Seminar series on étale cohomology

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This is the notes taken for the series of seminars, given by members of the Institute of Mathematics at University of Aberdeen at Fall 2011. Most of the seminars are given by P. Brito and J. Taylor. This notes is intended to give a picture and introduce the tools around the étale cohomology and does not provide the detailed mathematical rigour.

1 Motivations (from algebraic group)

Let X be a smooth projective variety over field $\mathbb{K} = \overline{\mathbb{F}_p}$ for $p > 0$
We usually denote $q = p^a$, and so $\mathbb{F}_q \subset \overline{\mathbb{F}_p}$

When we say X is <u>defined by $\mathbb{F}_q$</u>, this implies we have <u>Frobenius map</u> $F : X \to X$ induced by $\begin{array}{rcl}\phi : \mathbb{K} & \to & \mathbb{K} \\ x & \mapsto & x^q\end{array}$ (and so $\mathbb{K}^\phi = \mathbb{F}_q$)

Note that $|X^F| = |X(\mathbb{F}_q)|$

It turns out that $F^m : X \to X$ corresponds to X defined over $\mathbb{F}_{q^m}$; and $X = \bigcup_{m>0} X^{F^m}$.

Define the <u>Zeta function</u>

$$Z(X,t) := \exp(\sum_{n=1}^{\infty} |X^{F^n}| \frac{t^n}{n}) \quad \in \mathbb{Q}\,[[t]]$$

Weil Conjecture(s):

(1) $Z(X,t)$ is a rational function (i.e. $Z(X,t) = P(t)/Q(t)$ for some polynomials P, Q)

(2) $P(t) = \prod_{j=1}^{b} (1 - \beta_j t)$

$Q(t) = \prod_{i=1}^{a} (1 - \alpha_i t)$

where α_i, β_j are algebraic integers with modulus $q^{i/2}$ and $a + b = 2 \dim X$

Weil's idea:
Use some "cohomology", over field of characteristic 0.
But this is not possible for Zariski topology
Solution: Étale cohomology

Application:
ℓ-adic cohomology $H^i_c(X, \overline{\mathbb{Q}_\ell}) := H^i_c(X, \mathbb{Z}_\ell) \otimes \overline{\mathbb{Q}_\ell} \quad \ell(\neq p) > 0$ prime
Note Frobenius map F indeuces $F^* : H^i_c \to H^i_c$

Main Theorem:

$$
\begin{aligned}
|X^F| &= \sum_i (-1)^i \operatorname{Tr}(F^*, H^i_c(X, \overline{\mathbb{Q}_\ell})) \\
|X^{F^m}| &= \sum_{i=1}^{a} \alpha_i^m - \sum_{j=1}^{b} \beta_j^m
\end{aligned}
$$

2 Sheaves and Categories

2.1 Limits and colimits

Let I be a category

Definition 2.1
Let $\beta : I^{\mathrm{op}} \to \mathcal{C}$ be a functor (e.g. when $\mathcal{C} = k$-vector space, this functor can be thought of as right I-module where I is considered as a k-algebra)
A cone C over β is denoted by the diagram $\{C \to \beta(i)\}$ which means:
$\forall s : i \to j$ in I, the following diagram commutes

$$
\begin{tikzcd}
\beta(j) \arrow[rr, "\beta(s)"] \arrow[dr, "\phi_j"'] & & \beta(i) \arrow[dl, "\phi_i"] \\
& C &
\end{tikzcd}
$$

i.e. C is the collection of natural transformations from $(i \mapsto C) \forall i$ to β.
The limit, $\lim \beta$ is the terminal object in the category of cone. i.e. we have following commutative diagram:

$$
\begin{tikzcd}
\beta(j) \arrow[rr, "\beta(s)"] \arrow[dr] \arrow[ddr, "\phi_j"'] & & \beta(i) \arrow[dl] \arrow[ddl, "\phi_i"] \\
& \lim \beta & \\
& C \arrow[u, dashed, "\exists!"'] &
\end{tikzcd}
$$

The dual notion of cone is cocone C of $\alpha : I \to \mathcal{C}$, represented by diagram $\{\alpha(i) \to C\}$ such that there is commutative diagram as above.
The dual notion of limit is colimit, which is the initial object in the category of cocone.

When $\mathcal{C} = \textbf{Sets}$:

$$
\begin{aligned}
\lim \beta &= \Big\{\{x_i\{\in \prod_{i \in I} \beta(i) \Big| \beta(s)(x_j) = x_i \ \ \forall s : i \to j\Big\} && (2.1) \\
\mathrm{colim} \alpha &= \coprod_{i \in I} \alpha(i) / \sim && (2.2)
\end{aligned}
$$

where $\sim$ is generated by $\alpha(i) \ni x \sim y \in \alpha(y)$ if $\alpha(s)(x) = y$

Example 2.2

(1) $I = \{\bullet, \bullet, \ldots\}$ *(category with objects and no arrows)*
$\beta : I^{\mathrm{op}} \to \mathbf{Sets} \qquad \alpha : I \to \mathbf{Sets}$

$$\begin{aligned} \lim \beta &= \prod_{i \in I} \beta(i) =: \underline{\textit{product}} \\ \mathrm{colim}\alpha &= \coprod_{i \in I} \alpha(i) =: \underline{\textit{coproduct}} \end{aligned}$$

(2) $I = \{i \leftarrow j \to k\}$

$$\begin{aligned} \lim(A \xrightarrow{f} C \xleftarrow{g} B) &= \{(a,b) A \times B | f(a) = g(b)\} \\ &=: \underline{\textit{pullback}} =: A \times_C B \\ \mathrm{colim}(A \xleftarrow{f} C \xrightarrow{g} B) &= A \sqcup B / f \sim g \\ &=: \underline{\textit{pushout}} =: A \sqcup_C B \end{aligned}$$

(3) $I = \{\bullet \rightrightarrows \bullet\}$

$$\begin{aligned} \lim(\, A \overset{f}{\underset{g}{\rightrightarrows}} B \,) &= \{(x,y) \in A \times B \mid f(x) = y = g(x)\} \\ &= \underline{\textit{equaliser}} \text{ “ = ” } \textit{kernel} \\ &\quad \textit{(in abelian categories, this is ker}(f-g)) \\ \mathrm{colim}(\, A \overset{f}{\underset{g}{\rightrightarrows}} B \,) &= B / f \sim g \\ &= \underline{\textit{coequaliser}} \text{ “ = ” } \textit{cokernel} \end{aligned}$$

We denote $K \to A \rightrightarrows B$ as the equaliser above, where K is the limit object; and $A \rightrightarrows B \to C$ as the coequaliser above, with C being the colimit object.

2.2 Grothendieck topology and sheaves

Definition 2.3
A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ is representable if $F = Hom_{\mathcal{C}}(-, X)$ for some $X \in \mathcal{C}$; and X is called the representative of F

A presheaf is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$.
The category of presheaves is denoted as $\mathrm{PSh}(\mathcal{C})$

There is an obvious functor $\begin{array}{ccc} \mathcal{C} & \to & \mathrm{PSh}(\mathcal{C}) \\ X & \mapsto & \mathrm{Hom}_{\mathcal{C}}(-, X) \end{array}$

Yoneda Lemma
$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, X), F) \simeq F(X)$

Consequently, the functor above is an embedding $r : \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$
We usually denote the image of $X \in \mathcal{C}$ as rX or even just X itself if there is no ambiguity.

Definition 2.4
A category which has all small limits (resp. colimits) is called complete (resp. cocomplete).

When $\mathcal{C}$ is not cocomplete, we want to approximate $\mathcal{C}$ by a cocomplete category $\widetilde{\mathcal{C}}$ such that

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[dr, "\forall\gamma"'] & \widetilde{\mathcal{C}} \arrow[d, dashed, "\exists \text{ colim. preserving}"] \\
& \forall\mathcal{D}
\end{tikzcd}$$

It turns out that $\widetilde{\mathcal{C}} = \mathrm{PSh}(\mathcal{C})$
The slogan is $\mathrm{PSh}(\mathcal{C})$ is the "free cocompletion" of $\mathcal{C}$

<u>Note</u>: The embedding $r : \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ does not necessarily commute with colimits
i.e. $r(U \coprod V) \neq (rU) \coprod (rV)$ in general. In another words, by embedding $\mathcal{C}$ into $\mathrm{PSh}(\mathcal{C})$, we lost some geometric information (See later).

Definition 2.5
Let $\mathcal{C}$ be a category. A <u>Grothendieck topology</u> $\mathcal{T}$ on $\mathcal{C}$ is an assignment of a set of <u>covering families</u> $\mathrm{Cov}_{\mathcal{T}}(X) = \{\mathcal{U} = \{U_i \to X\}_{i\in I}\}$ *(where $\mathcal{U}$ are called <u>cover</u>) for each object $X \in \mathcal{C}$ satisfying:*

*(1) If $\{U_i \to X\}_{i\in I}$ is a cover and $\{V_{i,j} \to U_i\}_{j\in J}$ is a refinement $\forall i \in I$
then $\{V_{i,j} \to X\}_{i,j}$ is a cover*

*(2) If $f : Y \to X$ and $\{U_i \to X\}$ is a cover
then $U_i \times_X Y$ exists and $\{U_i \times_X Y \to Y\}$ is a cover of Y*

*(3) If $f : Y \to X$ is an isomorphism
then $\{Y \to X\}$ is a cover of X*

Example 2.6
Let T be a topological space, and $\mathcal{C}$ = open sets of T
$\mathrm{Cov}_{\mathcal{T}}(X) = \{$*usual open cover of X*$\}$ *gives Grothendieck topology $\mathcal{T}$*

The following example demonstrate how colimits are lost through embedding to $\mathrm{PSh}(\mathcal{C})$

Example 2.7
$\mathcal{C} = \mathbf{Top}$, $\{U_i \to X\}$ *be cover of $X \in \mathcal{C}$*

$$\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i \to X \tag{2.3}$$

is an coequaliser in $\mathcal{C}$ but <u>NOT</u> coequaliser in $\mathrm{PSh}(\mathcal{C})$

In order to turn (2.3) into a coequaliser, we need the Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$:

$$(\mathcal{C} \ni) \quad X \mapsto \mathrm{Cov}_{\mathcal{T}}(X) = \left\{ \begin{array}{l} \text{covers of } X \\ \mathcal{U} := \{U_i \to X\}_{i\in I} \end{array} \right\}$$

Given cover $\mathcal{U} = \{U_i \to X\}_{i\in I}$, can form the <u>distinguished cocone</u>

$$\coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i \to X$$

which will be (see later) our coequaliser in the enlarged category

Proposition 2.8
$\exists$ category $\mathrm{Sh}(\mathcal{C})$ together with a functor $\mathcal{C} \to \mathrm{Sh}(\mathcal{C})$ which sends distinguished cocones into colimits and universal with respect to this property. i.e. we have commutative diagram:

$$\begin{tikzcd}
\textit{d.cocones} \arrow[dr] & & \mathcal{C} \arrow[r] \arrow[dr] & \mathrm{Sh}(\mathcal{C}) \arrow[d, dashed, "\exists\,!\textit{(up to isom.)}"] \\
& \textit{colimits} & & \mathcal{D}
\end{tikzcd}$$

Definition 2.9
A <u>*sheaf*</u> *is a presheaf satisfying:*
$\forall$ *cover* $\mathcal{U} := \{U_i \to X\}_{i\in I}$*, we have an equaliser diagram:*

$$F(X) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$$

More explicitly,

(1) *if* $s, t \in F(X)$ *with the same image in* $\prod_i F(U_i)$*, then* $s = t$

(2) *(Gluing property)*
$\forall$ *collection* $\{s_i\}$ *where* $s_i \in F(U_i)\forall i$*, such that* $s_i|_{U_i\times_X U_j} = s_j|_{U_i\times_X U_j}$
then they extend to $s \in F(X)$

<u>Terminology</u>
Let $\mathcal{D}$ be category. We say that $Z \in \mathcal{D}$ <u>sees a cocone</u> $\{D(i) \to X\}$ <u>as a colimit</u>, if $\mathrm{Hom}_{\mathcal{D}}(X, Z) \xrightarrow{\sim} \lim_i \mathrm{Hom}_{\mathcal{D}}(D(i), Z)$

<u>Note</u>:
If every object Z sees a cocone $\{D(i) \to X\}$ as a colimit, then X is the limit itself.

Claim: Sheaf = presheaf which sees distinguished cocones as colimits
<u>Note</u>: This is actually obvious from the definition, but we want to demonstrate more.
Proof
We write $U_{i,j} := U_i \times_X U_j$. The definition of sheaf gives:

$$\begin{aligned} F(X) &\cong \lim\left(\prod F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j})\right) \\ &\cong \lim\left(\prod \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(rU_i, F) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(rU_{i,j}, F)\right) \qquad \text{(by Yoneda)} \end{aligned}$$

By applying Yoneda on the left hand side:

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(X, F) \cong \lim\left(\mathrm{Hom}_{\mathrm{PSh}}(\coprod rU_i, F) \rightrightarrows \mathrm{Hom}_{\mathrm{PSh}}(\coprod_{i,j} rU_{i,j}, F)\right)$$

i.e. F sees distinguished cocones as colimits $\coprod U_{i,j} \rightrightarrows \coprod_i U_i \to X$. we now get:

$$\mathrm{Hom}_{\mathrm{PSh}}(X, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}}(\mathcal{S}_{\mathcal{U}}, F)$$

where $\mathcal{S}_{\mathcal{U}} := \mathrm{colim}(\coprod_{i,j} rU_{i,j} \rightrightarrows \coprod_i rU_i) \in \mathrm{PSh}(\mathcal{C})$. Also note that

$$\mathcal{S}_{\mathcal{U}}(V) \cong \{f : V \to X | f \text{ factors through } U_i \to X \text{ some } i\}$$

because this is $\coprod_{i,j} \mathrm{Hom}(V, U_{i,j}) \rightrightarrows \coprod_i \mathrm{Hom}(V, U_i)$
$\mathcal{S}_{\mathcal{U}}$ is called the <u>sieve</u> generated by $\mathcal{U}$. It keeps all the essential information from topology.

<u>Question</u>: When can we say (Grothendieck) topologies $\mathcal{T}$ and $\mathcal{T}'$ have the same sheaves?

2.3 Refinements and Sheafification

Definition 2.10
Let $\mathcal{U} := \{U_i \to X\}$ *and* $\mathcal{V} := \{V_j \to X\}$

we say $\mathcal{V}$ refines $\mathcal{U}$ if we have commutative diagram:

$$\begin{tikzcd} V_j \arrow[rr, "\forall f"] \arrow[dr, dashed, "\exists"'] & & X \\ & U_i \arrow[ur] & \end{tikzcd}$$

We say $\mathcal{T}'$ refines $\mathcal{T}$ if every cover in $\mathcal{T}$ has refinement by a cover in $\mathcal{T}'$ and say $\mathcal{T}'$ and $\mathcal{T}$ are equivalent if they refines each other.

Proposition 2.11
Suppose $\mathcal{T}'$ refines $\mathcal{T}$. If F is a sheaf for $\mathcal{T}'$, then F is a sheaf for $\mathcal{T}$

(Recommended exercise: prove this)

As there is the obvious (forgetful) functor $\mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$, we take the left adjoint of this and this is in fact the sheafification functor. The consequence of the existence of this functor is that we can compute colimits of sheaves in the category of presheaves and then sheafify.

2.4 Étale motivation

Let $\mathcal{C} = \mathbf{Top}$ (or category of manifolds, etc.)
Take a "local homeomorphism" (this is like taking a covering map) $E \xrightarrow{p} X$
$\rightsquigarrow \quad E \times_X E \rightrightarrows E \to X \quad$ coequaliser in $\mathcal{C}$
$\rightsquigarrow \quad r(E \times_X E) \rightrightarrows rE \to rX \quad$ coequaliser in $\mathrm{PSh}(\mathcal{C})$ (using the fact that p is a local homeo.)
Any sheaf F sees the last coequaliser as a colimit.

However, if $\mathcal{C}$ is the category of schemes, where the analogue of the "covering map" is the étale map then the étale map $E \xrightarrow{p} X$ giving $rE \times_X E \rightrightarrows rE \to X$ is not a coequaliser in $\mathrm{PSh}(\mathcal{C})$
In order to force this to be colimits, we need Grothendieck topology, where we include $\{E \xrightarrow{p} X\}$ as a cover.

The étale topology is the smallest topology which contains Zariski covers $\{U_i \to X\}$ and $\{E \xrightarrow{p} X | p \text{ étale map}\}$

2.5 Derived functor

Reader could refer to Fall-2011 graduate course notes on "Derived Equivalence of Blocks of Finite Groups". Commonly used source for this section is Gelfand,Manin's Methods of Homological Algebra.

The reader needs to know the following keywords:

(1) Additive and abelian categories
e.g. $\mathrm{PSh}^{\mathbf{Ab}}(\mathcal{C}) := \{\mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}\}$ and $\mathrm{Sh}^{\mathbf{Ab}}(\mathcal{C})$

(2) Complex

(3) Category of complexes over $\mathcal{C}$, denoted $\mathbf{C}(\mathcal{C})$

(4) Homotopy of complex, and chain homotopies, quasi-isomorphism (quism)

(5) Homotopy category $\mathbf{K}(\mathcal{C})$, constructed by taking homotopy classes of maps in $\mathbf{C}(\mathcal{C})$

(6) Derived functors:

Let $F : \mathcal{C} \to \mathcal{A}$ be additive functor between abelian categories. e.g. The global section functor:

$$\begin{array}{rcl} \Gamma_X : \mathrm{PSh}^{\mathbf{Ab}}(\mathcal{C}) & \to & \mathbf{Ab} \\ \mathcal{F} & \mapsto & \mathcal{F}(X) \end{array}$$

Sometimes we will write $\Gamma(-, X)$ for the global section functor. We often ask following questions for these kind of functors:

(1) Is the functor exact?

(2) If not, can you approximate it by exact functor?

(3) F induces functor on homotopy categories, $\mathbf{K}(F) : \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{A})$
How can you approximate $\mathbf{K}(F)$ by a functor which sends quasi-isomorphism (quism) to quasi-isomorphism?

The answer is to use derived functors and consider the corresponding derived categories $\mathbf{D}(\mathcal{C})$, $\mathbf{D}(\mathbb{A})$

$$\begin{tikzcd} \mathbf{K}(\mathcal{C}) \arrow[r, "\mathbf{K}(F)"] \arrow[d] & \mathbf{K}(\mathcal{A}) \arrow[d] \\ \mathbf{D}(\mathcal{C}) \arrow[r, "RF"] & \mathbf{D}(\mathcal{A}) \end{tikzcd}$$

RF above is the (right) derived functor. This is somehow the best (initial) approximation to $\mathbf{K}(F)$. The formula we often use is:

$$RF(X) = F(I)$$

where I is injective resolution of X (assuming it always exists for all $X \in \mathbf{K}(\mathcal{C})$)
i.e., $\forall X\ \ \exists X \to I$ which is a quism and $I \in \mathcal{I}$ where $\mathcal{I}$ is a subcategory of $\mathcal{C}$ s.t. $F|_{\mathcal{I}}$ exact.

The reason for such formula is as follows:
As $F|_{\mathcal{I}}$ exact
$\Rightarrow \quad \mathbf{K}(F)|_{\mathbf{K}(\mathcal{I})}$ sends quism to quism
As $\forall X \in \mathcal{C}$, $\exists I \in \mathcal{I}$ with $X \to I$ quism
$\Rightarrow \quad \mathbf{D}(\mathcal{C}) \cong \mathbf{K}(\mathcal{I}) \qquad$ (this is non-trivial)
Hence we have following diagram:

$$\begin{tikzcd} X \arrow[d, mapsto] & \mathbf{K}(\mathcal{C}) \arrow[r, "F"] \arrow[d] & \mathbf{K}(\mathcal{A}) \arrow[d] \\ I & \mathbf{D}(\mathcal{C}) \cong K(\mathcal{I}) \arrow[r, "F"] & \mathbf{D}(\mathcal{A}) \end{tikzcd}$$

with the F on the second row sending quism to quism. As well as giving the above formula. We denote $R^iF(X) := H^i(F(X))$

The sheaf cohomology is the (right) derived functor of $\Gamma_X (= \Gamma(-, X))$ above.

3 Varieties and Schemes

Fix a field $K = \overline{K}$. $\qquad \mathbb{A}^n = K^n$ affine n-space.
Projective space $\mathbb{P}^n$ is defined to be the sapce $\mathbb{A}^{n+1} \setminus \{0\}$ s.t. $x \sim y$ iff $\exists \lambda \in K^\times$ s.t. $x_i = \lambda y_i\ \ \forall i$ (i.e. $x = \lambda y$)
All rings are commutative with 1.

3.1 Affine and projective algebraic sets

Theorem 3.1 (Hilber's Basis Theorem)
If R is a Noetherian ring, then $R[X]$ is Noetherian.
(Note: R is Noetherian iff every ideal of R is finitely generated.)

Definition 3.2
Let $S \subseteq K[X_1, \dots, X_n]$ be a set of polynomials, we define

$$V(S) := \{x \in \mathbb{A}^n | f(x) = 0 \ \ \forall f \in S\}$$

to be the affine algebraic set asociated to S. Conversely, for any subset $V \subseteq \mathbb{A}^n$, we define

$$I(V) := \{f \in K[X_1, \dots, X_n] | f(x) = 0 \forall x \in V\}$$

to be the vanishing ideal associated to V. We define

$$K[V] := K[X_1, \dots, X_n]/I(V)$$

to be the affine coordinate ring of V.

Example

(1) $V(\{1\}) = \emptyset$
$V(\{0\}) = \mathbb{A}^n$

(2) $x = (x_1, \dots, x_n) \in \mathbb{A}^n$ then $V(\{(X_1 - x_1, \dots, X_n - x_n)\}) = \{x\}$

(3) Twisted cubic. Take $f = X_2 - X_1^2, \ \ g = X_3 - X_1^3$
$\Rightarrow f, g \in [X_1, X_2, X_3]$
$\Rightarrow V(\{f, g\})\{(a, a^2, a^3) \in \mathbb{A}^3 | a \in \mathbb{A}^1\}$

Remark
If $S \subseteq K[X_1, \dots, X_n]$ and $J = \langle S \rangle$, then $V(S) = V(J) \overset{\text{Hilbert Basis}}{=} V(\{f_1, \dots, f_r\})$.

Assume R is a (positively) graded ring $R = \bigoplus_{d \geq 0} R_d$ of abelian groups such that $R_d R_e \subseteq R_{d+e} \ \ \forall d, e \geq 0$. Then an element of R_d is a homogeneous element of degree d. We say an ideal $J \lhd R$ is a homogeneous if $J = \bigoplus_{d \geq 0}(J \cap R_d)$

Check that the product, sum, intersection and radical of homogeneous ideals is again homogeneous.

Let $R := K[X_0, \dots, X_n]$ then R is graded where R_d is all linear combinations of monomials in $X_0, \dots X_n$ with total weight d.

Definition 3.3
$S \subseteq K[X_0, \dots, X_n]^H$ then

$$V(S) = \{p \in \mathbb{P}^n | f(p) = 0 \ \ \forall f \in S\}$$

to be the projective algebraic set associated to S. Conversely, $V \subseteq \mathbb{P}^n$, define

$$I(V) = \langle f \in K[X_0, \dots, X_n]^H | f(x) = 0 \ \ \forall x \in V \rangle$$

to be the homogeneous vanishing ideal associated to V.
Then $K[V] := K[X_0, \dots, X_n]/I(V)$ is homogeneous coordinate ring

Remark: "$f(p) = 0 \;\; \forall p \in \mathbb{P}^n$" is only well-defined if f is homogeneous.
$f \in K[X_0, \dots, X_n]^H$

$$\widetilde{f}(p) = \begin{cases} 0 & \text{if } f(p) = 0 \\ 1 & \text{if } f(p) \neq 0 \end{cases}$$

Proposition 3.4
Let $S, TS_i \subseteq R$ (or R^H), $i \in I$ a (possibly infinite) indexing set. Then

(1) $V(ST) = V(S) \cup V(T)$

(2) $V(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} V(S_i)$

Corollary 3.5
The affine/projective algebraic sets in $\mathbb{A}^n$ (or $\mathbb{P}^n$ resp.) form the closed sets of a topology called the Zariski topology

Remark:

(1) $S_1 \subseteq S_2 \subseteq R$ or R^H, then $V(S_2) \subseteq V(S_1)$

(2) $S \subseteq R$ or R^H then $S \subseteq I(V(S))$

(3) $V_1 \subseteq V_2 \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$, then $I(V_2) \subseteq I(V_1)$

(4) $V \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$, then $V \subseteq V(I(V))$

So I and V are order reversing operators between subsets of $\mathbb{A}^n$ ($\mathbb{P}^n$ resp.) and ideals (homog. ideals resp.) of R (R^H resp.).

Lemma 3.6
If $V \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$, then $V(I(V)) = \overline{V}$ the topological closure of V in the Zariski topology.

So this says I takes injective affine algebraic sets to ideals of R.
What is $I(V(S))$ for $S \subseteq R$? Or when is I injective?

Theorem 3.7 (Hilbert's Nullstellensatz)
If $S \subseteq K[X_1, \dots, X_n]$ then $I(V(S)) = \operatorname{rad}(S) := \sqrt{S} := \{f \in R | f^m \in S$ for some $m \geq 1\}$

This says I and V are inverse bijection between

$$\{\text{affine algebraic sets of } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals of } K[X_1, \dots, X_n]\}$$

(Radical ideals means ideal S such that $S = \operatorname{rad}(S)$)

In the projective case, we have the following. Let $R_+ = \bigoplus_{d>0} R_d$, we have

$$\{\text{projective algebraic sets of } \mathbb{P}^n\} \leftrightarrow \{\text{homogeneous radical ideals of } K[X_0, X_1, \dots, X_n] \text{ except } R_+\}$$

Reason: If $S \subseteq R^H$, then $V(S) = \emptyset$
$\Rightarrow \;\; S = R$ or $R_+ \;\; \Rightarrow \;\; I(\emptyset) = R$

We say a topological space is Noetherian if it satisfies DCC on closed sets.

Proposition 3.8
V a non-empty algebraic set

(1) V is Noetherian topological space: $\exists$ unique decomposition $V = V_1 \cup \dots \cup V_r$ where V_i are maximal (wrt includsion) closed irreducible subsets of V.

(2) V is irreducible $\Leftrightarrow$ $I(V)$ is prime.

Example

(1) $\{a\} \in \mathbb{A}^n$. Then $I(\{a\}) = \{f | f(a) = 0\}$
If $fg \in I(\{a\})$ then $f(a)g(a) = 0$, so $f(a)$ or $g(a) = 0$, so $f \in I(\{a\})$ or $g \in I(\{a\})$

(2) $p(X_1, X_2) = X_1X_2$
$V(\langle p \rangle) = \{(a_1, a_2) \in \mathbb{A}^2 | p(a_1, a_2) = 0\}$
$\quad = V(\{X_1\}) \cup V(\{X_2\})$
$I(V(\{p\})) = \langle X_1X_2 \rangle$

Let V be an algebraic set and $f \in R$ or R^H a polynomial, then the principal open set determined by f is

$$V_f = V \setminus V(\{f\}) = \{x \in V | f(x) \neq 0\}$$

THe principal open sets form a basis for the topology. If $U \subseteq V$ is open, then the closed set $V \backslash U$ closed, so there is an algebraic set W s.t. $V \backslash U = W \cap V$. By Hilbert's basis theorem $W \cap V = V(\{f_1, \ldots, f_r\})$, so $U = \bigcup_{i=1}^r V_{f_i}$.

Example
Let $= \mathbb{C}$ and let $f \in K[X]$, then $V(\{f\})$ is a finite set consisting of the roots of f. Then every closed set is finite because they are a finite union of closed sets $V(\{f\})$. Conversely, the open sets are large (they are dense). This is main problem when we use Zariski topology. Étale topology (or even Grothendieck topology) is the way to (try to) resolve this issue (by including more open sets than the Zariski topology).

3.2 Morphisms

Definition 3.9
If $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ affine algebraic sets, then a map $\phi : V \to W$ is a morphism if $\exists f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ s.t. $\phi(x) = (f_1(x), \ldots, f_m(x))$ $\forall x \in V$. We say ϕ is an isomorphism if ϕ is bijective and ϕ^{-1} is a morphism.

Remark: $V(S) \subseteq W$, then $\phi^{-1}(V(S)) = V(\{\psi \circ \phi | \psi \in S\})$
So ϕ is continuous w.r.t. the Zariski topology

$\phi : V \to W$ is a morphism. Take $\phi^* : K[W] \to K[V]$ s.t. $\phi^*(\overline{f}) = \overline{f} \circ \phi$

Example
$\phi : V \to \mathbb{A}^1$, then $\exists f \in K[X_1, \ldots, X_n]$ s.t. $\phi(x) = f(x)$ $\forall x \in V$
but if $g \in K[X_1, \ldots, X_n]$ s.t. $\phi(x) = g(x)$ $\forall x \in V$, then $(f - g)(x) = 0$ $\forall x \in V$, so $f - g \in I(V)$
In particular, $\overline{f} = \overline{g}$. Given $f + I(V) \in K[V]$, we get a well-defined morphism $\phi(x) = f(x)$ $\forall x \in V$

Proposition 3.10
Let $\phi : V \to W$ be a morphism then $\phi \mapsto \phi^*$ gives a bijection

$$\{\psi : V \to W \textit{ morphisms}\} \leftrightarrow \{\textit{-algebra hom.} \psi^* : K[W] \to K[V]\}$$

Furthermore, ϕ is isom $\Leftrightarrow$ ϕ^* is isom.

Remark: $V \to K[V]$ can be extended to a contravariant functor between

$$\{\text{irred. aff. alg. sets + morphisms}\} \leftrightarrow \{\text{f.g. int. dom. over } K \text{ \& } K\text{-alg. hom.}\}$$

which is an equivalence of categories.

Example

(1) Let $= \overline{\mathbb{F}_p}$ for $p > 0$ a prime. Then take $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ to be $\phi(x) = x^p$. Then $\phi^* : K[X] \to K[X]$ is the map $X \mapsto X^p$.
So ϕ^* is the embedding $K[X^p] \hookrightarrow K[X]$

(2) $= \mathbb{C}$. Then $\phi : \mathbb{A}^1 \to \mathbb{A}^2$. $\phi(x) = (x^2, x^3)$. Then image is $V(\{p\})$ where $p = \langle X_1^3 - X_1^2 \rangle$.
$\phi^* : K[X^2, X^3] \hookrightarrow K[X]$

Question: Do we have correspondence $\{?\} \leftrightarrow \{\text{commutative rings over } K\}$
Answer: $? =$ Affine schemes

Affine variety is a pair $(V, K[V])$ where V is an irreducible affine algebraic set and $K[V]$ is its affine coordinate ring.

Note: we always assume all rings are commutative with 1 and K algebraically closed.

$$\begin{tikzcd}
\{\text{irred. aff. alg. sets + morphisms}\} \arrow[r, leftrightarrow] \arrow[d] & \{\text{f.g. int. dom. over } K \ \& \ K\text{-alg. hom.}\} \arrow[d] \\
\{\text{aff. alg. sets + morphisms}\} \arrow[r, leftrightarrow] \arrow[d] & \{\text{f.g. nilpotent free rings over } K\} \arrow[d] \\
? \arrow[r, leftrightarrow] & \{\text{commutative rings with 1}\}
\end{tikzcd}$$

3.3 Spectrum of Rings

Let R be a ring then we define the spectrum of R to be

$$\mathrm{Spec} R = \{\mathfrak{p} \lhd R | \mathfrak{p} \text{ is a proper prime idea of } R\}$$

We say a prime ideal $\mathfrak{p}$ is a point of the spectrum which we denote $[\mathfrak{p}] \in \mathrm{Spec} R$

If $\mathfrak{p} \in \mathrm{Spec} R$ then $R/\mathfrak{p}$ is an integral domain so we can consider the quotient field of $R/\mathfrak{p}$ which we denote $\kappa(\mathfrak{p})$. If $f \in R$ then we can consider the image of f under the sequence of maps

$$R \to R/\mathfrak{p} \to \kappa(\mathfrak{p})$$

and we define this to be $f(\mathfrak{p})$ or f evaluated at $\mathfrak{p}$. If $S \subseteq R$, we define

$$\begin{aligned} V(S) &= \{[\mathfrak{p}] \in \mathrm{Spec} R | f(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p}) \ \ \forall f \in S\} \\ &= \{[\mathfrak{p}] \in \mathrm{Spec} R | S \subseteq \mathfrak{p}\} \end{aligned}$$

These $V(S)$ forms closed set of a topology on $\mathrm{Spec} R$, which we call Zariski topology (c.f. previous section)

Example
$R = K[V]$ of some irreducible affine algebraic set V. A consequence of the Nullstellensats is that $S \subseteq V$ is a singelton (point) $\iff$ $I(S)$ maximal
If $\mathfrak{p}$ is maximal in $K[V]$ then $V(\mathfrak{p}) = \{[\mathfrak{p}]\}$, and $K[V]/\mathfrak{p} \cong K$, and under this association $f(\mathfrak{q})$ (for some prime ideal $\mathfrak{q}$) is really what we expect.

Example
$R = K[X]$. Every prime ideal in R is of the form $\langle f_a \rangle$ where $f_a = X - a$ for $a \in K = \overline{K}$, or is the zero

ideal. So $V(\langle f_a \rangle) = \{[\langle f_a \rangle]\}$, which corresponds to a point $a \in K$ which is a point in the affine 1-space in the sense we used previously. However, the zero ideal $[0]$ is not closed because 0 is contained in every ideal. It is also not open because its complement from $\mathbb{A}^1$ is infinite. We call $[0] \in \mathrm{Spec}R$ a generic point of the spectrum. (This is a subset whose topological closure is the whole space). We denote $\mathbb{A}^1$ to mean $\mathrm{Spec}K[X]$.

Let$f \in R$, then we define the principal open set of $\mathrm{Spec}R$ determined by f as

$$\begin{aligned} X_f &= \mathrm{Spec}R \setminus V(\{f\}) \\ &= \{\text{all proper prime ideal of } R \text{ not containing } f\} \end{aligned}$$

Recall R_f is the ring localised at f. If $T = \{f^n | n \geq 0\}$ then

$$R_f = \{a/t | a \in R, t \in\}$$

where $(a_1/t_1) \sim (a_2/t_2) \iff \exists s \in T$ s.t. $s(a_1 t_2 - a_2 t_1) = 0$

Assume $[\mathfrak{p}] \in X_f$ then these correspond one-to-one to the prime ideals of R_f by sending $\mathfrak{p}(\lhd R) \mapsto \mathfrak{p}R_f(\lhd R_f)$. So we can associate X_f to $\mathrm{Spec}R_f$.

3.4 Affine schemes and schemes

Definition 3.11
An affine scheme X is a pair $(|X|, \mathcal{O}_X)$ where $|X|$ is $\mathrm{Spec}R$ of some ring and $\mathcal{O}_X$ is a sheaf of rings on $|X|$ which we will define in the following.

Recall to define a sheaf $(\mathcal{O}_X)$ we must associate to every open set $U \subseteq X$ a ring $\mathcal{O}_X(U)$ and for every pair $U \subseteq V \subseteq X$ a restruction map $\mathrm{res}_{V,U} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ which satisfies

(1) $\mathrm{res}_{U,U} = \mathrm{id}_U$

(2) $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$ for all $U \subseteq V \subseteq W \subseteq X$

(3) sheaf axiom: $U \subseteq X$ and an open cover $\mathcal{U} = \cup_{a \in I} U_a$ then for all $f_a \in \mathcal{O}_X(U_a)$ s.t. f_a and f_b agrees on restriction to $U_a \cap U_b$, then $\exists! f \in \mathcal{O}_X(U)$, s.t. $f|_{U_a} = f_a \ \ \forall a \in I$

If X_f is a principal open set of $X = \mathrm{Spec}R$, then we want to set $\mathcal{O}_X(X_f) = R_f$.
Let's assume X_g is a principal open set of X s.t. $X_g \subseteq X_f$, then g is divisible by f. We take the restriction map to be the natural map $\mathrm{res}_{X_f, X_g} : \mathcal{O}_X(X_f) = R_f \to (R_f)_g = R_g = \mathcal{O}_X(X_g)$

Proposition 3.12
Using the description above, $\mathcal{O}_X$ extends uniquely to a sheaf of rings on $|X|$ called the structure sheaf of X (or sheaf of regular functions).

Definition 3.13
A scheme X is a pair $(|X|, \mathcal{O}_X)$ where $|X|$ is a topological space and $\mathcal{O}_X$ is a sheaf of rings on $|X|$ which is locally affine. By this, we mean $|X|$ is covered by open sets $U_i \subseteq |X|$ s.t. $\exists$ ring R_i and homeomorphisms $U_i \cong |\mathrm{Spec}R_i|$ such that $\mathcal{O}_X|_{U_i} \cong \mathcal{O}_{\mathrm{Spec}R_i}$

If X is a scheme and $U \subseteq X$ then a regular function on U is a section of $\mathcal{O}_X$ over U, i.e. an element of $\mathcal{O}_X(U)$

A global regular function is a global section of $\mathcal{O}_X$

3.5 Subschemes

If X is a scheme and $U \subseteq X$ an open set, then we define $(|U|, \mathcal{O}_X|_U)$ to be the open subscheme determined by U. If X_f is a principal open set of X contained U, then $(|X_f|, \mathcal{O}_{X_f}) = (|X_f|, \mathcal{O}_X|_{X_f}) = \mathrm{Spec}R_f$. As $U = \bigcap_{a \in I} U_a$ for some finite indexing set I. Then use the fact that these principal open sets contained in U cover U, we get $(|U|, \mathcal{O}_X|_U)$ as a scheme.

Assume near $X = \mathrm{Spec}R$ and $I \lhd R$ is an ideal, then we can canonically identify $V(I)$ of $\mathrm{Spec}R$ with $|\mathrm{Spec}R/I|$. A closed subscheme is defined to be $\mathrm{Spec}R/I$ for some ideal $I \lhd R$. Let $Y = \mathrm{Spec}R/I$, then we define the ideal sheaf $\mathcal{J}_{Y/X}$ of Y in X to be the sheaf of ideals of $\mathcal{O}_X$ gives an distinguished open sets by $\mathcal{J}(X_f) = IR_f$.

We have an inclusion $j : |Y| \hookrightarrow |X|$ then the pushforward of $\mathcal{O}_Y$ is the sheaf $j_*\mathcal{O}_Y$ on $|X|$ given by $j_*\mathcal{O}_Y(U) = \mathcal{O}_Y(j^{-1}(U))$ for any open set $U \subseteq |X|$
We can identify $\mathcal{O}_Y$ with $j_*\mathcal{O}_Y$, which can be identified with the quotient sheaf $\mathcal{O}_X/\mathcal{J}$ (this may only be a presheaf, we need to take the its sheafification if necessary)

Remark: Not all sheaves of ideals in $\mathcal{O}_X$ come from ideals of R. If $\mathcal{J}$ comes from an ideal in R, we call it a quasi-coherent sheaf of ideals. If X is a scheme, we say $\mathcal{J} \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf of ideals if for every affine open subset $U \subseteq X$ then restriction $\mathcal{J}|_U$ is quasi-coherent on U.

Definition 3.14
A closed subscheme Y of X is a closed subspace $|Y| \subseteq |X|$ together with a sheaf of rings $\mathcal{O}_Y$ that is the quotient sheaf of $\mathcal{O}_X$ by a quasi-coherent sheaf of ideals.such that the intersection of $\mathcal{J}$ with any affine open set $U \subseteq X$ is the closed subscheme associated to $\mathcal{J}(U)$.

Example
$K[X]$, ideal $\langle X \rangle$
$V(\langle X \rangle) = \{[\langle X \rangle]\} = V(\langle X^i \rangle)$
So the closed set is not uniquely determined by one ideal, and the structure sheaf of $K[X]/\langle X^i \rangle$ is different for different i. Therefore, a closed subscheme does not uniquely determine the ideal, but it does uniquely determine the ideal sheaf.

Definition 3.15
If X is a scheme we can specifty the structure sheaf by its so called stalk. Let $x \in |X|$ then the stalk of $\mathcal{O}_X$ at x is

$$\begin{aligned} \mathcal{O}_{X,x} &= \mathrm{colim}_{U \ni x} \mathcal{O}_X(U) && (3.1) \\ &= \bigotimes_{U \ni x} \mathcal{O}_X(U)/I \quad \text{as } \mathbb{Z}\text{-module} && (3.2) \end{aligned}$$

where I is an ideal generated by the following relations: Assume $U \hookrightarrow V$ inclusion of open sets and let $r \in \mathcal{O}_X(U)$ and $s \in \mathcal{O}_X(V)$ be elements, then we identify r with s if $\mathrm{res}_{V,U}(s) = r$

Example
$X = \mathrm{Spec}R$ and $X_g \hookrightarrow X_f$ principal open sets, so f divides g and

$$\begin{aligned} R_f &\to (R_f)_g && (3.3) \\ a &\mapsto a/1 && (3.4) \end{aligned}$$

Example
$X = \mathrm{Spec}R$. The principal open sets form a basis for the topology on $|X|$. Let $[\mathfrak{p}] \in \mathrm{Spec}R$. If $U \subseteq |X|$ containing $\mathfrak{p}$ then $U = \cup X_f$ s.t. $f \notin \mathfrak{p}$. Considering the stalk

$$\begin{aligned} \mathcal{O}_{X,[\mathfrak{p}]} &= \mathrm{colim}_{f \notin \mathfrak{p}} \mathcal{O}_X(X_f) && (3.5) \\ &= \mathrm{colim}_{f \notin \mathfrak{p}} R_f \;=\; R_{\mathfrak{p}} && (3.6) \end{aligned}$$

which has elements of form r/f^i for some $f \notin \mathfrak{p}$. In fact, $R_\mathfrak{p}$ is a local ring, i.e. has unique maximal ideal. The unique maximal ideal is the image of $\mathfrak{p}$ under the embedding $R \hookrightarrow R_\mathfrak{p}$.

3.6 Morphisms

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on a space $|X|$, then a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is uniquely determined by its induced map on stalks (by sheaf axiom).

In classical algebraic geometry, if V, W are affine algebraic sets over algebraically closed field K, then a morphism $\phi : V \to W$ induces a morphism $\phi^* : K[W] \to K[V]$ of K-algebras.

If X and Y are schemes, there is no embedding of the structure sheaf into some bigger sheaf of functions. In particular, a continuous map $\psi : |X| \to |Y|$ does not induce a map of sheaves $\mathcal{O}_Y \to \psi_*\mathcal{O}_X$. So to define morphism, we need to define a pair $(\psi, \psi^\#) : X \to Y$, with $\psi : |X| \to |Y|$ continuous and $\psi^\# : \mathcal{O}_Y \to \psi_*\mathcal{O}_X$ with some appropriate conditions.

What we would like is to say that the value of a section $f \in \mathcal{O}_Y(U)$ at a point $q \in U$ agrees with the pullback $\psi^\#(f) = f \circ \psi$ at a point $p \in \mathcal{O}_X(\psi^{-1}U)$

Definition 3.16
A morphism between schemes X and Y is a pair $(\psi, \psi^\#)$, s.t. $\psi : |X| \to |Y|$ continuous and $\psi^\#$ is a map of sheaves $\psi^\# : \mathcal{O}_Y \to \psi_*\mathcal{O}_X$ s.t. for any point $p \in X$ and any nbhd U of $q = \psi(p)$ in Y, a section $f \in \mathcal{O}_Y(U)$ vanishes at q if and only if $\psi^\#(f)$ of $\psi_*\mathcal{O}_X(U)$ vanishes at p.

Assume $X = \mathrm{Spec}R$ and $Y = \mathrm{Spec}S$, then $\psi^\# : \mathcal{O}_Y \to \psi_*\mathcal{O}_X$ induces a map

$$\mathcal{O}_{Y,q} = \mathrm{colim}_{U \ni q}\mathcal{O}_Y(U) \to \mathrm{colim}_{U \ni q}\mathcal{O}_X(\psi^{-1}U) \to \mathrm{colim}_{V \ni p}\mathcal{O}_X(V) = \mathcal{O}_{X,p}$$

If $p = [\mathfrak{p}] \in \mathrm{Spec}R$ and $q = \psi([\mathfrak{p}]) = [\mathfrak{q}] \in \mathrm{Spec}S$, then this says we have induced map

$$\mathcal{O}_{Y,q} = S_\mathfrak{q} \to R_\mathfrak{p} = \mathcal{O}_{X,p}$$

The condition of a morphism says $f \in \mathcal{O}_Y(U)$vanishes at $q \iff \psi^\# f$ vanishes at p. Hence this just says $f \in \mathfrak{q} \iff \psi^\# f \in \mathfrak{p}$ which says $S_\mathfrak{q} \to R_\mathfrak{p}$ sends the unique maximal ideal to maximal ideal. (This is called a local homomorphism)

Theorem 3.17
For any scheme X and R a ring , the morphisms $(\psi, \psi^\#) : X \to \mathrm{Spec}R$ are in one-to-one correspondence with ring homomorphism $R \to \mathcal{O}_X(X) = \psi_*(\mathcal{O}_X)(\mathrm{Spec}R)$ given by $(\psi, \psi^\#) \to \psi^\#$

Corollary 3.18
The category of affine schemes with morphisms is equivalent to the (opposite) category of commutative rings

Remark: There is a unique ring homomorphism $\mathbb{Z} \to R$. By Theorem, there is a unique morphism of schemes $X \to \mathrm{Spec}\mathbb{Z}$ for all sheaves X. Therefore $\mathrm{Spec}\mathbb{Z}$ is a terminal object in the category of schemes.

Definition 3.19
Assume X, Y, Z are sets with maps $\psi : X \to Z$ and $\phi : Y \to Z$, then the fibre product or pullback $X \times_Z Y$ is defined to be

$$X \times_Z Y = \{(x, y) \in X \times Y | \psi(x) = \phi(y)\}$$

It is universal with respect to the existence of properties $X \times_Z Y \to X$ and $X \times_Z Y \to Y$, i.e. the *diagram commutes:*

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

Assume $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$ and $Z = \mathrm{Spec}R$, then the tensor product $A \otimes_R B$ is universal w.r.t. the existence of $A \hookrightarrow A \otimes_R B$ and $B \hookrightarrow A \otimes_R B$ making the diagram commutes:

$$\begin{array}{ccc} A \otimes_R B & \longleftarrow & A \\ \uparrow & & \uparrow \\ B & \longleftarrow & R \end{array}$$

From these diagrams we can see that the fibred product of affine schemes should be $X \times_Z = \mathrm{Spec}(A \otimes_R B)$. Also note that the pullback defined above is the pullback in category of schemes as introduced in Example 2.2, and $A \otimes_R B$ is the pushout in the category of commutative rings.

3.7 Types of morphisms

There are two main problems with morphisms of schemes:

(1) $\psi : X \to Y$ then the image of a closed set need not be closed or open.

(2) If $\phi : X \to Y$ is another morphism of schemes, then the set of points where ψ and ϕ agree is not necessarily closed

Let $\psi : X \to Y$ be a morphism, we define the diagonal morphism to be the unique morphism

$$\Delta : X \to X \times_Y X$$

whose composition with the canonical projection is the identity.

Definition 3.20
We say ψ is separated if $\Delta(|X|)$ is closed in $|X \times_Y X|$. We say X is separated if the unique morphism $\psi : X \to \mathrm{Spec}\mathbb{Z}$ is separated.

Note: A topological space is Hausdorff if and only if the image of the space under the diagonal map is closed in the product space. So separatedness is the analogue of Hausdorff (and in the language of categories, the two definitions, using diagonal morphism, are formally the same).

In the case of affine schemes, $X = \mathrm{Spec}R$, then it is always separated. Because $\Delta(|X|)$ is closed corresponding to $\mathrm{Spec}(R \otimes_{\mathbb{Z}} R/I)$, where $I = \ker(\text{multiplication}) = \langle a \otimes 1 - 1 \otimes a | a \in R\rangle$.

Let $\psi : A \to B$ ring homomorphism, this induces $\hat{\psi} : Y = \mathrm{Spec}B \to X = \mathrm{Spec}A$ (by equivalence of categories). Take distinguished open set Y_g in $\mathrm{Spec}B$ and X_f in $\mathrm{Spec}A$ such that $\hat{\psi}(Y_g) \subseteq X_f$ then this induces $\mathcal{O}_Y(Y_g) = B_g \to \mathcal{O}_X(X_f) = A_f$. By the equivalence of categories, we can just as well start from a morphism $Y \to X$.

For general schemes X, Y, we want a morphism $f : X \to Y$ such that $\forall x \in X$, $\exists$ affine nbhd $U \ni x \in X$ and affine $V \ni f(x) \in Y$ such that $f|_U$ is a morphism of affine schemes.

4 Étale morphisms

Theorem 4.1 (Inverse function theorem)
M, N be smooth manifolds of the same dimension, and $f : M \to N$ morphism. Then $d_x f : T_x M \to T_{f(x)} N$ is isomorphism $\Leftrightarrow$ f local isomorphism at $x \in X$, i.e. there exists nbhd $U \ni x$ such that $f|_U$ is a diffeomorphism onto its image.

So differential *df* encodes local behaviour, which is less apparent in the construction of morphisms of schemes.

Definition 4.2
An <u>étale morphism</u> $f : Y \to X$ is a <u>flat</u> and <u>unramified</u> morphism of schemes.

Roughly, a flat morphism f is a morphism which has its fibre $f^{-1}(x)$ vary continuously w.r.t. x, i.e. we want $\dim f^{-1}(x) = \dim Y - \dim_x X$. An unramified morphism is something more complicated. But all these will be explained in the following.

Definition 4.3
Let X be a scheme and $x \in |X|$, recall the residue field $\kappa(x)$ of $\mathcal{O}_{X,x}$ is $\mathcal{O}_{X,x}/\mathfrak{m}_x$ where $\mathfrak{m}_x$ is the (unique) maximal ideal in $\mathcal{O}_{X,x}$.
The inclusion map $x \hookrightarrow |X|$ and the map $\mathcal{O}_{X,x} \to \kappa(x)$ define a morphism of schemes $\mathrm{Spec}\kappa(x) \to X$.
The <u>fibre</u> of a morphism $f : Y \to X$ is the pullback $Y \times_X \mathrm{Spec}\kappa(x)$, which we denote $f^{-1}(x)$.

The notation can be justified by showing a fibre is homeomorphic to the preimage of x of the underlying topological space. With this construction, the fibre $f^{-1}(x)$ is a scheme over $\mathrm{Spec}\kappa(x)$ (via the map $f^{-1}(x) \to \mathrm{Spec}\kappa(x)$).

Definition 4.4
Let $\phi : A \to B$ be ring homomorphism, then we say B is of <u>finite type over A</u> if $B \cong A[x_1, \ldots, x_n]/I$ (i.e. B is a finitely generated A-algebra)
For schemes, $\phi : Y \to X$ is of finite type if $\exists$ affine covers $\{U_i = \mathrm{Spec}B_i\}$ of Y and affine covers $\{V_j = \mathrm{Spec}A_j\}$ of X such that $\forall i, \exists j$ with $\phi(U_i) \subseteq V_j$, and B_j is a A_i-algebra of finite type.

Recall that a morphism of ring $\phi : A \to B$ is flat if the functor $M \otimes_A - : A$-Mod$\to B$-Mod is exact.

Definition 4.5
A morphism $f : Y \to X$ is <u>flat</u> if for all affine open sets $U \subseteq Y$ and $V \subseteq X$ with $f(U) \subseteq V$, we have $\mathcal{O}_X(V) \to \mathcal{O}_Y(U)$ is flat.

Remark: this happens if and only if the local homomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are flat for all $y \in Y$

4.1 Relation of finiteness to dimension

(Krull) Dimension of a ring R, $\dim R$, can be defined as $\sup\{n | \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$.
Hence we can define $\dim \mathcal{O}_{X,x}$
Let $f : Y \to X$ be a flat morphism of schemes, $x \in X$, consider the scheme $f^{-1}(x)$ over $\mathrm{Spec}\kappa(x)$, and we have

$$\dim \mathcal{O}_{f^{-1}(x),|\kappa(x)|} = \dim \mathcal{O}_{Y,y} - \dim \mathcal{O}_{X,x}$$

where $y \in \fallingdotseq^{-1} (x)$, note that $|\kappa(x)|$ is a one-point-space, which is identified with y.

Let F be a finitely generated filed extension of field K

Definition 4.6
We say that $\{x_i\} \in K$ *is* *transcendentally independent* *if* $\{x_i\}$ *do not satisfy any polynomial relations with coefficients in* K

A *transcendental basis* *is a collection* $\{x_i\}$ *of transcendentally independent elements such that* F *is algebraic over* $K(\{x_i\})$*; i.e. every* $x \in F$ *is a solution of a polynomial with coefficients in* $K(\{x_i\})$

The *transcendental dimension* $tr\dim_K F :=$ *cardinality of the transcendental basis.*

Theorem 4.7 (Krull)
Let A *be integral domain which is finitely generated* K*-algebra, then* $\dim A = tr\dim_K A$

Definition 4.8
Let A *be a ring,* $\mathrm{rad}(A) = \{$ *nilpotent elements of* $A\}$*. We say* A *is* *reduced* *if* $\mathrm{rad}(A) = \{0\}$*. Denote* A_{rad} *as* $A/\mathrm{rad}(A)$

Note that the canonical homomorphism $A \to A_{\mathrm{rad}}$ induces $\mathrm{Spec}(A_{\mathrm{rad}}) \xrightarrow{\sim} \mathrm{Spec}A$

Definition 4.9
A scheme X *is* *reduced* *if for all open* $U \subseteq X$*,* $\mathcal{O}_X(U)$ *is reduced*
If X *is reduced irreducible scheme* X*, then it is called* *integral*

Example
$X = \mathrm{Spec}A$, then X is integral $\Leftrightarrow$ A is integral domain (Exercise)

Definition 4.10
X *scheme,* $\mathrm{rad}_X : U \mapsto \mathrm{rad}(\mathcal{O}_X(U))$ *defines a sheaf of ideals.*
Define X_{red} *with structure sheaf* $\mathcal{O}_X/\mathrm{rad}_X$*.*

It can be shown that $X_{\mathrm{red}} \to X$ is a closed immersion (equivalently, X_{red} is isomorphic to a closed subscheme of X)

Definition 4.11
Let $\psi : A \to B$ *be homomorphism of local rings. Then* ψ *is* *unramified* *if* $B/\psi(\mathfrak{m}_A)B$ *is a finite, separable field extension of* $A/\mathfrak{m}_A$*. We say* B *is unramified over* A*.*
For $f : Y \to X$ *morphism of schemes of finite type, we say that it is* *unramified* *if* $\mathcal{O}_{X,f(y)}$ *is unramified over* $\mathcal{O}_{Y,y}$

The idea to keep in mind is that, in a "nice" situation, unramified morphism implies the preimage of any point is a finite set of the same size. In summary, an étale morphism is a morphism which is well-behaved with respect to tensor products, and preimage of each point is relatively nice and easy to work with.

4.2 Tangent Spaces

Assume $V \subseteq \mathbb{A}^n$ is an affine algebraic set. Assume $f \in I(V) \lhd K[X_1, \dots, X_n]$, then we associate to f the linear polynomial given by

$$d_p(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(p) X_i \quad \in K[X_1, \dots, X_n]$$

Definition 4.12
We define the *tangent space* *of* V *at* p *to be*

$$T_p(V) = \{v \in \mathbb{A}^n | d_p(f)(v) = 0 \ \ \forall f \in I(V)\}$$

Example
$f(X,Y) = X^2 + Y^2 - 1$ and $V = V(\langle f \rangle)$, $I(V) = \langle f \rangle$. Let $p = (0,1)$, then
$d_p(f) = \dfrac{\partial f}{\partial X_1}(0,1)X_1 + \dfrac{\partial f}{\partial X_2}(0,1)X_2 = 2X_2$
$T_p(V) = \{(v_1, 0) \in \mathbb{A}^2\}$ $(\dim T_p(V) \geq \dim V \ \ \forall p \in V)$

If A is an R-algebra and M is an A-module then <u>A-derivation</u> is a map $D : A \to M$ s.t.

- $D(a + a') = D(a) + D(a')$
- $D(aa') = aD(a') + D(a)a'$
- $D(r \cdot 1) = 0 \ \ \forall r \in R$

The space of such map is denoted $\mathrm{Der}_R(A, M)$.

Lemma 4.13
We have a K-linear isomorphism

$$\begin{array}{rcl} \Phi : T_p(V) & \to & \mathrm{Der}_K(K[V], K_p) \\ v & \mapsto & D_v \end{array}$$

where $D_V(\bar{f}) = d_p(f)(v) \ \ \forall \bar{f} \in K[V]$
So here K_p (a copy of the field K) is a module for $K[V]$ where the action is given by $f \cdot x = f(p)x$.

N.B. This isomorphism is also true for singular points.

4.3 The differential of a morphism

If $\phi : V \to W$ is a morphism of algebraic sets, then for $p \in V, q = \phi(p) \in W$, we denote the <u>differential</u> of ϕ at p to be

$$\begin{array}{rcl} d_p(\phi) : \mathrm{Der}_K(K[V], K_p) & \to & \mathrm{Der}_K(K[W], K_q) \\ D & \mapsto & D \circ \phi^* \end{array}$$

on the level of $T_p(V)$ and $T_q(W)$, we have the following. Assume $\phi = (f_1, \ldots, f_m)$, then the map $d_p(\phi) : T_p(V) \to T_q(W)$ is given by multiplication of the Jacobian

$$J_p(\phi) = \left(\frac{\partial f_i}{\partial X_j}(p) \right)_{1 \leq i \leq m; 1 \leq j \leq n}$$

Proposition 4.14
Assume V, W are irreducible and $d_p(\phi)$ is surjective where p and $q = \phi(p)$ are non-singular, then ϕ is <u>dominant</u>. i.e. $\overline{\phi(V)} = W$

4.4 A "local" version of tangent space

Assume $V \subseteq \mathbb{A}^n$ is irreducible and $p \in V$ then we form the local ring

$$K[V]_p = \{f/g | f, g \in K[V], g(p) \neq 0\}$$

Let $\epsilon_p : K[V]_p \to K$ be the evaluation homomorphism, i.e.$\epsilon_p(f) = f(p)$, then $\mathfrak{m}_p = \ker(\epsilon_p)$

Let $\kappa(p) = K[V]_p/\mathfrak{m}_p$ then $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a $\kappa(p)$-vector space. Note $\kappa(p) = K[V]_p/\mathfrak{m}_p$ then the action of $\kappa(p)$ on $\mathfrak{m}_p/\mathfrak{m}_p^2$ is given by

$$(f + \mathfrak{m}_p)(g + \mathfrak{m}_p^2) = fg + \mathfrak{m}_p^2$$

Proposition 4.15
There is a well-defined K-linear isomorphism

$$\begin{aligned} Hom_{\kappa(p)}(\mathfrak{m}_p/\mathfrak{m}_p^2, \kappa(p)) &\to \mathrm{Der}_K(K[V], K_p) && (4.1) \\ \mu &\mapsto D_\mu && (4.2) \end{aligned}$$

such that $D_\mu(a) = \mu(a + \mathfrak{m}_p^2)$ for all $a \in K[V]$ with $a(p) = 0$.

Corollary 4.16
There is a well-defined K-linear isomorphism $T_p(V) \to Hom_{\kappa(p)}(\mathfrak{m}_p/\mathfrak{m}_p^2, \kappa(p))$

4.5 Tangent spaces for schemes

Assume A is an R-algebra then the module of relative differentials of A over R is an A-module $\Omega_{A/R}$ together with an R-derivation $D : A \to \Omega_{A/R}$ which satisfies the following universal property:
For any A-module M and any R-derviation $D' : A \to M$, $\exists!$ A-module homomorphism $f : \Omega_{A/R} \to M$ s.t. $D' = f \circ D$.

This does exist. Define the free module generated by all symbols $D(a)$ and quotient by the appropriate relations for a derivation.

Example
If $A = K[X_1, \ldots, X_n]$ and $R = K$, then $\Omega_{A/R}$ is just the free module generated by $dX_1, \ldots, dX_n$.

Assume $I \lhd A$, let $\overline{A} = A/I$, then we have an exact sequence

$$I/I^2 \to \Omega_{A/R} \otimes_A \overline{A} \to \Omega_{\overline{A}/R} \to 0 \tag{4.3}$$

where the first map is induced by $f \mapsto D(f) \otimes 1$

Example
Again $A = K[X_1, \ldots, X_n]$, then (4.3) shows $\Omega_{\overline{A}/R}$ is the quotient of $\Omega_{A/R} \otimes_A \overline{A} = \bigoplus_{i=1}^n \overline{A} \cdot dX_i$ by the submodule generate by $df = \sum_{i=1}^n \frac{\partial f}{\partial X_i} dX_i$ for all $f \in I$ (Note I is a point).

Corollary 4.17
If A is local with residue field $k = A/\mathfrak{m}$ contained in A then we have an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/R} \otimes_A R$$

induced by (4.3).

Definition 4.18
Let $x \in X$ be a point of a scheme of finite type over K, write $\mathfrak{m}_x$ for the maximal ideal of the stalk $\mathcal{O}_{X,x}$. Then we define the tangent space at x to be

$$T_{X,x} = Hom_K(\mathfrak{m}_x/\mathfrak{m}_x^2, K)$$

4.6 Alternative definition of étale morphisms

Previously, étale morphism is motivated by the attempting to acquire an analogue of Inverse Function Theorem on schemes. Now we also want to have an analogue of Implicit Function Theorem for schemes.

Recall Implicit Function Theorem says that:
Let $x \in \mathbb{R}^{n+k}$, $f_1, \ldots, f_k$ continuous s.t. $\det_{1 \le i,j \le k} \left(\frac{\partial f_i}{\partial X_j}(x) \right) \neq 0$, then the restriction

$$\begin{aligned} \{\text{Locus of } f_1 = \cdots = f_k\} &\to \mathbb{R}^n && (4.4) \\ (x_1, \ldots, x_{n+k}) &\mapsto (x_{k+1}, \ldots, x_{k+n}) && (4.5) \end{aligned}$$

is locally an isomorphism at x

Example
$(2X = Y) \cong \mathbb{A}^1$
$\phi : V(X^2 - Y) \to \mathbb{A}^1$ via $(x, y) \mapsto y$
Let $p = (1, 1)$. For all but a finite number of values a contains both $(\sqrt{a}, a)$ and $(-\sqrt{a}, a)$.

Definition 4.19
Let $S := R[X_1, \ldots, X_n]/\langle f_1, \ldots, f_n \rangle$ and $X := \mathrm{Spec}S$. The morphism

$$\phi : X \to \mathrm{Spec}R \tag{4.6}$$

is étale at a point $x \in X$ if the

$$\det \left(\frac{\partial f_i}{\partial X_j}(x) \right) \neq 0 \tag{4.7}$$

(Equivalently the Jacobian is a unit in R) So (4.7) means exactly that $\Omega_{S/R} = 0$. Indeed, $\Omega_{S/R} =$ module gen by dX_i quotient out b relations $\sum_{i=1}^n \frac{\partial f_i}{\partial X_j} dX_j = 0$
$\Omega_{S/R}$ is zero when $\det(\frac{\partial f_i}{\partial X_j})$ is a unit.

Nevertheless, in general situation, we should use the original definition of étale morphism, i.e. a flat and unramified morphism. For convenience, we recall the definition of unramified morphism: a morphism $\phi : Y \to X$ such that the induced map on stalks $\phi_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ with maximal ideals $\mathfrak{m}_{X,f(y)}$ and $\mathfrak{m}_{Y,y}$ respectively satisfy (1) $\phi_y(\mathfrak{m}_{X,f(y)})\mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$ and (2) $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$ is finite separable field extension of $\mathcal{O}_{X,f(y)}/\mathfrak{m}_{X,f(y)}$. The following propositions says the two definitions of étale morphism is the same:

Proposition 4.20
ϕ_y is unramified at all $y \in Y$ $\Leftrightarrow$ $\Omega_{X/Y} = 0$

Proposition 4.21
A morphism $\phi : Y \to X$ is flat and unramified if and only if it locally is of the form of in Definition 4.19. Namely, for all $y \in Y$, then $\exists V = \mathrm{Spec}S \ni y$ and $U = \mathrm{Spec}R \ni f(y)$, such that $\phi|_V$ is of that form, and $S \cong R[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$ with $\det(\partial f_i/\partial X_j)$ is a unit in R.

This justifies that (4.6) is étale in the original definition.

Flatness of (4.6) comes from $R \to S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$ (i.e. that S is a finitely generated R-algebra). If R is a field, then S is an R-vector space, which is free and hence implies flatness.

Unramified comes from the Proposition 4.20: It suffices to see $\Omega_{S/R} = 0$. $\Omega_{S/R}$ is a S-module with generators $dX_1, \ldots, dX_n$ with relations $\sum \frac{\partial f_i}{\partial X_j} dX_j = 0$, this is equivalent to saying $\det(\frac{\partial f_i}{\partial X_j})$ is a unit.

4.7 Examples

More general examples:
(1) Open immersions are étale (since they induce an isomorphism on stalks)

(2) Closed immersions are usually NOT étale . The problem that usually comes up is flatness (see later).

Note that étale morphism has finite fibre, as unramified morphism have finite fibres

Unramified but not flat
(1) Assume K a field. Inclusion of a point in $\mathbb{A}^1$.

Geometric map: $\text{Spec}(K[X]/(X)) \to \text{Spec}(K[X])$
Ring hom: $K[X] \to K[X]/(X)$ via $f \mapsto f(0)$

Claim: $K[X]/(X) \otimes_{K[X]} -$ is not exact
Proof:
The functor is exact $\Leftrightarrow$ $K[X]/(X)$ as a $K[X]$-module is not flat
$\Leftrightarrow$ $K[X]/(X)$ is torsion. (Note this is true for K PID)
But $K \cong K[X]/(X)$ is a torsion $K[X]$-module as $K[X]/(X) = \{f \in K | f \cdot r = 0$ for some non-zero $r \in K[X]\}$.

Alternatively, we can use the short exact sequence $0 \to K[X] \xrightarrow{\cdot X} K[X] \to K \to 0$, then tensor with $K[X]/(X)$ we get $0 \to K[X]/(X) \xrightarrow{0} K[X]/(X) \to K \to 0$ which is not exact. ■

(2) $\mathbb{A}^1 \to \{y^2 = x^2 + x^3\} \subset \mathbb{A}^2$.

The later space has a singularity at 0 (The fibre at 0 contains 2 points and everywhere else is 1). Therefore this is not flat as flatness preserve the size of fibre everywhere.

Algebraically:
Geometric map: $\text{Spec}(K[t]) \to \text{Spec}(K[X,Y]/(Y^2 - X^2(X+1)))$
Ring hom: $S = K[X,Y]/(Y^2 - X^2(X+1)) \to B = K[t]$ via $X \mapsto t^2 - 1$ and $Y \mapsto t^3 - t$.

Claim: $K[t]$ is not a flat S-module.
Proof:
We use the fact that an A-module M is flat $\Leftrightarrow$ $\forall$ f.g. ideal I of A, $I \otimes M \to M$ is injective.

Take $I = \langle X, Y \rangle$, then

$$\begin{array}{rcl} I \otimes K[t] & \to & K[t] \\ x \otimes t^n & \mapsto & (t^2-1)t^n \\ y \otimes t^n & \mapsto & (t^3-t)t^n \end{array}$$

This is not injective as the non-zero element $x \otimes t - y \otimes 1 \mapsto (t^2-1)t - (t^3-t) = 0$.
So now we have the map $\mathbb{A}^1 \to \text{Spec}S$ is not flat (at 0).

However, this map is unramified as we will show that $\Omega_{B/S} = 0$. Note that $\Omega_{B/S} = \{db | b \in B \text{ etc.}\}$
$\Rightarrow$ we need $0 = dx = d(t^2-1) = 2tdt$ and $0 = dy = d(t^3-t) = (3t^2-1)dt$
But the gcd of $2t$ and $3t^2 - 1$ is 1, so $dt = 0$, hence $\Omega_{B/S} = 0$.

To conclude, $\mathbb{A}^1 \to \text{Spec}S$ is étale everywhere but 0.

A ramified example
(1) Suppose char $K \neq n$, $\mathbb{A}^1 \to \mathbb{A}^1$ via $X \mapsto X^n$. This is ramified at 0 as $\frac{dX^n}{dX}|_0 = nX^{n-1}|_0 = 0$.

Intuition
Étale maps $X \to \text{Spec}K$ are simply $\coprod_i \text{Spec}K_i \to \text{Spec}K$ where K_i are finite separable extension of

K
If X is affine, i.e. we have Spec$A \to$ Spec$K$ (induced by/to $K \to A$).

5 Étale site and cohomology

5.1 Sites

Definition 5.1
A site is a category $\mathcal{C}$ with Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$. Recall $\mathcal{T}$ is an assignment of Ob($\mathcal{C}$)$\to$ **Sets**, *assigning object X to a covering families of X.*

Zariski site is $\mathcal{C} = \mathrm{Op}(|X|)$ category of open sets of X where X is a scheme. Coverings in the Zariski site are given by $\{f_i : U_i \hookrightarrow U\}_{i\in I}$ such that $\bigcup_i f_i(U_i) = U$.

Étale site for X a scheme is given by the category $\mathcal{C}$, where objects are étale morphisms $U \to X$ (i.e. given by a pair (U, f) s.t. $f : U \to X$ is étale), the morphisms of $\mathcal{C}$ are given by the commutative diagram
$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$
where all the arrows in this diagram are étale .

The Grothendieck topology of $\mathcal{C}$ are defined by coverings $\{\text{étale} \quad f_i : U_i \to U\}_{i\in I}$ and $\bigcup_i f_i(|U_i|) = |U|$.

If U_i Zariski, then $U_i \hookrightarrow U$ are open immersion, which are étale .
$\therefore$ any Zariski covering is an étale covering.

Given a Grothendieck topology, we can define sheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ (where $\mathcal{D}$ can be category of abelian groups or category of rings, or etc.) by $F(U) \xrightarrow{\sim}$ equaliser($\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$) for all $\mathcal{U} = \{U_i \to U\}$ covering of $\mathcal{C}$.

The following proposition gives a very easy criterion to check whether a sheaf is defined on the étale site of X.

Proposition 5.2
F is an étale sheaf (i.e. a sheaf on the étale site of X) if and only if:

(1) it is a sheaf for Zariski coverings (sheaf of Zariski site of X)

(2) it is a sheaf for coverings $\{\text{étale} \quad V \to U\}$ (where V and U affine)

In particular, F is an étale sheaf, then it is a Zariski sheaf.

5.2 Étale cohomology

Let S be a scheme and $S_{\text{ét}}$ be its étale site, where $\mathcal{U} = \{U_i \xrightarrow{\phi_i} U\}_{i\in I}$ denote cover where $U = \bigcup_{i\in I} \phi_i(U_i)$. The functor $\begin{array}{rcl} \Gamma(S,-) : \mathrm{Sh}(S_{\text{ét}}) & \to & \mathbf{Ab} \\ \mathcal{F} & \mapsto & \Gamma(S,\mathcal{F}) \end{array}$ is left exact, and its right derived functor gives the étale cohomology (group) $H^*_{\text{ét}}(S;\mathcal{F})$. Explicitly, form an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$$

then $H^*_{\text{ét}}(S;\mathcal{F}) = H^*(\Gamma(S), I^*)$.

Example 5.3
The following are examples of sheaves we should always keep in mind

(1) $\underline{\mathbb{Z}/n\mathbb{Z}}$ *as the constant sheaf with value in* $\mathbb{Z}/n\mathbb{Z}$

(2) $\mathbb{G}_a$ *be sheaf* $U \mapsto \Gamma(U, \mathcal{O}_U)$

(3) $\mathbb{G}_m$ *be sheaf* $U \mapsto \Gamma(U, \mathcal{O}_U^\times)$

Property 1: $H^0_{\text{ét}}(S; \mathcal{F}) = \Gamma(S, \mathcal{F})$
Property 2: If I is injective, then $H^*_{\text{ét}}(S; I) = 0$ for all $* > 0$

In what follows, we will look at
(1) étale cohomology of a sheaf on a point $H^*_{\text{ét}}(\text{Spec}(K); \mathcal{F})$ for K field
(2) The Čech cohomology $\check{H}^*(S; \mathcal{F})$, which is more "geometeric" and "computable"
(3) Show $\check{H}^*(S; \mathcal{F}) = H^*_{\text{ét}}(S; \mathcal{F})$ for $* = 0, 1$.
(4) Mayer-Vietoris sequence for $H^*_{\text{ét}}$
(5) Compute $\check{H}^*(S; \mathbb{G}_m)$

5.3 Étale cohomology of a point

Let K be a field, so we can regard $\text{Spec}(K)$ as a point.
Define $\text{Spec}(K)_{\text{ét}}$ be category with objects being $\coprod_{i\in I} \text{Spec}(K_i)$ where K_i/K are finite separable extension and the indexing set I is finite. $\text{Spec}(K)_{\text{ét}}$ is a site with covers being surjective families.
Take K^{sep} be separable closure of K, and $G := \text{Gal}(K^{\text{sep}}/K)$ (strictly speaking, this should be the K-invariant automorphism group of K^{sep})
If we take intermediate field $K^{\text{sep}}/L/K$ s.t. L is (finite) Galois over K, we get homomorphism $G \to \text{Gal}(L/K)$.
We equip G with the profinite topology (the smallest topology making all of these homomorphisms continuous)

Galois theory gives a bijection:

$$\begin{array}{rcl} \left\{ \begin{array}{c|c} \text{intermediate field} & L/K \\ K^{\text{sep}}/L/K & \text{finite separable} \end{array} \right\} & \leftrightarrow & \{H \subseteq G \text{ open}\} \\ L & \mapsto & \text{Stab}_G(L) \\ (K^{\text{sep}})^H & \leftarrow\!\shortmid & H \end{array}$$

and equivalence of sites:

$$\begin{array}{rcl} \text{Spec}(K)_{\text{ét}} & \to & G\text{-}\mathbf{Sets} \\ X & \mapsto & \{\text{Spec}(K^{\text{sep}}) \to X \text{ which fixes Spec}K\} \\ \coprod_{i\in I} \text{Spec}((K^{\text{sep}})^{H_i}) & \leftarrow\!\shortmid & \coprod_{i\in I} G/H_i \end{array}$$

where G-**Sets** is a category of sets acted on by G for which G acts continuously and objects are finite union of orbits and morphisms are G-equivariant maps. This further gives equivalence of sheaf categories:

$$\begin{array}{ccccc} \text{Sh}(\text{Spec}(K)_{\text{ét}}) & \leftrightarrow & \text{continuous } G\text{-modules} & \leftrightarrow & \text{Sh}(G\text{-}\mathbf{Sets}) \\ \mathcal{F} & \mapsto & M_{\mathcal{F}} := \text{colim}_L \mathcal{F}(\text{Spec}(L)) & & \\ & & M & \mapsto & (A \mapsto \text{Map}_G(A, M)) \end{array}$$

where the colimit is taken over intermediate fields L which are finite Galois over K.

Theorem 5.4
$H^*_{ét}(\mathrm{Spec}(K), \mathcal{F}) \cong H^*(G; M_{\mathcal{F}})$ where the later (group) cohomology is computed by resolving the G-module $M_{\mathcal{F}}$ by continuous G-modules. (It is called the continuous group cohomology of profinite groups)
$\mathrm{Spec}(K)$ *is a "geometric point" if K is separably closed. In this case $K^{\mathrm{sep}} = K$, then $G = \{1\}$, then $H^*_{ét}(\mathrm{Spec}(K), \mathcal{F}) = 0 \ \forall * > 0$.*

5.4 Čech Cohomology

We now work over general site $\mathcal{C}$. Let $\mathcal{F} \in \mathrm{Sh}(\mathcal{C})$, $U \in \mathcal{C}$, $H^*(U; \mathcal{F}) = H^*(\Gamma(U, I^*))$, with $\mathcal{F} \to I^0 \to I^1 \to \cdots$ is an injective resolution.

Definition 5.5
Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a cover of U, and let $\mathcal{F}$ be a presheaf. The Čech complex $C^*(\mathcal{U}; \mathcal{F})$ is

$$\prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{d^0} \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \xrightarrow{d^1} \cdots$$

where $d^l = \sum_{k=1}^{l} (-1)^k d^l_k$, where

$$d^l_k : \prod \mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_l}) \to \prod \mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_{l+1}})$$

is induced by the morphism

$$U_{i_0} \times_U \cdots \times_U U_{i_{l+1}} \to U_{i_0} \times_U \cdots \times \widehat{U_{i_k}} \times \cdots \times U_{i_{l+1}}$$

Definition 5.6
The Čech cohomology of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is $\check{H}^*(\mathcal{U}; \mathcal{F}) = H^*(C^*(\mathcal{U}; \mathcal{F}))$

5.5 Changing the cover

Recall that, we say $\mathcal{V} = \{V_j \to U\}_{j \in U}$ refines $\mathcal{U}$ if for $j \in J$, we can find

$$\begin{tikzcd} V_j \arrow[rr, "\phi_j"] \arrow[dr] & & U_{\tau_j} \arrow[dl] \\ & U & \end{tikzcd}$$

$\tau_j \in I$. The morphisms
$$V_{j_0} \times_U \cdots \times_U V_{j_l} \to U_{\tau_{j_0}} \times_U \cdots \times_U U_{\tau_{j_l}}$$
induces $C^*(\mathcal{U}; \mathcal{F}) \to C^*(\mathcal{V}; \mathcal{F})$ which induces $\check{H}^*(\mathcal{U}; \mathcal{F}) \to \check{H}^*(\mathcal{V}; \mathcal{F})$, which is independent of choices.

Definition 5.7
The Čech cohomology of U with coefficient in $\mathcal{F}$ is $\check{H}^*(U; \mathcal{F}) = \mathrm{colim}_{\mathcal{U}} \check{H}^*(\mathcal{U}; \mathcal{F})$

5.6 The properties

Lemma 5.8
If $\mathcal{F}$ is a sheaf then $\check{H}^0(U; \mathcal{F}) = \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.
More generally, if $\mathcal{F}$ is a presheaf, $\check{H}^0(U; \mathcal{F}) = a\mathcal{F}(U)$ where $a\mathcal{F}$ is the sheafification of $\mathcal{F}$.

Proof
$\check{H}^0(\mathcal{U};\mathcal{F}) = \ker(\prod \mathcal{F}(U_{i_0}) \xrightarrow{d^0=d^0_0-d^0_1} \prod \mathcal{F}(U_{i_0} \times_U U_{i_1})) = \text{equaliser}(\prod \mathcal{F}(U_{i_0}) \rightrightarrows \prod \mathcal{F}(U_{i_0} \times_U U_{i_1}))$
If $\mathcal{F}$ is a sheaf, then this equaliser is precisely $\mathcal{F}(U)$. Generally, $a\mathcal{F}(U)$ is the colimit of all these equalisers. □

Lemma 5.9
If I is injective then $\check{H}^*(U;I) = 0$ for $* > 0$.

Proof
In fact $\check{H}^*(\mathcal{U};I) = 0$ for all $* > 0$ whenever $\mathcal{U}$ is a cover and I is an injective presheaf.

Define a chain complex of presheaves

$$\mathbb{Z}\mathcal{U}_* := (\mathbb{Z}_{\coprod U_{i_0}} \leftarrow \mathbb{Z}_{\coprod U_{i_0}\times_U U_{i_1}} \leftarrow \cdots)$$

where $\mathbb{Z}_V$ is presheaf defined by $\mathbb{Z}_V(W) := \mathbb{Z}\,\mathrm{Hom}(W,V)$
$\Rightarrow \quad C^*(\mathcal{U};I) = \mathrm{Hom}(\mathbb{Z}\mathcal{U}_*, I)$
As I injective, $\mathrm{Hom}(-,I)$ exact. So it suffices to show that $\mathbb{Z}\mathcal{U}_*$ is exact in positive degrees.
As we are working with presheaves, it suffices to prove $\mathbb{Z}\mathcal{U}_*(V)$ is exact in positive degrees.

$$\begin{aligned}
\mathbb{Z}\mathcal{U}_*(V) &= \mathbb{Z}\,[\coprod \mathrm{Hom}(V,U_{i_0})] \leftarrow \mathbb{Z}\,[\coprod \mathrm{Hom}(V,U_{i_0}\times_U U_{i_1})] \leftarrow \cdots \\
&= \mathbb{Z}\,[\coprod \mathrm{Hom}(V,U_{i_0})] \leftarrow \mathbb{Z}\,[\coprod \mathrm{Hom}(V,U_{i_0}) \times_{\mathrm{Hom}(V,U)} \mathrm{Hom}(V,U_{i_1})] \leftarrow \cdots \\
&= \bigoplus_{\phi\in\mathrm{Hom}(V,U)} \Big(\mathbb{Z}\,[\coprod \mathrm{Hom}_\phi(V,U_{i_0})] \leftarrow \mathbb{Z}\,[\coprod \mathrm{Hom}_\phi(V,U_{i_0}) \times \mathrm{Hom}_\phi(V,U_{i_1})] \leftarrow \cdots\Big)
\end{aligned}$$

where $\mathrm{Hom}_\phi(V,U_j)$ means space of homomorphism for which ϕ factors through U_j.
The complex then becomes a sum of complexes

$$\mathbb{Z}\,[S] \leftarrow \mathbb{Z}\,[S\times S] \leftarrow \mathbb{Z}\,[S\times S\times S] \leftarrow \cdots$$

which are always exact in positive degrees. □

Theorem 5.10
There is a spectral sequence $\check{H}^s(U;H^t_{\acute{e}t}(\mathcal{F})) \Rightarrow H^{s+t}_{\acute{e}t}(U;\mathcal{F})$. In fact, this is true if we replace étale cohomology $H^*_{\acute{e}t}$ with cohomology on any site H^*.

Remark: For $\mathcal{F}$ sheaf, $H^q(\mathcal{F})$ is the presheaf $V \mapsto H^q(V;\mathcal{F})$

Corollary 5.11
$\check{H}^*(U;\mathcal{F}) = H^*_{\acute{e}t}(U;\mathcal{F})$ for $* = 0,1$ (Again, we can replace $H_{\acute{e}t}$ by H)

5.7 Introduction to spectral sequences

Definition 5.12
A (first quadrant, cohomological) spectral sequence, denote (E_r, d_r) consists of

- *Abelian groups $E_r^{s,t}$ for $s,t \geq 0, r \geq 1$*
  For a fixed r, the collection $E_r^{s,t}$ is called an E_r-page
- *Differentials $d_r^{s,t} : E_r^{s,t} \to E_r^{s+r,t-r+1}$, which satisfy $d_r^{s+r,t-r+1} d_r^{s,t} = 0$*

We can the form cohomology $H^{s,t}(E_r) = \frac{\ker d_r^{s,t}}{\operatorname{Im} d_r^{s-r,t+r-1}}$
These satisfy $H^{s,t}(E_r) = E_{r+1}^{s,t}$
Each page is the cohomology of the previous one.

Definition 5.13
Observe that if $r > \max(t+1, s)$, then $d_r^{s,t} = 0$ and $d_r^{s-r,t+r-1} = 0$. So $H^{s,t}(E_r) = E_r^{s,t}$.
In other words, for s, t fixed, $E_r^{s,t} = E_{r+1}^{s,t} = E_{r+2}^{s,t} = \cdots$. The common value is called $E_\infty^{s,t}$. The collection of $E_\infty^{s,t}$ is called the $\underline{E_\infty\text{-page}}$

Definition 5.14
Let F^* be a graded abelian group. We say that (E_r, d_r) converges to F^*, and write $E_1^{s,t} \Rightarrow F^{s+t}$, if there is a filtration $F^* = FF_0^* \geq F_1^* \geq F_2^* \geq \cdots$ with $F_q^p = 0$ for $q > p$, such that $E_\infty^{s,t} = F_s^{s+t}/F_{s+1}^{s+t}$.

Theorem 5.15
Let $E \xrightarrow{\pi} B$ be a fibration, fibre F, and $\pi, B = 0$. There is a spectral sequence $H^*(B; H^*F) \Rightarrow H^*(E)$

5.8 Double complexes

A double complex is an array of abelian groups $\{C^{s,t}\}_{s,t\in\mathbb{N}_0}$ and maps $d_h : C^{s,*} \to C^{s+1,*}$ (the horizontal differential) and $d_v : C^{*,t} \to C^{*,t+1}$ (the vertical differential) such that $d_h^2 = 0 = d_v^2$ and $d_v d_h = d_h d_v$.

Definition 5.16
$H_h^{*,\bullet}$ is the bigraded group obtained by taking vertical cohomology, then horizontal cohomology, of $C^{*,\bullet}$

$\operatorname{Tot}(C)^* = \bigoplus_{s+t=*} C^{s,t}$ *equipped with differential $d_h + (-1)^t d_v$ is the total complex of C*

Theorem 5.17
There is a spectral sequence $H_h^{*,\bullet}(C) \Rightarrow H^*(\operatorname{Tot}(C))$

Remark We can flip $C^{s,t}$ (diagonally) to obtain another double complex $C^{t,s}$, which we get us another spectral sequence converging to the same thing. Equivalently, the second spectral sequence comes from taking $H_v^{*,\bullet}$ of $C^{*,\bullet}$. This will help us to obtain our Theorem 5.10.

Proof of Theorem 5.10
Let $\mathcal{F} \to I^*$ be an injective resolution and consider the double complex $C^*(\mathcal{U}, I^*)$.

(1) What is $H_h^{s,t}(C(\mathcal{U}; I))$?
Vertical part:

$$\begin{tikzcd}
\vdots & \vdots & \\
\prod_{i_0} I^1(U_{i_0}) \arrow[u] & \prod_{i_0,i_1} I^1(U_{i_0} \times_U U_{i_1}) \arrow[u] & \cdots \\
\prod_{i_0} I^0(U_{i_0}) \arrow[u] & \prod_{i_0,i_1} I^0(U_{i_0} \times_U U_{i_1}) \arrow[u] & \cdots
\end{tikzcd}$$

Vertical cohomology

$$\begin{tikzcd}
\vdots & \vdots & \\
\prod_{i_0} H^1(U_{i_0}, \mathcal{F}) \arrow[u] & \prod_{i_0,i_1} H^1(U_{i_0} \times_U U_{i_1}; \mathcal{F}) \arrow[u] & \cdots \\
\prod_{i_0} H^0(U_{i_0}, \mathcal{F}) \arrow[u] & \prod_{i_0,i_1} H^0(U_{i_0} \times_U U_{i_1}; \mathcal{F}) \arrow[u] & \cdots
\end{tikzcd}$$

which is:

$$\begin{tikzcd}
\vdots & \vdots & \\
C^0(\mathcal{U}, H^1(\mathcal{F})) \arrow[u] & C^1(\mathcal{U}, H^1(\mathcal{F})) \arrow[u] & \cdots \\
C^0(\mathcal{U}, H^0(\mathcal{F})) \arrow[u] & C^1(\mathcal{U}, H^1(\mathcal{F})) \arrow[u] & \cdots
\end{tikzcd}$$

Take horizontal cohomology:

$$\begin{tikzcd}
\vdots & \vdots & \\
\check{H}^0(\mathcal{U}, H^1(\mathcal{F})) \arrow[u] & \check{H}^1(\mathcal{U}, H^1(\mathcal{F})) \arrow[u] & \cdots \\
\check{H}^0(\mathcal{U}, H^0(\mathcal{F})) \arrow[u] & \check{H}^1(\mathcal{U}, H^1(\mathcal{F})) \arrow[u] & \cdots
\end{tikzcd}$$

which is $\check{H}^*(\mathcal{U}, H^\bullet(\mathcal{F}))$

(2) What is $H^*(\mathrm{Tot}(C(\mathcal{U}; I)))$?
To answer this, we map flip the double complex diagonally. Taking vertical cohomology:

$$\begin{tikzcd}
\vdots & \vdots & \\
\check{H}^1(\mathcal{U}, I^0) \arrow[u] & \check{H}^1(\mathcal{U}, I^1) \arrow[u] & \cdots \\
\check{H}^0(\mathcal{U}, I^0) \arrow[u] & \check{H}^0(\mathcal{U}, I^1) \arrow[u] & \cdots
\end{tikzcd}$$

By Lemma 5.8, the bottom row is $I^0(U), I^1(U), I^2(U), \ldots$; and second row is $\check{H}^1(\mathcal{U}; I^0), \check{H}^1(\mathcal{U}, I^1), \ldots$. By Lemma 5.9, every rows above the bottom one are zero. Hence when we take the horizontal cohomology we get $H^0(\mathcal{U}; \mathcal{F}), H^1(\mathcal{U}, \mathcal{F}), \ldots$. These are E_2-term of a spectral sequence converging to $H^*(\mathrm{Tot}(C))$. But $E_2 = E_\infty$ for positioning reason (all differential are zero, each diagonal has only one non-zero term). i.e. $E_\infty^{*,0} = H^*(\mathrm{Tot}(C))$ since $E_\infty^{*,t} = 0$ for $t > 0$
$\Rightarrow \quad H^*(\mathrm{Tot}(C)) = H^*(U; \mathcal{F})$

(3) Conclusion:
The spectral sequence of the original double complex now has the form

$$\check{H}^*(\mathcal{U}; H^*(\mathcal{F})) \Rightarrow H^*(U; \mathcal{F})$$

which is our theorem. $\square$

Remark: Taking colimits, we can replace $\mathcal{U}$ by U. (Exercise/Think carefully)

Proof of Corollary 5.11
Use $\check{H}^*(U; H^*(\mathcal{F})) \Rightarrow H^*(U;\mathcal{F})$
First, $\check{H}^0(U; H^t(\mathcal{F})) = aH^t(\mathcal{F})(U) = 0$ for $t > 0$. This is because the sheaf $aH^t(\mathcal{F})$ is zero, the reason for this is that

$$aH^t(\mathcal{F}) = aH^t(iI^*) = H^t(aiI^*) = H^t(I^*) = 0$$

where i is the inclusion functor sending sheaves to presheaves.

Also $\check{H}^0(U; H^0(\mathcal{F})) = H^0(\mathcal{F})(U) = H^0(U;\mathcal{F})$.
Now $\check{H}^0(U; H^*(\mathcal{F})) = 0$ for $* > 0$
and $\check{H}^*(U; H^0(\mathcal{F})) = \check{H}^*(U;\mathcal{F})$
So we get E_∞ has

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \cdots \\
0 & ? & ? & \cdots \\
\check{H}^0(U;\mathcal{F}) & \check{H}^1(U;\mathcal{F}) & ? & \cdots
\end{array}$$

So $\check{H}^0(U;\mathcal{F}) = H^0(U;\mathcal{F})$ and $\check{H}^1(U;\mathcal{F}) = H^1(U;\mathcal{F})$. $\square$

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