leftarrow}{\rightleftarrows} A\text{-mod} \overset{\leftarrow}{\rightleftarrows} A/AeA\text{-mod}$$

Philosophically, one should further generalise qh algebras in order to unify with objects we want to study, for example the algebra for which the qh algebra covers. The two main such generalisations are cellular algebras and standardly stratified algebras. The experts in these fields usually view cellular algebras as the "combinatorial generalisation" of qh algebras, although we should note that not all qh algebras are cellular. On the other hand, standardly stratified algebras are usually viewed as "homological generalisation" of qh algebras, as there also exists recollement for standarly stratified algebras, except that each stratification layer is made up of the endomorphism of the standard modules, which is not necessarily the underlying filed k of the algebra. Recently, it has been known that for most, and probably all, explicitly defined cellular algebra, stratification exists with each stratification layer comes from (symmetric) cellular algebras. In general, it is not known whether we can stratify any given cellular algebra.

1.4 Quasi-hereditary covers

The motivation to study HWC or qh algebras for people with the Lie theory background is usually to get more insight of the BGG category $\mathcal{O}$ via purely abstract and algebraic (and homological) approach. On the other hand, people also like to see how closely related are between different qh algebras and HWC; e.g. are there any relation between category $\mathcal{O}$ and $S(n,r)$-mod. However in recent years, we realise there is yet another reason why we want to, or we like to, study qh algebras. Definition of quasi-hereditary algebras says that there is a very nice relation between the structure of the projective indecomposable modules, the standard modules (we will define this later) and the simple modules; if somehow another algebra of interest which has a more complicated structure behaves almost the same as a quasi-hereditary algebra, then we could study this complicated beast via qh algebra. Such a connection is called (1-faithful) *quasi-hereditary cover*, meaning there is a quasi-hereditary algebra $S(A)$ "covering" your algebra of interest, A say, in a very nice way which is called the idempotent truncation:

$$A \cong eS(A)e \cong \mathrm{Hom}_{S(A)}(S(A)e, S(A)e)$$

for some idempotent e (also note $S(A)e$ is projective). This induces a functor

$$e \cdot - \;=\; eS(A) \otimes_{S(A)} - : S(A)\text{-mod} \to A\text{-mod}$$

For people who know Schur algebra well, this is the Schur functor when $A = k\mathfrak{S}_r$ and $S(A) = S(n,r)$ given $n \geq r$.

Ringel has proved that any algebra admit a quasi-hereditary cover, its proof is by constructing the qh algebra out of A, and this construction usually results in a qh algebra way larger than a "nice and small enough" cover of A for some special class of algebras A. The actual meaning for being "nice and small enough" (i.e. 1-faithful) is defined by Rouquier only very recently in 2008 during his study for complex reflection group and q-Schur algebras. The byproduct of his paper is that he found such covering is unique (up to isomorphism), and this should motivate the study of qh algebras (or HWC) even further. On the other hand, even for some very nice class of algebras, most notably cellular algebras, whether 1-faithful qh cover exists is not known; although it should also be noted that 1-faithful qh cover has been found for most of the well-known cellular algebras. Usually when we say qh cover, we already implicitly saying it is 1-faithful unless otherwise specified. In the above list of examples of HWC, we have already mentioned 1-faithful qh cover for some algebras; we should also mention that

(1) regular block of $\mathcal{O}$ $\xrightarrow{\text{qh cover}}$ coinvariant algebras [Soergel]

(2) $\text{Rep}^r(GL_n(k))$ or Schur algebra $\xrightarrow{\text{qh cover}}$ symmetric group algebra [Green, Parshall, Donkin]

(3) q-Schur algebra $\xrightarrow{\text{qh cover}}$ Hecke algebra of type A [Donkin]

(4) cyclotomic q-Schur algebra $\xrightarrow{\text{qh cover}}$ cyclotomic Hecke algebra of type A

One can also view qh cover as being "half" of Schur-Weyl duality or the double centraliser property.

2 Prerequisites

We list some terminology and results you should know before we start, we will give brief explanation to the terminology whenever possible:

Primitive idempotents, projective and injective modules, indecomposable modules, simple (irreducible) modules. Semisimple algebra.

Theorem 2.1
A be finite dimensional k-algebra. Then $1 = e_1 + \cdots + e_n$ where e_i are primitive idempotents. This corresponds to the decomposition $A = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ where $P_i = Ae_i$ are projective indecomposable A-modules. Simple A-modules are given by $S_i = \text{top}(P_i) := P_i/\,\text{rad}(P_i)$.

It would be good to know these as well: Artin-Wedderburn Theorem, Krull-Schmidt Theorem.

Self-injective algebra: algebra A such that the left regular representation is injective. i.e. for such algebra A, projectives = injectives.

Symmetric algebra: self-injective algebra A which satisfy more restrictive condition (there exists a symmetric Frobenius form on A). Consequence: $S_i = \text{top}(P_i) \cong \text{soc}(P_i)$

Lemma 2.2 (Schur's Lemma)
$\text{End}_A(S_i) = k$ *and* $\text{Hom}_A(S_i, S_j) = 0$ *if* S_i *not isomorphic to* S_j.
More generally, $\text{Hom}(P_i, M) = e_iM$*, its dimension is the multiplicity of* S_i *as composition factor of* M*, denote by* $[M : S_i]$.

(Jacobson) radical and (Loewy) socle of a module, filtration and composition factor
Tensor product and functor, Hom functor, and the fact that for A-B bimodule M; $M \otimes_B - : B\text{-mod} \to A\text{-mod}$ is left adjoint to $\text{Hom}_A(M, -) : A\text{-mod} \to B\text{-mod}$
Complexes, projective/injective resolutions of an object, (co)homology $\text{Ext}^i_{\mathcal{C}}(M, N)$.

It would be particular nice if you know: derived categories

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