Topics in Groups Theory

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Books:
Alperin and Bell (Springer), John S. Rose (Dover), M. Suzuki
Robert A. Wilson, The Finite Simple Groups, Springer 2009
This course will follow book of Wilson's, which is relatively harder to read.

1 Introduction

G is a group, finite; $|G|$ its order.

Theorem 1.1 (Lagrange)
G finite, $H \leq G \quad \Rightarrow \quad |H| \big| |G|$

Sketch proof
Distinct right cosets $Hg = \{hg | h \in H\}$ cover G and all have size $|H|$
$(G : H)$ the set of all right cosets of H in G. Then $|G| = |H| \cdot |G : H|$ $\square$

The map $\theta : G \to H$ is a homomorphism if $(g_1 \cdot g_2)\theta = g_1\theta \cdot g_2\theta$
It is an isomorphism!of groups if also bijective
$\ker(\theta) = \{g \in G | g\theta = e_H\}$
θ is injective $\Leftrightarrow \ker\theta = \{e_G\}$
$\ker\theta \trianglelefteq G$

$K \leq G$ is a normal subgroup of G, write $K \trianglelefteq G$, if $Kg = gK \; \forall g \in G$
G is simple if there are no normal subgroups other than $\{1\}$ and G, equivalently, G simple $\Leftrightarrow$ any non-trivial homom from G is injective
$K \trianglelefteq G$, can form G/K, a quotient group (or factor group) on the set $(G : K)$, by $Kg_1Kg_2 = Kg_1g_2$
$|G/K| = |G|/|K|$

Theorem 1.2 (Isomorphism Theorem)
If $\theta : G \to H$ homomorphism, then $\ker\theta \trianglelefteq G$, $\operatorname{Im}\theta = \{g\theta | g \in G\} \leq H$, and there is an isomorphism $G/\ker\theta \cong \operatorname{Im}\theta$
Sketch proof
Writing $K = \ker\theta$, check $K \trianglelefteq G$, $\operatorname{Im}\theta \leq H$ and

$$\begin{aligned} \overline{\theta} : G/K &\to \operatorname{Im}\theta \\ Kg &\mapsto g\theta \end{aligned}$$

is a well-defined isomorphism $\square$

So, homomorphic images of G are just quotients of G
Note: If $K \trianglelefteq G$, then

$$\begin{aligned} \pi : G &\twoheadrightarrow \overline{G} := G/K \\ g &\mapsto \overline{g} := Kg \end{aligned}$$

i.e. quotients of G are homomorphic images of G

$$\begin{tikzcd} \theta : G \arrow[rr] \arrow[dr, "\pi"'] & & \operatorname{Im}\theta \\ & G/K \arrow[ur, dashed, "\overline{\theta}"'] & \end{tikzcd}$$

Theorem 1.3 (Second Isomorphism Theorem - Correspondence Theorem)
Let $K \trianglelefteq G$. Then every subgroup $\overline{H}$ of G/K is of the form H/K for some unique H with $K \leq H \leq G$
We get a lattice isomorphism between $\{$all subgroups of $G/K\}$ and $\{$all subgroups of G containing $K\}$

$$\begin{tikzcd} G/K \arrow[r, dotted, no head] \arrow[d, no head] & G \arrow[d, no head] \\ H/K = \overline{H} \arrow[r, dotted, no head] \arrow[d, no head] & H \arrow[d, no head] \\ \{1_{G/K}\} \arrow[r, dotted, no head] & K \end{tikzcd}$$

Moreover, $H/K \trianglelefteq G/K \Leftrightarrow H \trianglelefteq G$ and if so $\frac{G/K}{H/K} \cong G/H$
Proof
If $H \trianglelefteq G$, define a homomorphism

$$\begin{aligned} G/K &\to G/H \\ Kg &\mapsto Hg \end{aligned}$$

This is a homomorphism surjective onto G/H, kernel H/K $\square$

Theorem 1.4 (Third Isomorphism Theorem)
Let $K \trianglelefteq G$, $H \leq G$. Then $HK \leq G$ $(HK := \{hk | h \in H, k \in K\})$
$H \cap K \trianglelefteq H$ and $H/H \cap K \cong HK/K$
Proof
The homomorphism

$$\begin{aligned} \pi|_H : H &\to G/K \\ h &\mapsto Kh \end{aligned}$$

has image HK/K, kernel $H \cap K$ $\square$

The group of automorphisms is

$$\operatorname{Aut}(G) = \{\theta : G \to G \text{ isomorphisms}\}$$

with group operation being composition, we write g^x for the image of $g \in G$ under $x \in \operatorname{Aut}(G)$

Inner automorphisms (conjugation autos) are, for $g \in G$, $\theta_g : x \mapsto g^{-1}xg$ for $x \in G$

Then θ_g is an auto of G and the map $\begin{aligned} \theta : G &\to \operatorname{Aut}(G) \\ g &\mapsto \theta_g \end{aligned}$ is a homomorphism.

Define the followings:

$$\begin{array}{rrcl} \underline{\text{Group of inner automorphisms}} & \mathrm{Inn}(G) & := & \mathrm{Im}\,\theta \trianglelefteq \mathrm{Aut}(G) \\ \underline{\text{Outer automorphism group}} & \mathrm{Out}(G) & := & \mathrm{Aut}(G)/\mathrm{Inn}(G) \\ \underline{\text{Centre of } G} & Z(G) & = & \ker\theta = \{z \in G | zx = xz \; \forall x \in G\} \end{array}$$

(Note that for G abelian, $Z(G) = G$, $\mathrm{Inn}(G) = \{e\}$, $\mathrm{Aut}(G) = \mathrm{Out}(G)$)

Exercise: $\mathrm{Aut}(D_8) \cong D_8, \mathrm{Aut}(Q_8) \cong S_4, \mathrm{Aut}(A_5) \cong S_5$
Exercise: $G = p^d$-elementary abelian of order p^d (i.e. an abelian group of order p^d with $x^p = 1 \; \forall x \in G$), then $\mathrm{Aut}(G) = GL_d(p)$

G finite.
If $Z(G) = \{1\}$, then $G \cong \mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$
If G is simple, we have Schreier "conjecture "

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G) \text{ is "small" and solvable}$$

This is a theorem now, as a consequence of the classification of finite simple groups.
E.g. $\mathrm{Aut}(A_n) = S_n$ for $n = 5$ or $n > 6$
$\mathrm{Aut}(A_6) = \Sigma_6$, group of order $2 \cdot 6!$

Important definition: G^* is almost simple if there is a simple group G with $G \cong \mathrm{Inn}(G) \trianglelefteq G^* \leq \mathrm{Aut}(G)$ (G is the unique minimal normal subgroup of G^*, and $G^*/G \leq \mathrm{Out}(G)$)

Example: $H \trianglelefteq K \trianglelefteq G \not\Rightarrow H \trianglelefteq G$ (Exercise: give an example)

Definition
K char G (K is characteristic subgroup of G) if $K^\alpha (:= \alpha(K)) = K \quad \forall \alpha \in \mathrm{Aut}(G)$
i.e. K is α-invariant $\forall \alpha \in \mathrm{Aut}(G)$
G is characteristically simple group if it has no proper non-trivial characteristic subgroup

K char $G \quad \Rightarrow \quad K \trianglelefteq G$, since $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$

Exercise:
(1) H char $K \trianglelefteq G \quad \Rightarrow \quad H \trianglelefteq G$
(2) H char K char $G \quad \Rightarrow \quad H$ char G

Example: $Z(G)$ char G, G' char G
$G' = \langle [x, y] | x, y \in G \rangle$ – commutator subgroup

Exercise: $K \trianglelefteq G, G/K$ abelian $\Leftrightarrow G' \leq K$

G is perfect if $G = G'$

2 Series, Jordan-Hölder Theorem

G is finite (or, if infinite, need some DCC)

Definition
A series is a chain of subgroups G_i:

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

A series is normal if also $G_i \trianglelefteq G \ \forall i$

A proper series for G has all quotients G_i/G_{i-1} nontrivial.

A proper series is a composition series if all quotients G_i/G_{i-1} are simple (so, a composition series is one that cannot be properly refined)

Theorem 2.1
Any finite group has a composition series
Proof
If G simple, stop.
Otherwise, let K be a maximal normal subgroup of G; then G/K simple and $|K| < |G|$ so contained in K $\square$

Remark. This is constructive; in fact $1 \trianglelefteq H \trianglelefteq G$ is a series for any $H \trianglelefteq G$

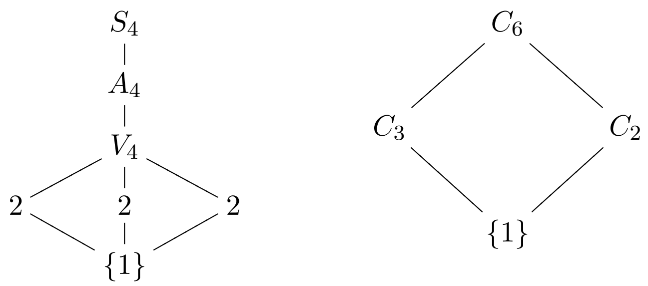
E.g.: $\mathbb{Z}$ has no (finite) composition series

Theorem 2.2 (Jordan-Hölder)
Let G be finite. Then any two composition series have the same quotients, counted with multiplicity

$$\begin{array}{rcccccccccc} & 1 & = & G_0 & \trianglelefteq & G_1 & \trianglelefteq & \cdots & \trianglelefteq & G_a & = & G \\ \text{and} & 1 & = & H_0 & \trianglelefteq & H_1 & \trianglelefteq & \cdots & \trianglelefteq & H_b & = & G \end{array}$$

two composition series, then $a = b$ and the collectors $\{G_i/G_{i-1}\}$ and $\{H_i/H_{i-1}\}$ are the same up to isomorphism

Example:



Proof
Put $H = H_{b-1}$, $K = G_{a-1}$

Case: $H = K$
Since $|H| < |G|$, induction applies:
We have two composition series of H, so they are "isomorphic"

Case: $H \neq K$

Note that $HK \trianglelefteq G \quad \Rightarrow \quad G = HK$ by maximality

$$\begin{tikzcd}
& G = HK \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] \arrow[dd, dash] & & K \arrow[dl, dash] \arrow[dd, dash] \\
& H \cap K \arrow[dd, dash] & \\
H_{b-2} \arrow[dd, dotted, dash] & & G_{a-2} \arrow[dd, dotted, dash] \\
& L_{c-2} \arrow[dd, dotted, dash] & \\
H_1 \arrow[dr, dash] & & G_1 \arrow[dl, dash] \\
& \{1\} &
\end{tikzcd}$$

Find a composition series for $H \cap K$
Then we have four composition series for G. They are isomorphic series "naturally defined"

Firstly, by induction, the two composition series for H are isomorphic (induction hypothesis), and so are those for K

Hence, the two left series for G are isomorphic, and so are the two on the right.
But also the two in the middle are isomorphic:
This is because, $G/H = HK/H \cong K/H \cap K$, and $G/K \cong H/H \cap K$ $\square$

Remark. All finite groups can be "broken up" into simples, i.e. have series with all quotients simple. On the other hand, building all finite groups out of simple groups is complicated.

E.g. There are, up to isomorphism, 26 different groups of order 2^6, all have six composition factor C_2
E.g. Composition factors C_2, A_5: $\quad C_2 \times A_5, \quad S_5 = \mathrm{Aut}(A_5), \quad SL_2(5)$
$(Z(SL_2(5)) = \{\pm I\}, SL_2(5)/\{\pm I\} \cong A_5)$

A chief series is a proper normal series which cannot be properly refined to a proper normal series. i.e. $\forall i \nexists A \trianglelefteq G$ s.t. $G_i \trianglelefteq A \trianglelefteq G_{i+1}$. The chief factors are characteristically simple

E.g. $S_4 \overset{C_2}{-} A_4 \overset{C_3}{-} V_4 \overset{C_2^2}{-} 1$

Any finite group has a chief series.
Any normal series can be refined to a chief series.
Any two chief series are "isomorphic"

Theorem 2.3
Let N be minimal normal subgroup of G. Then N is direct product of isomorphic simple groups, all conjugate in G
Proof
Let K be a minimal normal subgroup of N.
If $K = N$, stop
If $K \lneqq N$, then $K \ntrianglelefteq G$
$\Rightarrow \quad \exists g_1 \in G$ with $K^{g_1} \neq K \quad \Rightarrow \quad K \cap K^{g_1} = \{1\}$
Let $K_2 := KK^{g_1}$ (which is $\trianglelefteq N$)
Since $K \cap K^{g_1} \trianglelefteq N \quad \Rightarrow \quad K_2 \cong K \times K^{g_1}$

If $K_2 = N$, stop
Otherwise, $K_2 \lneqq N \quad \Rightarrow \quad K_2 \ntrianglelefteq G \quad \Rightarrow \quad \exists h \in G$ with $K_2^h \neq K_2$
i.e. $\exists k \in K$ s.t. $(kk^{g_1})^h \notin K_2$
Notice $(kk^{g_1})^h = k^h \cdot k^{g_1 h}$ (Aim: $\exists g_2 \in G$ s.t. $K_2 \cap K^{g_2} = \{1\}$)
$$\Rightarrow \begin{cases} \text{either} \quad k^h \notin KK_1 & \Rightarrow K^h \nleq K_2 \quad g_2 := h \\ \quad\ \text{or} \quad k_1^{g_1^h} \notin KK_1 & \Rightarrow K^{g_1 h} \nleq K_2 \quad g_2 := g_1 h \end{cases}$$

$\Rightarrow \quad K_3 := K_2 K^{g_2} \trianglelefteq N$, and $K_2 \cap K^{g_2} = 1$, so $K_3 \cong K \times K_2$.

Continue this and we get $N \cong K \times K^{g_1} \times K^{g_2} \times \cdots \times K^{g_k}$.
Finally, K is simple, since, if $X \trianglelefteq K$, then $X \trianglelefteq N$ (as $N \cong K \times K^{g_2} \times \cdots \times K^{g_k}$), so $X = K$ by minimality of K as normal in N □

Remark.

(1) The proof applies to characteristically simple groups N i.e. Characteristically simple group N are direct products of isomorphic simple groups
In particular, all chief factors of a finite group are also direct products of isomorphic simple groups.

(2) If N is characteristically simple group, then $N = T_1 \times \cdots \times T_k$, with each $T_i \cong T$ some simple group T. Either $T = C_p$ prime p, or T is non-abelian simple

Exercise:

(1) If $T = C_p$ then $N = V_k(p)$, a vector space over $\mathbb{F}_p$ of dimension k. There are $\frac{p^k-1}{p-1} + \frac{p^k-1}{p^2-1} \cdot \frac{p^k-p}{p^2-p} + \cdots$ normal subgroup

(2) If T is non-abelian simple, there are precisely 2^k normal subgroup. Namely, $T_{i_1} \times T_{i_2} \times \cdots \times T_{i_l}$

2.1 Groups with operators, X-groups

(see J.S. Rose book for more details)

Definition
X group, G is an X-group if $\phi : X \to \mathrm{Aut}(G)$ a homomorphism
Define operation of X on G by $g^x := g^{(x^\phi)}$

$H \leq G$ is an X-subgroup of G, write $H \leq_X G$, if H is X-invariant (Note: $H \leq_X G \Rightarrow H$ is an X-group)
If $H \trianglelefteq G, H \leq_X G \Rightarrow G/H$ is an X-group via $(Hg)^x := Hg^x$
If G_1, G_2 are X-groups (X fixed) a homomorphism $\alpha : G_1 \to G_2$ is an X-homomorphism if

$$(g^x)^\alpha = (g^\alpha)^x \text{ for } x \in X, g \in G_1$$

An X-group is X-simple if it has no non-trivial normal X-invariant subgroups
If G is an X-group, an X-composition series is a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq Ga = G$$

with each $G_i \leq_X G_{i+1}, G_i \trianglelefteq G_{i+1}$, which cannot be properly refined.

E.g.

G	X	X-subgroup	X-series	X-comp. series
G	1	subgroup	series	comp. series
G	$\mathrm{Inn}(G)$	normal subgroup	normal series	chief series
G	$\mathrm{Aut}(G)$			

Another example: G-modules
G any group, $\rho : G \to GL(V)$ a finite dimensional representation over some field. Set $X = \mathrm{Im}\,\rho$, then V is an X-group

In this setup, one proves existence of X-composition series, X-isomorphism theorems and X-Jordan-Hölder theorem

3 (Finite) Nilpotent Groups

Definition
The normal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_a = G$ is a central series if $G_i/G_{i-1} \leq Z(G/G_{i-1})\ \forall i$

G is nilpotent (of class a) if it has a central series (the shortest such of length a)

Upper central series is the series of groups $Z_i(G)$ s.t. $Z_0(G) = 1, Z_1(G) = Z(G), Z_{i+1}(G) \trianglelefteq G$ with $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$
so, G nilpotent (of class a) $\Leftrightarrow$ $G/Z(G)$ nilpotent of class $a-1$
G nilpotent of class 1 $\Leftrightarrow$ G abelian

Theorem 3.1
Finite p-groups are nilpotent. If $|G| = p^e$, then class of $G < e$
Proof
$|G| = p^e$. If $e = 2$ then G is abelian, so class 1.
Now $Z(G) \neq 1$
$\Rightarrow \quad |G/Z(G)| < p^e \quad \Rightarrow \quad$ nilpotent of class $< e-1$
$\Rightarrow \quad$ G nilpotent of class $< e$

$$\begin{tikzcd}[arrows=dash]
1 \arrow[r] & Z(G) \arrow[r] \arrow[d, dotted] & Z_2(G) \arrow[r] \arrow[d, dotted] & Z_3(G) \arrow[r, dashed] & G \arrow[d, dotted] \\
 & 1 \arrow[r] & Z(G/Z(G)) \arrow[rr, dashed] & & G/Z(G)
\end{tikzcd}$$

$\square$

E.g. $G = D_{2^e}$ is nilpotent of class $e-1$, $(e \geq 2)$
If $e = 2$, $D_{2^e} = \langle \alpha, \beta | \alpha^2 = 1, \beta\alpha\beta = \alpha \rangle = V_4$, so clear
If $e > 2$, $Z(D_{2^e}) = 2$ and $D_{2^e}/Z(D_{2^e}) \cong D_{2^{e-1}}$

Theorem 3.2
For finite groups, TFAE:

(1) G is nilpotent

(2) $H \lneqq G \quad \Rightarrow \quad H \lneqq N_G(H)$

(3) Each Sylow subgroup of G is normal in G

(4) G is a direct product of its Sylow subgroups, one for each prime dividing $|G|$

Exercise:
- G nilpotent, $H < G \quad \Rightarrow \quad H$ nilpotent
- Any homomorphic image of G is nilpotent
- Direct product of nilpotent groups is nilpotent group

Proof
(1) $\Rightarrow$ (2) Let $H \lneqq G$. Take i maximal s.t. $Z_i(G) \leq H$
May assume $i = 0$ (factor out $Z_i(G)$)
$\Rightarrow \quad Z(G) \nleq H$. But $Z(G) \leq N_G(H) \Rightarrow H \lneqq N_G(H)$

(2) $\Rightarrow$ (3) Let P be a Sylow p-subgroup of G

Claim: For any G, $N_G(N_G(P)) = N_G(P)$ for P Sylow in G

Proof of Claim:
$P \in \mathrm{Syl}_p(N_G(P))$ and $P \trianglelefteq N_G(P) \quad \Rightarrow \quad P$ char $N_G(P)$
This is because P is the only subgroup of $N_G(P)$ with order $|P|$, this implies all automorphism of $N_G(P)$ sends P to itself

Also, since $N_G(P) \trianglelefteq N_G(N_G(P))$
$\Rightarrow \quad P \trianglelefteq N_G(N_G(P))$
$\Rightarrow \quad N_G(N_G(P)) = N_G(P)$ ∎

By hypothesis of (2), have $N_G(P) = G$

(3) $\Rightarrow$ (4) Let $P_1, \ldots, P_k$ be the Sylow p-subgroups of G, one for each prime $p \big| |G|$. Each $P_i \trianglelefteq G$ by hypothesis of (3).
$\Rightarrow \quad P_1P_2\cdots P_k \trianglelefteq G$ s.t. $|P_1P_2\cdots P_k| = |G|$
$\Rightarrow \quad G = P_1P_2\cdots P_k$, and $P_i \cap \prod_{j\neq i} P_j = \{1\}$ by considering their orders
$\Rightarrow \quad G \cong P_1 \times P_2 \times \cdots \times P_k$

(4) $\Rightarrow$ (1) Follows from Exercise above and Theorem 3.1

□

Corollary 3.3
G nilpotent, $H < G$ maximal $\quad \Rightarrow \quad H \trianglelefteq G$ index p, some prime p
Proof
$N_G(H) \gneq H \quad \Rightarrow \quad H \trianglelefteq G$
G/H has no proper subgroups $\Rightarrow$ cyclic of prime order for some prime p □

Note
If $K \trianglelefteq G$ and $K \leq H \leq G$, then $\quad H/K \leq Z(G/K) \Leftrightarrow \langle [h,g] | h \in H, g \in G\rangle = [H,G] \leq K$

Definition
Lower central series for G nilpotent is the chain $\Gamma_i(G)$ s.t. $\Gamma_1(G) = G, \Gamma_{i+1}(G) = [\Gamma_i(G), G]$
If $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_a = G$ is a central series, then $\Gamma_{a-i+1}(G) \leq G_i \leq Z_i(G)$
In particular, $\forall c$, $\Gamma_{c+1}(G) = 1 \Leftrightarrow Z_c(G) = G$

3.1 Two nilpotent characteristic subgroups of a finite group G

G any finite groups

Definition
The Fitting subgroup (nilpotent radical of G), $F(G)$, is the maximal subgroup amongst all normal nilpotent subgroups of G

Proposition 3.4
In a finite group, there is a unique maximal normal nilpotent subgroup:
If H, K are nilpotent normal subgroups of G, so is HK
(i.e. $F(G)$ well-defined)
Proof
The last claim is clear if H, K are both p-groups
$\Rightarrow \quad O_p(G)$, the unique maximal normal p-subgroup of G exist

Claim: $F(G) = O_{p_1}(G) \times \cdots \times O_{p_k}(G)$, where the p_i are the prime divisors of $|G|$

Proof of Claim:
RHS is nilpotent and normal
If $K \trianglelefteq G$, nilpotent, and P is a Sylow p-subgroup of K, then P char $K \trianglelefteq G \Rightarrow P \trianglelefteq G \Rightarrow P \leq O_p(G)$
$\Rightarrow \quad K \leq$ RHS

$\square$

Definition
G any finite group, the Frattini subgroup of G, $\Phi(G)$, is the intersection of all maximal subgroups of G

Note: Φ char G

Definition
$g \in G$ is a non-generator of G if whenever $G = \langle X, g \rangle$ we have $G = \langle X \rangle$

Lemma 3.5
$\Phi(G) = \{g \in G | g \text{ non-generator } \}$
Proof
If $g \notin \Phi$, then $\exists M < G$ maximal with $g \notin M$
$\Rightarrow \quad G = \langle M, g \rangle$, but $\langle M \rangle = M < G$
$\Rightarrow \quad g$ is not a non-generator

Conversely, assume $\exists X < G$ with $G = \langle X, g \rangle$ but $G > \langle X \rangle$
Let $M < G$ maximal with $\langle X \rangle \leq M$
Then $g \notin M \Rightarrow g \notin \Phi(G)$ $\square$

Denote $\mathrm{Syl}_p(G)$ as the set of Sylow p-subgroup of G

Proposition 3.6
For any G finite, $\Phi(G)$ is nilpotent ($\Rightarrow \Phi(G) \leq F(G)$)
Proof
Uses an important general lemma (Lemma 3.7) known as Frattini argument

Let $P \in \mathrm{Syl}_p(\Phi(G))$
Then $G = N_G(P)\Phi(G)$ so by Lemma 3.5, $G = N_G(P)$,
so $P \trianglelefteq \Phi(G)$. Thus $\Phi(G)$ nilpotent. $\square$

Lemma 3.7 (Frattini argument)
G finite group, $K \trianglelefteq G$, $P \in \mathrm{Syl}_p(K)$
Then $G = N_G(P)K$. Hence $G/K \cong N_G(P)/N_K(P)$
Proof
Let $g \in G$. By normality, $\exists k' \in K$ s.t. $P^g = P^{k'}$
$\Rightarrow \quad (k' =: k^{-1}) \;\; P^{gk} = (P^g)^k = P$
$\Rightarrow \quad gk \in N_G(P)$, so $g \in N_G(P)K$

$G/K = N_G(P)K/K \cong N_G(P)/K \cap N_G(P) = N_G(P)/N_K(P)$ $\square$

Lemma 3.8
If G is a p-group, then $G/\Phi(G)$ is an elementary abelian group, and hence a vector space over $\mathbb{F}_p$. In fact, $\Phi(G) = G'G^p$
Proof
If M is maximal subgroup of G, then $M \trianglelefteq G$ of index p so $G' \leq \Phi(G)$ (as G/M abelian), and $G^p = \langle g^p | g \in G \rangle \leq \Phi(G)$ so $G'G^p \leq \Phi(G)$

Equality: If $g \in G \setminus G'G^p$, consider its image in $\overline{G} = G/G'G^p$
Then $\overline{g} \neq \overline{0}$, a non-zero vector, so let $\overline{g}, \overline{g_2}, \dots, \overline{g_k}$ be a basis.
Then g is not a non-generator ($G = \langle g, g_2, \dots, g_k, \cancel{\Phi} \rangle$) $\square$

Definition
Minimal generating set for G: no element of the set can be deleted and still generate G

Theorem 3.9 (Burnside's Basis Theorem)
If G is a finite p-group, any two minimal generating sets for G have the same size, $\dim_{\mathbb{F}_p} G/\Phi(G)$.
Proof
exercise: if $g_1, \dots g_k$ is a minimal generating set for G, then $\overline{g_1}, \dots, \overline{g_k}$ is a basis for $G/\Phi(G)$ $\square$

Remark. $S_5 = \langle (12), (12345) \rangle = \langle (12), (23), (34), (45) \rangle$
both are minimal generating set, with different size
Question What is the maximal size of a minimal generating set? What do minimal generating sets of maximal size look like?

4 Soluble (Solvable) groups

Definition
A <u>derived series</u> for G is a series $G^{(i)}$ s.t. $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ (so $G' = G^{(1)} = \Gamma_2(G)$)

A group is <u>soluble</u> if $G^{(v)} = 1$ for some v

Remark.
- Nilpotent $\Rightarrow$ soluble
- S_3 is soluble but not nilpotent
- simple soluble $\Rightarrow$ C_p for some prime p

Lemma 4.1
TFAE:

(1) G is soluble

(2) The chief factors are elementary abelian

(3) The composition factors are cyclic of prime order

Proof
(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear

(3) $\Rightarrow$ (1):
If $1 \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = G$ with abelian factors, then $G^{(i)} \leq H_i$, by indicator:
$G^{(0)} = H^{(0)}$, and $G^{(i+1)} = (G^{(i)})' \leq H_i' \leq H_{i+1}$ $\square$

<u>Exercise</u>: Subgroups, quotients, direct products of soluble groups are soluble
<u>Exercise</u>: $H, K \trianglelefteq G$, both H, K soluble $\Rightarrow$ HK is a soluble normal subgroup of G. Hence, using the fact that if N, G/N soluble then G soluble, we can define the <u>soluble radical of G</u>, i.e. the maximal normal soluble subgroup of G

Theorem 4.2 (Galois)
G finite soluble group, $M < G$ maximal subgroup of G
$\Rightarrow$ $|G : M| = p^a$ for some prime power p^a
Proof
Let $K \trianglelefteq G$ minimal normal. Then K is elementary abelian p-group, for some prime p. (The proof here is much easier than before: K is soluble, and K' char K so $K' = 1$, so K is abelian. $O_p(K)$ char K and K^p char K, so K elementary abelian p-group for some p)

If $K \leq M$, induction applies to $M/K < G/K$, so assume not. Then $G = KM$ (since $M \lneqq KM \leq G$)
And $K \cap M = 1$: $K \cap M \trianglelefteq M$, $K \cap M \trianglelefteq K$ (as K is abelian) so $K \cap M \trianglelefteq G$, so $K \cap M = 1$ by minimality of $K \trianglelefteq G$

So $|G : M| = |K| = p^a$, for some a $\square$

Remark. Here K is a regular normal subgroup of G on $(G : M)$ (<u>regular</u> = transitive and only identity fixes all point)

4.1 Hall's Theorem on Finite Soluble Groups

π = Set of primes
$n \in \mathbb{N}$, write $n = n_\pi n_{\pi'}$ where

$$
\begin{aligned}
(p_1^{l_1}\cdots p_k^{l_k})_\pi &= \prod_{p_i\in\pi} p_i^{l_i} && (\underline{\pi\text{-part}}) \\
(p_1^{l_1}\cdots p_k^{l_k})_{\pi'} &\notin \pi && (\underline{\pi'}\text{-part})
\end{aligned}
$$

A $\underline{\pi\text{-group}}$ H is a finite group with $|H| = |H|_\pi$
A subgroup H of G is a $\underline{\text{Hall } \pi\text{-subgroup}}$ of G if $|H| = |G|_\pi$, so subgroup of G of maximal possible π-order

Remark. $\pi = \{p\}$
Hall π-subgroup = Sylow p-subgroup
Hall π' subgroup = $\underline{\text{Hall } p\text{-complement subgroup}}$

Theorem 4.3
If G is a finite soluble group, π is a set of primes

(1) G has a Hall π-subgroup

(2) Any two are conjugate in G

(3) Any π-subgroup of G is in some Hall π-subgroup

Example: $GL_3(2)$ has no Hall 3-complement, and has two non-conjugate Hall 7-complement
$\left\{\begin{pmatrix}1 & * & * \\ 0 & & \\ 0 & & \end{pmatrix}\right\}, \left\{\begin{pmatrix}1 & 0 & 0 \\ * & & \\ * & & \end{pmatrix}\right\}$

Proof
Let $|G| = mn$, with $m = |G|_\pi$, $n = |G|_{\pi'}$
Induction on n. If $n = 1$, then trivial. So assume $n > 1$

Case 1:
Assume $\exists 1 \neq K \trianglelefteq G$ with $|K| = m_1 n_1$ with $n_1 < n$

(1) Now $|G/K| = \frac{m}{m_1}\frac{n}{n_1}$, so by induction, G/K has a subgroup S/K of order $\frac{m}{m_1}$ with $K \le S < G$
Then $|S| = mn_1 < |G|$. By induction, S (and hence G) has a subgroup of order m

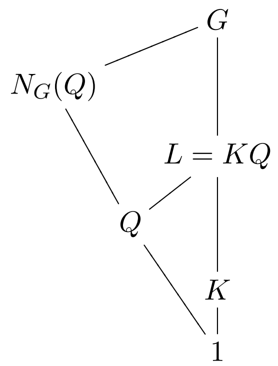
(2) If H_1, H_2 are subgroups of G, order m, then in $\overline{G} = G/K$, then images $\overline{H_1}, \overline{H_2}$ are Hall π-subgroups. By induction, $\exists \overline{x} = Kx \in \overline{G}$ with $\overline{x}^{-1}\overline{H_2}\,\overline{x} = \overline{H_1}$.
$\Rightarrow \quad x^{-1}H_2Kx = H_1K \quad \Rightarrow \quad x^{-1}H_2x, H_1$ are Hall π-subgroup in H_1K, so conjugate. Apply induction.

(3) Let P be any π-subgroup of G. Then $\overline{P} = PK/K$ is a π-subgroup in $\overline{G} \Rightarrow \overline{P} \subseteq$ some Hall π-subgroup $\overline{S} = S/K$ of $G/K \quad (K \le S < G)$
S order mn_1. By induction in S, we see that $P \subseteq$ some Hall π-subgroup in G

Case 2:
Any non-identity normal subgroup of G has order divisible by n ($\forall 1 \neq N \trianglelefteq G, n \big| |N|$)
Let K be maximal normal in G. Then $|K| = p^a$ for some prime power p^a, K elementary abelian.
Then $n | p^a$, whence $n = p^a$ as $p^a | nm$ and $(n,m) = 1$

Let $L \trianglelefteq G$ containing K s.t. $\overline{L} = L/K$ minimal normal in $\overline{G} = G/K$. Then $|L| = q^b$ for some prime power q^b with $p \neq q$

Let Q be a Sylow p-subgroup of L. $|L| = p^a q^b$, $L = KQ$ (by Frattini argument)



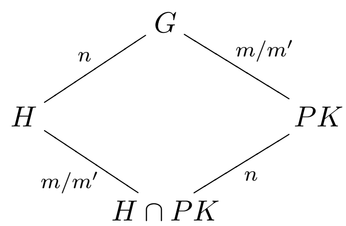
(1) **Claim:** $N_G(Q)$ is a Hall π-subgroup
Proof of Claim:
Firstly, Frattini argument: $G = LN_G(Q) = KQN_G(Q) = KN_G(Q)$
$\Rightarrow \quad m \big| |N_G(Q)$
$K \cap N_G(Q) = 1$, as

- $K \cap N_G(Q) \trianglelefteq K$ (since K abelian)
- $K \cap N_G(Q) \trianglelefteq N_G(Q)$ (since $K \trianglelefteq G$)
- $\Rightarrow \quad K \cap N_G(Q) \trianglelefteq G$, but K minimal normal, and $K \not\leq N_G(Q)$ as $Q \ntrianglelefteq G$

$\Rightarrow \quad |N_G(Q)| = m \quad \Rightarrow \quad H = N_G(Q)$ is a Hall subgroup ■

(2) Let H_2 be any other Hall π-subgroup of G
Since $LH_2 \leq G$, with $|G| \Big| |LH_2|$
$\Rightarrow \quad G = LH_2$
Now $|L \cap H_2| = q^b$ (as $LH_2/L \cong H_2/L \cap H_2$)
$\Rightarrow \quad L \cap H_2$ is a Sylow q-subgroup of L
$\Rightarrow \quad Q^x = L \cap H_2$ for some $x \in L$
$H_2 \leq N_G(L \cap H_2) = N_G(Q)^x = N_G(Q^x) = H^x \quad \Rightarrow \quad H_2 = H^x$ (since they are of same order)

(3) Let P be a π-subgroup of G, with $|P| = m' < m$
Now $|G : H| = n$ which is π'-number
$|G : PK| = m/m'$ which is π-number
$\Rightarrow \quad P, H \cap PK$ have the same order: $|G : H \cap PK| = |G : H||G : PK|$ (by Lemma 4.4, to be shown) $= \frac{nm}{m'}$



So $P, H \cap PK$ are both Hall π-subgroups in PK
$\Rightarrow \quad$ conjugate there by (2)
$\Rightarrow \quad P$ is conjugate to a subgroup of H

□

Lemma 4.4
G any finite group let A_1, A_2 be subgroups of coprime indices n_1, n_2.
Then $G = A_1 A_2$ (a <u>factorization</u> of G) and $|G : A_1 \cap A_2| = n_1 n_2$

Proof
$|G : A_1 \cap A_2| = \underbrace{|G : A_1|}_{n_1} |A_1 : A_1 \cap A_2| = \underbrace{|G : A_2|}_{n_2} |A_2 : A_1 \cap A_2|$
$\Rightarrow \quad n_1 n_2 \big| |G : A_1 \cap A_2|$
$|A_1 A_2| = \dfrac{|A_1||A_2|}{|A_1 \cap A_2|} = \left(\dfrac{(|G|/|G : A_1|)(|G|/|G : A_2|)}{|G|/|G : A_1 \cap A_2|} \right) = \dfrac{|G||G : A_1 \cap A_2|}{n_1 n_2}$
$\Rightarrow \quad G = A_1 A_2$ $\square$

$A_1, A_2 \leq G$. G is factorisable as $G = A_1 A_2$ if every $g \in G$ is $g = a_1 a_2$ with $a_i \in A_i$
Note: $G = A_1 A_2 \Leftrightarrow G = A_2 A_1 \Leftrightarrow G = A_1^g A_2; \forall g \in G$
A_1, A_2 conjugate $\Rightarrow G \neq A_1 A_2$
Exercise: $G = A_1 A_2 \Leftrightarrow A_2$ is transitive on $(G : A_1) \Leftrightarrow A_1$ is transitive on $(G : A_2)$
(this result immediately implies:) $G = A_1 A_2 \Leftrightarrow |G : A_1| = |A_2 : A_1 \cap A_2|$ (by using $|A_1 A_2| = \frac{|A_1||A_2|}{|A_1 \cap A_2|}$)

Theorem 4.5 (Theorem of Ore)
If G is finite soluble, A_1, A_2 maximal subgroup of G, not conjugate then $G = A_1 A_2$

Proof as Exercise (hard)

Factorisation in almost simple groups are interesting, important and rare. See later.

Definition
If $|G| = p_1^{e_1} \cdots p_k^{e_k}$ with the p_i distinct primes, a Sylow basis of G is $\{P_1, \ldots, P_k\}$ with $|P_i| = p_i^{e_i}$ s.t. $P_i P_j = P_j P_i \ \forall i, j$ (so that $P_i P_j \leq G \ \forall i, j$)

Note: $P_i P_j = P_j P_i \Rightarrow P_i P_j \leq G \ \forall i, j$ and in fact
$P_{i_1} P_{i_2} \cdots P_{i_l} \leq G \ \forall \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\}$ – we get Hall subgroups this way

Theorem 4.6
If G is a finite soluble group, G has Sylow basis, and any two are conjugate
Proof
Let $|G| = p_1^{e_1} \cdots p_k^{e_k}$, with the p_i distinct primes.
Let H_i be a (Hall) p_i-complement. (i.e. has order $|G|/p_i^{e_i}$)
Put $P_j = \bigcap_{i \neq j} H_i$
$\Rightarrow \quad |P_j| = p_j^{e_j}$, using Lemma 4.4
And $P_i P_j = \bigcap i \neq l \neq j H_l = P_j P_i$

Conjugacy: Let $\{P_1, \ldots, P_k\}$ and $\{P_1^*, \ldots, P_k^*\}$ be Sylow bases with $|P_i| = |P_i^*|$, so $P_i, P_i^* \in \mathrm{Syl}_{p_i}(G)$
Put $H_i = P_1 \cdots P_{i-1} P_{i+1} \cdots P_k$
and $H_i^* = P_1^* \cdots P_{i-1}^* P_{i+1}^* \cdots P_k$

Claim: $\exists g \in G$ with $H_i^g = H_i^* \ \forall i$

Proof of Claim:
First, $\exists g_1$ with $H_1^{g_1} = H_1^*$ (Hall)
Now assume we found g_{i-1} with $H_j^{g_{i-1}} = H_j^* \ \forall j \leq i-1$

Change notation if necessary so that $H_j = H_j^* \ \forall \leq i-1$
Now $G = H_i P_i$, and let $x \in G$ with $H_i^x = H_i^*$ (Hall)
$\Rightarrow \quad x = hz$ with $h \in H_i, z \in P_i$
$\Rightarrow \quad H_i^z = H_i^*$ and $z \in P_i \leq H_i \ \forall j < i$
$\Rightarrow \quad H_j^z = H_j^*$ for $j \leq i$
After k steps, finished. $\blacksquare$

Given claim, we have $P_i^g = P_i^* \ \forall i$ $\square$

Theorem 4.7 (Wielandt)
G finite, H_1, H_2, H_3 soluble subgroups of G with $|G : H_i| = n_i$ pairwise coprime. Then G soluble
Proof
Looking for $N \trianglelefteq G$ soluble with G/N soluble by induction (then G soluble)
Note that $G = H_1H_2 = H_1H_3 = H_2H_3$, by Lemma 4.4 May assume $H_i \neq 1\ \forall i$. Let K minimal normal in H_1
$\Rightarrow \quad K$ elementary abelian p-group.

WLOG, $p \nmid n_2$

Claim: This implies $K \leq H_2$

Proof of Claim:
By Lemma 4.4
$$\underbrace{\left|K(H_1 \cap H_2) : H_1 \cap H_2\right|}_{\text{divides } n_2} = \left|K : K \cap H_1 \cap H_2\right|$$
so $K \leq H_1 \cap H_2$

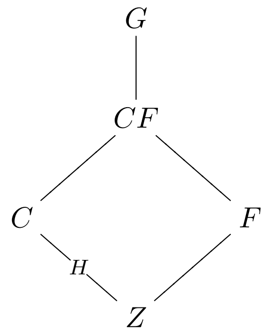
Now let $N = \langle K^g | g \in G\rangle$ - the <u>G normal closure of K</u>
$\Rightarrow \quad N = \langle K^{h_1h_2} | h_i \in H_i\rangle = \langle K^{h_2} | h_2 \in H_2\rangle \leq H_2$
$\Rightarrow \quad N \trianglelefteq G, N \leq H_2$
It follows that N is soluble (as H_2 is), and G/N is soluble by induction hypothesis.

$\square$

Theorem 4.8 (P.Hall)
If the finite group G has a Hall p-complement for each prime $p \big| |G|$, then G is soluble
Proof
$|G| = p_1^{e_1} \cdots p_k^{e_k}$
If $k = 1$, G is nilpotent. Done
If $k = 2$, this is Burnside's p^aq^b Theorem, this uses representation theory

Assume $k \geq 3$, let H_i be a Hall p_i-complement.
By induction, H_i is soluble $\forall i$ (Lemma 4.4)
Now use Theorem 4.7 $\square$

Lemma 4.9
G finite soluble group $\quad \Rightarrow \quad C_G(F(G)) \leq F(G)$
(So $G/Z(F(G)) \leq \mathrm{Aut}(F(G))$)
Proof
Put $F = F(G)$, $Z = Z(F(G))$, $C = C_G(F(G))$. So $Z = C \cap F$
Assume $C \nleq F$. Let $H \trianglelefteq G$ with $Z < H$



$\overline{H}$ minimal normal in $\overline{C} = C/Z$: Since $\overline{H}$ is elementar yabelian, have $H' \leq Z$
Then H is nilpotent (so $H \leq F$ - not so): minimal normal in H,
$\Gamma_3(H) = [H', H] \leq [Z, C] = 1$ $\square$

5 Interlude

In finite simple groups here been classified:
(q denote a power of prime)

C_p,	p prime
A_n,	$n \geq 5$
$L_d(q)$,	$d \geq 2$, if $d = 2$ then $q \geq 4$
$U_d(q)$,	$d \geq 3$, if $d = 3$ then $q \geq 3$
$Sp_{2m}(q)$,	$m \geq 2$, if $m = 2$ then $q \geq 3$
$PSL_d^{\epsilon}(q)$,	$d \geq 7$
	if d even then G doubly infinite families, denoted by $\epsilon = \pm$;
	if d odd then ϵ is empty

10 families, exceptional groups of Lie type (q denote prime power):
$G_2(q)$, $F_4(q)$, $E_6^+(q)$, $E_6^-(q)$, $E_7(q)$, $E_8(q)$
${}^3D_4(q)$
${}^2B_2(q)$ with $q = 2^{2a+1}$
${}^2G_2(q)$ with $q = 3^{2a+1}$
${}^2F_4(q)$ with $q = 2^{2a+1}$

26 sporadic simple groups
$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, \cdots, M$

In applications, have usually a finite groups acting in some way, .e.g as a group of permutations, or a group of matrices (in this case, this is representation theory, mainly used to study the first 5 families mentioned)

We will be consider the first case mainly (i.e. acting as groups of permutations)
e.g.1: as a group of automorphisms of a graph, e.g. Peterson graph e.g.2: as a group acting on roots of polynomial

Recall about <u>permutation actions</u>:

G finite group, Ω a finite set. G <u>acts on</u> if there is a map $\begin{array}{ccc} \Omega \times G & \to & \Omega \\ (\omega, g) & \mapsto & \omega g \end{array}$
s.t. $\omega 1 = \omega$, $\omega gh = (\omega g)h \;\; \forall \omega \in \Omega, g, h \in G$
If so, for $g \in G$, the map $\begin{array}{ccc} \phi_g : \Omega & \to & \Omega \\ \omega & \mapsto & \omega g \end{array}$ is a permutation on Ω
and the map $\begin{array}{ccc} \phi : G & \to & \mathrm{Sym}(\Omega) \\ g & \mapsto & \phi_g \end{array}$ is a homomoprhism, called a <u>permutation representation</u> of G

The <u>kernel</u> of the action, $G_{(\Omega)} := \ker \phi$. Then $G\phi = G^{\Omega} := G/G_{(\Omega)} \leq \mathrm{Sym}(\Omega)$
The action is <u>faithful</u> if $G_{(\Omega)} = 1$

If G acts on Ω, the <u>orbit</u> of G containing α is $\alpha G = \{\alpha g | g \in G\} \quad \Rightarrow \quad \Omega$ splits into G-orbits
G is <u>transitive</u> on Ω if $\alpha G = \Omega \;\; \forall \alpha \in \Omega$

Problems about actions are usually easily reduced to problems about transitive actions (consider the actions on orbits). So the assumption of G transitive is common.

If G acts on Ω_1, Ω_2, then $\theta : \Omega_1 \to \Omega_2$ is a <u>G-homomorphism</u> if $(\alpha g)\theta = (\alpha\theta)g \;\; \forall \alpha \in \Omega_1, g \in G$
θ is a <u>G-isomorphism</u> if it is also bijective.

Also note that $(G : G_\alpha)$ is the G-space of all right cosets of $G_\alpha = \{g \in G | \alpha g = \alpha$, with action obtained by $(G_\alpha x)g = G_\alpha(xg)$

Lemma 5.1
If G is transitive on Ω, and $\alpha \in \Omega$, then actions of G on Ω and on $(G : G_\alpha)$ are G-isomorphic
Proof

$$\begin{aligned} \theta : \Omega &\to (G : G_\alpha) \\ \alpha x &\mapsto G_\alpha x \end{aligned}$$

is bijective, commutes with the actions of G $\square$

Example:
$(G : H), (G : K)$ are G-isomorphic $\Leftrightarrow$ H, K are G-conjugate

Definition
A equivalence relation ρ is a G-congruence on Ω if $\alpha\,\rho\,\beta \Rightarrow \alpha g\,\rho\,\beta g\ \forall g \in G$
Trivial means either equality or universal.
If G is transitive on Ω, then G is primitive if $\nexists$ non-trivial G-congruence on Ω

Remark. Kernels of G-homomorphism are "blocks" of G-congruences

Lemma 5.2
G transitive on Ω, ρ a non-trivial G-congruence on Ω, $\alpha \in \Omega$, $\rho(\alpha)$ the (equiv.) class of α: $\rho(\alpha) := \{\beta \in \Omega | \alpha\,\rho\,\beta\}$

(1) The (setwise) stabilizer of $\rho(\alpha)$ in G acts transitively on $\rho(\alpha)$

(2) $\rho(\alpha)$ is the union of some G_α-orbits, including $\{\alpha\}$

(3) G is transitive on the set Ω/ρ of ρ-classes (blocks), so all the ρ-classes have the same size

(4) $G_\alpha \lneqq G_{\rho(\alpha)} \lneqq G$
Conversely, if $G_\alpha \lneqq H \lneqq G$, can define a non-trivial G-congruence on Ω by:

  (a) $\gamma\,\rho\,\beta \Leftrightarrow \beta = \gamma h$, some $h \in H$
  (b) $\gamma\,\rho\,\beta \Rightarrow \gamma g\,\rho\,\beta g\ \forall g$

Proof

(1) Let $\beta \in \rho(\alpha)$, G transitive on Ω
$\Rightarrow \exists g \in G$ with $\beta = \alpha \Rightarrow \rho(\alpha)$ must be kept invariant by $g \Rightarrow g \in G_{\rho(\alpha)}$

(2) G_α keeps $\rho(\alpha)$ invariant

(3) $\alpha \in \rho(\alpha)$ and if $\rho(\beta)$ another block, $\exists g \in G$ s.t. $\alpha g = \beta \quad \Rightarrow \quad \rho(\alpha)g = \rho(\beta)$

(4) First claim is clear. Proof of converse:
If $G_\alpha \lneqq H \lneqq G$, the ρ defined is clearly non-trivial G-congruence.
$\Rightarrow \quad \Omega/\rho = \{\Gamma g | g \in G\}$ where $\Gamma = \alpha H$

Claim: The blocks form a partition of Ω
Proof of Claim:
They cover Ω. So we want: $(\Gamma \cap \Gamma g \neq 0 \Rightarrow \Gamma = \Gamma g)$

$\beta \in \Gamma \cap \Gamma g \Rightarrow \beta = \alpha h_1 = \alpha h_2 g$ some $h_1, h_2 \in H$
$\Rightarrow \quad h_2 g h_1^{-1} \in G_\alpha \lneqq H \quad \Rightarrow \quad g \in H \quad \Rightarrow \quad \Gamma g = \Gamma$ ■

$\square$

Corollary 5.3
Set up as Lemma 5.2, by (4), G primitive on $\Omega \quad \Leftrightarrow \quad G_\alpha$ maximal subgroup of G

Corollary 5.4
Set up as Lemma 5.2, by (3), all blocks have the same size, didviding $|\Omega|$.
If G transitive of prime degree (degree means size of Ω) $\Rightarrow$ G primitive

Lemma 5.5
G primitive on Ω, $\alpha \neq \beta \quad \Rightarrow \quad G = \langle G_\alpha, G_\beta \rangle$, unless G is C_p of degree p
In fact, if G transitive, $\alpha \in \Omega$, then fix(G_α)=$\{\beta | \beta h = \beta \ \forall h \in G_\alpha\}$ is a block of a congruence.
Proof
Let G be transitive. Define $\alpha \, \rho \, \beta$ if $G_\alpha = G_\beta$ (note: $\beta \in \text{fix}(G_\alpha) \Rightarrow G_\alpha = G_\beta$)

Claim: This is a G-congruence

Proof of Claim:
$g \in G$. $\alpha \, \rho \, \beta \Rightarrow G_\alpha = G_\beta \Rightarrow G_{\alpha g} = G_{\beta g} \Rightarrow \alpha g \, \rho \, \beta g$ ■

$\text{fix}(G_\alpha) = \rho(\alpha)$. So, if G is primitive on Ω, the congruence above is trivial (equlity or universal).
If equality, then $G_\alpha \neq G_\beta$, btoh maximal $\Rightarrow G = \langle G_\alpha, G_\beta \rangle$
If universal, then G_α fixes Ω pointwise $\Rightarrow G_\alpha = 1$
and as G_α maximal, then G is C_p for some prime p □

Lemma 5.6
Let G transitive on Ω, let $1 \neq N \trianglelefteq G$. The N-orbits form a G-invariant partition of Ω

Proof
Exercise □

Corollary 5.7
If G is primitive, $1 \neq N \trianglelefteq G \Rightarrow N$ transitive on Ω

Exercise: Let G be transitive of degree prime p, (i.e. $|\Omega| = p$ and hence $G \leq S_p$)
Let N be minimal normal subgroup of G. Then N is simple (possibly C_p), and G/N is abelian of order dividing $p-1$ (use Frattini argument and $G \leq S_p$. Note if P is order p, $N_{S_p}(P)$ has order $p(p-1)$, so $G' = N$ is simple)

Dichotomy for primitive groups: 2-transitive groups, simply-primitive groups

2-transitive groups

Definition
G on Ω is 2-transitive on Ω if G is transitive on ordered pairs of distinct points of Ω, i.e. if $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$, then $\exists g \in G$ with $\alpha_i g = \beta_i$
k-transitive is defined similarly on k-tuples

Remark. G is k-transitive on Ω of degree n, then $n(n-1)(n-k+1) \big| |G|$

Example:
S_n on $[1, n]$ is n-transitive
A_n is $(n-2)$-transitive but not $(n-1)$-transitive

Example:
$G = PGL_d(q)$ - projective general linear group $= GL_d(q)/\{\text{scalars}\}$ acts on $\Omega = \{$1-dim. subspaces of

$V_\alpha(q)\} =: \mathbb{P}_{d-1}(q)$
$|\Omega| = \frac{q^d-1}{q-1}$
G is 2-transitive on Ω
G is 3-transitive $\Leftrightarrow d = 2$
G is 4-transitive $\Leftrightarrow d = 2, q = 3, G = PGL_2(3) \cong S_4$

Example: $G = AGL_d(p)$, $(d \geq 1)$ the group of "symmetries" of $\Omega = V_d(p)$
i.e. $\langle$translations, linear transformations$\rangle \leq \mathrm{Sym}(V)$
$K = \{t_v | v \in V\}$ where $t_v : x \mapsto x + v$, subgroup of translation
$G_0 = GL_d(p)$ is acting on Ω as linear transformation
$\Rightarrow$ transitive on $\Omega \setminus \{0\}$
$\Rightarrow$ 2-transitive on Ω

(Exercise) G is 3-transitive $\Leftrightarrow d = 2$, or $d = 1, p = 3$(i.e. $G \cong S_3$)
(Exercise) G is 4-transitive $\Leftrightarrow d = 2 = p, AGL_2(2) \cong S_4$

In fact:

(1) $K \trianglelefteq AGL$, since $\forall h \in G_0$, $h^{-1}t_v h = t_{vh}$ $\forall x \in V$

(2) K is a regular normal elementary abelian subgroup

(3) $K \cap G_0 = 1$, $AGL = G_0 K$
Any $g \in G$ is $g = hk$ for some unique $h \in H, k \in K$
$(h_1k_1)(h_2k_2) = (h_1h_2)(k_1^{h_2}k_2)$

Definition
Semidirect product $H \ltimes K$:
H, K groups, $\theta : H \to \mathrm{Aut}(K)$
Elements of $H \otimes K$ are of form (h, k) s.t.

$$(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1^{h_2\theta}k_2) \quad \forall h_i \in H, k_i \in K$$

This is a group, with normal subgroup $\cong K$ and a subgroup $\cong H$, s.t. $H \cap K = 1, G = HK$
(Direct product is a special case)

Lemma 5.8
Let G be primitive on Ω, let $K \trianglelefteq G$ a regular normal subgroup (so K is transitive with $K_\alpha = 1$ $\forall \alpha$)

(1) K is minimal normal in G (so it is a direct product of isomorphic simple groups, possibly C_p)

(2) Let $\alpha \in \Omega$, we can identify Ω and K in such a way that the actions of G_α on Ω and on K is isomorphic

(3) If G is 2-transitive, then K is elementary abelian

(4) If K is elementary abelian, then K is in fact a vector space $V_d(p)$ over $\mathbb{F}_p$ and action of G on Ω is isomorphic to a subgroup of $AGL_d(p)$ acting as above
In fact, G_α ($\leftrightarrow G_{\mathbf{0}}$,stabliser of vector $\mathbf{0}$) is an irreducible subgroup of $GL_d(p)$
(This is because if some proper subspace L of K is G_α-invariant, then $G_\alpha \lneqq LG_\alpha \lneqq G$)

Proof

(1) If $1 \neq L \trianglelefteq G \Rightarrow L$ is transitive $\Rightarrow |K| = |\Omega| \big| |L|$
$\Rightarrow$ if $L \leq K$, then $L = K$

(2) $\alpha \in \Omega$, any β is αk for unique k
$\alpha k \leftrightarrow k$
$\Omega \leftrightarrow K$
If $g \in G_\alpha$, $\alpha k g = \alpha g^{-1} k g = \alpha k^g \leftrightarrow k^g$

(3) Let G be 2-transitive. By (2), G_α is transitive on $\Omega \setminus \{\alpha\}$
$\Rightarrow$ G_α transitive on $K \setminus \{1\}$ by conjugations
$\Rightarrow$ $\forall k \in K \setminus \{1\}$, they have the same order p, some prime p
$\Rightarrow$ K is a p-group $\Rightarrow$ elementary abelian

(4) Notation: K is a vector space written multiplicatively.
$G = G_\alpha K$.

Action of K:
Take $k' \in K$, we have the correspondence

$$(k)k' \leftrightarrow (\alpha k)k' = \alpha(kk') \leftrightarrow kk'$$

$\Rightarrow$ action by k' is just by translation

Action of G_α:
Take $h \in G_\alpha$, h acts on K by $k \mapsto h^{-1}kh$
This is a linear transformation: $h^{-1}(k_1k_2)h = h^{-1}k_1h \cdot h^{-1}k_2h$ (and scalar multiple)

□

Theorem 5.9 (Burnside)
Let G be a 2-transitive primitive group on Ω, $|\Omega| = n$. Let N be minimal normal subgroup of G. Then

either (a) N elementary abelian p-group (some p), $G \leq AGL_d(p)$ $(p^d - 1 \big| |G_0|)$
or (b) N is non-abelian simple

Proof
If N is regular, then get (a) by Lemma 5.8. So we assume N is not regular (and aim to get (b))

Claim: N is primitive on Ω

Proof of Claim:
Assume not. Let Γ be a minimal non-trivial block of N on Ω
$\Rightarrow$ $|\Gamma \cap \Gamma g| \leq 1, g \in G$, unless $\Gamma = \Gamma g$
Γg is another N-block (possibly in a different system of imprimitivity)
$\Rightarrow$ $\Gamma \cap \Gamma g$ is also an N-block, so either $\Gamma = \Gamma g$ or $|\Gamma \cap \Gamma g| \leq 1$

Write $B = \{\Gamma g | g \in G\}$ - "lines". Any 2 points in Ω are on a unique line
$\Rightarrow$ all pairs of points are (by G 2-transitivity)

Let $\alpha \in \Omega$. Now N_α fixes setwise each line on α

Claim: $N_{\alpha\beta} = 1$ for $\alpha \neq \beta$
Proof of Claim:
$N_{\alpha\beta}$ fixes all points not on line(α, β)
Repeat with α, γ and get $N_{\alpha\beta} = 1$ ⊡

Hence N is a Frobenius groups (i.e. transitive, $N_\alpha \neq 1$ $\forall \alpha$, and $N_{\alpha\beta} = 1$ $\forall \alpha, \beta$)

$$\Rightarrow \quad \exists \text{ characteristic regular subgroup } K \leq N \tag{5.1}$$

(see Remark for proof of this in our case)
$\Rightarrow$ $K \trianglelefteq G$ $\Rightarrow$ $N = K$ $\Rightarrow$ N is reugalr # ■

Recall Theorem 2.3: N minimal normal $\Rightarrow N = T_1 \times \cdots \times T_k$ with $T_i \cong T \; \forall i$ and T simple.
Also, by Corollary 5.7, each T_i is transitive on Ω

Claim: $k = 1$, N is simple, non-abelian

Proof of Claim:
If $k > 1$, then T_i is regular.
$\Rightarrow \quad |N| = |T|^k, n = |T|$ (note n is size of Ω), by Lemma 5.10
this orbits of N_α on $\Omega \setminus \{\alpha\}$ have all the same size dividing $n-1$ (by Corollary 5.4)
$N_\alpha \trianglelefteq G_\alpha$, G_α transitive on $\Omega \setminus \{\alpha\}$
$|N_\alpha| = |T|^{k-1}$

On the other hand, $(n-1, |N_\alpha|) = 1$
$\Rightarrow \quad N_\alpha = 1 \quad \Rightarrow \quad N$ regular
$\Rightarrow \quad k = 1 \quad \Rightarrow \quad N$ simple, non-abelian (if abelian, then regular by the next Lemma 5.10) ■

We have now completed the proof. □

Remark. We give a proof of the statement (5.1) under the set up of the lemma
Claim: G is 2-transitive, $N \trianglelefteq G$ and N Frobenius $\Rightarrow \exists K \leq N$ characteristic regular subgroup of N
Proof
Let $K = \{1\} \cup \{x \in N | x \text{ have no fixed points on } \Omega\}$
Let $n = \Omega$
$|K| = n, |N| = nc$ where $c = |N_\alpha|$

$$N = K \cup \underbrace{(\bigcup_{\alpha \in \Omega} N_\alpha \setminus \{1\})}_{n(c-1)}$$

K is a transitive set:
If $k \in K \setminus \{1\}$ takes α to β, let $g \in G_\alpha$ take β to γ
then $k^g \in K \setminus \{1\}$ with $k^g : \alpha \mapsto \gamma$
$K \leq N$
If not, let $k_1, k_2 \in K$ with $k_1 k_2^{-1} \notin K \Rightarrow k_1 \neq k_2$ and $k_1 k_2^{-1}$ fixes some α - then $\alpha k_1 = \alpha k_2$
$\Rightarrow \quad K$ cannot be transitive
$\Rightarrow \quad K \trianglelefteq G$ (as $k^g \in K \; \forall k \in K, g \in G$) □

Definition
Action of G on Ω is semi-regular if $g \in G$ fixes any point of $\Omega \Rightarrow g = 1$

Lemma 5.10
If M transitive subgroup of $\mathrm{Sym}(\Omega)$, then $C_{\mathrm{Sym}(\Omega)}(M)$ is a semi-regular

Proof
Exercise □

Simply Primitive Groups

Definition
G is simply primitive on Ω if it is primitive but not 2-transitive

Orbits of G_α : $\quad \Gamma_0(\alpha) = \{\alpha\}, \Gamma_1(\alpha), \ldots, \Gamma_{r-1}(\alpha)$
where r is the number of these for G_α, called the rank of G on Ω
Note $r = 2 \Leftrightarrow G$ is 2-transitive

For $r > 2$
the <u>subdegree</u> $n_i = |\Gamma_i(\alpha)|$ $\quad 1 = n_0 \le n_1 \le \cdots \le n_{r-1} \quad \sum_0^r n_i = n$

We can consider the induced action of G on $\Omega \times \Omega$:

$$(\alpha, \beta)g = (\alpha g, \beta g) \qquad g \in G; \alpha, \beta \in \Omega$$

Get orbits $\Gamma_0, \Gamma_1, \ldots, \Gamma_{r-1}$, these are called <u>orbitals</u> of G
$\Rightarrow \quad \Gamma_i(\alpha) = \{\beta \in \Omega | (\alpha, \beta) \in \Gamma_i\}$
$\Gamma_0 = \{(\alpha, \alpha) | \alpha \in \Omega\}$ is called the <u>diagonal orbital</u>
$|\Gamma_i| = n \cdot n_i$

The orbitals give orbital graphs on Ω (a directed graph):
If Γ is an orbital, then $\Gamma^* = \{(\beta, \alpha) | (\alpha, \beta) \in \Gamma\}$ is also an orbital
Γ and Γ^* are <u>paired</u>
We get complete G-invariant r-colouring of the complete graph on the vertices in Ω
$\mathcal{G}_1$ = Peterson graph
$|\Omega| = \binom{5}{2} = 10 \quad G$ =Aut(Peterson)=S_5
(Automorphism group of graph $\mathcal{G}$, Aut($\mathcal{G}$), acts on the set of vertices preserves edge)

Rank=3: $n_0 = 1, \Gamma_0(\{1,2\})$ (trivial edges from $\{i,j\}$ to $\{i,j\}$)
$n_1 = 3, \Gamma_1(12)$ (edges of a Petersen graph, i.e. edges with vertex $\{i,j\}$ and $\{k,l\}$ with $\{i,j\} \cap \{k,l\} = \emptyset$)
$n_2 = 6, \Gamma_2(12)$ (all other edges of K_10, i.e. edges with vertex $\{i,j\}$ and $\{i,k\}$, $j \ne k$)

<u>Exercise</u>: S_n acts on $\binom{n}{2}$ (stablizer $S_{n-1} \times S_2$) as a primitive rank 3 group, subdegrees 1 ($\leftrightarrow$ 12), $2n-2$, $\binom{n-2}{2}$
S_n acts on $\binom{n}{k}$, $\frac{n}{2} > l \le 1$, rank is $l+1$
Remark. If $\mathcal{G}$ is an undirected graph, we have the notion of <u>distance</u>
$G \le \text{Aut}(\mathcal{G})$, G preserves distance
Say G is <u>distance transitive</u> on $\mathcal{G}$ if given α_1, α_2 has distance d and β_1, β_2 has distance d, then $\exists g \in G$ s.t. $\alpha_i g = \beta_i$
What are these on regular symmetric graphs?

<u>Exercise</u>: S_n is distance transitive on the graph $\Omega = \binom{n}{l}$ = l-subsets of $[1, n]$, edges $A - B$ if $|A \cap B| = l - 1$

If G is primitive, $n_1 = 1 \quad \Rightarrow \quad n_i = 1\ \forall i \quad \Rightarrow \quad G$ regular (see 5.4 or 5 or 6)
<u>Exercise</u>: $n_1 = 2 \quad \Rightarrow \quad n_i = 2\ \forall i \quad \Rightarrow \quad G = D_{2p}$ on p points
<u>Sim's Conjecture</u>: n_1 fixed $\Rightarrow |G_\alpha$ bounded

Proposition 5.11 (D.G. Higman)
Let G be transitive on Ω. Then G is primitive $\Leftrightarrow$ all the non-diagonal orbital graphs are connected
Proof
Exercise (very very hard) $\square$

Permutation Character

G acts on Ω, a permutation group, $\pi(g) = |\operatorname{fix}_\Omega(g)|$ is a character of G of the permutation representation.

Lemma 5.12
If $G \leq \operatorname{Sym}(\Omega)$, permutation character π
Then $\langle \pi, \mathbf{1} \rangle_G = \frac{1}{|G|} \sum_g \pi(g) = \#(\operatorname{orb}(G, \Omega))$
So G is transitive $\Omega \Leftrightarrow \langle \pi, \mathbf{1} \rangle_G = 1$
Proof

$$\#(\operatorname{orb}(G,\Omega)) = \sum_{\alpha \in \Omega} |G_\alpha| = \#\{(\alpha, g) \in \Omega \times G | \alpha g = \alpha\} = \sum_{g \in G} \pi(g)$$

$\square$

Lemma 5.13
G acts on Ω_1, Ω_2, with characters π_1, π_2
Then $\langle \pi_1, \pi_2 \rangle_G = \#$ orbits of G on $\Omega_1 \times \Omega_2$
In particular, if G acts transitively on Ω, with character π, then $\langle \pi, \pi \rangle = \operatorname{rank}_\Omega(G)$
Proof
First part:

$$\langle \pi_1, \pi_2 \rangle_G = \langle \pi_1 \pi_2, \mathbf{1} \rangle_G$$

Note $\pi_1 \pi_2$ is the permutation character of G on $\Omega_1 \times \Omega_2$
G is 2-transitive on Ω: $\pi = \mathbf{1} + \chi$, χ irreducible
G is rank 3: $\pi = \mathbf{1} + \chi_1 + \chi_2$, χ_i distinct irreducible $\square$

Definition
G acts on Ω is <u>multiplicity-free</u> if its permutation character is $\pi = \mathbf{1} + \chi_1 + \cdots + \chi_{v-1}$ with the χ_i distinct

<u>Question</u>: Classify such permutation groups

Remark. If G is distance-transitive on an undirected graph than its permutation character is multiplicity-free

<u>Exercise</u>: Show that if G has permutation rank ≤ 5 then G is multiplicity-free
<u>Exercise</u>: If G is transitive on Ω, $\exists\, g \in G$ with $\pi(g) = 0$

6 Alternating Groups

$|S_n| = n!$, $|A_n| = n!/2$, A_n consist of all the even permutations.
S_n, A_n acts naturally on $\Omega = [1, n]$
A_n is $(n-2)$-transitive on $[1, n]$, S_n is n-transitive
Also recall: A_n is generated by the 3-cycles (ijk) on $[1, n]$
(e.g., (12)(34)=(124)(134), (12)(13)=(123), also note multiplication is defined such that left one act first)

Theorem 6.1
A_n is (non-abelian) simple for $n \geq 5$ (Note for $n = 3$, A_3 is abelian simple)
Proof I
Induction on n:
A_5 is simple (prove this). Let $n > 5$, assume true for $n - 1$

Let $G = A_n$, let $1 \neq K \trianglelefteq G \Rightarrow \quad K_\alpha \trianglelefteq G_\alpha (\cong A_{n-1})$ for $\alpha \in [1, n]$
Now G_α is simple by induction hypothesis $\Rightarrow K_\alpha = 1$ or $K_\alpha = G_\alpha$

Case $K_\alpha = 1$: Impossible. Because K is transitive regular, so by Lemma 5.8 and the statement above it about AGL being at most 3-transitive (except if $n = 4$) gives a contradiction

Case $K_\alpha = G_\alpha$: This implies that $K = G$ (as K would contains a 3-cycles, and hence all 3-cycles, on $[1, n]$, as these conjugate in A_n) $\square$

Proof II
Start as before, get K regular on $[1, n] \Rightarrow |K| = n$
$\Rightarrow \quad K$ contains a whole A_n-ccls of elements.
But, in fact, the A_n-ccls are larger than $n - 1$ (see later) $\square$

Proof III
(This proof is elementary) Suppose $1 \neq K \trianglelefteq G$

Claim: K contains a 3-cycle

Proof of Claim:
Let $1 \neq g \in K$, fixing as many points of $[1, n]$ as possible.
We claim g is a 3-cycle, suppose this is not true, there are 2 possibilities:

(1) all cycles of g have size 2: $g = (12)(34)\cdots$, let $x = (345)$ Then $[g, x] \neq 1$, but fixes $\{1, 2\} \cup \mathrm{fix}(g) \setminus \{5\}$ #
(2) $g = (123\ldots45\ldots) \quad \Rightarrow \quad [g, x] \neq 1$
But $\mathrm{fix}([g, x]) = \mathrm{fix}(g) \cup \{2\}$ (check) ■

Claim $\Rightarrow K$ contains all 3-cycles (all conjugate) $\Rightarrow K = A_n$ $\square$

Theorem 6.2
Let $G \leq A_n$, G primitive, G containing a 3-cycle. Then $G = A_n$
Proof
Define G-congruence ρ on $[1, n]$:

$$\alpha \,\rho\, \beta \text{ if } \alpha = \beta \text{ on 3-cycle } (\alpha\beta\gamma) \in G \text{ for some } \gamma$$

Then ρ is reflexive, symmetric, G-invariant
And, ρ is transitive: since $\alpha \,\rho\, \beta \,\rho\, \gamma \quad \Rightarrow \quad \exists(\alpha\beta\delta) \in G$ if $\delta = \gamma$
if $\delta \neq \gamma$ $\exists(\beta\gamma\epsilon)$ so $\langle(\alpha\beta\delta), (\beta\gamma\epsilon)\rangle \leq G$ is A_4 or A_5 $\Rightarrow$ (by $\alpha \,\rho\, \beta$) $(\alpha\beta\gamma) \in G$

$\Rightarrow$ α, β lie in a 3-cycle in G for any α, β (as ρ universal)
$\Rightarrow$ Let $\alpha, \beta, \gamma \in [1, n]$ distinct, let $(\alpha\beta\gamma), (\beta\gamma\epsilon)$ be suitable 3-cycles (using $\alpha\,\rho\,\beta, \beta\,\rho\,\gamma$) $\Rightarrow (\alpha\beta\gamma) \in G$ by above
$\Rightarrow$ have all 3-cycles of A_n in $G \Rightarrow G = A_n$ $\square$

G primitive on Ω, say Γ is Jordan set if the pointwise stabiliser in G of $\overline{\Gamma} = \Omega \setminus \Gamma$ is transitive on Γ
Γ is a primitive Jordan set if G is primitive on Γ
e.g. if G contains a p-cycle $(1 \cdots p)$ then $\Gamma = \{1, \ldots, p\}$ is a primitive Jordan set
Note: If k-transitive on Ω, then any subset of size $n - k + 1$ is a Jordan set

Exercise: If G is a primitive group with primitive Jordan set (size m), then show that G is $(n-m+1)$-transitive
(Hint: Consider Γ, Γg, $\Gamma \setminus (\Gamma_g \cap \Gamma)$ is a block of non-primitive of $G_{\overline{\Gamma}}$ on Γ, so just a singleton)

Corollary 6.3
G primitive on Γ, $|\Omega| = n$, if $\frac{n}{2} < p < n - 2$, if $p \big| |G| \Rightarrow G \geq A_n$ (such prime exists for $n \geq 8$)
Proof
Exercise $\square$

Exercise*: If G is primitive on S_n, and p is a prime with $\frac{n}{2} < p < n - 2$. If G contains a p-cycle then $G \geq A_n$
Hence: if $\frac{n}{2} < p < n - 2$ a prime, then $p \nmid |G|$

Recall, conjugacy classes of elements of S_n corresponds to partitions of n:

Two elements of S_n are conjugate in $S_n \Leftrightarrow$ they have same cycle-type (in disjoint cycle notation)
Notation: $n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}$ means a_i of n_i-cycles, $n_1 > n_2 > \cdots > n_k \geq 1$, $n = \sum a_i n_i$

Lemma 6.4
The size of S_n-conjugacy class of elements of type $n_1^{a_1} \cdots n_k^{a_k}$ is

$$\frac{n!}{a_1!(n_1)^{a_1} \cdots a_k!(n_k)^{a_k}}$$

Centralizer of one such element is a direct product of k wreath products:

$$\underbrace{(C_{n_1}^{a_1} \rtimes S_{a_1})}_{\underline{\text{wreath product}}} \times (C_{n_2}^{a_2} \rtimes S_{a_2}) \times \cdots \times (C_{n_k}^{a_k} \rtimes S_{a_k})$$

Proof
$n!$ permutation of n numbers, hence explain the numerator.
We need to quotient out those which determine the same cycle:
There are n_i ways to write the same n_i-cycle, and we have a_i of those, so have $n_i^{a_i}$ in the denominator
Also there are $a_i!$ ways to permute these a_i lot of n_i-cycle. $\square$

Lemma 6.5
Conjugacy classes of elements in A_n: elements in A_n have type $n_1^{a_1} \cdots n_k^{a_k}$ with $(\sum_{n_i} a_i)$ even
For such elements, the conjugacy classes under A_n is the whole of its S_n-ccl, unless all the n_i are odd, all the $a_i = 1$; this is the only case where $C_{S_n}(x) = C_{A_n}(x)$ so then the S_n-ccl splits into two A_n-ccls of equal length
Proof
Exercise $\square$

6.1 Structure of $\mathrm{Aut}(A_n)$

Note, $A_m \hookrightarrow A_k \;\Rightarrow\; k \geq m$
$m \geq 5 \;\Rightarrow\;$ any action of A_m is on $\geq m$ points

Lemma 6.6
If $n > 6$, then $\forall\, H \leq A_n$ s.t. $H \cong A_{n-1} \;\Rightarrow\; H = (A_n)_\alpha$ of some $\alpha \in [1, n]$
Proof
$n = 4, 5$ Use Sylow. So assume $n > 6$ now.
Now A_{n-1} is simple, so H has no permutation of degree k with $1 < k < n - 1$
Assume H is not a stabilizer of some point.
$\Rightarrow$ H is transitive on $[1, n]$ an in fact primitive there.

Now $n > 7$ (as $7 \nmid |A_6|$). Let $x \in H$ which goes to 3-cycles in A_{n-1} under our isomorphism $\phi : H \xrightarrow{\sim} A_{n-1}$
Now $C_{A_{n-1}}(x\phi) = \langle x\phi \rangle \times A_{(n-1)-3}$
$\Rightarrow$ $C_H(x) > A_{n-4}$
$\Rightarrow$ $C_{A_n}(x) \geq C_H(x) > A_{n-4}$
$\Rightarrow$ x is a 3-cycle on $[1, n]$ (other elements of order 3 have smaller centralizer)
$\Rightarrow$ $H = A_n$ by Corollary 6.3 # $\square$

Remark. This fails for $n = 6$ because
(1) A_5 acts on transitively on the set of size 6 of its Sylow 5-subgroups
(2) $C_{A_6}(1^3 3^1) = C_{A_6}(3^2)$
$|\operatorname{ccl}_{A_6}(123)| = |\operatorname{ccl}_{A_6}(123)(456)|$
$|\operatorname{ccl}_{S_6}(123)(45)| = |\operatorname{ccl}_{S_6}(123456)|$
$|\operatorname{ccl}_{S_6}(12)| = |\operatorname{ccl}_{S_6}(12)(34)(56)|$

Theorem 6.7
$\mathrm{Aut}(A_n) = S_n$, for $n > 3$, unless $n = 6$
Proof
Assume $n \geq 4$, $n \neq 6$. Any automorphism permutes the subgroups of A_n isomorphic to A_{n-1}. There are precisely n of these, one corresponding to each point of $[1, n]$, so any automorphism induces a permutation of $[1, n]$.
$\Rightarrow$ have the injection $\mathrm{Aut}(A_n) \hookrightarrow S_n$
This is obviously surjective because conjugation by element of S_n is an automorphism. $\Rightarrow$ $\mathrm{Aut}(A_6) \cong S_6$
$\square$

Theorem 6.8
$\mathrm{Aut}(A_6) = \Sigma_6$, a group of order 1440, containing S_6 as a subgroup of index 2.
Proof
<u>Existence</u>:
Let G e a simple group of order 60 (e.g. A_5)
$\Rightarrow$ G has 6 Sylow 5-subgroup, on which it acts.
$\Rightarrow$ $\exists$ faithful permutation representation $\phi : G \to A_6$

$G\phi \leq A_6$ of index 6, this gives a permutation representation $A_6 \to A_6$ which must be an isomorphism. This automorphism takes $G\phi$ (now a transitive subgroup of A_6) to the stabiliser of a point of A_6.
Now the elements of order 3 on the left are type 3^2 (as $3 \nmid \frac{60}{6}$), but those on the right are of the type $1^3 3^1$
$\Rightarrow$ this automorphism is not induced by conjugations
$\Rightarrow$ $\mathrm{Aut}(A_6) > S_6$

<u>Index 2</u>:
Any automorphism fixing the ccl of 3-cycles is induced by conjugation from S_6

There are only two ccls of elements order 3
If θ_1, θ_2 are two automorphism swapping these two classes (i.e. $\theta_1, \theta_2 \in \Sigma_6 \setminus S_6$), then $\theta_1^{-1}\theta_2$ fixes each of these ccls, so is in S_6
$\Rightarrow \quad |\Sigma_6 : S_6| = 2$ □

Corollary 6.9
G simple order 60 $\Rightarrow G \cong A_5 \cong PSL_2(5) \cong PSL_2(4)$
Proof
The proof of Theorem 6.8 uses an arbitrary simple group of order 60. □

Example: (Bochart)
H a primitive subgroup of S_n, not containing A_n
Then $|H| \leq n!/[\frac{n+1}{2}]!$ (e.g. $n = 6, S_5$ on Syl_5)
Let k be maximal s.t. $H \cap S_k = 1; \Rightarrow |H||S_{n/2}| = |HS_{n/2}| \leq n!$ and $|\Gamma| = k$. Take Γ s.t. $|\Gamma| = k$ and $H \cap \mathrm{Sym}(\Gamma) = 1$

> **Claim:** $k \geq \frac{n}{2}$
> **Proof of Claim:**
> $\exists g (\neq 1) \in H \cap \mathrm{Sym}(\overline{\Gamma})$, $\alpha g \neq \alpha, \alpha \in \overline{\Gamma}$
> $[g, h]$ is 3-cycle (check)
> $\exists h (\neq 1) \in H \cap \mathrm{Sym}(\Gamma \cup \{\alpha\})$
> $\Rightarrow \quad H \geq A_n \quad \#$

Sylow p-subgroups of S_n:
$n = n_1 p^{e_1} + n_2 p^{e_2} + \cdots + n_{e_1} p + n_{e_1+1} \qquad 0 \leq n_i < p$

Concentrating on the case $p^{e_1}, e_1 = 1$:
$p_2, e_1 = 2$: $\qquad (C_p)^p \rtimes C_p$
$\vdots$
p_n
$p_{n+1} \qquad (P_n)^p \rtimes C_p$
Any Sylow p-subgroup is

$$\prod_{1 \leq i \leq e_1} \underbrace{P_{e_i} \times \cdots \times P_{e_i}}_{n_i \text{ times}}$$

Thus has the right order

How about maximal subgroups H of A_n, S_n (and other subgroups G with $A_6 < G \leq \Sigma_6$)?
What if H is intransitive on $[1, n]$? Then H has an orbit of size $1 \leq k \leq \frac{n}{2}$ and $H \leq S_k \times S_{n-k}$
So the maximal subgroups intransitive on $[1, n]$ are precisely $S_k \times S_{n-k}$ for $1 \leq k \leq \frac{n}{2}$

Lemma 6.10
The maximal intransitive subgroups of S_n are $S_k \times S_{n-k}$ for $1 \leq k \leq \frac{n}{2}$
These are maximal in S_n unless $k = \frac{n}{2}$ ($n = 2k, S_k \times S_k < (S_k \times S_k) \rtimes S_2 < S_{2k}$)
Proof
Let $S_k \times S_{n-k} < X \leq S_n$
Then X on $[1, n]$ is transitive, in fact primitive, (unless $n = 2k$), since X is then 2-transitive. if not, take α from k, X_α has suborbits of size 1, $k-1$, $n-k$. Take β from $n-k$, X_β has suborbits size $1, k, n-k-1$ Impossible unless $n = 2k$
But X contains a transposition and 3-cycle.
$\Rightarrow \quad X \geq A_n$ □

Addition to previous lectures:
$A_n \trianglelefteq S_n \leq \mathrm{Aut}(A_n)$
$\theta : \mathrm{Aut}(A_n) \to S_n$
$K = \ker\theta$, $K \cap A_n = \{1\} \Rightarrow [K, A_n] = 1 \Rightarrow K = 1$ so θ injective
Exercise: Sylvester's construction of an outer automorphism of S_6 (Wilson Ex 2.19)

6.2 Wreath Products

Let C be a group, let D be a permutation group on Δ
The wreath product $C \,\mathrm{wr}_\Delta\, D$ is the semidirect product of the base group C^Δ (the direct product of $|\Delta|$ copies of C) by D, with D acting by permuting the components:

$$(c_{\delta_1}, c_{\delta_2}, \dots, c_{\delta_l})d = (c_{\delta_1 d^{-1}}, c_{\delta_2 d^{-1}}, \dots, c_{\delta_l d^{-1}})$$

i.e. $C \,\mathrm{wr}\, D = C^\Delta \rtimes D$

Suppose now C acts on Γ as a permutation group. The imprimitive action of $C \,\mathrm{wr}\, D$ on $\Gamma \times \Delta$:
Action of base group: $(\mu, \delta)(c_{\delta_1}, \dots, c_{\delta_l}) = (\gamma c_\delta, \delta)$
Action of D: $(\mu, \delta)d = (\mu, \delta d)$
For $\delta \in \Gamma$, let $\Gamma_\delta = \{(\mu, \delta) | \mu \in \Gamma\}$ (block of impriimitivity), then $B = \{\Gamma_\delta | \delta \in \Delta\}$ is the set of blocks
$\Rightarrow \quad C \,\mathrm{wr}\, D$ is imprimitive on $\Gamma \times \Delta$
In particular, have $S_k \,\mathrm{wr}\, S_l$ imprimitive on $\Gamma \times \Delta$, with $|\Gamma| = k, |\Delta| = l$

Lemma 6.11
Let G be (transitive but) imprimitive on Γ, $|\Gamma| = n$ so $G \leq S_n$, with l blocks of size k. Then $G \leq S_k \,\mathrm{wr}\, S_k$ (the full stabilizer of this partition of Γ into blocks). Moreover, these groups $S_k \,\mathrm{wr}\, S_l$ are maximal in S_n $(n = kl)$
Proof
(Proof of moreover part): $S_k \,\mathrm{wr}\, S_l < H \leq S_n$. Then H on $[1, n]$ is transitive, in fact primitive.
H_α contains $S_{k-1} \times S_k \,\mathrm{wr}\, S_{l-1}$
But H contains transpositions $\Rightarrow H = S_n$ (c.f. Theorem 6.2) □

Remark. If we are in A_n rather than S_n, use the 3-cycles in H, unless $k = 2$. The case $k = 2$ is harder and have to use elements of type 2^2, in fact it is false if $n = 8$: $S_2 \,\mathrm{wr}\, S_4 \overset{\text{index } 7}{<} AGL(3, 2) \overset{\text{index } 15}{<} A_8$

How about G primitive on $[1, n]$?
Let N_1, N_2 be minimal normal in G and $N_1 \neq N_2$
Then $[N_1, N_2] = 1$, and the N_1, N_2 are both transitive on $[1, n]$, so regular on $[1, n]$ and $C_G(N_1) = N_2, C_G(N_2) = N_1$ (c.f. Lemma 5.10)
Also, the N_i are non-abelian.
It follows that G has at most two minimal normal subgroups.

If two, these are non-abelian, isomorphic to each other:
Put $N = N_1 \times N_2$ (the socle of G)
Put $H = N_\alpha \quad \Rightarrow \quad N = HN_1 = HN_2$ and $H \cap N_i = 1$
$\Rightarrow \quad H \cong H/H \cap N_1 \cong HN_1/N_1 = N_1N_2/N_1 \cong N_2$ and by symmetry $H \cong N_1$
In fact, by Lemma 5.8, can identify Ω and N_2 so that N_2 acts by right regular action: $n *_R n_2 = nn_2$ for $n \in \Omega, n_2 \in N_2$
$\Rightarrow \quad N$ acts by left regular action on Ω as above since $N_1 = C_{S_n}(N_2)$: $n *_L n_1 = n_1^{-1} n$ for $n \in \Omega, n_1 \in N_1$
These actions commutes: $n *_L n_1 *_R n_2 = n_1^{-1} n n_2 = n *_R n_2 *_L n_1$
So N acts on Ω by $n * (n_1, n_2) = n_1^{-1} n n_2$
This is so-called diagonal action, due to Burnside

E.g. $A_5 \times A_5$ acting on A_5: $x(g,h) = g^{-1}xh$, $G_1 = \{(g,g) | g \in A_5\}$
The G_1 orbits are ccls in A_5

Now $G \leq N_{S_n}(N)$, but also $\exists g \in N_{S_n}(N) \setminus G$:
g acts as $n \mapsto n^{-1}$
$\Rightarrow$ G is never maximal in S_n (or A_n)
$\Rightarrow$ $N = T^{2k}$ where T is non-abelian simple and $n = |T|^k$

Summarizing:

Lemma 6.12
If G is primitive subgroup of S_n, then G has at most two minimal normal subgroup.
If have two, they have to be regular, non-abelian, isomorphic to (each other) T^k with T non-abelian simple, $k \geq 1$, with $n = |T|^k$
And G is not maximal in S_n (or A_n)

Exercise: $G \times G$ on G by $g * (g_1, g_2) = g_1^{-1} g g_2$ is transitive; it is primitive $\Leftrightarrow$ G simple

Lemma 6.13
(c.f. Lemma 5.8) If G has a regular elementary abelian normal subgroup, then $G \leq AGL_d(p) \leq S_n$, $n = p^d$
In addition, if G transitive, G is primitive $\Leftrightarrow$ G_0 is an irreducible linear group on $V_d(p)$

Let G primitive with a unique minimal normal subgroup N
If that is abelian, then $|N| = p^d$ and have G_0 being an irreducible linear group on $V_d(p)$

6.3 The primitive (product) action of wreath products

C acts on Γ, D acts on Δ $\quad |\Gamma| = k, |\Delta| = l$
$C \operatorname{wr} D$ acts on Γ^Δ, degree k^l (not $\Gamma \times \Delta$ as $C \operatorname{wr} D = C^\Delta \rtimes D$)
Action by base group C^Δ:

$$\begin{aligned} (\gamma_{\delta_1}, \ldots, \gamma_{\delta_l})(c_{\delta_1}, \ldots) &= (\gamma_{\delta_1} c_{\delta_1}, \ldots, \gamma_{\delta_l} c_{\delta_l}) \quad \text{(coordinate-wise)} \\ (\gamma_{\delta_1}, \ldots, \gamma_{\delta_l})d &= (\gamma_{\delta_1 d^{-1}}, \ldots, \gamma_{\delta_l d^{-1}}) \end{aligned}$$

Lemma 6.14
If D transitive on Δ, C primitive on Γ but not regular C_p, then $C \operatorname{wr} D$ is primitive on Γ^Δ, degree k^l
Proof
Let $\gamma \in \Gamma$. The stailizer of the constant point $(\gamma, \gamma, \ldots, \gamma)$ is $H = C_\gamma \operatorname{wr} D$

Claim: H is maximal in $C \operatorname{wr} D$

Proof of Claim:
Exercise

□

Example:
$W(k,l) = S_k \operatorname{wr} S_l$, degree $n = k^l$, with $k \geq 3$ is primitive
If $k = 3$ or 5, we have an elementary abelian regular normal subgroup
$\Rightarrow$ previous case $\Rightarrow$ assume $k \geq 5$
This is usually maximal in S_k, $n = k^l$

Exercise:
$W(k,l)$ with $k \geq 2$ has rank $l+1$, subdegrees are $1, l(k-1), \binom{l}{2}(k-1)^2, \ldots, \binom{l}{l}(k-1)^l$

6.4 Diagonal Actions

$N = T^m, T$ non-abelian simple, $n = |T|^{m-1}$, $m \geq 2$

$$D := \{(t, \ldots, t) \in T^m | t \in T\} < N$$

diagonal subgroup of N, $\Omega = (N : D)$
$D(T,m) := N_{S_n}(T^m)$. In fact, $D(T,m)/T^m \cong \mathrm{Out}(T) \times S_m$ (Note that $\mathrm{Out}(T) = \mathrm{Aut}(T)/T$ here)
Action of N: $D(x_1, \ldots, x_m)(t_1, \ldots, t_m) = D(x_1 t_1, \ldots, x_m t_m)$
Action of $\mathrm{Aut}(T)$: $D(x_1, \ldots, x_m)\alpha = D(x_1^\alpha, \ldots, x_m^\alpha)$
Action of S_m: $D(x_1, \ldots, x_m)\pi = D(x_{1\pi^{-1}}, \ldots, x_{m\pi^{-1}})$

Example:
$m = 2, D(T,2) = (T \times T) \cdot (\mathrm{Aut}(T) \times C_2)$
may identify Ω with T:

$$\begin{array}{rccl} (t_1, t_2): & t & \mapsto & t_1^{-1} t t_2 \\ \alpha: & t & \mapsto & t^\alpha \\ \pi (\in S_2): & t & \mapsto & t^{-1} \end{array}$$

$G < S_n$, $n = |T|^{m-1}$ is of diagonal type if $T^m \trianglelefteq G \leq D(T,m)$. Such G is primitive $\Leftrightarrow$ the action of the subgroup of S_m on coordinates is primitive.
The groups $D(T,m)$ is usually maximal in S_n or A_n

Theorem 6.15 (O'Nan Scott)
If $G < S_n$ primitive, then G is either almost simple or G is a subgroup of one of:

(1) $S_k \times S_{n-k}$

(2) $S_k \operatorname{wr} S_{n/k}$

(3) $AGL_d(p)$ (affine action), $n = p^d$, a prime power

(4) $W(k,l)$ (product action), $n = k^l, k \geq 5$

(5) $D(T,m)$ (diagonal action), $n = |T|^{m-1}$, $m \geq 2$, T non-abelian simple

(Proof omitted, see Wilson's book, also, a more precise revision for primitive permutation groups giving a structure)

Corollary 6.16
Maximal subgroups of S_n (or A_n):

(1) $S_k \times S_{n-k}$, $k < \frac{n}{2}$, intransitive

(2) $S_k \operatorname{wr} S_l$, $n = k \times l$, imprimitive

(3) $AGL_d(p), n = p^d$

(4) $D(T,m)$, $n = |T|^{m-1}$

(5) G primitive almost simple

Proof
May appear later (or not)
In fact, with a stronger version for G primitive subgroups of S_n:
G is almost simple except in cases we have seen (plus one other) $\square$

But which are maximal in S_n (or A_n)?
Example:
$S_5 < S_{10}$ on $\binom{5}{2}$, $(S_5 : S_3 \times S_2)$, maximal? Do not know
$S_6 < S_{10}$ on $3|3$ (partition into three 3s), $(S_6 : S_3 \operatorname{wr} S_2)$
$S_5 < S_6 < S_{10}$
Let $G = S_6$, $H = S_5$, have a factorisation of S_6 :

$$\begin{tikzcd}
 & G = S_6 \arrow[dl, dash, "10"'] \arrow[dr, dash, "6"] & \\
G_\alpha = S_3 \operatorname{wr} S_2 \arrow[dr, dash, "6"'] & & S_5 = H \arrow[dl, dash, "10"] \\
 & S_3 \times S_2 &
\end{tikzcd}$$

H is transitive, on S_6: $S_3 \operatorname{wr} S_2$
So need to know factorisation

$$\begin{tikzcd}
 & G = AB \arrow[dl, dash, "\text{maximal}"'] \arrow[dr, dash, "\text{can take maximal,inductively}"] & \\
A \operatorname{wr} S_2 \arrow[dr, dash, "6"'] & & B \arrow[dl, dash] \\
 & A \cap B &
\end{tikzcd}$$

these are now known for G almost simple. The question is answered

Upshot: G almost simple and primitive of degree $n \Rightarrow G$ maximal in A_n or S_n unless in a list

Example:
$G = S_m \quad |A||B|$ large
Let $\frac{m}{2} < p < m - 2$, a prime (exists for $m \geq 8$, by Chebychev)
$\Rightarrow \quad p|m! \Rightarrow p \big| |A|$ or $\cancel{p \big| |B|}$
$m! = |AB| = \frac{|A||B|}{|A \cap B|}$
By remarks earlier, if A is primitive on $[1, m]$ then $A \geq A_m \quad \#$
$\Rightarrow \quad A$ is intransitive on $[1, m]$ (cannot be transitive but imprimitive as $p \big| |A|$)
$\Rightarrow \quad A = S_k \times S_{m-k}$ for some k with $1 \leq k < \frac{m}{2}$

Case $k = 1$ is not very interesting.
Assume $k > 1$. Then B is transitive on $\binom{m}{k}$ (k-subsets of $[1, m]$). We say B is <u>k-homogoeneous</u> on $[1, m]$

$k = 2 : B$ is 2-transitive on $|B|$ odd (in fact, it is solvable)
$k > 2 : G$ is k-homogeneous $\Rightarrow G$ is $(k-1)$-homogeneous
Exercise: If π_k is the permutation character of S_m on $\binom{m}{k}$, then $\pi_k = \pi_{k-1} + \chi_k$ where χ_k is irreducible (use $\langle \pi_k, \pi_l \rangle = 1 + \min(l, k)$)
$\Rightarrow \quad B$ a is known group

So, to find maximal subgroup of S_m (or A_m), it is necessary and sufficient to find the maximal subgroup of all smaller almost simple groups

6.5 Doubly transitive representations of S_m or A_m

Theorem 6.17 (Maillet)
If S_m (or A_m) is 2-transitive degree n, then $n = m$, or one of the below ($m \leq 8$)

(1) $m = 5; n = 6$

(2) $m = 6; n = 6$ or 10

(3) $m = 7, 8; n = 15$ and this only happens in A_m

Lemma 6.18
If G is a transitive permutation group on Ω, and G has a ccl $\mathcal{C} \neq \{1\}$, then G has a non-trivial subdegree at most $|\mathcal{C}|$
Proof
Fix $c \in \mathcal{C}$. Let $\alpha \in \Omega$ with $\beta = \alpha c \neq \alpha$. Consider βG_α:
If $\omega \in \beta G_\alpha$, have $\omega = \beta h$ some $h \in G_\alpha$
$\Rightarrow \quad \omega = \alpha h^{-1} c h$ with $h^{-1} c h \in \mathcal{C}$ $\square$
Sketch Proof of Theorem 6.17
In S_m, take $\mathcal{C} = \{\text{transposition}\}$
In A_m, take $\mathcal{C} = \{\text{3-cycles}\}$
$\Rightarrow \quad n - 1 \leq \frac{1}{2} m(m-1)$
Let H be the stabiliser of a point in $[1, n]$
$\Rightarrow \quad |S_m : H| = n$ and H maximal on S_m
If H is primitive on $[1, m]$, then "small" by Bochert (see Example after Corollary 6.9):
$\left[\frac{m+1}{2}\right]! \leq 1 + \frac{1}{2} m(m-1) \Rightarrow m \leq 6$
(If in A_m, get $m \leq 8$)
If H is transitive but not primitive on $[1, m]$, then $H = S_k \operatorname{wr} S_{m/k}$, the action is on $k|k| \cdots |k$, not 2-transitive on $m > 6$
If H is intransitive on $[1, m]$, the action of S_m is on $\binom{n}{k}$, not 2-transitive unless $k = 1$ $\square$

7 Linear Groups

Let F be a field, here usually finite $F = \mathbb{F}_q$, $q = p^l$
$GL_d(F)$, group of invertible linear transformation on $V = V_d(F)$
For $F = \mathbb{F}_q$, write $GL_d(q)$, group of all non-singular $d \times d$-matrices over F
$|GL_d(q)| = (q^d - 1)(q^d - q) \cdots (q^d - q^{d-1})$

$SL_d(q) \trianglelefteq GL_d(q)$ matrices of determinant 1
$|SL_d(q)| = |GL_d(q)|/(q-1)$

$PGL_d(q) := GL_d(q)/Z$, where $Z = \{\text{scalar matrices in } GL\}$
$L_d(q) = PSL_d(q) := SL_d(q)/Z \cap SL_d(q)$, has order $|SL_d(q)|/e$, where $e = (d, q-1)$

$PGL_d(q)$ acts naturally on the set of 1-subspaces of $V_d(q)$ (i.e. the projective space of dimension $d-1$, $\mathbb{P}_{d-1}(q)$, of size $\frac{q^d-1}{q-1}$)
Recall: $PGL_d(q)$ is 2-transitive, and is 3-transitive $\Leftrightarrow d = 2$
$PSL_2(q)$ is 2-transitive, and is 3-transitive $\Leftrightarrow q = 2^l$

More on $d = 2$: "Möbius action" of $PGL_2(q)$ of degree $q+1$, $\mathbb{F}_q \cup \{\infty\}$

Lemma 7.1

$$\langle (x,y) \rangle \leftrightarrow \begin{cases} x/y & y \neq 0 \\ \infty & y = 0 \end{cases}$$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \tfrac{az+b}{cz+d} \text{ ??????}$$

$PSL_2(q) = \{z \mapsto \frac{az+b}{cz+d} | ad - bc \text{ is a non-zero square}\}$
$AGL_1(q) = G_\infty \{z \mapsto az + b | a \neq 0\}$
$G_{\alpha\beta\gamma} = 1$, any α, β, γ distinct
$G_{0\infty} = \{z \mapsto \alpha z | a \neq 0\}$
G is 3-transitive, with $G_{\alpha\beta\gamma} = 1 \;\; \forall \alpha\beta\gamma$

Generators of $PGL_2(q)$:
$z \mapsto z + 1$
$z \mapsto \lambda z \qquad (\langle \lambda \rangle = \mathbb{F}_q^\times)$
$z \mapsto -1/z$

Lemma 7.2

q	Group	order	degree
2	$L_2(2) \cong S_3$	6	3
3	$L_2(3) \cong A_4$	12	4
4	$L_2(4) \cong A_5$	60	5
5	$L_2(5) \cong A_5$	60	6

Also: $L_2(9) \cong A_6 \cong \Omega_4^-(3), L_2(7) \cong L_3(2), L_4(2) \cong A_8 \cong \Omega_6^+(2), L_4(2) \ncong L_3(4)$

Proof

Exercise $\square$

Theorem 7.3

$L_d(q)$ is simple for $d \geq 2$, unless $d = 2$, $q \leq 3$

To prove this, we need Iwasawa's Lemma:

Lemma 7.4 (Iwasawa's Lemma)

Let G be a perfect group (i.e. $G' = G$) acting primitively on a set Ω, let $\alpha \in \Omega$, and assume G_α has a normal subgroup K which is abelian (or just solvable) with $G = \langle K^g | g \in G \rangle$
If $N \trianglelefteq G$, then $N \leq G_{(\Omega)}$, the kernel of G on Ω, or $N = G$
So $G/G_{(\Omega)}$ is simple

Example:

A_n is simple: Consider $G = A_n$ acting on $\binom{n}{3}$, the 3-subsets, (for $n = 6, 3|3$)
$G_\alpha = (S_{n-3} \times S_3) \cap A_n, K = \langle (123) \rangle$

Proof

Assume $N \nleq G_{(\Omega)}$, so N transitive on Ω
$\Rightarrow \quad G = NG_\alpha$. Hence $NK \trianglelefteq NG_\alpha = G$
Now, $\langle K^g | g \in G \rangle = G$
$\Rightarrow \quad NK = G$
$\Rightarrow \quad G/N \cong NK/N \cong K/K \cap N$ is abelian
$\Rightarrow \quad G' \leq N$, but $G = G'$
$\Rightarrow \quad N = G$ $\square$

Definition
$t \in SL_d(q)$ is a <u>transvection</u> if $t-1$ has rank 1 and $(t-1)^2 = 0$
(JNF has 1 block size 2, all other size 1) <u>Note</u>: If t is a transvection on V, then wrt some basis $\mathcal{B}$ of V, t has matrix of form

$$\begin{pmatrix} 1 & & 0 \\ 0 & \ddots & \\ 1 & 0 & 1 \end{pmatrix}$$

This is because, let $W = \ker(t-1)$, let $v_d \in V \setminus W$, let $v_1 = v_d(t-1)$
extend the basis $v_1, \ldots, v_d - 1$ of W, get $\mathcal{B} = \{v_1, \ldots, v_{d-1}, v_d\}$
Also, the elementary matrices $E^{(*)}$ are transvections $X_{ij}(\lambda), X_{ij}(\lambda)^{-1} = X_{ij}(-\lambda)$
<u>Transvection subgroup</u>: $X_{ij} = \{X_{ij}(\lambda) | \lambda \in \mathbb{F}_q\}$

<u>Exercise</u>: Al transvections are conjugate in $SL_d(q)$ if $l = 2$. If $d = 2, q$ odd then two ccls

Proposition 7.5
$SL_d(q)$ is generated by the transvections
Proof
Linear algebra: any matrix of det 1 can be reduced to I_A by applying elementary row operations $v_i := v_i + \lambda v_j,\ i \neq j$
Each of these operations is just multiplication on the left by a matrix $E^{(*)}$
$E^{(n)} \cdots E^{(1)} A = I \quad \Rightarrow \quad A = (E^{(1)})^{-1} \cdots (E^{(n)})^{-1}$ □

Proof of Theorem 7.3
Let $G = SL_d(q)$. Then G is 2-transitive, hence primitive, on $\Omega = \{\text{1-subspaces of } V_d(q)\}$
Kernel $G_{(\Omega)} = \{\text{scalar matrices}\}$
Let $\alpha = \langle (1, 0 \cdots, 0) \rangle$

$$\Rightarrow \quad G_\alpha = \left\{ \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ \lambda_2 & & & \\ \vdots & & H_{d-1} & \\ \lambda_d & & & \end{pmatrix} \lambda_i \in \mathbb{F}_q \right\} \quad , \quad K = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \lambda_2 & & & \\ \vdots & & I_{n-1} & \\ \lambda_d & & & \end{pmatrix} \Big| \lambda_i \in \mathbb{F}_q \right\}$$

elementary abelian order q^{d-1} normal in G_α
The elements of $K \setminus \{1\}$ are transvections.
We shall check they generate G, and each is a commutator in G (shown in next proposition) □

Proposition 7.6
$(d > 1)$ All transvections are commutators, except when $d = 2, q \leq 3$
Proof
If $d \geq 3$

$$\begin{pmatrix} 1 & & \\ \alpha & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -\alpha & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -\beta & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ -(\alpha\beta) & & 1 \end{pmatrix}$$

so if $\alpha \neq 0$, take $\beta = \alpha^{-1}\gamma$

$d = 2$:

$$\begin{pmatrix} \alpha^{-1} & \\ & \alpha \end{pmatrix} \begin{pmatrix} 1 & \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha^2\beta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta(\alpha^2 - 1) & 0 \end{pmatrix}$$

So given γ, take $\alpha \neq 0$ with $\alpha^2 \neq 1$ $(q > 3)$
and $\beta = \gamma(\alpha^2 - 1)^{-1}$, to get $\begin{pmatrix} 1 & \\ \gamma & 1 \end{pmatrix}$ as a commutator □

7.1 Some subgroup of $GL_d(q)$ or $SL_d(q)$

Parabolic:

(1) $B = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ \# & & * \end{pmatrix} \middle| * \in F^*, \# \in F \right\}$ - Borel subgroup

(2) lower traingular

(3) $U = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \# & & 1 \end{pmatrix} \middle| \# \in F \right\}$ - lower unitriangular matrix $|U| = q^{\frac{1}{2}d(d-1)}$ a Sylow p-subgroup of GL, $q = p^t$

$B = U \rtimes T$, with $T = \{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} | \lambda_i \in F^\times \}$ diagoonal matrices or split torus
$V = V_d(q)$ B is the stabilizer of a complete flag: $0 = V_0 < V_1 < V_2 \cdots < V = V_d$
$V_i = \langle v_1, \dots, v_i \rangle$

Other flags: $0 < V_i < V_j < \cdots < V = V_d$
Stabilisers of flags are parabolic subgroups
maximal parabolic is just a stabilizer of a subspace of W of V
P_k is the stabilizer of a k-subgroup W of V:
$P_k = \{ \left(\begin{array}{c|c} A_k & 0 \\ \hline C & B_{d-k} \end{array} \right) | (A, B) \in GL_k \times GL_{d-k}, C$ any of $k \times k$ matrix$\}$
(choose basis $e_1, \dots, e_k$ of W extend to a basis $e_1, \dots, e_k, \dots, e_h$ for V)

The subgroup $U_k = \{ \left(\begin{array}{c|c} I_k & \\ \hline * & I_{d-k} \end{array} \right) \}$ is a normal subgroup of P_k;
$L_k = \left(\begin{array}{c|c} A_k & 0 \\ \hline 0 & B_{d-k} \end{array} \right)$ - complement to U_k in P_k, the Levi subgroup!of linear group of P_k
Then $P_k = U_k \rtimes L_k$

By the way, N= all monomial matrices, then $N = N_G(T)$
$T = B \cap N$, $W = N/T$ - Weyl group
Example: $GL_d(q)$, $W = \mathrm{Sym}_d$

Remark. The subgroups of $GL_d(q)$ containing B are precisely the parabolics obtained by stabilizing the flags obtained by deleting some members of the complete flag for B

Exercise:
$G = GL_d(q)$ has permutation rank $k+1$ for $(G : P_k)$ (P_k stabiliser of W, dim k), $1 \leq k \leq \frac{d}{2}$
And, P_k is a maximal subgroup of $GL_d(q)$

Some other subgroups:
some families of geometric subgroup
Then theorem of Aschbache tells you that any subgroup H of SL lies in one of these or $H/Z(H)$ is almost simple, irreducible, etc....

(1) P_k - the stabilizer of a k-subspace W of V

(2) $V = V_1 \oplus \cdots \oplus V_k$, $d = ka$, $\dim V_i = a$
$GL_a(F) \,\mathrm{wr}\, \mathrm{Sym}_k$

(3) $V = V_1 \otimes V_2$, $\dim V_i = d_i$, $d = d_1 d_2, d_1 \neq d_2$
$GL_{d_1}(F) \otimes GL_{d_2}(F)$

(4) $V = V_1 \otimes \cdots \otimes V_k$, $\dim V_i = a$, $d = a^k$
$GL_a(F) \operatorname{wr} \operatorname{Sym}_k$

(5) Subfield subgroup $\mathbb{F}_{q'} < \mathbb{F}_q$
$V = W \otimes \mathbb{F}_q$, $\dim_{\mathbb{F}_{q'}} W = d$
$GL_d(q') < GL_d(q)$

(6) Extension field subgroups $GL_{d/e}(q^e) < GL_d(q)$

(7) "Extraspecial"

(8) Classical

Upslot: Questions about maximal subgroups of linear groups can be reduced to question about modular representation theory of quasi-almost simple groups

7.2 Automorphism group of $PSL_d(q)$

$PSL_d(q), q = p^f$, Outer automorphisms come in different flavours:

$$\begin{array}{cccccccc} & L_d(q) & \trianglelefteq & PGL_d(q) & \trianglelefteq & P\Gamma L_d(q) & \trianglelefteq & \mathrm{Aut} \\ \text{(index)} & & \substack{(d,q-1) \\ \text{Inn diag}} & & \substack{f \\ \text{field auto}} & & \substack{2 \text{ if } d>2 \\ 1 \text{ if } d=2 \\ \text{graph auto}} & \end{array}$$

$\Gamma L_d(q)$: semilinear transfromation of $V = V_d(q)$
$(r+w)\theta = r\theta + w\theta$
$\exists \sigma \in \Gamma = \mathrm{Gal}(\mathbb{F}_q \,/\, \mathbb{F}_p) (\lambda v)\theta = \lambda^\sigma v\theta$ (σ depends on θ)
Fix a basis,

$$\begin{pmatrix} a_{11} & \dots & a_{1d} \\ \vdots & \dots & \end{pmatrix}^\sigma = \begin{pmatrix} a_{11}^\sigma & \dots & a_{1d}^\sigma \\ \vdots & \dots & \end{pmatrix}$$

$\Gamma L_d(q) = GL_d(q) \rtimes \Gamma$
finally, a graph auto: $A \mapsto A^{-t}$ (inverse transpose of A) an auto of GL Exercise: Graph auto is inner if $d = 2$, outer otherwise
$\mathrm{Aut} := P\Gamma L \rtimes \langle \tau \rangle$

Theorem 7.7 (Steinberg, for groups of Lie type)
The Aut. group of $L_2(q)$ is $P\Gamma L_2(q)$
The Aut. group of $L_d(q)$ with $d > 2$ in $P\Gamma L_d(q) : C_2$ (semicolon denotes semidirect product)
Sketch
Need to show no more automorphisms. Let $G = PSL_d(q)$, $V = V_d(q)$, $\phi \in \mathrm{Aut}(L_d(q))$, $q = p^f$
Any p-local subgroup (i.e. normaliser of a a p-subgroup) stabilise (set-wsie) a subspace
$\Rightarrow$ in P_k for some k:
$H = N_G(Q), Q$ a p-subgroup, let

$$W := \mathrm{fix}_V(Q)$$

is a subspace kept invariant by $N_G(Q)$. Note that if the ccls of stabilisers of hyperplanes is not ϕ-invariant, adjust ϕ by a graph automorphism. (note that it would have been swapped with the ccls of stabilisers of 1-spaces, as the only possibility by structure of P_k).

So assume ϕ keeps the set of hyperplanes invariant, and then it follows that it keeps the set of k-spaces invariant for all k. In particular, ϕ acts on the set of 1-spaces.

$G = SL_d(q)$ is transitive on the set of complete flags, so may assume that ϕ stabilises a complete flags, and in fact, may assume that ϕ stabilises $\langle v_1\rangle, \langle v_2\rangle, \dots, \langle v_d\rangle$, with $\{v_1, \dots, v_d\}$ a basis. Moreover, using a diagonal automorphism, if necessary, to adjust ϕ, we may assume ϕ fixes $v_1, \dots, v_d$.

We now claim that such ϕ is induced by a Galois automorphism of $\mathbb{F}_q \,/\, \mathbb{F}_p$, i.e. ϕ is semilinear on V

$$\begin{array}{cccccc}
 & \begin{pmatrix}1 & & \\ \lambda & 1 & \\ & & \ddots\end{pmatrix} & \begin{pmatrix}1 & & \\ \mu & 1 & \\ & & \ddots\end{pmatrix} & = & \begin{pmatrix}1 & & \\ \lambda+\mu & 1 & \\ & & \ddots\end{pmatrix} & \\
\phi: & \downarrow & \downarrow & & & \\
 & \begin{pmatrix}1 & & \\ \lambda^\sigma & 1 & \\ & & \ddots\end{pmatrix} & \begin{pmatrix}1 & & \\ \mu^\sigma & 1 & \\ & & \ddots\end{pmatrix} & = & \begin{pmatrix}1 & & \\ \lambda^\sigma+\mu^\sigma & 1 & \\ & & \ddots\end{pmatrix} & \lambda^\sigma \in \mathbb{F}_q, (\lambda+\mu)^\sigma = \lambda^\sigma + \mu^\sigma \\
\text{and} & \begin{pmatrix}\lambda & & \\ & \lambda^{-1} & \\ & & I\end{pmatrix} & \begin{pmatrix}\mu & & \\ & \mu^{-1} & \\ & & I\end{pmatrix} & & \to (\lambda\mu)^\sigma = \lambda^\sigma\mu^\sigma &
\end{array}$$

$\Rightarrow \quad \sigma \in \mathrm{Gal}(\mathbb{F}_q)$ A general proof: In Carter's book, Groups of Lie type $\square$

7.3 Some isomorphisms and interesting actions

Exercise: Any simple group of order 168 is isom. to $L_2(7)$

Lemma 7.8
$L_2(2) \cong L_2(7)$
Proof
$L_3(2)$ acts on $\mathbb{P}_2(2)$: $V = V_3(2)$

points: 1-subspace of V

lines: 2-subspace of V

incidence: subspace of V

Fano plane (any 2 points on a unique line (of size 3)) (see picture)

In fact, can ??? mod 7 points so that 013, 124, 235, … are lines

$\mathbb{F}_7$
$g : u \mapsto u + 1$ on $\mathbb{F}_7$ (points to points, lines to lines), $g = (0123456)$
$h : u \mapsto 2u$ in $N(\langle g\rangle)$, $h = (124)(365)$
$t = (12)(36) \in N_G(\langle h\rangle)$
These generate $L_3(2)$

Let $G = L_3(2)$ (or any simple group of order 168)
$n_p(G) = 8$, $|N_G(P)| = 21$
Let $P \in \mathrm{Syl}_7(G)$, say $P = \langle g\rangle$
Number the Sylow 7-subgroups as $P = \infty, 0, 1, \dots, 6$
Choose one of the Sylow 7-subgroup as 0 and $g : z \mapsto z + 1$, then all the numbering are determined

Let $h \in N_G(P)$, order 3, $h : z \mapsto 2z$ $(N_G(P) = N_{A_7}(P))$
$\exists t \in N_G(\langle h \rangle)$, order 2, inverting h (know this is subgroup $L_2(2)$)
Now the stabiliser of any point in $\mathrm{Syl}_7(G)$ has odd order, so t is fixed-point-free in this action of degree 8:

$$\underbrace{(0\infty)}_{\text{fix } h}(1x)(2y)(4z)$$

($x, y, z \in \{3, 5, 6\}$ but not known which yet)
Conjugating t by h or h^{-1}, we may ssume $t : 1 \leftrightarrow 6$
Then $2t = 3$, since $2t = 1ht = 1th^{-1} = 6h^{-1} = 3$, so we get

$$(0\infty)(16)(23)(45)$$

Thus, $t : z \mapsto -1/z^{-1}$
$\Rightarrow \quad L_3(2) \lesssim L_2(7)$, so $\cong$ by order $\square$

Lemma 7.9
$A_6 \cong L_2(9)$, $S_6 \cong P\Sigma L_2(9)$ (Σ field autos), $\Sigma_6 \cong P\Gamma L_2(9)$
Proof
Produce an action of A_6 degree 10 (3|3 - stab. is $(S_3 \operatorname{wr} S_2) \cap A_6$)
Put $\mathbb{F}_9 \cup \{\infty\}$ structure on it to get $A_6 \lesssim L_2(9)$
$H = N_{A_6}(\langle (123) \rangle, \langle (456) \rangle) \cong (S_3 \operatorname{wr} S_2) \cap A_6$
123|456 is fixed by H
An element h of order 4 in H: (14)(2536)
Let $\mathbb{F}_q = \{0, \pm 1, \pm i, \pm(1 \pm i)\}$
Let $g_1 : z \mapsto z + 1$, $g_2 : z \mapsto z + i$
Now h also fixes partition 156|234
Rest of notation is now fixed, $t : z \mapsto -1/z^{-1}$
$\Rightarrow \quad A_6 \lesssim L_2(9)$
The above part demonstrates actions of A_6 on $\mathbb{F}_9 \cup \{\infty\} = \mathbb{P}_1(9)$
It yields A_6 can be "embedded" into $L_2(9)$
Now (56) acts as $z \mapsto z^3$ a field automorphism
$\Rightarrow \quad S_6 \cong P\Sigma L_2(9)$ $\square$

Note: $\mathrm{Gal}(\mathbb{F}_{p^l} / \mathbb{F})$ is generated by $z \mapsto z^p$ and is cyclic of order l

Some 2-transitive actions of $L_d(q)$:
$L_d(q)$is 2-transitive on $\frac{q^d-1}{q-1}$ (2 actions for $d > 2$, 1 action for $d = 2$)
plus: $L_2(7), \deg 7$ (2 action)
$L_2(9), \deg 6$
$L_2(11), \deg 11$
$P\Sigma L_2(8) \cong {}^2G_2(3), \deg 28$
$L_4(2) \cong A_8, \deg 8$

8 symplectic Groups $Sp_d(q) \leq GL_d(q)$

Definition
If f a bilinear alternating non-singular form on $V = V_d(q)$, f is called symplectic form (We are taking definition $f(v, v) = 0 \;\; \forall v$ for alternating because $f(v, w) = -f(w, v)$ is weaker when char=2)
Non-degenerate (non-singular): $\forall v \neq 0, \exists w$ s.t. $f(v, w) \neq 0$
$\Rightarrow \quad V^{\perp} := \{w | f(v, w) = 0\} \neq V \;\; \forall v \neq 0$

Lemma 8.1
If f is a symplectic form on V, there exists basis $e_1, f_1, e_2, f_2, \dots e_m, f_n$ with $f(e_i, e_j) = 0 = f(f_i, f_j)$ and $f(e_i, f_j) = \delta_{ij}$
Proof
Construct:
Let $e_1 \neq 0$. Take $f_1 \in V \setminus e_1^\perp$ (non-empty)
with (after scaling) $f(e_1, f_1) = 1$. Continue this process in $\langle e_1, f_1 \rangle^\perp$ $\square$

Definition
Such e_i, f_i is a hyperbolic pair
Then the matrix of f under this basis is

$$\left(\begin{array}{cc|cc|c} & 1 & & & \\ -1 & & & & \\ \hline & & & 1 & \\ & & -1 & & \\ \hline & & & & \ddots \end{array}\right)$$

or taking $e_1, e_2, \dots, f_1, f_2, \dots$

$$\left(\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array}\right)$$

Let $G = Sp_{2m}(q)$ preserves this form, i.e. $G = \{g \in GL_{2m}(q), f(vg, wg) = f(v, w) \ \forall v, w \in V\}$
In terms of matrices $\{A \in GL_{2m}(q) | A^t J A = J\}$

$Sp_{2m}(q)$ is regular in its action on the symplectic bases. (transitive with trivial stabliser)

Lemma 8.2
$|Sp_{2m}(q)| = (q^{2m} - 1)(q^{2m-1})|Sp_{2m-2}(q)| = \cdots = q^{m^2} \prod_{i=1}^m q^{2i} - 1$

Lemma 8.3
$Sp_2(q) \cong SL_2(q)$,
Proof
$A^t \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} A = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ for $ad - bc = 1$ $\square$

$Z(Sp_{2m}(q)) = \{\pm I\}$, $PSp_{2m}(q) \cong Sp_{2m}(q)/\{\pm I\}$

Definition
symplectic transvections are those transvections in Sp

$$t_{v,\lambda} : x \mapsto x - \lambda f(x, v) v$$

E.g. $v = e_1$:
$f(x, e_1) = 0$ for other basis

$f(x, f_1) = 1$, so under basis $e_1, e_2, \dots, e_m, f_m, \dots, f_1$:

$$\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & \lambda & & & \ddots & \\ & & & & & 1 \end{array}\right)$$

Proposition 8.4
$Sp_{2n}(q)$ is generated by these transvections
Proof
$G = \langle$ these transvections $\rangle$, will show G is transtive on the set of symplectic basis

(1) G is transitive on the set of all vectors:
To go from v to w. If $v \not\perp w$, then $vt_{v-w,\lambda} = v + \lambda f(v,w)(v-w)$
Take $\lambda = -f(v,w)^{-1}$ we get $vt_{v-w,\lambda} = w$
If $v \perp w$, choose $x \in V \setminus (v^\perp \cup w^\perp)$
and go $v \mapsto x \mapsto w$ by the above method

(2) G is transitive on the set of hyperbolic pairs:
(v, w_1) and (v, w_2) are hyperbolic pair
If $w_1 \not\subseteq w_2$, take $t_{w_1-w_2,\lambda}$ as in is to send $w_1 \mapsto w_2$
If $w_1 \perp w_2$, go $w_1 \mapsto w_1 + v \mapsto w_2$ which fixing v

(3) G transitive on the set of symplectic basis by induction
If $\mathcal{B}' = \{e_1', f_1', \dots\}$ may assume $e_1' = e_1, f_1' = f_1$ by (2)
Then work in $\langle e_1, f_1\rangle^\perp - V_{2m-2}(q)$ with symplectic form obtained by restriction. Induction do the rest

$\square$

Corollary 8.5
$Sp_{2m}(q) \subseteq SL_{2m}(q)$

Proposition 8.6
Transvections in $Sp_{2m}(q)$ are commutators, unless $(2m, q)$ is one of $(2,2), (2,3)(4,2)$
Proof
This is true as $Sp_2(q) = SL_2(q)$ for $q \geq 3$
So only have to check cases $Sp_4(3)$ and $Sp_6(2)$ (exercise)
$(Sp_{2m}(q) = [SL_{2m}(q), SL_{2m}(q)])$ $\square$

Theorem 8.7
$PSp_{2m}(q)$ is simple, except for $PSp_2(2) \cong S_3, PSp_2(3) \cong A_4, PSp_4(2) \cong S_6$
Proof
Use Iwasawa's Lemma 8.8 $\square$

Lemma 8.8 (Iwasawa)
The action of $Sp_{2m}(q)$ on the $\frac{q^{2m}-1}{q-1}$ points of $\mathbb{P}_{2m-1}(q)$ is rank 3, with subdegrees $1, q(q^{2m-2}-1)/(q-1), q^{2m-1}$
This stabilizer of $\langle v\rangle$ is $P_1 = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ * & A & 0 \\ * & * & \lambda^{-1} \end{pmatrix} \middle| \lambda \in \mathbb{F}_q^\times, A \in Sp_{2m-2}(q) \right\}$ with normal subgroup $\left\{ \begin{pmatrix} 1 & & \\ 0 & I & \\ \lambda & 0 & 1 \end{pmatrix} \right\} =$ $\{t_{v,\lambda} | \lambda \in \mathbb{F}_q\}$

The action is primitive, kernel is $\{\pm I\}$
Proof
$v \in V$, suborbits $\{\langle v\rangle\}$ (size 1), $\{\langle w\rangle | w \in v^\perp - \langle v\rangle\}$, $\{\langle w\rangle | w \in V \setminus v^\perp\}$ (size q^{2m-1})

The action is primitive since any non-trivial block would consist of $\{\langle v\rangle\}$ together with one of the $G_{\langle v\rangle}$-orbits, but the size does not divide $\frac{q^d-1}{q-1}$

Finally, $G_{\langle v\rangle} = P_1$ - see below about P_k $\square$

Proposition 8.9
$Sp_4(2) \cong S_6$, so not perfect
Proof
S_6 on $U = V_6(2)$ natural action as permutations of coordinates.
$W = \{(x_1, \ldots, x_6) | \sum x_i \equiv 0 \mod 2\}$
$f(x, y) = \sum x_i y_i$, bilinear form on U
$W^\perp = \{(1, 1, \ldots, 1)\}$
$\Rightarrow \quad W \supseteq W^\perp$
$V = W/W^\perp$ a 4-dimensional space over $\mathbb{F}_2$ with a symplectic form preserved by S_6
$\Rightarrow \quad S_6 \leq Sp_4(2)$ on V
$\Rightarrow \quad S_6 \cong Sp_4(2)$ by order. $\square$

<u>Exercise</u>: $S_{2m+2} \leq Sp_{2m}(2)$

8.1 Parabolic subgroups in $Sp_{2m}(q)$

$V = V_{2m}(\mathbb{F}_q)$
A complete symplectic flag: $0 < W_1 = \langle e_1\rangle < W_2 = \langle e_1, e_2\rangle < \cdots < W_m = W_m^\perp = \langle e_1, \ldots, e_m\rangle < W_{m-1}^\perp = \langle e_1, \ldots, e_m, f_m\rangle$ W_i isotropic, so $f|_{W_i}$ is 0, for each $1 \leq i \leq m$
B Borel stabilises such a complete flag
The parabola are stabilisers of some flags
Maximal parabolic subgroup $P_k :=$ Stabilisers of W_k with $W_k \subseteq W_k^\perp$

P_k stabilise $W = \langle e_1, \ldots, e_k\rangle$ and $W^\perp = \langle e_1, \ldots, e_m, f_{k+1}, \ldots, f_m\rangle$ (dim $2m - k$)
$W^\perp/W = \langle \overline{e}_{k+1}, \ldots, \overline{e}_m, \overline{f}_{k+1}, \ldots, \overline{f}_m\rangle$ -Symplectic space dim $2(m-k)$
P_k contains a <u>Levi subgroup</u> $GL_k \times Sp_{2(m-k)}$: $\left(\begin{array}{c|c|c} A & & \\ \hline & B & \\ \hline & & A^{-t} \end{array}\right)$ (on W, $W^\perp/W$, $V/W^\perp$)
Kernel of the homomorphism $P_k \to L_k$ is $Q_k = \begin{pmatrix} I & & \\ * & I & \\ * & * & I \end{pmatrix}$
Q_k is the <u>unipotent radical</u> of P_k, order $q^{k(k+1)/2} q^{2k(m-k)}$, often a "special" p-group:
$Q' = Z(Q) = \Phi(Q)$, order $q^{k(k+1)/2}$

<u>Example</u>
<u>$2m = 4, k = 1$</u>
$Q_1 = \left\{\begin{pmatrix} 1 & & & \\ \alpha & 1 & & \\ \beta & 0 & 1 & \\ \gamma & \beta & -\alpha & 1 \end{pmatrix}\right\}$, e.g. $q = 3$, can get $\left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}\right\}$ as a product of commutator
$k = 2, Q_2 = \left\{\begin{pmatrix} I & \\ C & I \end{pmatrix} | C = C^t\right\}$

Exercise: P_1, P_m in general?

Remark. $Sp_{2m}(q)$ is $C_m(q)$ as group of Lie type

The other interesting stabilisers of subspace W have $W \cap W^\perp = 0$
$\Rightarrow \quad V = W \oplus W^\perp$ $(\dim W = 2k, \dim W^\perp = 2(m-k))$

$$N_k = Sp_{2k}(q) \times Sp_{2(m-k)}(q) < Sp_{2m}(q)$$

$(Sp_4(2) \cong S_6$ deg $6, 10)$
2-transitive actions of $Sp_{2m}(q)$: $2m = 2$, deg $q+1$ and deg q if $q \in \{5, 7, 11\}$
and $Sp_{2m}(2)$ deg $2^{m-1}(2^m \pm 1)$ on the two classes of orthogonal forms.

Automorphism groups: $PSp_{2m}(q) \trianglelefteq PGSp_{2m}(q)$ (diagonal $(2, q-1)$), $g : f(vg, wg) = \lambda_g f(v, w) \in \mathbb{F}_q$
$PSp_{2m}(q) \trianglelefteq PGSp_{2m}(q) \trianglelefteq P\Gamma Sp_{2m}(q) \trianglelefteq \mathrm{Aut}$ (this last $\trianglelefteq$ has index 1 except for $Sp_4(2f)$ then index 2)

9 Unitary Groups

$F = \mathbb{F}_{q^2} > \mathbb{F}_q$ a quadratic extension, $V = V_d(q^2)$

$$\begin{array}{rcll} \sigma : \alpha & \mapsto & \alpha^q & \langle \sigma \rangle = \mathrm{Gal}(\mathbb{F}_{q^2} / \mathbb{F}_q) \\ \alpha & \mapsto & \overline{\alpha} & N(\alpha) = \alpha^{q+1}, N : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2} \text{ onto} \end{array}$$

Let f be a σ-Hermitian, left linear, non-singular form

$$GU_d(q) = \{g \in GL_d(q^2) \mid f(vg, wg) = f(v, w)$$

If $\dim V \geq 2 \Rightarrow \exists$ isotropic vectors:
Since, let $v \in V$ with $f(v, v) \neq 0$
Let $u \in v^\perp$
If u not isotropic, consider $f(u + \lambda v, u + \lambda v) = f(u, u) + \lambda\overline{\lambda} f(v, v)$ - can make it zero by choice of λ

Unitary bases:

(1) $d = 2m$ $\quad e_1, f_1, \ldots, e_m, f_m$ as usual: $f(e_i, f_j) = \delta_{ij}$,etc.
$d = 2m+1$ $\quad e_1, f_1, \ldots, f_m, v \quad v \perp e_i, f_j, f(v, v) = 1$
$\Rightarrow$ all forms are equivalent, for each dimension

(2) Also, have an orthonormal bases

Lemma 9.1

$$|GU_d(q)| = q^{\frac{1}{2}d(d-1)}(q^d - (-1)^d)(q^{d-1} - (-1)^{d-1}) \cdots (q^2 - 1)(q + 1)$$

Proof
Write

$$\begin{array}{rcl} z_d & = & \#\text{ isotropic vectors in dimension } d \\ y_d & = & \#\text{ vectors norm 1} \quad \Rightarrow \quad q^{2d} = 1 + z_d + (q-1) y_d \end{array}$$

Also, $z_{d+1} = z_d + (q^2 - 1) y_d = -q z_d + (q^{2d} - 1)(q + 1)$

Since $z_0 = 0 = z_1$, can solve recurrence: $z_d = (q^d - (-1)^d)(q^{d-1} - (-1)^{d-1})$
$\Rightarrow \quad y_d = q^{d-1}(q^d - (-1)^d)$
The order of GU follows, by induction along an orthonormal basis $\square$

$$|SU| = |GU|/(q+1) \quad , \quad |PSU| = |GU|/(q+1)\gcd(d,q+1)$$

In particular, $PSL_2(q) \cong PSU_2(q)$

Lemma 9.2
$PSU_3(q)$ acts 2-transitively on the isotropic points of $\mathbb{P}_2(q^2)$. There are q^3+1 of these; for larger d, this leads to a primitive rank 3 action.
Proof
e isotopic. Let f_1, f_2 isotropic in $V \setminus \langle e^\perp \rangle$ (Note: no isotropic vectors in $e^\perp$ as $d = 3$)
Can swap e, f_1, to e, f_2, using an element in $GU_3(q)$
$\Rightarrow \quad (GU_3)_{\langle e \rangle}$ transitive of degree q^3, and SU_3 has index $q+1$ in GU_3
$\Rightarrow \quad$ still transitive $\square$

For $d \geq 3$, $PSU_d(q)$ is simple except for $PSU_3(2)$ (order 72, 2-transitive on 9), c.f. Iwasawa Lemma

Maximal parabolic subgroups: P_k = stabiliser of a totally isotropic k-vector space $= GL_k(q^2) \times GU_{d-2k}(q)$

10 Orthogonal Groups

Quadratic form $Q : V \to F = \mathbb{F}_q$
s.t. $Q(\lambda v) = \lambda^2 Q(v)$
$Q(u+v) = Q(u) + Q(v) + f(u,v)$, with f bilinear symmetric form
$\Rightarrow \quad 2Q(v) = f(v,v)$
In char $F \neq 2$, have $Q \leftrightarrow f$ symmetric
If char $F = 2$, f does *not* determine Q, and f is alternating

10.1 Case: Odd characteristic

work with f symmetric, bilinear, non-degenerate

Lemma 10.1
Two equivalence classes of such form:

$$\begin{array}{rll} \text{Either} & \exists \text{ orthonormal basis for } V & \\ \text{Or} & \exists \text{ orthonormal basis with} & \begin{array}{rcl} f(v_i, v_i) & = & 1 \ \forall i < d \\ f(v_d, v_d) & = & \alpha \ \text{ some non-square } \alpha \end{array} \end{array}$$

Proof
Find v_1 s.t. $f(v_1, v_1) \neq 0$. "Normalize" to either 1 or α (α fixed non-square). Now continue in $v_1^\perp$

If $f(v_1,v_1) = f(v_2,v_2) = \alpha$, can replace inside $\langle v_1, v_2 \rangle$
Choose $\lambda^2 + \mu^2$ s.t. *not* a square in F; normalize to α^{-1}
$\Rightarrow \quad \lambda v_1 + \mu v_2, \mu v_1 - \lambda v_2$ are orthonormal $\square$

<u>$d = 2m+1$ odd</u>: Get the same group both forms $GO_{2m+1}(q)$ <u>$d = 2m$ even</u>: The groups different.
More useful distinction:

(1) maximal totally isotropic space has dimension (called <u>Witt index</u>) m, Q_{2m}^+

(2) Witt index $m-1$, Q_{2m}^-

Write

$$GO^{\epsilon}_{2m}(q) = \{g \in GL_{2m}(q) | Q^{\epsilon}_{2m}(vg) = Q(v) \ \ \forall v\} \qquad (\epsilon = \pm)$$

Lemma 10.2
If $d = 2$, there may or may not exists isotropic vectors: according to type of Q and $q \equiv \pm 1 \mod 4$
Proof
$q \equiv 1 \mod 4$: If $f(u,u) = 1 = f(v,v)$ an $u \perp v$, let $i = \sqrt{-1} \in F$
$f(u+iv, u+iv) = 0 \quad \Rightarrow \quad u + iv$ isotropic
If $f(u,v) = 1, f(v,v) = \alpha$, with $u \perp v$
$\Rightarrow \quad f(u + \lambda v) = 1 + \lambda^2 \alpha$
$\Rightarrow \quad f(u + \lambda v) \neq 0$ with $\lambda \in F$

$q \equiv 3 \mod 4$: Other way round $\square$

If $d > 2$, isotropic vector exist, so the forms are:

(1) $Q^+_{2m} \ \leftrightarrow \ f^+ : \quad e_1, f_1, \dots, e_m, f_m$

(2) $Q^-_{2m} \ \leftrightarrow \ f^- : \quad e_1, f_2, \dots, e_{m-1}, f_{m-1}, d, d'$ with $\langle d, d' \rangle = O^-_2$, $d, d' \perp e_i, f_j \ \forall i, j$, $f(d,d) = 1$, (in fact have $f(d, d') = 1$, see later)

Related groups:

$$GO^+_2(q) = D_{2(q-1)} \quad , \quad GO^-_2(q) = D_{2(q+1)}$$

<u>$d = 2$</u>:
$GO^+_2(q) = D_{2(q-1)}$:
e_1, f_1, cyclic group order $q - 1$: $e_1 \mapsto \lambda e_1, f_1 \mapsto \lambda^{-1} f_1$
t inverting: $e_1 \leftrightarrow f_1$
No more elements: $\langle e_1 \rangle, \langle f_1 \rangle$ are the only isotropic points

$GO^-_2(q) = D_{2(q+1)}$:
Let $V = \mathbb{F}_{q^2}$ - dim 2 over $\mathbb{F}_q$
$Q(v) = N(v) = v^{q+1}$ This is a quadratic form, no isotropic vectors
Cyclic group order $q + 1$: $N(\lambda) = 1 \Rightarrow v \mapsto \lambda v$ is an isometry
inverted by $t : v \leftrightarrow v^q$
No others- e.g. the stabiliser of 1 conssits of i and $v \leftrightarrow v^q$ (check)

Lemma 10.3

$$\begin{aligned} |GO_{2m-1}(q)| &= 2q^{m^2}(q^{2m} - 1)(q^{2m-2} - 1) \cdots (q^2 - 1) \\ |GO^{\epsilon}_{2m}(q) &= 2q^{m(m-1)}(q^2 - 1)(q^4 - 1) \cdots (q^{2m-2} - 1)(q^m - \epsilon 1) \end{aligned}$$

Proof
By induction, going up in steps of 2.
Let $z_m = \#$(non-zero) isotropic vectors in dimension $2m + 1$ or $2m$

Claim: $\quad z_m = q^{2m} - 1$ for Q_{2m+1} and $z_m = (q^m - \epsilon)(q^{m-1} + \epsilon)$ for Q^{ϵ}_{2m}

Proof of Claim:
Correct for dimension 1, 2 (both ϵ)

Let $\dim V = n + 2$, $V = U \oplus W$ (i.e. $U \perp W$) with U dimension 2 with some type, W dimension n of same type.

Isotropic $v \in V$ is $u + w$ - both norm 0 but not both zero vector, OR, norms are $\lambda(\neq 0), -\lambda$
$\Rightarrow \quad z_{m+1} = (2q-1)(1+z_m) + (q-1)(q^n - 1 - z_m) - 1 = qz_m + (q-1)(q^n+1)$ - now obtained formula in each case $\blacksquare$

Having chosen e_1, need f_1 with e_1, f_1 hyperbolic:

Claim: This can be done in q^{n-1} ways (hence formulae follow)
Proof of Claim:
Number of isotropic vectors in $e_1^\perp$:
$e_1^\perp/\langle e_1\rangle$ is a space of same type - so z_{m-1} isotropic vectors $\langle e_1\rangle + v$
$\Rightarrow \quad qz_{m-1} + (q-1)$ isotropic vectors in $e_1^\perp$
$\Rightarrow \quad \frac{1}{q-1}(z_m - qz_{m-1} - q + 1)$ choices of f_1 $\blacksquare$

$\square$

10.2 Even characteristic

Work with quadratic form Q over $F = \mathbb{F}_q$, char 2
Let f be the associated bilinear alternating form - $f(v,v) = 0$

$$\begin{aligned} \operatorname{rad} f &= \{w \mid f(v,w) = 0 \ \ \forall v\} \\ \operatorname{rad} Q &= \{w \in \operatorname{rad}(f) \mid Q(w) = 0\} \end{aligned}$$

$\operatorname{rad} Q \leq \operatorname{rad} f \leq V$ (subspaces), codimension of $\operatorname{rad} f$ over $\operatorname{rad} Q$ is 0 or 1:
$Q|_{\operatorname{rad} f} : \operatorname{rad} f \to F$ semilinear $Q(v+w) = Q(v) + Q(w), Q(\lambda v) = \lambda^2 Q(v)$
Norm of v=$Q(v)$, v is isotropic if $Q(v) = 0$
Q non-singular: $\operatorname{rad} Q = 0$
Q non-degenerate: $\operatorname{rad} f = 0$

Let Q non-singular $\Rightarrow \operatorname{rad} f$ has dimension ≤ 1
Look at $\dim V = 2m+1$ odd $\Rightarrow \operatorname{rad} f \dim 1$ because $V/\operatorname{rad}(V)$ has dimension $2m$ even, with a non-singular alternating form

$$GO_{2m+1}(q) \leq Sp_{2m}(q)$$

Note: in fact, $|P\Omega_{2m+1}(q)| = |PSp_{2m}(q)|$ for q odd, NOT isomorphic if $2m > 4$

Now look at $\dim V = 2m$ even
Now handle things as before, choose a symplectic type basis as much as poss with $Q(v) = 0$
Note: $Q(e) = 0, f(e,f) = 1 \Rightarrow$ can adjust f to have $Q(f) = 0$ (as $Q(f + \lambda e) = Q(f) + \lambda$, so replace f by $f + Q(f)e$)

If $\dim V > 2$, $\exists$ isotropic vectors: there are vectors $u \perp v$
if $Q(v) \neq 0$, then $Q(v + \lambda u) = Q(v) + \lambda^2 Q(u)$
$\Rightarrow \quad$ Take $\lambda^2 = Q(v)/Q(u)$, then $v + \lambda u$ isotropic

So consider $\dim V = 2$: v, w basis
Let $Q(v) = 1 = f(v,w)$, $Q(w + \lambda v) = Q(w) + \lambda^2 + \lambda$
$\Rightarrow \quad$ at most 2 forms, and in fact, exactly two:
$Q_2^+ : e_1, f_1$ hyperbolic pair (isotropic vectors exists)
$Q_2^- : v, w$ s.t. $Q(v) = 1, f(v,w) = 1, Q(w) = \mu \in F$ s.t. $X^2 + X + \mu$ irreducible (no isotropic vectors)

Now get:
$Q_{2m}^+ : e_1, f_1, \ldots, e_m, f_m$ -isotropic, etc.
$Q_{2m}^- : e_1, f_1, \ldots, e_{m-1}, f_{m-1}, v, w$ where v, w are as in Q_2^-

$GO^{\epsilon}_{2m}(q)$
$|SO_{2m}| = \frac{1}{2}|GO_{2m}|$ for q odd
$SO = GO$ for q even
PSO^{ϵ}_{2m} NOT simple, has a subgroup index $P\Omega^{\epsilon}_{2m}(q)$ which is simple if dim > 4 (hard to prove the existence of this, see Wilson)

Some isomorphisms:

(1) $P\Omega_3(q) \cong L_2(q)$, q odd, $W = V_2, V = S^2 W$ (see Wilson page 96, and example sheet)

(2) $P\Omega_4(q) \cong L_2(q) \times L_2(q)$

(3) $P\Omega^{-}_4(q) \cong L_2(q^2)$

(4) $L_4(q) \cong P\Omega^{+}_6(q)$ on $\bigwedge^2 V_4$

$GO^{\epsilon}_{2m} \leq Sp_{2m}(q)$
$q = 2 \Rightarrow$ two 2-transitive actions

$$\begin{tikzcd}
 & Sp_{2m}(q) \arrow[dl, no head] \arrow[dr, no head] & \\
GO^{+}_{2m} \arrow[dr, no head] & & GO^{-}_{2m} \arrow[dl, no head] \\
 & Sp_{2(m-1)}(q) \times 2 &
\end{tikzcd}$$

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