Topics in Representation Theory
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Introduction

Modular Representation Theory
R commutative ring, e.g. k field, $\mathbb{Z}$ or $\mathbb{Z}_p$
G group, Group ring/algebra RG, elements of form $\sum f_g g$ where $r_g \in R, g \in G$
finite sums, i.e. all but finitely many r_g are zero
addition: $(\sum r_g g) + (\sum r'_g g) = \sum (r_g + r'_g)g$
multiplication: $(\sum r_g g)(\sum r'_g g) = \sum s_g g$, where $s_g = \sum_{hk=g} r_h r'_k$
Group hom:
$\theta : G \to \text{Aut}(M)$ (M is R-module) $\leftrightarrow$ left RG-modules $(\sum r_g g)m = \sum r_g(\theta(g)(m))$

If k field of characteristic $p \nmid |G|$ (including characteristic 0)
then $kG = \bigoplus$ simple kG-modules
In general, any finitely generated kG module is a direct sum of simples.
If char$k \big| |G|$ then we are in modular representation theory
e.g. G cyclic of order p=char k
$\quad kG \cong k[X]/(X^p - 1)$
since char $k = p \qquad (X^p - 1) = (X - 1)^p$
kG has a unique maximal ideal J generated by $X - 1$ and the ideals of kG are precisely J^i
$kG/J \cong k$ as vector space, trivial G-action.
Note that, $0 \to J/J^2 \to kG/J^2 \to kG/J \to 0$ does NOT split
(J/J^2 and kG/J are 1-dimensional trivial)
kG/J^2 is NOT a direct sum of two 1-dimensional trivials
So in modular representation theory one must understand the simple modules and how they are put together (Here, $\text{Ext}^1(kG/J, J/J^2) \neq 0$)

Quivers A quiver is a directed graph with multiple arrows and loops e.g. Ext-quiver:
vertices x_i correspond to (isom. class of) simple modules S_i
No. of arrows from x_i to x_j correspond to the k-vector space dimension of $\text{Ext}^1(S_i, S_j)$

In general, for a quiver Q, we can define a path algebra kQ:
it has a basis indexed by paths (including paths of length zero)
multiplication on basis elements is by composition of paths, zero if paths does not compose
(Warning: if we want to think about left modules, we have to reverse order of composition)
kQ is finitely generated if Q is finite
kQ is a finite dimensional vector space $\Leftrightarrow Q$ finite and $\nexists$ oriented cycles

Representations of quivers Q:
Associate a k-vector space V_x with each vertex x and a linear map $V_x \to V_y$ to each arrow
There is a natural correspondence between representations of the quiver and kQ-modules:
Given a representation of Q, we can form a kQ-module whose underlying vector space is $\bigoplus_x V_x$ and where the basis elements $x_1 \cdot \to \cdots \to \cdot x_2$ acts like the composite

$$\bigoplus V_x \twoheadrightarrow V_{x_1} \to \cdots \to V_{x_2} \hookrightarrow \bigoplus V_x$$

Note that paths of length 0 (at vertex x) are idempotents e_x in kG (i.e. $e_x^2 = e_x$)
Conversely, given a kQ-module V we can form a representation of Q by setting $V_x = e_x$ 'applied to' V (either $e_x V$ or $V e_x$ depends on the direction we will choose later...)

When studying representation of algebras it is natural to think about indecomposable modules (those are modules that can not be expressed non-trivially as a direct sum)
Algebras are of finite representation type if they only have finitely many (isom. classes of) indecomposables
The quivers whose path algebras have finite representation type are precisely those associated with Dynkin diagrams

For kG, G finite, k infinite field of characteristic p
kG has finite representation type $\Rightarrow$ G has a cyclic Sylow p-subgroup

Cohomology
Hochschild cohomology: cohomology theory concerning bimodules of algebras (c.f. Topics in Calculus and Algebras)

Group cohomology: specialisation of Hochschild to bimodules for group algebras where on one side the action of G is trivial. (and so it really is about one-sided modules)

Low degree cohomology groups have good interpretations. In particular, for a group G and $\mathbb{Z}G$-module M,
$H^2(G, M) \leftrightarrow$ equiv. classes of group extensions $1 \to M \to H \to G \to 1$
(here M is embedding in H as an abelian normal subgroup acted on by H via conjugation which is induced by action of G on M)

There is an application to the construction of finite dimensional algebras that are simple (i.e. have no non-trivial 2-sided ideal)
Such k-algebras are central if the centre is k
Brauer group: abelian group whose elements correspond to (Morita) equiv. classes of central simple algebras
e.g. $k = \mathbb{R}$, there are 2 equiv. classes of central simple $\mathbb{R}$-algebra, with representatives $\mathbb{R}, \mathbb{H}$ (quaternions)
All equiv. classes can be represented by crossed products
The Brauer group can be interpreted via a union of cohomology groups of the form $H^2(G, M)$ with $G = \mathrm{Gal}(F/k)$, with F/k a finite Galois extension, $M = F$ with Galois action.

Books
Alperin - *Local Representation Theory*, Good for modular representation theory, but may put off by its misprints
Benson - *Representations and Cohomology vol. I*, Good after we have done the course
Pierce - *Associative algebra*, the course is built on this

1 Artinian Rings and Algebras

Definition 1.1
A ring A is left Artinian if it satisfies the descending chain condition (DCC) on left ideals. (i.e. $I_1 \supseteq I_2 \cdots$ must terminate)
Similarly for 'right' and Artinian will mean both left and right.
Similarly for ascending chain condition we have left/right Noetherian and (2-sided) Noetherian

Exercise: Let $A = \left\{ \begin{pmatrix} q & r \\ 0 & s \end{pmatrix} : q \in \mathbb{Q}, r, s \in \mathbb{R} \right\}$
Show that it is not left Artinian but is right Artinian, and similarly for Noetherian.

A is a left A-module - left regular representation
left ideals are left A-submodules
Ideals will mean 2-sided ideals

Definition 1.2
M left A-module, the annihilator of $m \in M$ is $\mathrm{Ann}(m) = \{a \in A | am = 0\}$ left ideal of A
The annihilator of M is $\mathrm{Ann}(M) = \{a \in A | am = 0 \; \forall m \in M\} = \bigcap_{m \in M} \mathrm{Ann}(m)$ 2-sided ideal of A

$\mathrm{Ann}(m)$ is maximal left ideal $\Leftrightarrow$ submodule Am generated by m is irreducible/simple

Definition 1.3
A module M is simple/irreducible if the only submodules are 0 and M

Definition 1.4
The Jacobson radical $J(A)$ of A is the intersection of the maximal left ideals

Remark. This looks to be dependent on 'left' but in fact one gets the same thing using 'right'

Definition 1.5
A is semisimple if $J(A) = 0$
Note that $A/J(A)$ is semisimple

Lemma 1.6 (Nakayama's Lemma)
TFAE for a left ideal I:

(1) $I \subseteq J(A)$

(2) If M f.g. left A-module and $N \leqslant M$ satisfies $N + IM = M$ then $N = M$

(3) $\{1 + x | x \in I\} = G$ is a subgroup of the unit group of A (i.e. they have multiplicative inverses)

Proof
(1)$\Rightarrow$(2):
Suppose $N \lneqq M$. Then there is a maximal submodule N_1 with $N \leqslant N_1 \lneqq M$.
But $J(A)$ annihilates M/N_1 and so $J(A)M \leqslant N_1$. Hence $IM \leqslant N_1$ and $N + IM \leqslant N_1$ #

(2)$\Rightarrow$(3):
Let $x \in I$. Set $y = 1 + x \quad \Rightarrow \quad 1 = y - x \in Ay + I \quad \Rightarrow \quad Ay + I = A$. From (2), $Ay = A$.
Thus, $1 = zy$ for some $z \in A$ But $zy = z + zx$
$\Rightarrow \quad z = 1 - zx$
$\Rightarrow \quad z$ is a unit of form $1 + x'$ where $x' \in I$
We have shown every element of $G = \{1 + x | x \in I\}$ has a left inverse in G, i.e. subgroup of unit group of A.

(3)⇒(1):
Suppose I_1 is a maximal left ideal of A. Let $x \in I$. If $x \notin I_1$, then $I_1 + Ax = A$ (by maximality of I_1)
$\Rightarrow \quad 1 = y + zx$ for some $y \in I_1, z \in A$
$\Rightarrow \quad y = 1 - zx$
But assuming (3), y is a unit and so $I_1 = A$ #
So $I \leqslant I_1$ for all maximal left ideals of A. Hence $I \leqslant J(A)$ □

Corollary 1.7
M f.g. $J(A)M = M \quad \Rightarrow \quad M = 0$

Proof
Take $N = 0$ and apply Nakayama's Lemma □

Remark. Thus $J(A)$ is largest ideal in A s.t. $\{1 + x | x \in I\}$ is a subgroup of unit group of A and so we would have ended up with the same thing if we started with 'right'.

Definition 1.8
A module is completely reducible if it is a sum of simple/irreducible modules. This is (by Zorn's Lemma) equivalent to every submodule having a complement. If M satisfy the DCC in submodules then

$$M \text{ is completely reducible} \quad \Leftrightarrow \quad \text{Rad}(M) = 0$$

where the radical is

$$\text{Rad}(M) = \text{ intersection of its maximal submodules}$$

In this case M is a finite *direct* sum of simples. Note that $\text{Rad}({}_AA) = J(A) = \text{Rad}(A_A)$ where ${}_AA$ is A considered as left A-module and A_A is A considered as right A-module

The socle is defined as

$$\text{Soc}(M) = \text{ sum of its simple submodules}$$

Lemma 1.9
If A is semisimple and left Artinian, then every left A-module is completely reducible.
Conversely, if A is left Artinian and ${}_AA$ is completely reducible then A is semisimple

Proof
⇒:
If A semisimple then $\text{Rad}({}_AA) = J(A) = 0$ and so ${}_AA$ is completely reducible.
Choosing generating set $\{m_i\}$ of a module M gives

$$\begin{aligned} \bigoplus_i {}_AA &\rightarrow M \\ (a_i) &\mapsto \sum a_i m_i \end{aligned}$$

and so $M \cong$ quotient of a direct sum of copies of ${}_AA$ and so is completely reducible.

⇐:
Conversely, if ${}_AA$ is completely reducible then ${}_AA = {}_AA/J(A) \oplus J(A)$
Multiplying on left by $J(A)$ we get $J(A) = (J(A))^2$. Nakayama ⇒ $J(A) = 0$ □

Lemma 1.10
Let A be left Artinian and M be f.g. A-module. Then $J(A)M = \text{Rad}(M)$

Proof
If M' is a maximal submodule of M then $J(A)(M/M') = 0$

$\Rightarrow \quad J(A)M \subseteq M'$
$\Rightarrow \quad J(A)M \subseteq \text{Rad}(M)$

Conversely, $M/J(A)M$ is completely reducible by Lemma 1.9.
$\Rightarrow \quad \text{Rad}(M/J(A)M) = 0$
$\Rightarrow \quad \text{Rad}(M) \subseteq J(A)M$ □

Proposition 1.11
Let A be left Artinian, then

(1) $J(A)$ is nilpotent

(2) If M is a f.g. A-module, then it is both left Artinian and left Noetherian

(3) A is left Noetherian

Proof

(1) Since A is left Artinian. $J(A)^r = J(A)^{2r}$ for some r
If $J(A)^r \neq 0$ then again using DCC, $\exists$ minimal left ideal I with $J(A)^r I \neq 0$
Choose $x \in I$ with $J(A)^r x \neq 0$. In particular, $x \neq 0$
Then $I = J(A)^r x$ by minimality and so for some $a \in J(A)^r$ we have $x = ax$
But then $(1-a)x = 0$. Hence $x = 0$ since $1 - a$ $(a \in J(A))$ is a unit #

(2) Let $M_i = J(A)^i M$. Then M_i/M_{i+1} is annihilated by $J(A)$ and hence is completely reducible by Lemma 1.9
Since M is a f.g. module over an left Artinian ring, (Exercise:) it satisfies the DCC on submodules
$\Rightarrow \quad M_i/M_{i+1}$ satisfies DCC on submodules
$\Rightarrow \quad M_i/M_{i+1}$ is finite direct sum of simple modules
$\Rightarrow \quad M_i/M_{i+1}$ satisfies ACC
$\Rightarrow \quad M$ satisfies ACC

(3) Apply (2) to ${}_A A$

□

Exercise:
A left Noetherian (and hence for left Artinian), any left inverse in necessarily a right inverse

Theorem 1.12 (Wedderburn, Artin)
Let A be a semisimple Artinian ring. Then

$$A = \bigoplus_{i=1}^{r} A_i \quad \text{where} \quad \begin{array}{l} A_i \cong \text{Mat}_{n_i}(\Delta_i) \\ \Delta_i \text{ is a division ring} \end{array}$$

and the A_i are uniquely determined.
A has exactly r isomorphism classes of simple modules M_i and $\text{End}_A(M_i) \cong \Delta_i^{op}$ and $\dim_{\Delta_i}(M_i) = n_i$
If A is simple then $A \cong \text{Mat}_n(\Delta)$
(Here, multiplication in Δ^{op} is opposite to that in Δ)

To prove Wedderburn, we need following lemmas.

Lemma 1.13 (Schur)
If M_1 and M_2 are simple A-modules, then for $M_1 \not\cong M_2$, $\text{Hom}_A(M_1, M_2) = 0$, otherwise, $\text{Hom}_A(M_1, M_1) = \text{End}_A(M_1)$ is a division ring

Proof
Exercise ☐

Lemma 1.14

(1) If M is an A-module and e is an idempotent, then $eM \cong \mathrm{Hom}_A(Ae, M)$

(2) $eAe \cong \mathrm{End}_A(Ae)^{op}$

Proof

(1) Define maps

$$\begin{array}{rcl} f_1 : eM & \to & \mathrm{Hom}_A(Ae, M) \\ em & \mapsto & (ae \mapsto aem) \\ f_2 : \mathrm{Hom}_A(Ae, M) & \to & eM \\ \alpha & \mapsto & \alpha(e) \end{array}$$

These are inverse to each other

(2) Apply (1) to $M = Ae$. Note that f_1, f_2 reverses order of multiplication.

☐

Lemma 1.15
Let M be a finite direct sum of simple A-modules, say $M = \bigoplus_{i=1}^r M_i$ with each M_i direct sum of n_i modules.

$$M_i = M_{i,1} \oplus \cdots \oplus M_{i,n_i}$$

each summand of M_i isomorphic to a simple module S_i and $S_i \ncong S_j \ \forall i \neq j$
Let $\Delta_i = \mathrm{End}_A(S_i)$. Then Δ_i is a division ring

$$\begin{array}{rcl} \mathrm{End}_A(M_i) & = & \mathrm{Mat}_{n_i}(\Delta_i) \\ \mathrm{End}_A(M) & = & \bigoplus_i \mathrm{End}_A(M_i) \quad \text{is semisimple} \end{array}$$

Proof
Schur's Lemma 1.13 $\Rightarrow$ Δ_i division ring
Choose isomorphism $\theta_{ij} : M_{i,j} \to S_i$
Given $\lambda \in \mathrm{End}_A(M_i)$, we define $\lambda_{jk} \in \Delta_i$ as the composite map

$$S_i \xrightarrow{\theta_{ik}^{-1}} M_{ik} \hookrightarrow M_i \xrightarrow{\lambda} M_i \twoheadrightarrow M_{i,j} \xrightarrow{\theta_{ij}} S_i$$

The map $\lambda \mapsto (\lambda_{jk})$ is injective: $\mathrm{End}_A(M_i) \to \mathrm{Mat}_{n_i}(\Delta_i)$

Conversely, given (λ_{jk}) we can construct λ as the sum of the composites

$$M_i \twoheadrightarrow M_{i,k} \xrightarrow{\theta_{ij}} S_i \xrightarrow{\lambda_{jk}} S_i \xrightarrow{\theta_{ij}^{-1}} M_{i,j} \hookrightarrow M_i$$

Finally $\mathrm{End}_A(M) = \bigoplus \mathrm{End}_A(M_i)$ since if $i \neq j$, $\mathrm{Hom}_A(S_i, S_j) = 0$ by Schur's Lemma 1.13 ☐

Proof of Wedderburn's Theorem 1.12
By 1.9. A is completely reducible.
Take $e = 1$ in 1.14, $A \cong \mathrm{End}_A({}_A A)^{op}$
Apply 1.15 to ${}_A A$. Note that the opposite ring of a matrix over a division algebra is again a matrix ring, over the opposite division algebra ☐

Remark. For a general left Artinian ring $A/J(A)$ is semisimple left Artinian, so has the structure given by 1.12 – it is a direct sum of matrix rings, the number of which is equal to the number of non-isomorphic simple A-modules

Example:
kG with k field, G finite group

Theorem 1.16
kG is semisimple $\Leftrightarrow$ char $k \nmid |G|$

Proof
Let char$k$=p
Suppose $p \big| |G|$
Consider the trivial module S – one dimensional as k-vector space, G acting trivially, $\mathrm{End}(S) = k$ If kG were semisimple, then number of times a copy of S appears in ${}_{kG}kG$ is 1
But in kG we have a chain of submodules

$$kG \gneqq I_1 \geqslant I_2 \gneqq 0$$

where I_1 = augmentation ideal = $\ker \begin{pmatrix} kG & \to & k \\ \sum r_g g & \mapsto & \sum r_g \end{pmatrix}$
This map is a kG-homomorphism, called the augmentation map
$I_2 = kG\sigma$ where $\sigma = \sum_{g\in G} g$

Note that $kG/I_1 \cong k$, 1-dimensional trivial modules
and $I_2 \cong k$, 1-dimensional trivial modules
$\sigma \in I_1$ since $p\big||G|$ (image of σ under augmentation map is zero in this case)
So we know we had get at least 2 copies of S in decomposition of ${}_{kG}kG$ #

Conversely, we have augmentation from complex representation theory (Maschke)
Suppose $p \nmid |G| \quad \Rightarrow \quad |G|$ invertible in k
One shows that any submodule V of a module U has a (direct) complement.
Let $\pi : U \twoheadrightarrow V$ vector space projection, so $U = V \oplus \ker \pi$ as k-vector spaces. Set

$$\pi' = \frac{1}{|G|} \sum_{g\in G} g\pi g^{-1}$$

π' is a kG-module homomorphism $U \twoheadrightarrow V$ and so $U = V \oplus \ker \pi'$ (i.e. $\ker \pi'$ is a kG-module complement to V in U) □

Theorem 1.17
Let k be algebraically closed field, char k=p, G finite
The number of (isom. class of) simple kG-modules = number of conjugacy classes of elements of order not divisible by p

Proof
In semisimple case (e.g. $\mathbb{C}\,G$)
Here $kG \cong$ direct sum of matrix algebra over division rings and the division rings are finite extensions of k
But k algebraically closed and so those division ring are k
Consider centre $Z(kG)$ of kG spanned by $e_1, \ldots, e_c$, where e_i correspond the identity matrix in its matrix subalgebra

$$\begin{aligned} \dim_k Z(kG) &= c \qquad \text{since the } e_i \text{ are linearly independent} \\ &= \text{no. of (isom. class of) simple modules} \end{aligned}$$

But $Z(kG)$ contains class sums $\sum_{g\in G} ghg^{-1}$, sum of elements in the conjugacy class of h. The class sums form a basis of $Z(kG)$
$\Rightarrow$ no. of class sums = no. of conjugacy classes = c $\square$

Example: $p = 2$, no. of simples = no. of ccls of odd order elts

Corollary 1.18
If G is a p-group (i.e. $|G| = p^r$), then the trivial module is the only simple module

Remark. We have already saw this in the case of cyclic group of order p

Aim: Investigate G be cyclic group of order n, $G = SL(2, p)$, $G = SL(2, \mathbb{Z}/p^i\mathbb{Z})$, and infinite groups $SL(2, \mathbb{Z}_p)$

1.1 G cyclic of order n

$n = p^a r$, $p \nmid r$, k algebraically closed, generator g
$X^r - 1$ is separable and so there are r distinct roots of unity.
Let λ be such a root, and take a 1-dimensional vector space and let g act by multiplication by λ. This gives a 1-dimensional kG-module
Different choices of λ give different modules (up to isom.)

G abelian $\Rightarrow$ each conjugacy class consists of one element only
G cyclic $\Rightarrow \exists$ exactly r elements of order not divisible by p
Hence Theorem 1.17 $\Rightarrow$ we are looking for r simple modules, which we have already found.

In fact, we do not need to use 1.17:
Any simple module is going to have to be 1-dimensional (since k algebraically closed)
$\Rightarrow$ g must act by multiplication by a root of unity
But $g^n - 1 = (g^r - 1)^{p^a}$ since char$k = p$ and so g is acting on a simple module as an r-th root of unity

1.2 $G = SL(2, p)$

G has exactly p conjugacy classes of elements of order not divisible by p (exercise)
1.17 $\Rightarrow$ we are looking for p simple kG-modules
V_2 vector space of columns $\begin{pmatrix} x \\ y \end{pmatrix}$, set

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{basis of } V_2$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{s.t.} \quad \begin{cases} gX = aX + cY \\ gY = bX + dY \end{cases}$$

The action of $g \in G$ extends to automorphism of $k[X, Y]$ polynomial algebra

$V_n :=$ subspace of homogeneous polynomials of degree $n - 1$ is a kG-module under this action, basis as a vector space is $X^{n-1}, X^{n-1}Y, \ldots, Y^{n-1}$ dimension n
$V_1 :=$ 1-dimensional module spanned by 1, i.e. trivial kG-module

Claim: $V_1, \ldots, V_p$ are simple modules (non-isom since they have different dim.)

Proof
Let $1 \leqslant r < p$. Aim: Show V_{r+1} is simple.

Choose $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
and consider V_{r+1} as a $k\langle g\rangle$-module and as a $k\langle h\rangle$-module

> **Subclaim**: X^r is generator of V_{r+1} as a $k\langle g\rangle$-module and kY^r is the socle (sum of simple submodules) of V_{r+1} as a $k\langle g\rangle$-module
> Similarly Y^r is generator of V_{r+1} as $k\langle h\rangle$-module and kX^r is the socle (sum of simple submodules) of V_{r+1} as a $k\langle h\rangle$-module
>
> **Proof of Subclaim**:
> Let W_{i+1} be the $(i+1)$-dimensional subspace of V_{r+1}
> with basis $X^iY^{r-i}, X^{i-1}Y^{r-i+1}, \ldots, XY^{r-1}, Y^r$
>
> $\Rightarrow \qquad V_{r+1} = W_{r+1} \gneqq W_r \gneqq \cdots \gneqq W_1 \gneqq W_0 = 0$
>
> (Check the following) • Each W_i is a $k\langle g\rangle$-submodule
> • Each W_i/W_{i-1} is 1-dimensional trivial $k\langle g\rangle$-module
>
> Prove by easy induction (exercise) that any element in $W_i \setminus W_{i-1}$ generates W_i as a $k\langle g\rangle$-module
> $\Rightarrow$ in particular, X^r generates V_{r+1} as a $k\langle g\rangle$-mdoule
>
> Now consider the socle as $k\langle g\rangle$-module
> Note $g^p = 1$. Thus the trivial module is the only simple $k\langle g\rangle$-module and so the socle, i.e. sum of simples, must be a trivial $k\langle g\rangle$-module
>
> But if $w \in W_i \setminus W_{i-1}$ then it generates W_i as a $k\langle g\rangle$-module and so $k\langle g\rangle w$ can only be 1-dimensional if $w \in W_1 = kY^r$
> But kY^r is indeed fixed by $\langle g\rangle$ and so it is the socle.

With the subclaim, let W be a nonzero kG-module of V_{r+1}
Then as a $k\langle g\rangle$-module it must contain a simple module, and that has to be kY^r (the only simple submodule available)
$\Rightarrow \quad Y^r \in W$
But now apply $k\langle h\rangle$ we generate the whole of V_{r+1}
$\Rightarrow \quad W = V_{r+1} \quad \Rightarrow \quad V_{r+1}$ simple (by subclaim) □

1.3 $G = SL(2, \mathbb{Z}/p^s\mathbb{Z})$

We have just done $s = 1$ case
Note that each $G_i = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a \equiv d \equiv 1 \mod p^i, b \equiv c \equiv 0 \mod p^i \right\} \trianglelefteq G$

$$G > G_1 > G_2 > \cdots > G_s = 1$$

$G/G_1 \cong SL(2,p)$, $\quad G_i/G_{i+1}$ elementary abelian p-groups of rank 3 ($\cong C_p \times C_p \times C_p$)
Thus each G_i ($i \geqslant 1$) is a normal p-subgroup of G

Lemma 1.19
'Clifford Theory': For finite G, k any field. $H \trianglelefteq G$
If M is semisimple kG-module, then ${}_{kH}M$ is also semisimple

Proof
Sufficient to consider the case where the module is a simple kG-module S

Note that if V is a kH-submodule of S
then gV is a kH-submodule of S $\forall g \in G$ (using normality of H)

Now let T be a simple kH-submodule of S
For $g \in G$. gT is a simple kH-submodule of S
Hence, $\sum_{g\in G} gT$ is a semisimple kH-submodule of S which is a kG-submodule
$\Rightarrow \quad S = \sum_{g\in G} gT \quad$ by simplicity of S
i.e. S semisimple as kH-module $\square$

Now applying Lemma 1.19 to $G = SL(2, \mathbb{Z}/p^s\mathbb{Z})$
We see that any simple kG-module is semisimple as kG_1-module
But G_1 is a p-group $\Rightarrow$ only simple kG_1-module is the trivial one by 1.18
So any semisimple kG_1-module is acted on trivially by G_1

There is therefore a one-to-one correspondence

$$\{ \text{ simple } kG\text{-modules } \} \leftrightarrow \{ \text{ simple } \underbrace{k(G/G_1)}_{kSL(2,p)}\text{-modules } \}$$

1.4 $G = SL(2, \mathbb{Z}_p)$

$G = SL(2, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ is p-adic integers. Note this is a topological group
$G_i = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a \equiv d \equiv 1 \mod p^i \\ b \equiv c \equiv 0 \mod p^i \end{array} \right\}$

$$G > G_1 > G_2 > \cdots \bigcap G_i = \{I\}$$

such a set up arise naturally when looking at field extensions:

$$K < K_1 < K_2 \cdots \bigcup K_i = L$$

each K_i is a finite Galois extension of K, L is algebraic extension of K of infinite degree.
$H = \text{Gal}(L/K) = K$-automorphism of L, fixing K
$H_i = \text{Gal}(L/K_i)$ fixing K_i

$$H > H_1 > H_2 \cdots$$

This is how we can get $G = SL(2, \mathbb{Z}_p)$ as a Galois group,
additive group is a $\mathbb{Z}G$-module
multiplicative group is $\mathbb{Z}G$-module

Continuous actions of G on $\mathbb{F}_p$-vector spaces leads to representation theory of Iwasawa algebras, this is the completed group algebras $\widehat{\mathbb{F}_p G}$, a Noetherian algebra which has the feel of a non-commutative power series ring
(Unimportant remark: there is a lot of research in Cambridge on representation theory of Iwasawa algebras, this is also the primary reason of the lecturer giving this course)

We now go back to the study of theory.

Detailed Proof of Theorem 1.17
We give alternative proof of the theorem in the semisimple case. This is in fact a dual proof of what we had before.

For an algebra A, let $[A, A] = \langle ab - ba | a, b \in A \rangle$, subspace of A Consider the k-vector space, $A/[A, A]$ (instead of looking at $Z(kG)$)

(*Remark*: $A/[A,A] = HH_0(A)$, the 0-th Hochschild homology group of A and $Z(A)$ is the 0-th Hochschild cohomology group $HH^0(A)$)

If $A = A_1 \oplus \cdots \oplus A_c$ direct sum of algebras then $[A,A] = \bigoplus [A_i, A_i]$
For a matrix algebra A, $[A,A]$ has codimension 1, consisting of all matrices of trace zero (exercise).

<u>kG semisimple</u>:
Thus, if kG is a direct sum of c matrix algebras, $\Rightarrow$ $[kG,kG]$ has codimension c
In fact (prove later in general result), if $g_1, \ldots, g_c$ are conjugacy class representatives then $\{g_i + [kG,kG]\}$ form a basis of $kG/[kG,kG]$
This shows that no. of simples = no. of conjugacy classes

We adapt this argument in the general case, <u>kG not necessarily semisimple</u>:
Set $T = [kG,kG]$, $\quad S = T + J(kG)$
Note that $S/J(kG) = [kG/J(kG), kG/J(kG)]$
$\Rightarrow$ codimension of $S/J(kG)$ in $kG/J(kG)$ = no. of summands in the matrix algebra decomposition of the semisimple algebra $kG/J(kG)$ = no. of simple kG-module

We may assume char $k = p \big| |G|$
Let $g_1, \ldots, g_r$ be representatives of the conjugacy classes of elements of order not divisible by p (usually referred as p'-elements)
$g_{r+1}, \ldots, g_c$ be representatives of the other conjugacy classes

Claim: $g_1 + S, \ldots, g_r + S$ form a basis of kG/S

Proof of Claim:
We use Lemma 1.20 (prove after this theorem) that S consists of the elements a of kG s.t. $a^{p^i} \in T$ for some $i \geqslant 0$.

(Important!) Note that $g_1 + T, \ldots, g_c + T$ are a basis of kG/T:
If x, y are conjugate in G, say $y = gxg^{-1}$
$\Rightarrow$ $x - y = x - gxg^{-1} = g^{-1}(gx) - (gx)g^{-1} \in T$
$\Rightarrow$ $g_i + T$ span kG/T
For linear independence, let $(1 \leqslant i \leqslant c)$,

$$\begin{array}{rcll} \phi_i : kG & \to & k & \text{linear functional} \\ g & \mapsto & 1 & \text{if } g \text{ conjugate to } g_i \\ & & 0 & \text{otherwise} \end{array}$$

Thus ϕ_i is constant on conjugacy classes
$\Rightarrow$ $\phi_i(gh - hg) = \phi_i(g(hg)g^{-1} - hg) = 0$
$\Rightarrow$ $\phi_i(T) = 0$
If $\sum \lambda_j (x_j + T) = 0 \quad \Rightarrow \quad \sum \lambda_j x_j \in T$
$\Rightarrow$ $0 = \phi_i(\sum \lambda_j x_j) = \lambda_i$, proving linear independence

Next let $g \in G$ and express $g = ux$ where $\begin{cases} u^{p^i} = 1 \text{ for some } i \\ x \text{ has order not divisible by } p \\ ux = xu \end{cases}$
(just use cyclic group $\langle g \rangle$)
Hence $g^{p^i} = x^{p^i} \quad \Rightarrow \quad (g - x)^{p^i} = 0$ (using char $k = p$)
Lemma 1.20 $\Rightarrow$ $g - x \in S$
Apply this to the elements $g_{r+1}, \ldots, g_c$
Thus, they are conjugate mod S to one of the $g_1, \ldots, g_r$
$\Rightarrow$ $g_1 + S, \ldots g_r + S$ span kG/S
Finally, Suppose $\lambda_1 g_1 + \cdots \lambda_r g_r \in S$ with $\lambda_j \in k$
Lemma 1.20 $\Rightarrow (\lambda_1 g_1 + \cdots \lambda_r g_r)^{p^i} \in T$ for some i

But $(\lambda_1 g_1 + \cdots + \lambda_r g_r)^{p^i} \equiv \lambda_1^{p^i} g_1^{p^i} + \cdots \lambda_r^{p^i} g_r^{p^i} \mod T$
(using the fact $(a+b)^p \equiv a^p + b^p \mod T$, exercise)
Thus, $\lambda_1^{p^i} g_1^{p^i} + \cdots + \lambda_r^{p^i} g_r^{p^i} \in T$
But the element $g_1^{p^i}, \ldots, g_r^{p^i}$ are in different conjugacy classes:
If $|G| = p^a m$ with $p \nmid m$
Choose $t, u \in \mathbb{Z}$ s.t. $tp^i + um = 1$
we have $g_j^m = 1$ and so $g_j^{p^i t} = g_j$
Thus, they are linear independent mod T ■

Theorem follows from the claim. □

Lemma 1.20
$T = [kG, kG] \qquad S = T + J(kG)$
$\Rightarrow \quad S = \{a \in kG | a^{p^i} \in T \text{ for some } i \geqslant 0\}$

Proof
Let $S_0 = \{a \in kG | a^{p^i} \in T \text{ for some } i \geqslant 0\}$. We will show $S_0 = S$

$\subseteq$: In $\overline{kG} = kG/J(kG)$, $\overline{a} := a + J(kG)$ s.t. $\overline{a}^{p^i} \in \left[\overline{kG}, \overline{kG}\right]$
$\Rightarrow$ The p^i power of each component of $\overline{a}$ in the matrix algebra decomposition has trace zero

Since trace = sum of eigenvalues $\Rightarrow$ tr $M^p = (\text{tr } M)^p$
So in characteristic $p \Rightarrow \overline{a} \in [A, A] \Rightarrow a \in T + J(kG) = S$

$\supseteq$: Conversely, $S_0 \supseteq T$ and $S_0 \supseteq J(kG)$ as each element of $J(kG)$ is nilpotent
So if we prove S_0 is closed under addition, it follows $S_0 \supseteq T + J(kG) = S$ and we are done.
Closure under addition:
use the fact/exercise under the claim of Lemma 1.17 $(a+b)^p \equiv a^p + b^p \mod T$
and use

$$a \in T \Rightarrow a^p \in T \tag{1.1}$$

(To show this, it is enough to show $(ab - ba)^p \equiv (ab)^p - (ba)^p \mod T$
$= a(ba \cdots b) - (ba \cdots b)a \in T)$

Now if $a, b \in S_0$, from equation (1.1) it follows $\exists u \in \mathbb{Z}_+$ s.t. a^{p^u} and $b^{p^u} \in T$
$\Rightarrow (a+b)^{p^u} \equiv a^{p^u} + b^{p^u} \mod T$
$\Rightarrow a + b \in S_0$

□

2 Projectives, injectives and indecomposables

Definition 2.1
A non-zero A-module is indecomposable if M cannot be expressed as $M = M_1 \oplus M_2$ with both M_1 and M_2 non-zero

Definition 2.2
An A-module M has the unique decomposition property if

(1) M is a finite direct sum of indecomposable modules
(2) whenever $M = \bigoplus_{i=1}^m M_i = \bigoplus_{i=1}^n M_i{}'$ with each M_i and $M_i{}'$ (non-zero) indecomposable then $m = n$ and after reordering $M_i \cong M_i{}'$

A ring has the unique decomposition property if every f.g. A-module does

Definition 2.3
A ring A is local if it has a unique maximal left ideal (which is $J(A)$)
Note that in a local ring, $A/J(A)$ is a division ring (exercise)
and every element not in $J(A)$ is a unit (i.e. left AND right invertible)
Exercise: A is local $\Leftrightarrow$ the non-invertible elements form a left ideal

Theorem 2.4 (Krull-Schmidt)
Suppose M is a finite direct sum of indecomposable with each $\mathrm{End}_A(M_i)$ local. Then M has the unique decomposition property

(Will be proved after Corollary 2.7)

Lemma 2.5 (Fitting)
Suppose M is an A-module with ACC and DCC on submodules and $f \in \mathrm{End}_A(M)$
Then for large enough n. $M = \mathrm{Im}\, f^n \oplus \ker f^n$
In particular, if M indecomposable, then element of $\mathrm{End}_A(M)$ is either isomorphic or nilpotent.

Proof
By ACC/DCC, we may choose n large enough s.t. $f^n : f^n(M) \xrightarrow{\sim} f^{2n}(M)$ is an isomorphism
If $m \in M$ then write $f^n(m) = f^{2n}(m_1)$
$\Rightarrow\ m = f^n(m_1) + (m - f^n(m_1)) \in \mathrm{Im}\, f^n + \ker f^n$
If $f^n(x) \in \mathrm{Im}\, f^n \cap \ker f^n$ then $f^{2n}(x) = 0 \quad \Rightarrow \quad f^n(x) = 0$ □

Lemma 2.6
Suppose M is indecomposable with ACC and DCC on submodules. Then $\mathrm{End}_A(M)$ is local

Proof
Let $E = \mathrm{End}_A(M)$. Choose I maximal left ideal of E
Suppose $a \notin I$. Then $E = Ea + I$
Write $1 = \lambda a + \mu$ with $\lambda \in E, \mu \in I$
Since μ is not an isomorphism Fitting's Lemma 2.5 $\Rightarrow \mu^n = 0$ for some n (using indecomposability of M)
$\Rightarrow \quad (1 + \mu + \cdots \mu^{n-1})\lambda a = (1 + \mu + \cdots + \mu^{n-1})(1 - \mu) = 1$
$\Rightarrow \quad a$ is invertible □

Corollary 2.7
Let A be (left) Artinian. Then A has the unique decomposition property

Proof
A has ACC and DCC by Proposition 1.11. □

Proof of Krull-Schmidt Theorem 2.4
Let $M = \bigoplus_{i=1}^m M_i = \bigoplus_{i=1}^n M_i'$

Induction on n. Assume $m > 1$ ($m = 1$ case is trivial)
Let

$$\begin{aligned} \alpha_i : &\quad M_i' \hookrightarrow M \twoheadrightarrow M_i \qquad \text{for } 1 \leqslant i \leqslant n \\ \beta_i : &\quad M_i \hookrightarrow M \twoheadrightarrow M_i' \end{aligned}$$

$\Rightarrow \quad \mathrm{id}_{M_i} = \sum \alpha_i \circ \beta_i : M_i \to M_i$
Since $\mathrm{End}_A(M_1)$ is local by supposition, some $\alpha_i \circ \beta_i$ must be a unit. Renumber so that $\alpha_1 \circ \beta_1$ is a unit. Then $M_1 \cong M_1'$

Consider the map $\mu = 1 - \theta$ where

$$\theta : M \twoheadrightarrow M_1 \xrightarrow{\alpha_1^{-1}} M_1' \hookrightarrow M \twoheadrightarrow \bigoplus_{i=1}^m M_i \hookrightarrow M$$

$$\begin{cases} \mu(M_1') = M_1 \\ \mu(\bigoplus_{i=2}^m M_i) = \bigoplus_{i=2}^m M_i \end{cases}$$

$\Rightarrow \quad \mu$ is surjective

If $\mu(x) = 0$ then $x = \theta(x)$ and so $x \in \bigoplus_{i=2}^m M_i$
But then $\theta(x) = 0$
Thus μ is an automorphism of M with $\mu(M_1') = M_1$
$\Rightarrow \quad \bigoplus_{i=2}^n M_i' = M/M_1' \cong M/M_1 = \bigoplus_{i=2}^m M_i$
and we can apply inductive hypothesis and we are done □

Definition 2.8
An A-module P is projective if given modules M and M' and $\lambda : P \to M$ and a surjective map $\mu : M' \to M$ then $\exists \nu : P \to M'$ s.t.

$$\begin{array}{ccccc} & & P & & \\ & \overset{\nu}{\swarrow} & \downarrow{\scriptstyle \lambda} & & \\ M' & \xrightarrow{\mu} & M & \longrightarrow & 0 \end{array}$$

An A-module I is injective if, similarly

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \xrightarrow{\mu} & M \\ & & \downarrow{\scriptstyle \lambda} & \overset{\nu}{\swarrow} & \\ & & I & & \end{array}$$

If we have a ses

$$0 \to N \to M' \to M \to 0$$

then if P is projective

$$0 \to \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, M') \to \mathrm{Hom}(P, M) \to 0$$

is exact and if I is injective

$$0 \to \mathrm{Hom}(M', I) \to \mathrm{Hom}(M, I) \to \mathrm{Hom}(N, I) \to 0$$

is exact

Proposition 2.9
TFAE

(1) P is projective

(2) Every surjective map $\lambda : M \to P$ splits

(3) P is a direct summand of a free module

Proof as an Exercise.
We will prove (1)$\Leftrightarrow$(3) when we talk about relative H-projectives for kG when $H \leqslant G$

Usually projectives and injectives are very different e.g. $k[X]$ polynomial algebra (if commutative algebra)
but for some Artinian algebras e.g. kG, G finite, they are the same.

Definition 2.10
A finite dimensional k-algebra A is Frobenius if $\exists$ linear map $\lambda : A \to k$ s.t.

(i) $\ker \lambda$ does not contain a non-zero left or right ideal

If so, we can define a bilinear form $\langle\ ,\ \rangle : \begin{array}{ccc} A \times A & \to & k \\ (a, b) & \mapsto & \lambda(ab) \end{array}$
Property (i) ensures that $\langle\ ,\ \rangle$ is non-degenerate on both sides $\langle\ ,\ \rangle$ has the property

$$\langle ab, c\rangle = \langle a, bc\rangle \tag{2.1}$$

There is an equivalent definition of Frobenius algebras in terms of non-degenerate bilinear form satisfying 2.1

A is symmetric if it also satisfies

(ii)
$$\lambda(ab) = \lambda(ba) \tag{2.2}$$

in which case the bilinear form is symmetric

A is self-injective if the regular representation ${}_AA$ is an injective A-module

Examples

(1) kG for G finite $\lambda(\sum_{g\in G} r_g g) = r_1$ (projecting onto component indexed by 1)

(2) Matrix algebras $\lambda =$ trace

(3) Monster algebra, G monster group, V is a G-module of dimension $194884 = 194883 \oplus 1$
V can be given the structure of a Frobenius algebra with a composition and non-degenerate bilinear form

(4) Topological quantum field theory

Lemma 2.11

(1) If A is a finite dimensional Frobenius algebra, then we have A-module isomorphism $(A_A)^* \cong {}_AA$
(* denote the vector space dual)
In particular, A is self-injective

(2) Suppose A is self-injective then TFAE:

  (a) M projective
  (b) M injective
  (c) M^* projective
  (d) M^* injective

  Note: if M is a left A-module then M^* has a natural structure as a right A-module

$$(fa)(m) = f(am) \quad \text{for } f : M \to k$$

  and similarly switching 'left' and 'right'

Proof

(1) Define

$$\begin{array}{rcl} \phi : {}_AA & \to & (A_A)^* \\ x & \mapsto & (y \mapsto \lambda(yx)) \end{array}$$

  where λ is from definition of Frobenius algebra
  Then if $\gamma \in A$,

$$(\gamma(\phi(x)))y = \phi(x)(y\gamma) = \lambda(y\gamma x) = (\phi(\gamma x))y$$

  $\Rightarrow \quad \phi$ is a homomorphism
  Since $\ker \lambda$ does not contain a non-zero left ideal
  ϕ is injective. A is finite dimensional and so ϕ is an isomorphism

(2) From self-injectivity, M is projective $\Leftrightarrow$ M^* is projective
  And so (i) and (iii) are equivalent
  But (i)$\Leftrightarrow$(iv) and (ii)$\Leftrightarrow$(iii) for all finite dimensional algebras

□

Consider decomposition of finite dimensional algebra A.
Firstly as $A - A$ bimodules

$$A = \bigoplus B_j \quad \text{where } B_j \text{ are indecomposable } A - A \text{ bimodule}$$

(recall $A - A$ bimodule $\Leftrightarrow$ left $A \times A^{op}$-module)
Note that the $A - A$ sub-bimodules of A are precisely the (2-sided) ideals
Krull-Schmidt 2.4 $\Rightarrow$ decomposition is unique (up to reordering and isomorphism)

Definition 2.12
The indecomposable ideals B_j are <u>blocks</u>

Note that $1 = e_1 + \cdots e_s$ with e_i idempotent, $e_j \in B_j$

In fact, they are orthogonal: $e_ie_j = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$

The e_j are multiplicative identities for the algebra B_j (B_j is closed under multiplication)

Each block may be regarded as a left A-module and decomposes as left A-module $\bigoplus P_{ij}$ of indecomposable left A-modules

The direct summands P_{ij} are direct summands of ${}_AA$ and so are indecomposable projectives and by Krull-Schmidt 2.4 the decomposition is unique (up to reordering and isomorphism)

There is a 1-1 correspondence between expressions

$$1 = e_1 + \cdots + e_t$$

with e_i orthogonal idempotents and direct sum

$${}_AA = P_1 \oplus \cdots P_t$$

given by $P_i = Ae_i$ (e_i corresponds to projection onto summand)

Note: Endomorphism of ${}_AA$ are achieved by right multiplication by elements of A

P_i is indecomposable $\Leftrightarrow$ e_i is primitive

Definition 2.13
An orthogonal idempotent is primitive if it cannot be expressed as $e = e_1 + e_2$ with e_1, e_2 non-zero idempotent

Proposition 2.14 (Idempotent Lifting/Refining)
Let N be a nilpotent ideal in A and f be an idempotent of $A/N = \overline{A}$
Then there is an idempotent $e \in A$ with $f = \overline{e}$

Proof
Consider A/N^t and we will define idempotents $f_i \in A/N^i$ inductively. Let $f_1 = f$

For $i > 1$, let $x \in A/N^i$ with image $f_{i-1} \in A/N^{i-1} = (A/N^i)/(N^{i-1}/N^i)$
$\Rightarrow \quad x^2 - x \in N^{i-1}/N^i$
$\Rightarrow \quad (x^2 - x)^2 = 0$ in A/N^i
Let $f_i = 3x^2 - 2x^3$
$\Rightarrow \quad f_i^2 - f_i = -(3-2a)(1+2a)(a^2-a)^2 = 0$
and observe that f_i has image $f_{i-1} \in A/N^{i-1}$

Alternatively in characteristic p, we can use $f_i = x^p$ □

Corollary 2.15
Let N be an nilpotent ideal in A. Let

$$1 = f_1 + \cdots + f_n \qquad \text{with } f_i \text{ primitive orthogonal idempotents}$$

Then we can write

$$1 = e_1 + \cdots e_n \qquad \text{with } e_i \text{ primitive orthogonal idempotents with } \overline{e_i} = f_i$$

Proof
Define idempotents ${e_i}'$ inductively.

$e'_1 = 1$ and for $i > 1$, e'_i is any lift of $f_i + \cdots + f_n$ be an idempotent in the ring $e'_{i-1}Ae'_{i-1}$
$\Rightarrow \quad e'_ie'_{i+1} = e'_{i+1} = e'_{i+1}e'_i$

Let $e_i = e'_i - e'_{i+1} \quad \Rightarrow \quad \overline{e_i} = f_i$
If $j > i$, $e_j = e'_{i+1}e_je'_{i+1}$ and so $e_ie_j = (e'_i - e'_{i+1})e'_{i+1}e_je'_{i+1} = 0$

Similarly $e_je_i = 0$ □

Applying this to a finite dimensional algebra A with $N = J(A)$, nilpotent by 1.11.
We have $A/N =$ direct sum of simple A-module S_i
This decomposition correspond to $1 = f_1 + \cdots + f_t$ in $A/J(A)$
Lifting using Corollary 2.15 gives $1 = e_1 + \cdots + e_t$ primitive idempotents
and $A = \bigoplus Ae_i$ with each Ae_i indecomposable projectives
By Krull-Schmidt, every indecomposable projective is isomorphic to one of these P_i

Summarizing:
$A = \bigoplus$ blocks $= \bigoplus P_i$ where P_i indecomposable projectives left A-modules
$A/J(A) = \bigoplus S_i$ where S_i are simples, then (after reordering) $P_i/J(A)P_i \cong S_i$
Hence $\{$simples of decomposition of $A/J(A)\} \longleftrightarrow \{$projective indecomposable of $A\}$

Lemma 2.16
For a f.d. Frobenius algebra A, $\operatorname{Soc}(P)$ is simple for an indecomposable projective left A-module

Proof
Express A_A as a direct sum of indecomposable projective right A-modules Q_i
By Lemma 2.11, $(A_A)^* \cong {}_AA$, so

$$ {}_AA = (\bigoplus Q_i)^* \cong \bigoplus Q_i^* \tag{2.3} $$

Q_i^* are projective left A-modules, again indecomposable.
But dual of a simple is a simple and so $\operatorname{Soc}(Q_i^*) = (Q_i/\operatorname{Rad}(Q_i))^*$ is simple
By Krull-Schmidt, the decomposition (2.3) is unique, $P \cong Q_i^*$
$\Rightarrow \quad \operatorname{Soc}(P)$ simple □

Lemma 2.17
$\operatorname{Hom}(P_i, S_j) \cong \begin{cases} \Delta_i & S_i \cong S_j \\ 0 & \text{Otherwise} \end{cases}$
where $\Delta_i^{op} = \operatorname{End}_A(S_i)$

Proof
If non-zero $\theta : P_i \to S_j$ then $P_i/\ker\theta$ (simple quotient of P_i) $\cong S_j$ since S_j simple
But $P_i/J(A)P_i \cong S_i \quad \Rightarrow \quad S_i \cong S_j$ □

Theorem 2.18
Suppose P is an indecomposable projective for a symmetric algebra A. Then $\operatorname{Soc}(P) \cong P/\operatorname{Rad}(P)$

Proof
Let e be a primitive idempotent in A with $P = Ae$
Let $\lambda : A \to k$ be a map as in Definition 2.10
$\Rightarrow \quad \operatorname{Soc}(P) = \operatorname{Soc}(Ae) = \operatorname{Soc}(P)e$ can be regarded as a left ideal of A (or by simplicity of $\operatorname{Soc}(P)$)
$\Rightarrow \quad \exists x \in \operatorname{Soc}(P)$ with $\lambda(xe) \neq 0$ (since, $\ker\lambda$ is not a left or right ideal of A, hence $\operatorname{Soc}(P) \supsetneqq \ker\lambda$)
By symmetry of λ, $\lambda(ex) \neq 0$
$\Rightarrow \quad e\operatorname{Soc}(P) \neq 0$
But Lemma 1.14(1) says $e\operatorname{Soc}(P) \cong \operatorname{Hom}_A(P, \operatorname{Soc}(P))$
$\Rightarrow \quad \exists$ non-zero hom $P \to \operatorname{Soc}(P)$
But Lemma 2.16 says $\operatorname{Soc}(P)$ is simple and we know the only simple quotient of $P \cong P/\operatorname{Rad}(P)$
$\Rightarrow \quad P/\operatorname{Rad}(P) \xrightarrow{\sim} \operatorname{Soc}(P)$ □

So we now know about the 'top' $(P/\operatorname{Rad}(P))$ and 'bottom' $(\operatorname{Soc}(P))$ of P, what happens in between?

For a f.g. left A-module M we can produce a chain

$$ 0 < M_1 < M_2 < \cdots < M_n = M $$

with each factor M_{j+1}/M_j being simple

Definition 2.19
Such a chain is a composition series and the simple factors are composition factors
Jordan-Hölder Theorem (c.f. Topics in Groups Theory course) says that composition factors and their multiplicities are unique

Let P_i be an indecomposable projective A-module. Consider

$$0 \leq \mathrm{Hom}_A(P_i, M_1) \leq \cdots \leq \mathrm{Hom}_A(P_i, M_n) = \mathrm{Hom}(P, M) \tag{2.4}$$

Each factor $\mathrm{Hom}_A(P_i, M_{j+1})/\mathrm{Hom}_A(P_i, M_j) \cong \mathrm{Hom}_A(P_i, M_{j+1}/M_j)$
(this isom. is due to the fact that P_i is projective)
So Lemma 2.17 applies $\mathrm{Hom}(P_i, M_{j+1})/\mathrm{Hom}(P_i, M_j) = 0$ or Δ_i where $\Delta_i \cong \mathrm{End}_A(S_i)^{op}$ (division ring), and $M_{j+1}/M_j \cong S_i$

Thus the number of composition factors $\cong S_i$ = number of non-zero factors in the chain (2.4)

Lemma 2.20
If k is algebraically closed, then $\dim_k(\mathrm{Hom}(P_i, M))$ = no. of composition factors of M isomorphic to S_i (called the multiplicity of S_i in M)

Proof
$\Delta_i \cong k$ in this case □

Definition 2.21
Suppose k is algebraically closed
The Cartan invariants of A is

$$c_{ij} = \dim_k \mathrm{Hom}_A(P_i, P_j) = \dim_k(e_i A e_j)$$

(second equality is by 1.14) where e_i, e_j are primitive idempotents associated with P_i, P_j
Equivalently, c_{ij} = multiplicity of S_i in P_j
The matrix (c_{ij}) is the Cartan matrix

In the introduction, we talked about Ext^1 quiver for A

$$\text{vertices} \quad \leftrightarrow \quad \text{(isom. class of) left simple } A\text{-modules } S_i$$

Number of arrows $i\bullet \to \bullet j$ is $\dim_k \mathrm{Hom}_A(S_j, \mathrm{Rad}(P_i)/\mathrm{Rad}^2(P_i))$ where $\mathrm{Rad}^2(\cdot) = \mathrm{Rad}(\mathrm{Rad}(\cdot))$
($\mathrm{Ext}_A(S_i, S_j) \cong \mathrm{Hom}_A(\mathrm{Rad}(P_i)/\mathrm{Rad}^2(P_i), S_j)$, which is dual to $\mathrm{Hom}_A(S_j, \mathrm{Rad}(P_i)/\mathrm{Rad}^2(P_i))$, see Benson 2.4.3.)

Definition 2.22
For a f.g. left A-module M

$$M \geq \mathrm{Rad}(M) \geq \mathrm{Rad}^2(M) \geq \mathrm{Rad}^3(M) \geq \cdots$$

is called Lowey series

Note: $\mathrm{Rad}^i(M)/\mathrm{Rad}^{i+1}(M)$ is semisimple, a direct sum of simples
Note: $\mathrm{Rad}^i(M) = J^i(A)M$
So the Ext^1-quiver is telling us about the composition factors of the layer of P_j below the 'top'

Remark. $\mathrm{Hom}_A(S_i, \mathrm{Rad}(P_j)/\mathrm{Rad}^2(P_j)) \cong \mathrm{Hom}_A(P_i, \mathrm{Rad}\, P_j/\mathrm{Rad}^2(P_j))$
its dimension $= \dim_k(e_i J(A) e_j / e_i J^2(A) e_j)$ where $P_i = Ae_i$, $P_j = Ae_j$

Definition 2.23
A left Artinian algebra is basic if $A/J(A)$ is a direct sum of non-isom. simples
(This is equivalent to A is direct sum of non-isom indecomposable projectives)

We will see that any Artinian algebra A is Morita equivalent to a basic one, namely $A_1 = \mathrm{End}_A(\bigoplus P_i)^{op}$ (where the summation only take one copy of each isom. class of indecomposable projectives) We call A_1 the basic algebra of (associated to) A

Exercise: Show $\mathrm{End}_A(\bigoplus P_i)^{op}$ is basic

Note: Morita equivalence $\Rightarrow$ Equivalence of categories between $\{A\text{-modules}\}$ and $\{A_1\text{-modules}\}$

Theorem 2.24 (Gabriel)
Let A be a finite dimensional basic algebra over an algebraically closed field
Then $A \cong kQ/I$ where kQ is the path algebra of the Ext^1-quiver of A
where I is (an ideal) in the ideal of paths of length ≥ 2

Proof
Choose primitive idempotents e_i with $e_i e_j = 0 = e_j e_i$ if $i \neq j$
$e_1 + \cdots + e_r = 1$ (using that A is basic)
Recall that we have an idempotent x_i of the path algebra kQ for each vertex (path of length 0 at the vertex)
We are going to define a map $kQ \to A$ sending $x_i \mapsto e_i$

Choose a vector space complement of $e_j J^2(A) e_i$ in $e_j J(A) e_i$
and choose basis for this complement.
(this means choose a basis for $e_j J(A) e_i / e_j J^2(A) e_i \cong \mathrm{Hom}_A(P_j, \mathrm{Rad}(P_i)/\mathrm{Rad}^2(P_i)))$
send (the elements of kQ corresponding to arrows from vertex i to vertex j) to (these basis elements)

Every relation in kQ says that products of non-compatible paths is zero
But these relations are satisfied by the corresponding products in A by relations:
$e_i e_j = e_j e_i = 0$ if $i \neq j$
$\Rightarrow \quad \theta : kQ \to A$ well-defined map

We need a lemma (Lemma 2.25) to show it is surjective: Apply the lemma with $A' = \mathrm{Im}\,\theta$ to see that the image is A
(Note: $\overline{\theta} : kQ/J^2(kQ) \to A/J^2(A)$ is surjective) □

Lemma 2.25
Suppose A is an Artinian ring and A' an ideal of A s.t. $A' + J^2(A) = A$. Then $A' = A$

Proof
Show by induction that $A' + J^n(A) = A' + J^{n+1}(A)$ for $n \geq 2$
Using the nilpotency of $J(A)$ (by 1.11) we are done.
If $x \in J^{n-1}(A)$ and $y \in J(A)$, choose $x' \in J^{n-1}(A) \cap A'$ s.t. $x - x' \in J^n(A)$
and $y' \in J(A) \cap A'$ s.t. $y - y' \in J^2(A)$

$$\begin{aligned} \Rightarrow \quad xy &= x(y - y') + (x - x')y' + x'y' \\ &\in J^{n-1}(A)J^2(A) + J^n(A)(J(A) \cap A') + (J^{n-1}(A) \cap A')(J(A) \cap A') \subseteq J^{n+1}(A) + A' \end{aligned}$$
□

In fact, the finite dimensional case of kQ (i.e. when there are no directed cycles) there is a one-to-one correspondence of blocks.

Recall that the decomposition into blocks corresponds to a decomposition of 1 as a sum of orthogonal central idempotents (which are the multiplicative identities of the blocks)

Lemma 2.26
Suppose A is Artinian and I is a 2-sided ideal contained in $J^2(A)$.
Then the natural map $A \to \overline{A} = A/I$ induces a bijection:

$$\{\text{central idempotents in } Z(A)\} \longleftrightarrow \{\text{central idempotents in } Z(A/I)\}$$

Remark. Apply this to $A = kQ$ to get a 1-1 correspondence

$$\{\text{Blocks of } kQ\} \longleftrightarrow \{\text{Blocks of } kQ/I\}$$

Proof
If f idempotent in $Z(A) \Rightarrow \overline{f}$ idempotent in A/I
If f' is another idempotent in $Z(A)$ with $\overline{f} = \overline{f'} \Rightarrow f - ff'$ nilpotent (as $f, f' \in J^2(A)$) and idempotent $\Rightarrow f - ff' = 0 \Rightarrow f = ff'$, similarly, $f' = f'f$, so $f = f'$

Conversely, if e is an idempotent in $Z(A/I)$ then by idempotent lifting in Proposition 2.14, $\exists$ idempotent $f \in A$ with $\overline{f} = e$.

Claim: f is central

Proof of Claim:
Since $\overline{f} \in Z(A/I)$ we have $\overline{f}(A/I)(1-\overline{f}) = 0$ in A/I
$\Rightarrow \quad fA(1-f) \subseteq I \subseteq J^2(A)$

Since f and $(1-f)$ are idempotents, multiply on left and right: $fA(1-f) \subseteq fJ^2(A)(1-f)$

We show by induction that $fA(1-f) = fJ^n(A)(1-f)$
Assume $fA(1-f) = fJ^n(A)(1-f) = fJ^{n-1}(A)J(A)(1-f)$

$$\begin{aligned} &\subseteq fJ^{n-1}(A)fJ(A)(1-f) + fJ^{n-1}(A)(1-f)J(A)(1-f) \\ &\subseteq fJ^{n-1}(A)fJ^2(A)(1-f) + fJ^n(A)(1-f)J(A)(1-f) \\ &\subseteq fJ^{n+1}(A)(1-f) \end{aligned}$$

Since $J(A)$ is nilpotent by 1.11 $\Rightarrow fA(1-f) = 0$
$\Rightarrow \quad \forall a \in A,\ fa = faf + fa(1-f) = faf$
Similarly $af = faf$
$\Rightarrow \quad fa = af\ \forall a \in A$
$\Rightarrow \quad f$ central

□

Even in the case where kQ is finite dimensional, we can see that blocks of one (basic) algebra correspond to components of the underlying graph of the Ext^1-quiver (ignoring the direction of arrows)

A non-split extension (of B by A) is a s.e.s. $0 \to A \to U \to B \to 0$ where U is not a direct sum of A and B

Proposition 2.27
Let S and T be simple A-modules with A a left Artinian k-algebra and k algebraically closed. TFAE:

(1) S and T belongs to the same block

(2) There are simple A-modules $S = S_1, S_2, \ldots, S_m = T$ s.t. each pair S_i, S_{i+1} are composition factors of the same indecomposable projective A-module

(3) There are simple A-module $S = T_1, T_2, \ldots T_n = T$ s.t. each pair T_i, T_{i+1} are equal or there is a non-split extension of one of them by other

Remark. These three conditions define equivalence relations.
Condition (3) can be rephrased in terms of $T_i \bullet \to \bullet\, T_{i+1}$ or $T_i \bullet \leftarrow \bullet\, T_{i+1}$ in Ext^1-quiver

Example: k finite field, $G = \begin{pmatrix} k^\times & k \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in k^\times, b \in k \right\}$ as affine group $\cong k \rtimes k^\times$
(G acts on k via conjugation of k in k')
There is a canonical 2-dimensional (non-simple) kG-module $U = \{$column vectors$\}$

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ y \end{pmatrix}$$

There is 1-dimensional kG-submodule $S = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| x \in k \right\}$ and $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ acts by multiplication by a
$U/S \cong$ trivial kG-module. U is not a direct sum, S is not trivial kG-module

Exercise: Show that all the simple kG-modules are 1-dimensional and construct the Ext^1-quiver
Exercise: Try the same with other triangular groups
Exercise*:
$G = SL(2, p)$, k algebraically closed of characteristic p
Recall we produced kG-modules $V_1, V_2, \ldots, V_p$ and Theorem 1.17 told us this was the complete list and V_i is of dimension i
$p = 2$: V_1 and V_2. Decompose kG as a direct sum of blocks
p odd: Show V_p projective and the simple modules associated with blocks are (1) V_p, (2) $V_1, V_3, \ldots, V_{p-2}$ and (3) $V_2, V_4, \ldots V_{p-1}$
(Case $p = 2$ should be easy, others are hard, c.f. Alperin's book)

Proof of Theorem 2.27
(1) $\Leftrightarrow$ (2):

(2) $\Rightarrow$(1): Easy from Cartan invariants as $c_{ij} = 0$ if $e_i J(A) e_j = 0$
If e_i, e_j are in different blocks, then $c_{ij} = 0$

(1) $\Rightarrow$(2): Suppose S and T are in the same block but not related as in (2). Let

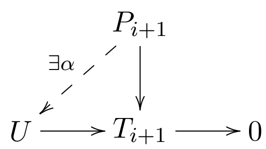
$$B = \sum P' \oplus \sum P'$$

where first summation is over P' projectives associated with simples related to S as in (2) and second summation is over P' projectives associated with simples NOT related to S as in (2)

But the first sum is closed under any endomorphism of B and such endomorphisms are all achieved by right multiplication by elements of A (using that $\mathrm{End}({}_A A)^{op} = A$)
$\Rightarrow$ the first sum is closed under right multiplication by elements of A $\Rightarrow$ first sum is an ideal
Similarly for the second sum, $\Rightarrow$ we have a non-trivial decomposition of B # (B is a block)

(2) $\Leftrightarrow$ (3):

(3)$\Rightarrow$(2): Suppose $0 \to T_i \to U \to T_{i+1} \to 0$ non-split
If we take the indecomposable projective P_{i+1} with 'top' T_{i+1} (i.e. $T_{i+1} \cong P_{i+1}/\operatorname{Rad}(P_{i+1})$), we have a canonical map $P_{i+1} \twoheadrightarrow T_{i+1}$. So by projectivity $\exists \alpha$ completing the diagram:



Claim: α is surjective
Proof of Claim:
Composition factors of U are T_i and T_{i+1}. Suppose $\operatorname{Im}\alpha \subsetneq U$, then $\operatorname{Im}\alpha$ is either T_i or T_{i+1}
$\Rightarrow$ $U = T_i \oplus T_{i+1}$, i.e. a split extension # ■

But $J^2(A)U = 0$ as it is extension by two simples
so in fact U is a quotient of $P_{i+1}/\operatorname{Rad}^2(P_{i+1}) = P_{i+1}/J^2(A)P_{i+1}$
$\Rightarrow$ P_{i+1} has T_i and T_{i+1} as a composition factor

(2)$\Rightarrow$(3): It is enough to show that the 'top' of any indecomposable projective P is related by (3) to any composition factor of P
Argue by induction on the Lowey series of P,i.e. $\operatorname{Rad}^i(P)(= J^i(A)P)$

Suppose simple S is a composition factor of the semisimple module $\operatorname{Rad}^i(P)/\operatorname{Rad}^{i+1}(P)$.
Thus S is a quotient of $\operatorname{Rad}^i(P)/\operatorname{Rad}^{i+1}(P)$
$\Rightarrow$ we have an extension

$$0 \to S \to V \to \underbrace{\operatorname{Rad}^{i-1}(P)/\operatorname{Rad}^i(P)}_{\text{maximal semisimple quotient of } \operatorname{Rad}^{i-1}(P)} \to 0$$

This is not split (if split then V semisimple, contradicting maximality)
$\Rightarrow$ $\exists$ simple submodule T of $\operatorname{Rad}^{i-1}(P)/\operatorname{Rad}^i(P)$ with $0 \to S \to U \to T \to 0$ non-split
$\Rightarrow$ S related to T as in (3), and now apply induction hypothesis to T

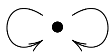
$\square$

3 Quivers

(See the Introduction section for definition)
Q quiver, kQ path algebra, basis are paths of finite length (inclduing length zero)
length zero path at vertex x is e_x idempotent in kQ
Multiplication is given by composition of paths (in reverse order) as in usual convention for maps

$$(\underset{y}{\bullet} \longrightarrow \underset{z}{\bullet})(\underset{x}{\bullet} \longrightarrow \underset{y}{\bullet}) = \underset{x}{\bullet} \longrightarrow \underset{y}{\bullet} \longrightarrow \underset{z}{\bullet}$$

Recall that left kQ-modules correspond to representations of Q
There is a simple kQ-module S_x 1-dimensional at vertex x, zero at other vertices.
In fact, if kQ is finite dimensional these are all simples, but otherwise there are others
Example: one vertex two loops



kQ is the free algebra on two variables $k[X,Y]$ (corresponding to the loops) and thus has simple modules of any dimension

The projective module kQe_x corresponding to the idempotents e_x:
basis - correspond to the paths in Q starting at x
action of kQ is by (reverse) composition as before

Theorem 3.1
Suppose kQ is finite dimensional. Then every submodule of a free kQ-module F is isomorphic to a direct sum of modules of the form kQe_x, and thus every submodule of a projective is projective

Remark. Condition kQ finite dimensional is in fact not needed. (i.e. theorem is true for any finite Q)

Definition 3.2
An algebra in which all left submodules of left projectives are left projectives are left hereditary

Remark. This is equivalent to having cohomological dimension ≤ 1

Definition 3.3
We can define a <u>filtration of kQ</u> consisting of the 2-sided ideal $kQ_{(n)}$, linear space in kQ of the paths of length $\geq n$

As a left kQ-module, $kQ_{(n)}/kQ_{(n+1)}$ is semisimple:
it is isomorphic to a direct sum of simple S_x
and the copies of S_x correspond to the paths of length n ending at x
$\Rightarrow \quad e_x kQ_{(n)}$ has a basis consisting of paths of length n ending at x

Note that $kQ_{(1)}$ is nilpotent ideal in kQ when kQ is finite dimensional (i.e. there are no oriented cycles) and $kQ/kQ_{(1)}$ is semisimple
$\Rightarrow \quad kQ_{(1)} = J(kQ) \quad \Rightarrow \quad kQ_{(n)} = J^n(kQ)$

A path of length n, $\quad \bullet_x \to \bullet \cdots \to \bullet_y$
induces an injective map

$$\underbrace{e_x kQ_{(n)}/kQ_{(n+1)}}_{\text{length } n \text{ path, end at } x} \hookrightarrow \underbrace{e_y kQ_{(n+m)}/kQ_{(n+m+1)}}_{\text{length } n+m \text{ path, end at } y}$$

and the images of these maps for distinct paths are linear independent

Similarly, if we now have a free kQ-module F again we have a filtration $F_{(n)}$ and if $\bullet_x \to \cdots \bullet \to \bullet_y$ then it induce the injective map

$$e_x F_{(m)}/F_{(m+1)} \hookrightarrow e_y F_{(n+m)}/F_{(n+m+1)}$$

and the images of these mas for distinct paths are linear independent

Proof of Theorem 3.1
Suppose P is a submodule of free kQ-module F
Define filtration $P_{(n)} = P \cap F_{(n)}$ where $F_{(n)}$ is the 'standard' filtration we have just had

$$\begin{array}{l}\text{If } \bullet_x \to \cdots \to \bullet_y \text{ is a path of length } n \\ \text{then the induced map } e_x P_{(m)}/P_{(m+1)} \hookrightarrow e_y P_{(n+m)}/P_{(n+m+1)} \text{ is injective}\end{array} \tag{3.1}$$

Recall $J(kQ) = kQ_{(1)}$ since kQ is finite dimensional
$P/kQ_{(1)}P$ is a module for $kQ/kQ_{(1)}$ and is a finite direct sum of copies of simples isomorphic to some S_x (recall S_x has 1 dimensional vector space at x and 0 at all other vertices)

$$P/kQ_{(1)}P \cong \bigoplus_{x,\alpha} S_{x,\alpha} \qquad \text{with } S_{x,\alpha} \cong S_x$$

Let $P_{x,\alpha} \cong kQe_x$ the indecomposable projective with S_x 'top'
Thus there is a surjective kQ-module homomorphism

$$\phi : \bigoplus P_{x,\alpha} \to P/kQ_{(1)}P$$

But we can complete

$$\begin{array}{ccccc} & & \bigoplus P_{x,\alpha} & & \\ & \overset{\psi}{\swarrow} & \downarrow \phi & & \\ P & \longrightarrow & P/kQ_{(1)}P & \longrightarrow & 0 \end{array}$$

to get the kQ-homomorphism $\psi : \bigoplus P_{x,\alpha} \to P$. (Think about this!)

Claim: ψ is an isomorphism

Proof of Claim:
Injective: Follow from the statement (3.1)

Surjective: write X for the cokernel

$$\bigoplus P_{x,\alpha} \xrightarrow{\psi} P \to X \to 0$$

tensoring over kQ with $kQ/kQ_{(1)}$ gives

$$\bigoplus S_{x,\alpha} \to P/kQ_{(1)}P \to X/kQ_{(1)}X \to 0$$

Hence, $X/J(kQ)X = X/kQ_{(1)}X = 0$
By Nakayama's Lemma 1.6, we get $X = 0$ ■

Thus $P \cong \bigoplus_\alpha P_{x,\alpha} \cong \bigoplus_\alpha kQe_x$ □

Remark. Note that $\bigoplus P_{x,\alpha}/kQ_{(1)}(\bigoplus P_{x,\alpha}) \cong \bigoplus S_{x,\alpha}$
The number of copies of each $S_{x,\alpha}$ is finite and well-defined by Krull-Schmidt for simple modules and so we deduce the number of each P_x in the sum is finite and well-defined.
Krull-Schmidt is true for finitely generated projectives

The hereditary property picks out those basic finite dimensional algebras that can be expressed as path algebras

Lemma 3.4
Let A be a finite dimensional hereditary basic algebra over an algebraically closed field
Then $A \cong kQ$, the path algebra of its Ext^1-quiver Q

To prove this, we need some lemmas:

Lemma 3.5
Let A be an algebra containing a (2-sided) ideal I with $I \subseteq J^2(A)$ with I finitely generated as a left and right ideal
Then A/I hereditary $\Rightarrow$ $I = 0$

Proof
Since I is f.g. as right ideal.

$$\text{If } I = IJ(A) \Rightarrow I = 0 \qquad \text{by Nakayama 1.6}$$

So it suffices to work mod $IJ(A)$ and so we may assume that $IJ(A) = 0$

Let $\overline{A} = A/I$ hereditary. Since I annihilate $J(A)$ we can regard $J(A)$ as a left $\overline{A}$-module
Hereditary property $\Rightarrow$ $J(\overline{A})$ is a projective left $\overline{A}$-module
$\Rightarrow$ $J(A) \to J(A)/I = J(\overline{A})$
$\Rightarrow$ as left $\overline{A}$-modules, $J(A) \cong J(\overline{A}) \oplus I$
$\Rightarrow$ $J^2(A) = J(A)J(\overline{A}) \oplus J(A)I = J^2(A)/I \oplus J(A)I$
$\Rightarrow$ $J(A)I = I$
Apply Nakayama on left this time $\Rightarrow I = 0$ □

Lemma 3.6
Suppose A is a finite dimensional hereditary algebra with a non-split extension of simples S_1 and S_2

$$0 \to S_1 \to U \to S_2 \to 0$$

Then the indecomposable projective P_2 with S_2 'top' contains a copy of P_1, the indecomposable projective with S_1 'top'

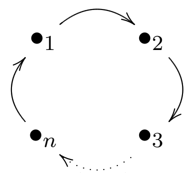
Proof
As last time, $\exists 0 \neq \theta : P_1 \to P_2$ (homomorphism) s.t. $\operatorname{Im}\theta \subseteq \operatorname{Rad}(P_2)$ and $\nsubseteq \operatorname{Rad}^2(P_2)$

The image $\theta(P_1)$ is a submodule of P_2
$\Rightarrow$ projective by hereditary property
$\Rightarrow$ The map splits 2.9
$\Rightarrow$ $P_1 \cong \theta(P_1) \oplus \ker\theta$

Indecomposability of $P_1 \Rightarrow \ker\theta = 0$
$\Rightarrow$ P_2 contains a copy of P_1 □

Lemma 3.7
If A is a finite dimensional hereditary algebra
Then the Ext^1-quiver Q of A contains no oriented cycles

Proof
Suppose Q is an oriented cycle:



$\Rightarrow$ (by Lemma 3.6) we have corresponding projectives

$$P_1 \lneqq P_2 \lneqq P_3 \lneqq \cdots \lneqq P_n \lneqq P_1$$

But those are all finite dimensional vector space # □

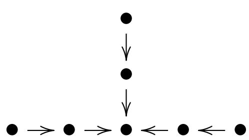
Proof of Lemma 3.4
Let A be finite dimensional, hereditary and basic
By Lemma 3.7, the Ext^1-quiver Q has finite dimensional path algebra kQ
By Gabriel's Theorem 2.24, $A \cong kQ/I$ where $I \leq J^2(kQ)$, and I is f.g. as A is finite dimensional
By Lemma 3.5, we get $I = 0$
$\therefore$ $A \cong kQ$ □

Indecomposables for kQ

Recall an algebra has finite representation type if there are only finitely many isomorphism classes of finitely generated indecomposables (not necessary projective)

Example:
Let Q be a n-vertices oriented cycles, k infinite, kQ does not have finite representation types:
Take a 1-dimensional subspace $\langle e_i \rangle$ at each vertex; linear maps representing the arrows :
${}_i\bullet \to \bullet_{i+1}$ is represented by linear map $e_i \mapsto e_{i+1}$
${}_n\bullet \to \bullet_1$ is represented by linear map $e_n \mapsto \lambda e_1$ $(\lambda \in k)$
These are indecomposable representations and if we have two such, one defined with λ and one with μ, then they are non-isomorphic if $\lambda \neq \mu$
$\Rightarrow$ (since k infinite field) we have infinitely many isomorphism classes of indecomposables

Exercise: Show this is true for k finite
Exercise:



does not have finite representation type (in fact the orientation of arrows does not matter)

Remark. If Q' is a subquiver of a quiver Q and we have a representation of Q'
then we can build a representation of Q by inserting zero vector spaces at any extra vertices and zero maps for extra arrows.
The outcome is indecomposable if the original representation of Q' is indecomposable. So we have:

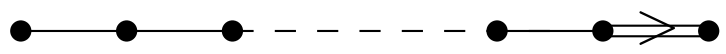
Lemma 3.8
If kQ has finite representation type then kQ' does

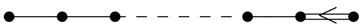
Lemma 3.9
If kQ has finite representation type then Q does not contain any oriented cycle

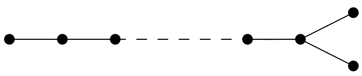
Proof
Above Example and Lemma 3.8 $\square$

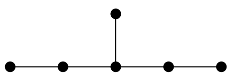
Theorem 3.10 (Gabriel, 1972)
kQ has finite representation type $\Leftrightarrow$ its underlying valued graph is a finite Dynkin diagram:

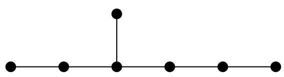
A_n:

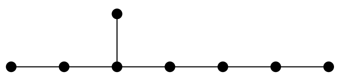
B_n: 

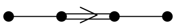
C_n: 

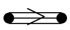
D_n: 

E_6 : 

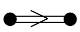
E_7: 

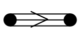
E_8: 

F_4: 

G_2: 

where

 denote an edge labelled $\bullet \overset{(2,1)}{\rule{2em}{0.4pt}} \bullet$

 denote an edge labelled $\bullet \overset{(3,1)}{\rule{2em}{0.4pt}} \bullet$

all other edges are labelled $\bullet \overset{(1,1)}{\rule{2em}{0.4pt}} \bullet$

$$_x\bullet \overset{({}_xd_y, {}_yd_x)}{\rule{3em}{0.4pt}} \bullet_y$$

Definition 3.11
The Cartan matrix of a labelled graph

$$c_{xy} = 2\delta_{xy} - \sum_{\gamma} {}_xd_y^{\gamma}$$

where γ is the label of edge $_x\bullet \overset{\gamma}{\rule{2em}{0.4pt}} \bullet_y$

Example:
For F_4,

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

We can form a symmetrised Cartan matrix (which is symmetric)

$$\overline{c_{xy}} = c_{xy} f_y$$

where we choose $f_y \in \mathbb{Z}_{>0}$ s.t. ${}_x d_y^{\gamma} f_y = {}_y d_x^{\gamma} f_x$ for each edge ${}_x\bullet \overset{\gamma}{\text{———}} \bullet_y$

Form a real vector space $\mathbb{R}^n$ with basis $\{u_x\}$ corresponding to vertices $\{x\}$, symmetric bilinear form

$$(u_x, u_y) = \overline{c_{xy}} \quad \text{(entry from symmetrised Cartan matrix)}$$

Weyl group of labelled the graph Γ, denoted $W(\Gamma)$ is generated by reflections:

$$w_x(u) = u - \frac{2(u, u_x)}{(u_x, u_x)} u_x$$

reflection in hyperplane perpendicular to u_x, it has order 2

Definition 3.12
The dimension vector of a representation V of a quiver Q is $\sum(\dim V_x)u_x$ where V_x=vector space at vertex x

Theorem 3.13
Let Γ finite connected valued graph without loops
Then, Γ is a finite Dynkin diagram $\Leftrightarrow$ (,) is positive definite
If Γ is finite Dynkin diagram then the Weyl group $W(\Gamma)$ is a finite group of automorphisms and there is no non-zero vector in $\mathbb{R}^n$ fixed by the whole of $W(\Gamma)$

Proof omitted. c.f. Lie algebras

Definition 3.14
Let Γ be a finite Dynkin diagram, the root system associated with Γ is the finite subset Φ of $\mathbb{R}^n$ of the images under $W(\Gamma)$ of the basis vectors u_x
The elements of Φ are called roots. They are integral combinations of the u_x
Reflection in the hyperplane perpendicular to a root permutes Φ
If v is a root write w_y for this reflection
A non-zero vector which is a non-negative linear combination of the u_x is a positive root, v is a negative root if $-v$ is positive

Theorem 3.15 (Gabriel, 1972)
Let Q be finite quiver whose underlying labelled (valued) graph is a finite Dynkin diagram.
Then there is a natural 1-1 correspondence:

$$\left\{ \begin{array}{c} \text{f.g. indecomposable} \\ \text{representation of } Q \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{positive roots} \\ \text{in } \mathbb{R}^n \end{array} \right\}$$

Thus kQ has finite representation type

(We will prove this theorem later)

Remark.

(1) This theorem provides $\Leftarrow$ direction of the proof of Theorem 3.10

(2) Q is of finite representation type $\Leftrightarrow$ any new quiver obtained by changing orientation on all edges between x and y has finite representation type

(3) The proof is constructive - we actually get an algorithm that produces indecomposables

Sketch Proof of ⇒ of Theorem 3.10

<u>⇒</u>:

First note that if we have multiple arrows going from i to j, e.g. $_i\bullet \rightrightarrows \bullet_j$, then kQ does not have finite representation type.

<u>Exercise</u>: Prove that $\bullet \rightrightarrows \bullet$ has infinite representation type.

We further assume that all the labels are $(1,1)$. (see Remark) This will restrict us to getting A_n, D_n and E_6, E_7, E_8

Suppose underlying graph Γ of Q is not a finite Dynkin diagram

⇒ (by Lemma 3.13) (,) is not positive definite

⇒ $\exists n_x \in \mathbb{Z}_{\geq 0}$ s.t. $(\mathbf{n}, \mathbf{n}) \leq 0$ where $\mathbf{n} = \sum_x n_x u_x$

⇒ $2\sum n_x^2 \leq 2 \sum_{x\bullet \to \bullet y} n_x n_y$

Each edge gets counted both ways around

$$\sum n_x^2 \leq \sum_{x\bullet \to \bullet y} n_x n_y \tag{3.2}$$

Let V_x be vector space of dimension n_x

We will show there are infinitely many isomorphism classes of representations of Q using these V_x (i.e. with dimension vector (n_x))

(We can then deduce there are infinitely many isomorphism classes of indecomposables)

We need to assign a linear map to each $_x\bullet \to \bullet_y$ arrow

Two such representations are isomorphic ⇔ ∃ automorphisms in $\Pi_x GL(V_x)$ taking one to another

⇒ we need to look at orbits of $\prod_x GL(V_x)$ on $\prod_{x\bullet \to \bullet y} \operatorname{Hom}(V_x, V_y)$

Suppose k is algebraically closed, $\prod GL(V_x)$ has dimension $\sum_x n_x^2$ as an algebraic variety

But action is as $\prod GL(V_x)/\{\text{scalars}\}$ since scalars acts trivially

⇒ $\prod GL(V_x)/\{\text{scalars}\}$ has dimension $(\sum n_x^2) - 1$

⇒ the orbits have dimension $\leq (\sum n_x^2) - 1$

But $\prod_{x\bullet \to \bullet y} \operatorname{Hom}(V_x, V_y)$ has dimension $\sum_{x\bullet \to \bullet y} n_x n_y$

⇒ (from equation (3.2)) the orbits have dimension $\lneqq$ space acted on

⇒ there are infinitely many orbits and hence infinitely many isom. classes. □

Remark. (1) The proof was due to Tits. There is a non-algebraic geometry proof in Pierce's book ruling out particular subquivers

(2) When k is algebraically closed, all the labelling will be $(1,1)$. Labellings are important if we want to talk about non-algebraically closed fields k

e.g. Ext^1-quiver $\operatorname{Ext}^1(S_i, S_j) \cong \operatorname{Hom}(\operatorname{Rad}(P_i)/\operatorname{Rad}^2(P_i), S_j)$ where S_i, S_j are simple A-modules and P_i indecomposable projective associated with S_i

Use labels on our Ext-quiver if k is not algebraically closed.

$\operatorname{Ext}^1(S_i, S_j)$ is acted on by $\operatorname{End}_A(S_i) = \Delta_i$, a division ring, and also by $\operatorname{End}_A(S_j)^{op} = \Delta_j$, another division ring.

Labels are used to retain the information about the relative dimensions of the Δ_i over k

Using the labelled quivers we can indeed end up with Dynkin diagrams other than A_n, D_n, E_6, E_7, E_8 (can have labels other than $(1,1)$ on the Dynkin diagram edges) But from now on, just consider unlabelled quivers.

Definition 3.16
For a vertex x of a quiver
x is a sink if all arrows between x and any other vertex point towards x
x is a source if all arrows between x and any other vertex point away from x
Clearly, every quiver without oriented cycles has sinks and sources

Definition 3.17
Given a quiver Q, define a new quiver s_xQ with same vertices but with the orientations of the edges meeting x reversed.

Suppose we order the vertices so that $i < j$ whenever ${}_{x_i}\bullet \to \bullet_{x_j}$

Lemma 3.18
With such ordering on Q with n vertices

(1) If $1 \leq j < n$ then x_j is a sink and x_{j+1} is a source of the quiver $s_j \cdots s_2 s_1 Q$

(2) If $1 < j \leq n$ then x_j is a source and x_{j-1} is a sink of the quiver $s_j s_{j+1} \cdots s_n Q$

(3) $s_n s_{n-1} \cdots s_1 Q = s_1 s_2 \cdots s_n Q = Q$

Proof
Follows from the choice of ordering on the vertices and that if $v_{j_1}, \ldots, v_{j_s}$ are distinct vertices then, $x_i \bullet \to \bullet x_j$ in $s_{j_1} s_{j_2} \cdots s_{j_s} Q \Leftrightarrow$
either $x_i \bullet \to \bullet x_j$ in Q and either none or both of i, j occur in $j_1, \ldots, j_s$
or $x_i \bullet \leftarrow \bullet x_j$ in Q and exactly one of the i, j occur in $j_1, \ldots, j_s$ □

Definition 3.19
An ordering of vertices is admissible if for each j, x_j is a sink for $s_{j+1} \cdots s_n Q$

Lemma 3.20
$\exists$ an admissible ordering for vertices of $Q \Leftrightarrow \nexists$ oriented cycles in Q

Proof
By Lemma 3.18. Clearly cannot be done if we do have oriented cycles. □

Exercise: If Q and Q' are two quivers with the same tree as underlying graph $\Rightarrow \exists$ some choice of $j_1, \ldots, j_s$ which converts Q into Q'

Now suppose y is a sink in finite quiver Q

Definition 3.21
We define functors

$$\begin{array}{rrcl} S_y^+ : & {}_{kQ}\,\mathbf{mod} & \to & {}_{k(s_yQ)}\,\mathbf{mod} \\ S_x^- : & {}_{k(s_xQ)}\,\mathbf{mod} & \to & {}_{kQ}\,\mathbf{mod} \end{array}$$

(where ${}_{kQ}\,\mathbf{mod}$ denotes the category of left kQ-modules) as follows:
Given a representation V of Q, we define $S_y^+(V) = W$ where

$$\begin{array}{rclc} W_x & = & V_x & \text{for } x \neq y \\ W_y & = & \ker \phi & (\phi \text{ as follows}) \end{array}$$

$$0 \to W_y \to \bigoplus_{\substack{x\bullet \to \bullet y \\ \text{in } Q}} V_x \xrightarrow{\phi} V_y \tag{3.3}$$

There are obvious maps $W_y \to W_x = V_x$ using projections
Thus W is a representation of $s_y Q$
A map of representations $V \to V'$ of Q gives us a map $S_y^+ V \to S_y^+ V'$ between representations of $s_y Q$
$\Rightarrow \quad S_y^+$ is a functor

Given a representation W of $s_y Q$

$$\begin{array}{rcll} V_x & = & W_x & \text{for } x \neq y \\ V_y & = & \operatorname{coker} \psi & (\phi \text{ as follows}) \end{array}$$

$$W_y \xrightarrow{\psi} \bigoplus_{\substack{y\bullet\to\bullet x \\ \text{in } s_y Q}} W_x \to V_y \to 0 \tag{3.4}$$

If V is a representation of Q for which ϕ in the sequence (3.3) is surjective then $S_y^- S_y^+(V) \cong V$
$\Rightarrow \quad S_y^+$ and S_y^- give a categorical equivalence

$$\begin{array}{c} \text{Subcategory of } {}_{kQ}\,\mathbf{mod} \\ \text{for which } \phi \text{ surjective} \end{array} \approx \begin{array}{c} \text{Subcategory of } {}_{k(s_y Q)}\,\mathbf{mod} \\ \text{for which } \psi \text{ is injective} \end{array}$$

Thus we have,

Proposition 3.22
S_y^- an S_y^+ give a bijection:

$$\left\{ \text{indecomposable representations of } Q \right\} \leftrightarrow \left\{ \begin{array}{c} \text{indecomposable representations of } s_y Q \\ \text{except for the simple } S_y \text{ corresponding to } y \end{array} \right\}$$

Corollary 3.23
kQ has finite representation type $\Leftrightarrow k(s_y Q)$ has finite representation type

Exercise (cont.): Using the earlier exercise we see that if the underlying graph is a tree then the representation type is independent of orientations of arrows

Now consider dimension vectors. If V is a representation of Q for which ϕ is surjective, then

$$\begin{array}{rcll} \dim W_y & = & \left(\displaystyle\sum_{\substack{x\bullet\to\bullet y \\ \text{in } Q}} \dim V_x \right) - \dim V_y & \\ \dim W_x & = & \dim V_x & (\text{if } x \neq y) \end{array}$$

$\Rightarrow \quad$ the effect of applying S_y^+ is to send

$$(\text{dim. vector of } V) \to (w_y(\text{dim. vector of } V)) = \text{dim. vector of } W$$

(recall w_y is a reflection in the root space of our Dynkin diagram)

Similarly if W is a representation of $s_y Q$ with ψ injective
$\Rightarrow \quad$ effect of S_y^- on the dimension vector of W is the same as applying the reflection w_y

Definition 3.24
Suppose $x_1, \ldots, x_n$ is an admiisible ordering for Q, then the Coxeter functor w.r.t. the ordering

$$C^+ := S_{x_1}^+ \cdots S_{x_n}^+ : {}_{kQ}\,\mathbf{mod} \to {}_{kQ}\,\mathbf{mod}$$

(Note that since each arrow gets reversed twice: $s_{x_1} \cdots s_{x_n} Q = Q$)
Also get

$$C^- := S_{x_1}^- \cdots S_{x_n}^- : {}_{kQ}\,\mathbf{mod} \to {}_{kQ}\,\mathbf{mod}$$

Lemma 3.25
Given any indecomposable kQ-module V.

either (i) $C^-C^+(V) \cong V$ and the effect of C^+ on the dimension vector of V is the same as the Coxeter transformation $c = w_{x_1} \cdots w_{x_n}$

or (ii) $C^+(V) = 0$

Proof
Relies on Proposition 3.22 □

Definition 3.26
A Coxeter transformation on $\mathbb{R}^\Gamma$ is a linear transformation obtained by applying each of reflection exactly once.

$$c = w_{x_1} \cdots w_{x_n}$$

Then c has finite order h, the Coxeter number, on $\mathbb{R}^\Gamma$
The transformation c has no non-zero fixed points in $\mathbb{R}^\Gamma$ and given any $v \in \mathbb{R}^\Gamma$ for some $m \geq 0$, the vector $c^m(v)$ is not positive
(this follows since if $c^m(v)$ were positive $\forall\, m$ then $\sum_{i=0}^{h-1} c^i(v)$ is positive vector fixed by c #)

Proof of Gabriel's Theorem 3.15, Bertnstein-Gel'fand-Ponomarev 1972
Choose an admissible ordering on the vertices of Q (since underlying Dynkin diagram is a tree)
Let C^+ be the corresponding Coxeter functor and transformation for ${}_{kQ}\,\mathbf{mod}$

Step 1: (indecomposables $\mapsto$ positive roots, proof of well-define-ness and injective)
Suppose V is an indecomposable representation of Q with dimension vector v in $\mathbb{R}^\Gamma$ From the above there is some $c^m(v)$ not positive for some $m \geq 1$
$\Rightarrow$ (by Lemma 3.25) $(C^+)^m(V) = 0$

Choose m as small as possible with $(C^+)^m(V) = 0$
Thus for some i

$$\begin{cases} S^+_{i+1}S^+_{i+2} \cdots S^+_n (C^+)^{m-1}(V) \neq 0 \\ S^+_i S^+_{i+1} \cdots S^+_n (C^+)^{m-1}(V) = 0 \end{cases}$$

(the symbol x is suppressed in above formula)
$\Rightarrow$ (by Proposition 3.22)

$$S^+_{i+1} \cdots S^+_n (C^+)^{m-1}(V) \cong S_i$$

and

$$V \cong (C^-)^{m-1} S^-_n \cdots S^-_{i+1}(S_i)$$

where S_i is the simple 1-dimensional vector space at vertex i
$\Rightarrow$ dimension vector of V is

$$c^{-m+1} w_n \cdots w_{i+1}(v_i)$$

where v_i is basis element of $\mathbb{R}^\Gamma$ associated with vertex i, this dimension vector is a positive root

Note that this argument also shows that any indecomposable with the same dimension vector as V is isomorphic to V

Step 2: (Proof of surjectivity)
Conversely, if v is a positive root
$\Rightarrow$ $c^m(v)$ is not positive for some $m \geq 1$
Choose the shortest such expression of the form $w_i \cdots w_n (w_1 \cdots w_n)^{m-1}(v)$ is not a positive root
$\Rightarrow$ $w_{i+1} \cdots w_n (w_1 \cdots w_n)^{m-1}(v) = v_i$ (using the fact about finite root system that if u is a positive root, w_x is a reflection then either $w_x(u)$ is positive or $u = v_x$)
$\Rightarrow$ the representation $(C^-)^{m-1} S^-_1 \cdots S^-_{i+1}(S_i)$ has dimension vector v □

Remark.

(1) Theorem (Kac, 1980): For an arbitrary quiver Q, the set of dimension vectors in $\mathbb{R}^\Gamma$ (Γ the underlying graph) of indecomposable representations does not depend on the orientations of the arrows.

Dimension vectors of indecomposables $\leftrightarrow$ Positive roots of the corresponding root system

Kac-Moody Lie algebras, infinite dimensional Lie algebras with root systems which are not finite

(2) There are several other important quivers arising from the representation theory of finite dimensional algebras apart from Ext-quiver, e.g. Auslander-Reiten quiver

4 (Co-)homology

Let G be any group, M, N be left $\mathbb{Z}G$-module

Definition 4.1
We can take a projective resolution of N as a left $\mathbb{Z}G$-module

$$\cdots \to P_1 \to P_0 \to N \to 0$$

P_i projecitve left $\mathbb{Z}G$-module Notation: $\mathbf{P} \to N$
this induces

$$0 \to \operatorname{Hom}(N, M) \to \operatorname{Hom}(P_0, M) \xrightarrow{\delta_0} \operatorname{Hom}(P_1, M) \xrightarrow{\delta_1} \cdots$$

and we define the $\mathbb{Z}$-module

$$\operatorname{Ext}^i_{\mathbb{Z}G}(N, M) = \frac{\ker \delta_i}{\operatorname{Im} \delta_{i-1}}$$

Remark.

(1) Standard results from homological algebra say that these $\mathbb{Z}$-modules are independent of the choice of projecitve resolution and also gets the same by taking an injective resolution of M

$$0 \to M \to I_0 \to I_2 \to \cdots$$

with each I_i injective, and consider

$$0 \to \operatorname{Hom}(N, M) \to \operatorname{Hom}(N, I_0) \to \operatorname{Hom}(N, I_1) \to \cdots$$

There are sometimes well chosen resolution which help calculations
and there are always minimal injective resolutions where for example I_0 is the injective hull of M (the minimal injective containing M as a submodule, unique up to isomorphism)
The same is not necessarily true for projective resolutions (it depends on G)

Definition 4.2
If we set $N = \mathbb{Z}$, the trivial left $\mathbb{Z}G$-module, we define the i-th cohomology group as the $\mathbb{Z}$-modules (abelian groups) (or similarly can define over kG, k field)

$$H^i(G, M) = \operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, M)$$

Remark. Hochschild cohomology for any algebra A is defined similarly for any $A - A$ bimodule M:

$$HH^i_A(M) = \operatorname{Ext}^i_{A-A}(A, M)$$

where A in the RHS is regarded as an $A - A$ bimodule. Use projective resolutions of A as a $A - A$ bimodule (terms are projective $A - A$ bimodules)
e.g. if $A = \mathbb{Z}G$, then $\mathbb{Z}G$ is a $\mathbb{Z}G - \mathbb{Z}G$ bimodule (left $\mathbb{Z}G \otimes \mathbb{Z}G^{op}$-module)
but $\mathbb{Z}G^{op} \cong \mathbb{Z}G$ (via $g \mapsto g^{-1}$) $\Rightarrow$ $\mathbb{Z}G - \mathbb{Z}G$ bimodules can be converted into left $\mathbb{Z}(G \times G)$-modules and $\mathbb{Z}G$ is a left $\mathbb{Z}(G \times G)$-module by $(g, h)a = gah^{-1}$ where $a \in \mathbb{Z}G$

Homology groups are defined as follows:
Let N be a right $\mathbb{Z}G$-module, M be a left $\mathbb{Z}G$-module
Take a projective resolution $\mathbf{P} \to M$ we tensor with M over $\mathbb{Z}G$

$$\to N \otimes_{\mathbb{Z}G} P_i \xrightarrow{\partial_i} \cdots \to N \otimes_{\mathbb{Z}G} P_0 \to N \otimes_{\mathbb{Z}G} M \to 0$$

Note: because $\mathbb{Z}G$ may be non-commutative we have to be careful about the tensor products

Definition 4.3
Define the $\mathbb{Z}$-module (abelian group):

$$\mathrm{Tor}_i^{\mathbb{Z}G}(N, M) = \frac{\ker \partial_i}{\operatorname{Im} \partial_{i+1}}$$

Remark. Different resolutions give same abelian groups and we could have taken a resolution of N instead of M to get the same

Definition 4.4
When $N = \mathbb{Z}$ is regarded as trivial right $\mathbb{Z}G$-module, define homology with coefficients in M:

$$H_i(G, M) := \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M)$$

and if we suppress M, we mean $M = \mathbb{Z}$, $H_i(G) := H_i(G, \mathbb{Z})$
Similarly for right $\mathbb{Z}G$-modules

Remark. Hochschild homology for an algebra A:

$$\begin{aligned} HH_i^A(M) &:= \mathrm{Tor}_i^{A-A}(A, M) && M \text{ as } A-A \text{ bimodule} \\ HH_i(A) &:= HH_i^A(A) && A \text{ as } A-A \text{ bimodule} \end{aligned}$$

Long exact sequences of (co-)homology

Standard from homological algebra and construction of derived functors.
If we have a ses $0 \to M \to V \to N \to 0$

Proposition 4.5
The ses induces long exact sequence

$$\cdots \longrightarrow H_1(G, M) \longrightarrow H_1(G, V) \longrightarrow H_1(G, N) \longrightarrow H_0(G, M) \longrightarrow H_0(G, V) \longrightarrow H_0(G, N) \longrightarrow 0$$

Proposition 4.6
The ses induces long exact sequence

$$0 \longrightarrow H^0(G, M) \longrightarrow H^0(G, V) \longrightarrow H^0(G, N) \longrightarrow H^1(G, M) \longrightarrow H^1(G, V) \longrightarrow H^1(G, N) \longrightarrow \cdots$$

Observe that from the definitions,

Proposition 4.7
If P is projective then $H_i(G, P) = 0 \ \ \forall i > 0$
If I is injective then $H^i(G, I) = 0 \ \ \forall i > 0$

Lemma 4.8 (Dimension Shifting)
Let $0 \to M \to V \to N \to 0$ be a ses
If V is projective then $H_i(G,N) \cong H_{i-1}(G,M) \ \ \forall i > 1$
If V is injective then $H^i(G,M) \cong H^{i-1}(G,N) \ \ \forall i > 1$

Proof
From the long exact sequences and plug in the zero (co-)homology groups □

Thus when calculating (co-)homology, at the expense of potentially making this coefficient module more complicated to understand, one can shift degrees/dimensions.

Calculation of (co-)homology

To calculate one needs to choose resolutions carefully – Gruenberg resolution and a specific case is the standard resolution

Definition 4.9
Recall the augmentation map is the ring homomorphism

$$\begin{array}{rrcl} & \epsilon : \mathbb{Z}G & \to & \mathbb{Z} \\ & g & \mapsto & 1 \\ (\Rightarrow) & \sum n_g g & \mapsto & \sum n_g \end{array}$$

Augmentation ideal $= \ker \epsilon = I_G$ (sometimes $\mathfrak{g}$)
It consists of $\sum n_g g$ with $\sum n_g = 0$, the elements can be rewritten as $\sum n_g(g-1)$
Thus I_G is the additive group generated by $\{(g-1) | 1 \neq g \in G\}$

Lemma 4.10
If F is a free group on a set X, then the augmentation ideal I_F is a free left $\mathbb{Z}F$-module on the set $\overline{X} = \{x - 1 | x \in X\}$

Proof
Let $\alpha : \overline{X} \to M$ be a mapping to some left $\mathbb{Z}F$-module M
It suffices to prove α extends to a $\mathbb{Z}F$-module homomorphism $\beta : I_F \to M$ (i.e. satisfies universal property of free module)

First let α' be the group homomorphism:

$$\begin{array}{rcl} \alpha' : F & \to & M \rtimes F \\ x & \mapsto & (\alpha(x-1), x) \end{array}$$

We have a well-defined *map* $\begin{array}{rcl} \delta : F & \to & M \\ f & \mapsto & m_f \end{array}$ with $x \mapsto \alpha(x-1)$, where $\alpha'(f) = (m_f, f)$

Next, for any $f_1, f_2 \in F$, we have

$$\begin{array}{rcl} \alpha'(f_1 f_2) & = & \alpha'(f_1)\alpha'(f_2) \\ & = & (\delta(f_1), f_1)(\delta(f_2), f_2) \\ & = & (f_1\delta(f_2) + \delta(f_1), f_1 f_2) \end{array}$$

$\Rightarrow \quad \delta(f_1 f_2) = f_1\delta(f_2) + \delta(f_1)$ (this means δ is a derivation $F \to M$)

We can construct a homomorphism of abelian group:

$$\begin{array}{rcl} \beta : I_F & \to & M \\ f-1 & \to & \delta(f) \end{array}$$

Now $\beta(x-1) = \delta(x) = \alpha(x-1)$
because $\alpha'(x) = (\alpha(x-1), x)$
$\Rightarrow \quad \beta$ is an extension of α

β is a $\mathbb{Z}F$-module homomorphism

$$\begin{array}{rcl} \beta(f(f_1-1)) & = & \beta((ff_1-1)-(f-1)) \\ & = & \beta((ff_1-1)) - \beta(f-1) \\ & = & \delta(ff_1) - \delta(f) \ = \ f\delta(f_1) \ = \ f\beta(f_1-1) \end{array}$$

□

Lemma 4.11
If F is a free group and M is any $\mathbb{Z}F$-module
then $H^i(F,M) = 0 = H_i(F,M) \ \ \forall i > 1$

Proof
The complex $\cdots \to 0 \to \cdots \to 0 \to\to I_F \to \mathbb{Z}F \xrightarrow{\epsilon} \mathbb{Z} \to 0$
is a free $\mathbb{Z}F$-module resolution of $\mathbb{Z}$ by Lemma 4.10
From the definition of (co-)homology we get the result □

The Gruenberg resolution concerns groups with presentations as F/R (R relations, F free)
and involves relative augmentation ideals
If N is a normal subgroup of a group G, we have a natural map $\mathbb{Z}G \to \mathbb{Z}(G/N)$
and $\overline{I_N} =$ ker of this map.
e.g. when $N = G$, we just get the augmentation ideal I_G

<u>Exercise</u>: Observe that $\overline{I_N} = I_N \mathbb{Z}G = \mathbb{Z}G I_N$
where $I_N =$ augmentation ideal in $\mathbb{Z}N$, i.e. the right (or left) ideal generated by $\{x-1 | 1 \neq x \in N\}$

We now generalise Lemma 4.10:

Lemma 4.12
Let R be a normal subgroup of a free group F
If R is free on X then $\overline{I_R}$ is free as a right (left) $\mathbb{Z}F$-module on $\{x-1 | x \in X\}$

Proof
Suppose $\sum_{x\in X}(x-1)a_x = 0$ where $a_x \in \mathbb{Z}F$
Choose a transversal T to the cosets R in F
$\Rightarrow \quad \mathbb{Z}F = \bigoplus_{t\in T}(\mathbb{Z}R)t$
$\Rightarrow \quad a_x = \sum_{t\in T} b_{x,t}t$ with $b_{x,t} \in \mathbb{Z}R$
$\Rightarrow \quad \sum_{t\in T}\left(\sum_{x\in X}(x-1)b_{x,t}\right)t = 0$
$\Rightarrow \quad \sum(x-1)b_{x,t} = 0$ for each $t \in T$
But I_R is free on the set of all $x-1$ by Lemma 4.10
$\Rightarrow \quad b_{x,t} = 0 \ \forall x,t \quad \Rightarrow \quad a_x = 0 \ \ \forall x$ □

Lemma 4.13
Let $1 \to R \to F \to G \to 1$ be a presentation of G
Suppose S and T are right ideals of $\mathbb{Z}F$ that are free as right $\mathbb{Z}F$-modules on X and Y respectively.
Then

(1) $S/S\overline{I_R}$ is a free $\mathbb{Z}G$-module on $\{x + S\overline{I_R} | x \in X\}$

(2) ST is free as a $\mathbb{Z}F$-module on $\{xy | x \in X, y \in Y\}$ provided T is a 2-sided ideal

Proof

(1) $S/S\overline{I_R}$ is a right $\mathbb{Z}G$-module since $\mathbb{Z}G \cong \mathbb{Z}(F/R) \cong \mathbb{Z}F/\overline{I_R}$
Since $S = \bigoplus_{x\in X} x\mathbb{Z}F$, we have

$$\begin{aligned} S\overline{I_R} &= \bigoplus_{x\in X} x\overline{I_R} \\ S/S\overline{I_R} &\cong \bigoplus_{x\in X} x(\mathbb{Z}F/\overline{I_R}) \cong \bigoplus_{x\in X} x\mathbb{Z}G \end{aligned}$$

(2) $$ST = \bigoplus_{x\in X} (x\mathbb{Z}F)T = \bigoplus_{x\in X} xT = \bigoplus_{\substack{x\in X\\ y\in Y}} xy\mathbb{Z}F$$

□

Theorem 4.14 (Gruenberg resolution)
Let $1 \to R \to F \to G \to 1$ be a presentation of G
Then there is a free right $\mathbb{Z}G$-resolution of $\mathbb{Z}$:

$$\begin{aligned} \cdots \to \overline{I_R}^n/\overline{I_R}^{n+1} \to I_F\overline{I_R}^{n-1}/I_F\overline{I_R}^n \to \overline{I_R}^{n-1}/\overline{I_R}^n \\ \cdots \to \overline{I_R}^2/\overline{I_R}^3 \to I_F\overline{I_R}/I_F\overline{I_R}^2 \to \overline{I_R}/\overline{I_R}^2 \\ \to I_F/I_F\overline{I_R} \to \mathbb{Z}G \to \mathbb{Z} \end{aligned}$$

Proof
Lemma 4.10 and Lemma 4.12 $\Rightarrow$ both I_F and $\overline{I_R}$ are free $\mathbb{Z}F$-modules (assuming we are happy that R is a free group)
Lemma 4.13 shows that $I_F\overline{I_R}^{n-1}/I_F\overline{I_R}^n$ and $\overline{I_R}^{n-1}/\overline{I_R}^n$ are free $\mathbb{Z}G$-modules

Exactness:
The kernel of $\mathbb{Z}G \to \mathbb{Z}$ is I_G the augmentation ideal
The image of $I_F/I_F\overline{I_R} \to \mathbb{Z}G$ is also I_G

The kernel of $I_F/I_F\overline{I_R} \to \mathbb{Z}G$ is $\overline{I_R}/I_F\overline{I_R}$ since $\overline{I_R}$ is the kernel of $\mathbb{Z}F \to \mathbb{Z}G$
The image of $\overline{I_R}/\overline{I_R}^2 \to I_F/I_F\overline{I_R}$ is $\overline{I_R}/I_F\overline{I_R}$
(and so on) □

We are assuming the freeness of subgroups of free groups:

Theorem 4.15 (Nielsen-Schreier)
If R is a subgroup of a free group F, then R is a free group
Moreover, if $|F : R| = n < \infty$, then R is free on $nm + 1 - m$ generators, where F is free on m generators.

Proof omitted. There is more theory about numbers of generators and relations (and presentations) of subgroups of finite index in a group G, if given a presentation for G

Standard Resolution

For any G, there is a <u>standard presentation</u> $1 \to R \to F \xrightarrow{\pi} G \to 1$ where F is free on $\{x_g | 1 \neq g \in G\}$ under which $\pi : x_g \mapsto g$

We look at the Gruenberg resolution for this presentation $\longrightarrow$ the <u>standard resolution</u>

Define $x_1 = 1$. Then $\{x_g | g \in G\}$ is a transversal to the cosets of R in F
The Schreier theory (which we are assuming) says that R is freely generated by

$$y_{g_1,g_2} := x_{g_1} x_{g_2} x_{g_1 g_2}^{-1} \qquad (1 \neq g_1 \in G)$$

So, by Lemma 4.12, the $y_{g_1,g_2} - 1$ freely generate the free $\mathbb{Z}F$-module $\overline{I_R}$
Define

$$\begin{aligned} (g_1, g_2) &:= x_{g_1 g_2} - x_{g_1} x_{g_2} \\ &= (1 - y_{g_1,g_2}) x_{g_1 g_2} \end{aligned}$$

Thus, $(g_1, g_2) \in \overline{I_R}$ and is 0 $\Leftrightarrow$ g_1 or g_2 is 1
The non-zero (g_1, g_2) freely generate the $\mathbb{Z}F$-module $\overline{I_R}$.
We define for $n > 0$ symbols

$$\begin{aligned} (g_1 | g_2 | \cdots | g_{2n}) &:= (g_1, g_2)(g_3, g_4) \cdots (g_{2n-1}, g_{2n}) + \overline{I_R}^{n+1} \\ (g_1 | g_2 | \cdots | g_{2n-1}) &:= (1 - x_{g_1})(g_2, g_3) \cdots (g_{2n-2}, g_{2n-1}) + I_F \overline{I_R}^n \end{aligned}$$

These are in $P_{2n} := \overline{I_R}^n / \overline{I_R}^{n+1}$, $P_{2n-1} := I_F \overline{I_R}^{n-1} / I_F \overline{I_R}^n$
$\Rightarrow$ setting $P_0 := \mathbb{Z}G$ we have the Gruenberg resolution for the standard presentation:

$$\cdots \to P_{2n} \to P_{2n-1} \to \cdots \to P_0 \to \mathbb{Z}$$

and all the P_i's are free $\mathbb{Z}G$-modules
the $(g_1 | g_2 | \cdots | g_{2n})$ freely generate P_{2n}
the $(g_1 | \cdots | g_{2n-1})$ freely generate P_{2n-1}
Note $(g_1 | \cdots | g_n) = 0 \Leftrightarrow$ some $g_i = 1$

Lemma 4.16
The homomorphism $\partial_n : P_n \to P_{n-1}$ int the standard resolution is given by

$$\partial_n(g_1 | \cdots | g_n) = (g_2 | \cdots | g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1 | \cdots | g_{i-1} | g_i g_{i+1} | g_{i+2} | \cdots | g_n) + (-1)^n (g_1 | \cdots | g_{n-1}) g_n$$

Proof
<u>$n = 1$</u>:

$$\begin{aligned} (g) &= 1 - x_g + I_F \overline{I_R} \\ \partial_1(g) &= 1 - g \end{aligned}$$

<u>$n = 2$</u>: $\partial_2 : P_2 \to P_1$

$$(g_1 | g_2) = x_{g_1 g_2} - x_{g_1} x_{g_2} + \overline{I_R}^2 \mapsto x_{g_1 g_2} - x_{g_1} x_{g_2} + I_F \overline{I_R}$$

But $x_{g_1 g_2} - x_{g_1} x_{g_2} = (1 - x_{g_2}) - (1 - x_{g_1 g_2}) + (1 - x_{g_1}) x_{g_2}$
$\Rightarrow$ $\partial(g_1 | g_2) = (g_2) - (g_1 g_2) + (g_1) g_2$

$\underline{n=3}$: $\partial : P_3 \to P_2$

$$(g_1|g_2|g_3) = (1-x_{g_1})(g_2,g_3) + I_F\overline{I_R}^2 \mapsto (1-x_{g_1})(g_2,g_3) + \overline{I_R}^2 = (1-x_{g_1})(g_2|g_3)$$

The formula predicts the image to be

$$\begin{aligned} & (g_2|g_3) - (g_1g_2|g_3) + (g_1|g_2g_3) - (g_1|g_2)g_3 \\ = \; & (x_{g_2g_3} - x_{g_2}x_{g_3}) - (x_{g_1g_2g_3} - x_{g_1g_2}x_{g_3}) + (x_{g_1g_2g_3} - x_{g_1}x_{g_2g_3}) - (x_{g_1g_2} - x_{g_1}x_{g_2})x_{g_3} + \overline{I_R}^2 \\ = \; & (1-x_{g_1})(x_{g_2g_3} - x_{g_2}x_{g_3}) + \overline{I_R}^2 \\ = \; & (1-x_{g_1})(g_2|g_3) \end{aligned}$$

For $n > 3$, use induction (exercise) □

Let $\mathbf{P} \to \mathbb{Z}$ be the standard resolution of $\mathbb{Z}$ as right module
M right $\mathbb{Z}G$-module
So we want to consider the homology of the complex $\mathrm{Hom}_{\mathbb{Z}G}(\mathbf{P}, M)$ over homomorphisms

$$\delta^{n+1} : \mathrm{Hom}(P_n, M) \to \mathrm{Hom}(P_{n+1}, M)$$

P_n free on the set $(g_1|g_2|\cdots|g_n)$
$\Rightarrow \quad \psi \in \mathrm{Hom}_{\mathbb{Z}G}(P_n, M)$ is determined by the values on the symbols
$\Rightarrow \quad \psi \in \mathrm{Hom}_{\mathbb{Z}G}(P_n, M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G^n, M)$ corresponds to <u>n-cochains</u>

$$\begin{aligned} \phi : \overbrace{G \times \cdots \times G}^{n \text{ times}} & \to M \\ \phi(g_1, \ldots, g_n) & = \psi(g_1|\cdots|g_n) \end{aligned}$$

and s.t. $\phi(g_1, \ldots, g_n) = 0$ if any $g_i = 1$ (note cochain is just a map of sets)

Definition 4.17
δ^{n+1} induces an action on cohains:

$$\begin{aligned} \delta^{n+1}\phi(g_1, \ldots, g_{n+1}) = \; & \phi(g_2, \ldots g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ & + (-1)^{n+1}\phi(g_1, \ldots, g_n)g_{n+1} \end{aligned}$$

$$H^n(G,M) = \frac{\ker \delta^{n+1}}{\mathrm{Im}\, \delta^n} \qquad \text{abelian group}$$

this is a finitely generated group if G is finite
$\ker \delta^{n+1}$ (often written $Z^n(G,M)$) elements are <u>n-cocycles</u>
$\mathrm{Im}\, \delta^n$ (often written $B^n(G,M)$) elements are <u>n-coboundaries</u>

e.g. $\underline{n=1}$: $\phi \in Z^1(G,M)$
$0 = \phi(g_2) - \phi(g_1g_2) + \phi(g_1)g_2$
That is $\phi(g_1g_2) = \phi(g_2) + \phi(g_1)g_2$

Remark. These δ are for cochains of right modules. g_i appear in reverse order (check) on the right hand side for left modules.

Definition 4.18
These maps from $G \to M$ are derivations
Notation: $\mathrm{Der}(G, M)$

If ϕ is a 0-cochain (i.e. a constant map a), then $\delta^1\phi(g) = a(1-g)$
$\Rightarrow$ coboundaries are maps of the form

$$\begin{array}{rcl} G & \to & M \\ g & \mapsto & a(1-g) \quad (\text{some } a \in M) \end{array}$$

called the inner derivations
Notation: $\mathrm{Inn}(G, M)$

Corollary 4.19

$$H^1(G, M) \cong \frac{\mathrm{Der}(G, M)}{\mathrm{Inn}(G, M)}$$

n=2:
If ϕ is a 2 cochain, then the 2 cocycle condition is

$$\phi(g_1, g_2g_3) + \phi(g_2, g_3) = \phi(g_1g_2, g_3) + \phi(g_1, g_2)g_3$$

such a 2 cocycle $\phi : G \times G \to M$ is often called a factor set in old fashioned books
2 cocycles are important in constructing crossed product algebras e.g. quaternions

Proposition 4.20
Let G be finite, $|G| = m$. Suppose M is any right $\mathbb{Z}G$-module
then $mH^n(G, M) = 0 \ \forall n > 0$

Proof
Let $\phi : \underbrace{G \times \cdots \times G}_{n \text{ times}} \to M$ be a n-cochain
Define an $(n-1)$-cochain by

$$\psi(g_2, \dots, g_n) = \sum_{g \in G} \phi(g, g_2, \dots, g_n)$$

Summing the induced map in Definition 4.17 over all $g_1 = g$ to get

$$\begin{array}{rcl} \sum_{g \in G} \delta^{n+1}\phi(g, g_2, \dots, g_{n+1}) & = & m\phi(g_2, \dots, g_{n+1}) \\ & & + \sum_{i=2}^{n} (-1)^i \psi(g_2, \dots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ & & -\psi(g_3, \dots, g_{n+1}) \\ & & +(-1)^{n+1}\psi(g_2, \dots, g_n)g_{n+1} \\ & = & m\phi(g_2, \dots, g_{n+1}) - \delta^n\psi \end{array}$$

$\Rightarrow$ If the n-cocycle $\phi \in Z^n(G, M)$ then LHS=0
$\Rightarrow$ $m\phi = \delta^n\psi \quad \Rightarrow \quad m\phi \in B^n(G, M)$ a coboundary
$\Rightarrow$ $mH^n(G, M) = 0$ □

Corollary 4.21
Let G be a finite group, order m
Suppose M is a right $\mathbb{Z}G$-module s.t. $\forall x \in M, \exists! y$ s.t. $my = x$ (unique m-divisibility)
Then $H^n(G, M) = 0 \ \forall n > 0$

Proof
Let n-cocycle $\phi \in Z^n(G, M)$
$\Rightarrow \quad m\phi \in B^n(G, M)$ by Proposition 4.20 (so $m\phi = \delta^n \psi$ some $(n-1)$-cochain)
Using unique divisibility in M, we can (well-)define

$$\overline{\psi} = \frac{1}{m}\psi \quad \text{composite } \psi \text{ with division by } m$$

$\Rightarrow \quad \overline{\psi}$ is a $(n-1)$-cochain and $\phi = \delta^n \overline{\psi} \in B^n(G, M)$
$\Rightarrow \quad H^n(G, M) = 0$ □

Remark.

(1) If k field of characteristic not dividing $|G|$
Then $H^n(G, M) = 0 \ \forall n > 0$ (where we are now working in kG rather than $\mathbb{Z}G$)
but this is what we expect. kG is semisimple - all modules are projective and injective

(2) However, if char $k \big| |G|$, then it is sensible to use the resolution using the theory we had earlier
Note that a (minimal) resolution of k will involve projectives associated with simples in same block as the trivial module k (i.e. the principal block)

Exercise Prove the analogous result to Proposition 4.20 and Corollary 4.21 for homology groups

Low degree (co-)homology

For left $\mathbb{Z}G$-module (and similarly for right)

Example 4.22
By definition $H_0(G, M) = \mathbb{Z} \otimes_{\mathbb{Z}G} M = M/I_G M$ the largest G trivial quotient of M (recall I_G is the augmented ideal)

Example 4.23
By definition, $H^0(G, M) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \cong \{a \in M | ga = a \ \forall g \in G\}$ set of G-fixed points
Notation: M^G

Lemma 4.24

$$H_1(G, M) \cong \ker \begin{pmatrix} I_G \otimes_{\mathbb{Z}G} M & \to & M \\ (g-1) \otimes a & \mapsto & (g-1)a \end{pmatrix}$$

Proof
Consider $0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$, inducing les:

$$\cdots \to \mathrm{Tor}_1(\mathbb{Z}G, M) \to \mathrm{Tor}_1(\mathbb{Z}, M) \to \mathrm{Tor}_0(I_G, M) \to \mathrm{Tor}_0(\mathbb{Z}G, M) \to \mathrm{Tor}_0(\mathbb{Z}, M) \to 0$$

since $\mathrm{Tor}_1(\mathbb{Z}G, M) = 0$ as $\mathbb{Z}G$ is flat, we have

$$H_1(G, M) = \mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, M) \cong \ker(I_G \otimes_{\mathbb{Z}G} M \to \mathbb{Z}G \otimes_{\mathbb{Z}G} M)$$

Then notice $\mathbb{Z}G \otimes_{\mathbb{Z}G} M \cong M$ □

Proposition 4.25
If M is a $\mathbb{Z}G$-module with G acting trivially, then

$$H_1(G, M) \cong G_{ab} \otimes_{\mathbb{Z}} M \qquad (G_{ab} = G/[G, G] = G/G' \text{ the abelianisation of } G)$$

e.g. if $M = \mathbb{Z}$ then $H_1(G) \cong G_{ab}$

For this, we prove

Lemma 4.26
For any G, $I_G/I_G^2 \cong G_{ab}$

Proof
I_G is free on $\{x-1 | 1 \neq x \in G\}$ as a $\mathbb{Z}$-module
$\Rightarrow \begin{array}{rcl} \theta : I_G & \to & G_{ab} = G/G' \\ x-1 & \mapsto & G'x \end{array}$ homomorphism of $\mathbb{Z}$-modules, but,

$$(x-1)(y-1) = (xy-1) - (x-1) - (y-1) \tag{4.1}$$

$\Rightarrow \quad I_G^2 \subseteq \ker\theta$
$\Rightarrow \quad \exists$ induced map

$$\begin{array}{rcl} \overline{\theta} : I_G/I_G^2 & \to & G_{ab} \\ (x-1) + I_G^2 & \mapsto & G'x \end{array}$$

But $\begin{array}{rcl} G & \to & I_G/I_G^2 \\ x & \mapsto & (x-1) + I_G^2 \end{array}$ is a homomorphism by equation 4.1
and it induces an inverse
$\Rightarrow \quad I_G/I_G^2 \cong G_{ab}$ $\square$

Proof of Proposition 4.25
By Lemma 4.24, $H_1(G, M) \cong I_G \otimes M$ since G acting trivially on M
But $I_G \otimes_{\mathbb{Z}G} M \cong I_G/I_G^2 \otimes_{\mathbb{Z}} M$ again because of the trivia action n M
Now apply Lemma 4.26 $\square$

We now go back to studying the cohomology group, recall $H^1(G, M) = \mathrm{Der}(G, M)/\mathrm{Inn}(G, M)$

Exericse:
Establish these directly:

$$\begin{array}{rcl} \mathrm{Der}(G, M) & \cong & \mathrm{Hom}(I_G, M) \\ \mathrm{Inn}(G, M) & \cong & \mathrm{Im}(\mathrm{res} : \mathrm{Hom}(\mathbb{Z}G, M) \to \mathrm{Hom}(I_G, M)) \end{array}$$

Alternatively, this arises from consideration of a long exact sequence associated with

$$0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

the les:

$$0 \to \mathrm{Hom}(\mathbb{Z}, M) \to \mathrm{Hom}(\mathbb{Z}G, M) \xrightarrow{\mathrm{res}} \mathrm{Hom}(I_G, M) \to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M) \to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, M) \to \mathrm{Ext}^1_{\mathbb{Z}G}(I_G, M) \to \cdots$$

But since $\mathbb{Z}G$ is projective, $\mathrm{Ext}^i(\mathbb{Z}G, M) = 0 \; \forall i > 0$
(since $\mathrm{Hom}(P, -)$ is an exact functor when P projective)
But $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M) = H^1(G, M)$
$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \cong M$
The inner derivators correspond to $\mathrm{Im}(\mathrm{res} : \mathrm{Hom}(\mathbb{Z}G, M) \to \mathrm{Hom}(I_G, M))$
There is also a group theoretic characterisation of $H^1(G, M)$

Lemma 4.27
If M is a (left) $\mathbb{Z}G$-module there is a bijection

$$\left\{\begin{array}{c}\text{ccl of}\\ \text{complements of } M\\ \text{in } M \rtimes G\end{array}\right\} \longleftrightarrow H^1(G,M)$$

$$\begin{array}{c}\text{ccl of the canonical}\\ \text{complement of } G\end{array} \longleftrightarrow 0 \in H^1(G,M)$$

Proof
If X is a complement to $M \Rightarrow \exists g \in G$ is necessarily uniquely of the form $u^{-1}x$ with $u \in M, x \in X$
Define:

$$\begin{aligned}\delta_X : G &\to M\\ g &\mapsto u \qquad \text{where } g = u^{-1}x\end{aligned}$$

$\Rightarrow \quad \delta(g)g \in X$

Let $g_1, g_2 \in G$
$\Rightarrow \quad X$ contains the element $\delta_X(g_1)g_1\delta_X(g_2)g_2$ (as X is a group)
Also $\delta_X(g_1)g_1\delta_X(g_2)g_2 = \delta_X(g_1)\delta_X(g_2)^{g_1}g_1g_2$
$\Rightarrow \quad \delta_X(g_1g_2) = \delta(g_1)(\delta_X(g_2))^{g_1} = \delta_X(g_1) + g_1\delta(g_2)$ written additively
$\Rightarrow \quad \delta_X$ is a derivation $G \to M$

Conversely, suppose $\delta : G \to M$
$\Rightarrow \quad \exists$ complement to M in $M \rtimes G$ given by $X_\delta = \{\delta(g)g | g \in G\}$. Check this is a group
$\Rightarrow \quad$ got the bijection $\{\text{complements}\} \leftrightarrow \mathrm{Der}(G,M)$

We now consider conjugacy.
Suppose X and Y are complements $X = Y^{uy_1} = (Y^{y_1})^u = Y^u$ for $y_1 \in Y, u \in M$
If $g \in G$, then $\delta_X(g)g \in X$
$\Rightarrow \quad \delta_X(g)g = y^u$ for some $y \in Y$
But $y^u = uyu^{-1} = [u,y]y$, where $[u,y] = uyu^{-1}y^{-1}$
$\Rightarrow \quad \delta_X(g)g = [u,y]y$
$\Rightarrow \quad [u,y] = [u,g] = \delta(u)(g)$ where

$$\begin{aligned}\delta(u) : G &\to M\\ g &\mapsto [u,g] = u(1-g) \qquad \text{(written additively)}\end{aligned}$$

is an inner derivation
$\Rightarrow \quad \delta_X(u)(g^{-1})\delta_x(g)g = y \in Y$
$\Rightarrow \quad \delta_Y = -\delta(u) + \delta_x$
$\Rightarrow \quad \delta_Y \equiv \delta_X \pmod{\mathrm{Inn}(G,M)}$

Reversing the argument one can show that if $\delta_Y = \delta_X - \delta(u)$
$\Rightarrow \quad \delta_X(g)g = (\delta_Y(g)g)^u$
$\Rightarrow \quad X = Y^u$ □

Corollary 4.28
If $H^1(G,M) = 0$, then all complements to M in $M \rtimes G$ are conjugate to the canonical copy of G in $M \rtimes G$

(Proof is immediate from above)

There is an analogous result for Hochschild cohomology:
If A is an algebra, M is an $A-A$ bimodule

we can define an algebra structure on $M \oplus A$ (direct sum of bimodules) with $mn = 0\ \forall m, n \in M$
$\Rightarrow \quad M$ is an ideal in $M \oplus A$ with square = zero
Note that $m + 1$ is a unit. Indeed, $M + 1 = \{m + 1 | m \in M\}$ is a multiplicative group
Then there is a bijection

$$\left\{ \begin{array}{c} \text{ccl of} \\ \text{complements of } M \\ \text{in } M \oplus A \end{array} \right\} \longleftrightarrow HH^1(A, M)$$

where conjugacy is under the group $1 + M$
Complements to M here are copies of A that have trivial interserction with M
$M \oplus A$ has multiplicative identity $0 + 1_A$
and the complements has multiplicative identity $m + 1_A$ for some $m \in M$
$\Rightarrow \quad$ if $HH^1(A, M) = 0$, then all complements are conjugate under the conjugation by $1 + M$
$\Rightarrow \quad$ happens for example if $HH^1(A, M) = 0\ \ \forall M$
i.e. when A is a projective $A - A$ bimodule
(equivalently A is a direct summand of $A \otimes A^{op}$)

Now we move on to $H^2(G, M)$

Lemma 4.29
Let $1 \to R \to F \to G \to 1$ be a presentation
Then

$$\begin{array}{rcl} R_{ab} & \xrightarrow{\sim} & \overline{I_R}/I_F\overline{I_R} \\ rR' & \mapsto & (r-1) + I_F\overline{I_R} \end{array}$$

is an isomorphism, where $R_{ab} = R/[R, R] = R/R'$

Proof
R_{ab} is a $\mathbb{Z}G$-module via conjugation: $f \cdot (rR') = (f^{-1}rf)R'\ \ \forall f \in F, r \in R$. Consider

$$\begin{array}{rcl} R & \to & \overline{I_R}/I_F\overline{I_R} \\ r & \mapsto & (r-1) + I_F\overline{I_R} \end{array}$$

If $r_1, r_2 \in R$,

$$\begin{array}{rcl} (r_1r_2 - 1) & = & -(\ (r_1 - 1) + (r_2 - 1)\ ) + (r_1 - 1)(r_2 - 1) \\ & \equiv & (r_1 - 1) + (r_2 - 1) \quad \text{mod } I_F\overline{I_R} \end{array}$$

$\Rightarrow \quad r \mapsto (r-1) + I_F\overline{I_R}$ is a group homomorphism, and it induces homomorphism $R_{ab} \to \overline{I_R}/I_F\overline{I_R}$
(Exercise: check it is a $\mathbb{Z}G$-module homomorphism)
Note that G acts on R via conjugation of F on R, this induces an action on R_{ab} as conjugation of R on R induces trivial action on R_{ab}

Now produce an inverse:
We know from Lemma 4.12:

$$\begin{array}{rcl} \overline{I_R} & = & \bigoplus_{x \in X} \mathbb{Z}F(x-1) \quad \text{free } \mathbb{Z}F\text{-module} \\ \overline{I_R}/I_F\overline{I_R} & = & \bigoplus_{x \in X} \dfrac{\mathbb{Z}F(x-1)}{I_F(x-1)} \\ & \cong & \bigoplus_{x \in X} (\mathbb{Z}F/I_F)(x-1) \\ & \cong & \bigoplus_{x \in X} \mathbb{Z}(x-1) \quad \text{free abelian group on } (x-1) + I_F\overline{I_R} \end{array}$$

$\Rightarrow \quad \exists$ group homomorphism $\begin{array}{rcl} \psi : \overline{I_R}/I_F\overline{I_R} & \to & R_{ab} \\ (x-1)+I_F\overline{I_R} & \mapsto & x \end{array}$ an inverse as required $\square$

Definition 4.30
R_{ab} is <u>relation module</u> associated with the presentation, it is a $\mathbb{Z}G$-module via conjugation within F
R_{ab} is a finitely generated $\mathbb{Z}G$-module if R is finitely generated as a F-group (i.e. as a normal subgroup of F)

Theorem 4.31 (Maclane)
Let $1 \to R \to F \to G \to 1$ be a presentation
Let M be a $\mathbb{Z}G$-module. Then we have an exact sequence

$$H^1(F,M) \to \mathrm{Hom}_{\mathbb{Z}G}(R_{ab}, M) \to H^2(G,M) \to 0$$

Remark. $H^1(F,M)$ can be non-zero

Proof
M is a $\mathbb{Z}$F-module via $\pi : F \to G$
We have

$$0 \to \overline{I_R}/I_F\overline{I_R} \to I_F/I_F\overline{I_R} \to I_G \to 0$$

So as part of the long exact sequence arising from this:

$$\cdots \mathrm{Hom}_{\mathbb{Z}G}(I_F/I_F\overline{I_R}, M) \to \mathrm{Hom}(\overline{I_R}/I_F\overline{I_R}, M) \to \mathrm{Ext}^1(I_G, M) \to \mathrm{Ext}^1(I_F/I_F\overline{I_R}, M) \to \cdots$$

but $I_F/I_F\overline{I_R}$ is a free $\mathbb{Z}G$-module
$\Rightarrow \quad \mathrm{Ext}^1(I_F/I_F\overline{I_R}, M) = 0$
Also, $\mathrm{Ext}^1(I_G, M) \cong \mathrm{Ext}^2(\mathbb{Z}, M) = H^2(G,M)$ by considering the long exact sequence arising from

$$0 \to I_G \to \underset{\text{free}}{\mathbb{Z}G} \to \mathbb{Z} \to 0$$

This is dimension shifting
By Lemma 4.29, $\overline{I_R}/I_F\overline{I_R} \cong R_{ab} \quad \Rightarrow \quad \mathrm{Hom}_{\mathbb{Z}G}(\overline{I_R}/I_F\overline{I_R}, M) = \mathrm{Hom}_{\mathbb{Z}G}(R_{ab}, M)$

Claim: $\mathrm{Hom}_{\mathbb{Z}F}(I_F, M) = \mathrm{Hom}_{\mathbb{Z}G}(I_F/I_F\overline{I_R}, M)$

Proof of Claim:
$\forall \alpha \in \mathrm{Hom}_{\mathbb{Z}F}(I_F, M)$, $\alpha(I_F\overline{I_R}) = 0$ as M is a $\mathbb{Z}G$-module and $\mathbb{Z}G \cong \mathbb{Z}F/\overline{I_R}$.
This induces $\alpha' \in \mathrm{Hom}_{\mathbb{Z}G}(I_F/I_F\overline{I_R}, M)$.
This induction is an isomorphism of groups. ■

Next, $\mathrm{Der}(F,M) \cong \mathrm{Hom}_{\mathbb{Z}F}(I_F, M)$ by remark before 4.27
$\Rightarrow \quad$ we have:

$$\mathrm{Der}(F,M) \xrightarrow{\alpha} \mathrm{Hom}_{\mathbb{Z}G}(R_{ab}, M) \to H^2(G,M) \to 0$$

Under α, derivation $\delta \mapsto \alpha(\delta) \in \mathrm{Hom}(R_{ab}, M)$
If δ is inner (i.e. $\delta(f) = u(1-f) \forall f \in F$, some $u \in M$), then $\exists u \in M$ s.t.

$$\alpha(\delta)(rR') = u(1-r)$$

But $a(1-r) = 0 \;\; \forall r \in R \quad \Rightarrow \quad \mathrm{Inn}(F,M) \subseteq \ker \alpha$
and we have an induced exact sequence

$$H^1(F,M) \to \mathrm{Hom}_{\mathbb{Z}G}(R_{ab}, M) \to H^2(G,M) \to 0$$

$\square$

Remark.

$$1 \to R_{ab} \to E \to G \to 1$$

is an extension of G by R_{ab} ("stick G on top of R_{ab}") induced by

$$1 \to R \to F \to G \to 1$$

(by $R \to R_{ab}$ and $F \to E$)
A surjective map $\beta : R_{ab} \twoheadrightarrow M$ (hence M abelian) gives an extension of G by an $\mathbb{Z}G$-module M

$$1 \to M \to E' \to G \to 1$$

since $R_{ab}/\ker\beta \cong M$

Corollary 4.32
Let M be $\mathbb{Z}G$-module, which is an abelian group, there is a bijection

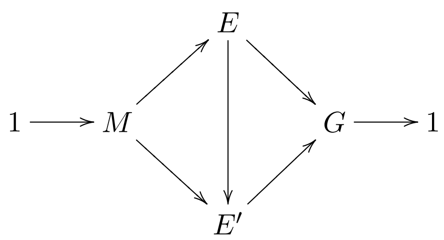
$$H^2(G,M) \longleftrightarrow \begin{array}{l}\text{equiv. class of extension of } G \text{ by } M \\ 1 \to M \to E \to G \to 1\end{array}$$

(See later)

Example
$1 \to R_{ab} \to F/[R,R] \to G \to 1$ is an extension of G by R_{ab}

Extension

Definition 4.33
Two extensions are equivalent if there is a map $E \to E'$ making



commute, such a map is necessarily an isomorphism (by Five Lemma)

For our extension:

$$1 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

as M abelian, and so can be written additively or multiplicatively
Choose a set-theoretic cross-section of π:
$s : G \to E$ s.t. $\pi s = \mathrm{id}_G$
(If s is actually a group homomorphism then the extension is split, i.e. $E \cong M \rtimes G$)

Also note that $s(G)$ is now the set of coset representatives of $i(M)$ in E. The action of G on M (regarded as a module) corresponds to conjugating $i(M)$ inside E (as a group), so we have

$$s(g)i(v)s(g)^{-1} = i(gv) \qquad \forall g \in G, v \in M$$

In general, there is a function $f : G \times G \to M$, which measures the failure of s to be a homomorphism
Note for $g,h \in G$, then $s(gh)$ and $s(g)s(h) \in E$, both map to $gh \in G$ under π

$\Rightarrow$ they differ by an element of $i(M)$ (using exactness)
$\Rightarrow$ can define $f(g,h) \in M$ by $s(g)s(h) = i(f(g,h))s(gh)$

Let suppose for simplicity $s(1) = 1$
$\Rightarrow$ $f(g,1) = 0 = f(1,g) \;\; \forall g \in G$
f is called <u>factor set</u>

Claim: Our extension can be completely determined from f and the $\mathbb{Z}G$-module structure of M

Proof of Claim:
$s(G)$ is a set of representatives for the cosets of $i(M)$ in E
We have injection:

$$\begin{array}{rcl} M \times G & \to & E \\ (u,g) & \mapsto & i(u)s(g) \end{array}$$

And surjection is easy. To compute the group law on $M \times G$ which makes the bijection into an isomorphism of groups, take $(u,g),(v,h) \in M \times G$

$$\begin{aligned} i(u)s(g)i(v)s(h) &= i(u)i(gv)s(g)s(h) \\ &= \underbrace{i(u+gv)}_{\text{writing } M \text{ additively}} \quad i(f(g,h))s(gh) \\ &= i(u+gv+f(g,h))s(gh) \\ \Rightarrow \quad (u,g)(v,h) &= (u+gv+f(g,h),gh) \end{aligned} \tag{4.2}$$

is the group law on $M \times G$

Call this group E_f
It looks like the product on $M \rtimes G$ determined by f Since $i(u) = i(u)s(1) \;\; \forall u \in M$, the composite $M \to E \cong E_f$ is the canonical inclusion $u \mapsto (u,1)$
and the composite $\begin{array}{rcl} \pi : E_f \cong E & \to & G \\ (u,g) & \mapsto & g \end{array}$ is the canonical projection.
$\Rightarrow$ the origin extension

$$1 \to M \to E \to G \to 1$$

is equivalent to

$$1 \to M \to E_f \to G \to 1$$

Note that if f is any function $f : G \times G \to M$
$\Rightarrow$ equation (4.2) defines an associative operation on $M \times G \Leftrightarrow f$ satisfies:

$$f(g,h) + f(gh,k) = gf(h,k) + f(g,hk) \;\; \forall g,h,k \in G \tag{4.3}$$

If this condition is satisfied then the operation does define a group structure on $M \times G$ (Exercise) $(0,1)$ is a 2-sided identity and $\exists$ inverses (Exercise)
Thus we have

$$\left\{ \begin{array}{c} \text{extensions with a} \\ \underline{\text{nomrlised section}} \\ \text{(i.e. } s(1)=1) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{functions } f : G \times G \to M \\ \text{satisfying (4.3) and} \\ f(g,1) = 0 = f(1,g) \end{array} \right\}$$
$$\leftrightarrow \left\{ \frac{\text{normalised 2-cocycles}}{G \times G \to M} \right\}$$

(second correspondence is due to equation (4.3) is the 2-cocycle condition) $\square$

So changing the choice of section s corresponds to modifying f by a coboundaries (Exercise)
$\Rightarrow$ this proves 4.32

Corollary 4.34
Let E be a finite group, $|E| = mn, (m,n) = 1$
Suppose M is an abelian normal subgroup of order m
Then E contains subgroups of order n and any two such are conjugate

Proof
Set $G = E/M$, M can be regarded as a $\mathbb{Z}G$-module
But $H^2(G,M) = 0 = H^1(G,M)$ by Lemma 4.20
$\Rightarrow$ all the extensions are split by 4.32
$\Rightarrow$ $\exists$ subgroups of order n (complements to M in E)
and by 4.28 all such complements are conjugate $\square$

Remark. There is an analogous result for Hochschild cohomology, looking at algebra extensions

$$0 \to M \xrightarrow{i} E \xrightarrow{\pi} A \to 0$$

where M is $A-A$ bimodule, $i(M)^2 = 0$, $i(M)$ is an ideal of E'

$$HH^2(A,M) \longleftrightarrow \begin{array}{l}\text{equiv. class of extension}\\ 0 \to M \to E \to A \to 0\end{array}$$

Here the 2-cocycle condition is equivalent to associativity on multiplication on $M \times A$ an we have a corollary that if $HH^1(A,M) = 0 = HH^2(A,M)$.
Then all such extension are split and all components to A in E are conjugate under group $1 + i(M)$

Exercise:

(1) $1 \to R \to F \to G \to 1$ be a presentation (of G) Let C be a trivial $\mathbb{Z}G$-module
Then there is an exact sequence:

$$\mathrm{Hom}_{\mathbb{Z}}(F_{ab}, C) \to \mathrm{Hom}_{\mathbb{Z}}(R/[R,F], C) \to H^2(G,C) \to 0$$

Thus each central extension of G (i.e. $1 \to C \xrightarrow{i} E \to G \to 1$ with $i(C)$ central in E) arises from a homomorphism $R/[R,F] \to C$

(2) Let G, C be abelian groups. $1 \to R \to F \to G \to 1$
Then there is an exact sequence:

$$\mathrm{Hom}_{\mathbb{Z}}(F_{ab}, C) \to \mathrm{Hom}_{\mathbb{Z}}(R/[F,F], C) \to \mathrm{Abext}(G,C) \to 0$$

where $\mathrm{Abext}(G,C)$ = equivalent classes of extension $1 \to C \to E \to G \to 1$
with E abelian and $E \cong$ subgroup of $H^2(G,C)$

$H_2(G) = H_2(G,\mathbb{Z})$ is called Schur multiplier

Lemma 4.35 (Hopf's Formula)
Let $1 \to R \to F \to G \to 1$ be a presentation

$$H_2(G) \cong (F' \cap R)/[F,R]$$

where $F' = [F,F]$ (Note that consequently RHS does not depend on presentation)

Proof
Tensor Gruenberg resolution with the trivial module $\mathbb{Z}$
$H_2(G) \cong \ker(\mathbb{Z} \otimes_{\mathbb{Z}G} \overline{I_R}/I_F\overline{I_R} \to \mathbb{Z} \otimes_{\mathbb{Z}G} I_F/I_F\overline{I_R}) = \ker(\mathbb{Z} \otimes_{\mathbb{Z}G} R_{ab} \to \mathbb{Z} \otimes_{\mathbb{Z}G} I_F/I_F\overline{I_R})$
Also $\mathbb{Z} \otimes_{\mathbb{Z}G} R_{ab} \cong R/[F,R]$ (as groups) via $n \otimes rR' \mapsto r^n[F,R]$
Similarly, $\mathbb{Z} \otimes_{\mathbb{Z}G} I_F/I_F\overline{I_R} \cong I_F/I_F^2 \cong F_{ab}$
$\Rightarrow$ $H_2(G) \cong \ker(R/[F,R] \to F_{ab} = F/[F,F])$ $\Rightarrow$ Hopf's formula $\square$

Lemma 4.36
If G is abelian, $H_2(G) \cong G \wedge G$

Proof
Exercise. Note $\wedge$ means exterior square of $\mathbb{Z}$-modules, i.e. $G \otimes G / \langle g \otimes g | g \in G \rangle$ $\square$

Exercise: $H_n(G)$ for G (f.g.) abelian

Theorem 4.37 (Universal Coefficient Theorem)
If G is a group, M a trivial $\mathbb{Z}G$-module. Then there exists an exact sequence

$$0 \to \text{Abext}(G_{ab}, M) \to H^2(G, M) \to \text{Hom}_{\mathbb{Z}}(H_2(G), M) \to 0$$

Thus every central extension give rise to a homomorphism $H_2(G) \to M$
If $G = [G, G]$ (i.e. G is a perfect group) then $H^2(G, M) \cong \text{Hom}(H_2(G), M)$

Proof
Exercise $\square$

Relative Projectivity

Definition 4.38
Let $H \leq G$, an RG-module M (where R is commutative, usually $\mathbb{Z}$ or k field) is relatively H-projective if whenever

$$\begin{array}{ccccc} & & M & & \\ & \overset{\exists\nu}{\swarrow} & \downarrow \lambda & & \\ M_2 & \xrightarrow{\mu} & M_1 & \longrightarrow & 0 \end{array}$$

commutes, with λ, μ $\mathbb{Z}G$-module maps and ν a $\mathbb{Z}H$-module map, then

$$\begin{array}{ccccc} & & M & & \\ & \overset{\exists\nu'}{\swarrow} & \downarrow \lambda & & \\ M_2 & \xrightarrow{\mu} & M_1 & \longrightarrow & 0 \end{array}$$

commutes for some ν', a $\mathbb{Z}G$-module map
Note: If $H = 1, R = k$ a field, then this agrees with projectivity of a kG-module
A short exact RG-sequence is H-split if it splits as a sequence of RH-modules

Definition 4.39
$H \leq G$ and M, M' are RG-modules. Choose a transversal T to the left cosets of H in G (T are 'coset representatives')
Define the trace or transfer:

$$\begin{aligned} \text{Tr}_{H,G} : \text{Hom}_{RH}(M' \downarrow_H, M \downarrow_H) &\to \text{Hom}_{RG}(M', M) \\ \phi &\mapsto \left(m' \mapsto \sum_{t \in T} t\phi(t^{-1}m') \right) \end{aligned}$$

where $M \downarrow_H$ means RH-module M: i.e. M restrict to H
Since ϕ is a RH-module map, $t\phi(t^{-1}m')$ only depends on the left coset tH and not the choice of the coset representatives. Thus this definition is independent of the choice of T

Remark. This depends on $|G : H| < \infty$

Lemma 4.40

(1) $\alpha \in \mathrm{Hom}_{RH}(M_1, M_2)$, $\beta \in \mathrm{Hom}_{RG}(M_2, M_1)$
Then $\beta \circ \mathrm{Tr}_{H,G}(\alpha) = \mathrm{Tr}_{H,G}(\beta \circ \alpha)$

(2) $\alpha \in \mathrm{Hom}_{RG}(M_1, M_2)$, $\beta \in \mathrm{Hom}_{RH}(M_2, M_1)$
Then $\mathrm{Tr}_{H,G}(\beta) \circ \alpha = \mathrm{Tr}_{H,G}(\beta \circ \alpha)$

(3) If $H \leq K \leq G$, then $\mathrm{Tr}_{K,G}\,\mathrm{Tr}_{H,K}(\alpha) = \mathrm{Tr}_{H,G}(\alpha)$

Proof
Trivial (exercise) □

Notation: We have just get the restriction of M to H: $M \downarrow_H$

Definition 4.41
Conversely, we have induced module: If V is RH-module, then

$$\begin{aligned} V \uparrow^G &= RG \otimes_{RH} V \\ &= \bigoplus_{t \in T} t \otimes V \end{aligned}$$

where T is a transversal to the left coset of H in G

Proposition 4.42
Let M be an RG-module, $H \leq G$ (of finite index), TFAE:

(1) M is (relatively) H-projective

(2) Every H-split surjective RG-module map $\lambda : M' \to M$ splits as a RG-module map

(3) M is a direct summand of $M \downarrow_H \uparrow^G$

(4) M is direct summand of some module induced from H

(5) (Higman's Criteria) The identity map on M is in the image of $\mathrm{Tr}_{H,G}$

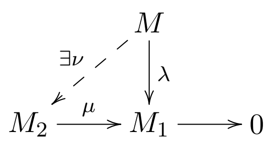
Proof
(1) ⇒ (2) ⇒ (3) ⇒ (4) obvious

(4) ⇒ (5):
If M is a direct summand of $N \uparrow^G$ for some RH-module N. Define

$$\begin{aligned} \rho : N \uparrow^G \downarrow_H &\to N \uparrow^G \downarrow_H \\ g \otimes n &\mapsto \begin{cases} g \otimes n & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

is a projection onto N as a summand of $N \uparrow^G \downarrow_H$
⇒ $\mathrm{Tr}_{H,G}(\rho) = \mathrm{id}_{N\uparrow^G}$ identity homomorphism on $N \uparrow^G$
Set $\theta : M \downarrow_H \to N \uparrow^G \downarrow_H \xrightarrow{\rho} N \uparrow^G \downarrow_H \to M \downarrow_H$
⇒ θ is an RH-endomorphism of M
⇒ (by Lemma 4.40 (1) and (2)) $\mathrm{Tr}_{H,G}(\theta) = \mathrm{id}$

(5) ⇒ (1):
Let



as in Definition 4.38. If $\theta \in \mathrm{End}_{RH}(M_1)$ with $\mathrm{Tr}_{H,G}(\theta) = \mathrm{id}$, then let $\nu' = \mathrm{Tr}_{H,G}(\nu \circ \theta)$
By Lemma 4.40 (1) and (2):

$$\begin{aligned} \mu \circ \nu' &= \mu \circ \mathrm{Tr}_{H,G}(\nu \circ \theta) = \mathrm{Tr}_{H,G}(\mu \circ \nu \circ \theta) \\ &= \mathrm{Tr}_{H,G}(\lambda \circ \theta) = \lambda \circ \mathrm{Tr}_{H,G}(\theta) = \lambda \end{aligned}$$

□

Remark. This proposition does *not* require $|G : H| < \infty$

Corollary 4.43
Suppose $|G : H|$ is invertible in R, then every RG-module M is relatively H-projective

Proof
$\mathrm{Tr}_{H,G}(\frac{1}{|G:H|}\,\mathrm{id}_M) = \mathrm{id}_M$, then apply Higman's criteria form Proposition 4.42 □

Corollary 4.44
Let k be a field of characteristic p (or $\mathbb{Z}_p$, p-adic integers)
Suppose H contains a Sylow p-subgroup of G. Then every kG-module is relatively H-projecitve
(Proof is immediate from above Corollary)

Remark. Maschke's Theorem is also a corollary of Corollary 4.43

Definition 4.45
Let M be an indecomposable RG-module, G is finite, R is a ring satisfying Krull-Schmidt.
Then D is a vertex of M if M is relatively D-projective, and not relatively D_1-projective $\forall D_1 \lneqq D$

A source of M is an indecomposable RD-module V where D is a vertex of M and M is an direct summand of the $V\uparrow^G$

Proposition 4.46 (Green)
Let M be an indecomposable RG-module

(1) The vertices are conjugate in G

(2) Let V_0 and V_1 are two sources, RD-modules.
Then $\exists g \in N_G(D)$ s.t. $V_0 \cong gV_1$, where gV_1 is a RD-module via $(\sum r_d d)(gv) = g((\sum r_d d^{g^{-1}})v)$.
(Here $d^{g^{-1}} = g^{-1}dg$)

(3) If the p'-part of $|G|$ is invertible in R, then the vertices are p-subgroups of G

Proof

(1) If M is a direct summand of $M\downarrow_H\uparrow^G$
an also M is a direct summand of $M\downarrow_K\uparrow^G$
$\Rightarrow$ M is a direct summand of $M\downarrow_K\uparrow^G\downarrow_H\uparrow^G$
But

$$M\downarrow_K\uparrow^G\downarrow_H\uparrow^G \cong \bigoplus_{HgK} M\downarrow_{H\cap K^g}\uparrow^G$$

where the sum is over double coset representatives

Note this is Mackey's Decomposition Theorem for N an RK-module:

$$N\uparrow^G\downarrow_H \cong \bigoplus_{HgK} (gN)\downarrow_{H\cap K^g}\uparrow^H$$

Hence, if H and K are both vertices the $H \cap K^g = H$
$\Rightarrow \quad H \leq K^g$
Similarly, the opposite direction $\Rightarrow H, K$ conjugate of each other

(2) M is D-projective if D is a vertex and so is a direct summand of $M \downarrow_D \uparrow^G$
But $M \downarrow_D$ is a direct sum of indecomposable RD-module, say $\bigoplus_\lambda N_\lambda$
$\Rightarrow \quad M$ is direct summand of $\bigoplus N_\lambda \uparrow^G$
But M is indecomposable and as we are supposing Krull-Schmidt applies
$\Rightarrow \quad M$ is a direct summand of one of the $N_\lambda \uparrow^G$
$\Rightarrow \quad \exists$ source V_2, RD-submodule of $M \downarrow_D$
$\Rightarrow \quad V_2$ is also direct summand of $V_0 \uparrow^G \downarrow_D$ and

$$V_0 \uparrow^G \downarrow_D \cong \bigoplus_{DgD} (gV_0) \downarrow_{D \cap D^g} \uparrow D$$

by Mackey.
$\Rightarrow \quad V_2 \cong gV_0$ for some $g \in N_G(D)$
Similarly for V_1 and hence required result

(3) Follows from Corollary 4.44

□

Theorem 4.47 (Green's Correspondence)
Let $H \leq G$ containing $N_G(D)$
There is a one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{indecomposable } RG\text{-module} \\ \text{with vertex in } D \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{indecomposable } RH\text{-module} \\ \text{with vertex in } D \end{array} \right\}$$

Thus, when the characteristic of field is p, the study of indecomposables boils down to looking at p-local subgroups (i.e. normalisers of p-groups)

Now consider kG as $G - G$ bimodule
left $kG \otimes kG^{op}$-module (notes $kG^{op} \cong kG$)
left $k(G \times G)$-module by action

$$(g, h)\xi = g\xi h^{-1} \qquad \text{for } \xi \in kG$$

Observe that $kG = k \uparrow^{G \times G}$ where k is trivial $k\Delta(G)$-module and $\Delta(G) = \{(g, g) | g \in G\} \cong G$

A block of kG is a direct summand of kG as $G - G$ bimodule, and so is a direct summand of $k \uparrow^{G \times G}_{\Delta(G)}$

Definition 4.48
One may choose a vertex for the block B, a subgroup of $\Delta(G)$ and thus it is of the form $\Delta(D)$ for $D \leq G$ and D is a p-subgroup (as vertex is p-subgroup)
D is the defect group of the block

In fact, (Green) the defect group of a block is an intersection of two Sylow p-subgroups

Remark. This section on relative projectivity and transfer does belong in cohomology chapter since trace induces a map

$$\mathrm{Tr}_{H,G} : \mathrm{Ext}^n_{RH}(M, M') \rightarrow \mathrm{Ext}^n_{RG}(M, M')$$

for $|G : H| < \infty$

Lemma 4.49
If $\alpha \in \mathrm{Ext}^n_{RG}(M, M')$, then $\mathrm{Tr}_{H,G}(\mathrm{res}_{G,H}(\alpha)) = |G : H|\alpha$

In particular for any $\alpha \in \mathrm{Ext}^n_{RG}(M, M')$ with $n > 0$, we have $|G|\alpha = 0$, i.e, Proposition 4.20 result for cohomology $\mathrm{Ext}^n(\mathbb{Z}, M)$ (Exercise for the keen)

Products

Cup products

(Notes: The following is sketch)

$$v : \mathrm{Ext}^n_{RG}(M, M') \times \mathrm{Ext}^n_{RG}(N, N') \to \mathrm{Ext}^{m+n}_{RG}(M \otimes_R N, M' \otimes_R N')$$

here $M \otimes_R N$ is a RG-module $g(m \otimes n) = gm \otimes gn$

Choose exact sequences:
$0 \to M' \to M_{m-1} \to \cdots \to M_0 \to M \to 0$
$0 \to N' \to N_{n-1} \to \cdots \to N_0 \to N \to 0$
representing the given $\xi \in \mathrm{Ext}^m(M, M')$ and $\eta \in \mathrm{Ext}^n(N, N')$
Then we take tensor product over R to get the *complex*:

$$0 \to M' \otimes N' \to \cdots \to M_0 \otimes N_0$$

where homology is $M \otimes N$ in degree 0, exact otherwise
So we get:

$$0 \to M' \otimes N' \to \cdots \to M_0 \otimes N_0 \to M \otimes N \to 0$$

and thus represents an element of $\mathrm{Ext}^{m+n}(M \otimes_R N, M' \otimes_R N')$

The cup product is graded commutative:

$$v(\xi, \eta) = (-1)^{mn} v(\eta, \xi)$$

In particular, the cohomology ring $\mathrm{Ext}^\bullet(\mathbb{Z}, \mathbb{Z})$ is a graded commutative ring, and $\mathrm{Ext}^\bullet(M, M)$ is a module (via cup product) for this cohomology ring. So one can study $\mathbb{Z}G$-modules via homological approach - look at $\mathrm{Ext}^\bullet(M, M)$ as a module in this way.
This will incorporate techniques from commutative algebra, (support) varieties, algebraic geometry (this approach is mainly done by Carlson and Benson)

In Chapter 1, we met $G_1 = \ker(SL_2(\mathbb{Z}_p) \to SL_2(\mathbb{Z}/p\mathbb{Z}))$
This is a pro-p group
Theory due to Lazard about (co-)homology in this context:
Poincaré duality: $H^i(G_1) = H_{n-i}(G)$ where we are using continuous maps and $n = 3 = \dim \mathfrak{sl}_2$

5 Central Simple Algebras

5.1 Crossed Products

We saw from Corollary 4.32:

$$H^2(G, M) \ni x \longleftrightarrow \begin{array}{l} \text{equiv. class of extension} \\ 1 \to M \to E \to G \to 1 \end{array}$$

For example, consider non-abelian groups E of order 8
We know that the centre Z is of order 2; $E/Z \cong C_2 \times C_2$
There are two isomorphism classes - dihedral and quaternion
These correspond to central extensions of $C_2 \times C_2$ by C_2

Alternatively, we can view them as extensions

$$1 \to C_4 \to E \to C_2 \to 1$$

where the C_2 is acting on C_4 by sending generators $g \mapsto g^{-1}$

$$\begin{aligned} \text{Dihedral Group} &\longleftrightarrow \text{zero element in } H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \\ \text{Quaternion Group} &\longleftrightarrow \text{non-zero element in } H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \end{aligned}$$

Remark. The following a more detailed explanation given by the typesetter, one can refer to *Abstract Algebra* by Dummit and Foote, or Pierce's book:

> Let $C_2 = \{1, h\}, C_4 = \langle g \rangle$, using the notation as in the Extension section (under Definition 4.33) $i : C_4 \to E$ and section $s : C_2 \to E$, the action of C_2 on C_4 is then:
>
> $$s(h)i(g)s(h)^{-1} = i(h \cdot g) = i(g^{-1})$$
>
> The zero element in $H^2(C_2, C_4)$ corresponding to the map sending $C_2 \times C_2$ to $1 \in C_4$, hence this gives the extension:
>
> $$D_8 = \langle \alpha, \beta | \alpha^4 = \beta^2 = 1, \beta\alpha\beta = \alpha^{-1} \rangle$$
>
> where we identified $i(g) = \alpha$, $s(h) = \beta$. The first condition comes from the normalised 2-cocycle, and the second condition comes from action of C_2 on C_4.
>
> For the other element of $H^2(C_2, C_4)$, note it suffices to find a normalised 2-cocycle satisfying coboundary condition, hence, we get the map:
>
> $$\begin{aligned} C_2 \times C_2 &\to C_4 \\ (1,1), (1,h), (h,1) &\mapsto 1 \\ (h,h) &\mapsto g^2 \end{aligned}$$
>
> This means $s(h)s(h) = g^2 s(h^2) = g^2$, and hence gives us the extension:
>
> $$Q_8 = \langle a, b | a^2 = b^2, b^{-1}ab = a^{-1} \rangle$$
>
> where we set $i(g) = a$, $s(h) = b$. So we can see the first condition comes from the 2-cocycle and the second condition, again, comes from the action of C_2 on C_4.

In general, given an extension $1 \to M \to E \to G \to 1$
We can form a group algebra kE, k field, $kM \hookrightarrow kE$ (commutative subring). In general, kE can be regarded as a crossed product of kM by G, denoted (kM, G, ϕ) with ϕ being the 2-cocycle correspond to E.

Take a maximal ideal $\mathfrak{p}$ of kM, invariant under the induced action of G on kM
$\Rightarrow \quad kM/\mathfrak{p}$ is a field, K, an extension of k, and we have maps:

$$\begin{array}{ccc} kM & \longrightarrow & kM/\mathfrak{p} \\ \downarrow & & \downarrow \\ kE & \longrightarrow & kE/(kE)\mathfrak{p} \end{array}$$

where the algebra $kE/(kE)\mathfrak{p}$ is called a crossed product of K by G.
(Note: $(kE)\mathfrak{p}$ is ideal of kE as $\mathfrak{p}$ is G-invariant)

Definition 5.1
A crossed product of a field K by a group G is an algebra of the following form:

each element is uniquely of the form $\sum \lambda_g t_g$ where $\{t_g | g \in G\} \leftrightarrow G$
such that the addition is termwise.

Multiplication ($\lambda_g \in K$):

$$\left(\sum_{g \in G} \lambda_g t_g\right)\left(\sum_{h \in G} \mu_h t_h\right) = \sum_{g,h} (\lambda_g \mu_h \phi(g,h)) t_{gh}$$

where $\phi : G \times G \to K^\times$ is a 2-cocycle and G is acting on K by k-automorphisms
(The crossed-product is sometimes termed as twisted group algebra in the literature)

Recall that 2-cocycle condition $\Leftrightarrow$ multiplication is associative
Notation: (K, G, ϕ)

Example:
$M \cong C_4 \cong \mathbb{Z}/4\mathbb{Z}, k = \mathbb{R}, G \cong C_2$
$1 \to M \to E \to G \to 1$ non-split (corresponding to ϕ) and maximal ideal $\mathfrak{p}$ of kM with $kM/\mathfrak{p} \cong \mathbb{C}$
$\mathbb{C} = K, G \cong C_2$ is acting as $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ one obtains the quaternions as a crossed product $(\mathbb{C}, \mathbb{Z}/2\mathbb{Z}, \phi)$.
(One could also obtain quaternion using $k = \mathbb{Q}$, $K = \mathbb{Q}(i)$)

One can also describe the quaternions as a crossed product of $\mathbb{R}$ by $C_2 \times C_2$ using $1 \to C_2 \to E \to C_2 \times C_2 \to 1$
but it is usual to use the previous description where $\mathbb{C}$ is a strictly maximal subfield - we will see later that central simple algebras are generally of the form (K, G, ϕ) with K strictly maximal subfield.

Definition 5.2
A k-algebra A is central simple k-algebra if

(1) A is simple (no non-trivial ideals)

(2) $\dim_k A < \infty$

(3) $Z(A) = k$

Digression 1

Consider the Heisenberg group: $\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$ upper unitriangular integral matrices
is a central extension:

$$1 \to \underset{\text{infinite cycle}}{C_\infty} \to E \to \underset{\text{free abelian rank 2}}{C_\infty \times C_\infty} \to 1$$

More generally, consider groups that are central extensions of free abelian groups (of finite rank)

$$1 \to Z \to E \to C_\infty^n \to 1$$

Embed $Z \hookrightarrow k^\times$ for field k, and so we have a canonical map $kZ \to k$ with kernel $\mathfrak{p}$, which is maximal
As before, we construct a crossed product of k by C_∞^n dependent on the 2-cocycle defining E

The crossed product is the (algberaic) quantum (n-)torus (k, C_∞^n, ϕ)
e.g. with the trivial extension - coordinate ring of the torus
'quantum' $\leftrightarrow$ 'non-commutative'

Theorem 5.3
Global dimension = $\max\{m|H^m(A,M)\neq 0$ for some left A-module $M\}$
= $\max\{$rank of B s.t. subcrossed product $(k,B,\phi|_{B\times B})$ is commutative$\}$

e.g. gl.dim. of commutative case = n
opposite extreme: e.g. for Heisenberg group, then gl.dim.=1 (hereditary ring)
These quantum torus are simple if $Z(A)=k$

Digression 2

The construction involved G-invariant maximal ideals. We can vary this to just consider G-invariant prime ideals $\mathfrak{p}$, and then take the fraction field of $kM/\mathfrak{p}$
(If $\mathfrak{p}$ maximal, then $kM/\mathfrak{p}$ is a finite field extension of k if M is finitely generated, and so G must act finitely on $kM/\mathfrak{p}$)

Theorem 5.4 (Roseblade)
$\mathfrak{p}$ is a G-invariant prime ideal of kM where M is free abelian of finite rank
If $(1+\mathfrak{p})M\cap M=1$, then $\mathfrak{p}=(\mathfrak{p}\cap kN)kM$ where $N=\{m\in M|m$ has finite orbit under $G\}$

One can repeat this in the pro-p world, $M\cong\mathbb{Z}_p^n$ to get similar result (Ardakov 2009)

(End of Digressions)

Recall from Wedderburn's Theorem 1.12 that a finite dimensional semisimple algebra is a direct sum of matrix algebra over division rings. Thus a simple algebra $\cong M_n(D)$ with D division ring. Its centre is $\{\lambda I|\lambda\in Z(D)\}$

Definition 5.5
$A^e := A\otimes A^{op}$ (e stands for envelop)

Note, A is an $A-A$ bimodule $\Rightarrow$ A is a left A^e-module

Lemma 5.6
A is simple algebra $\Leftrightarrow$ A is simple left A^e-module

Lemma 5.7
If A is a (finite dimensional) simple algebra, then $Z(A)$ is a field

Proof
Multiplying by $\lambda\in Z(A)$ gives λA ideal in A
If $\lambda\neq 0$ then $\lambda A=A$ and one has an inverse of μ (i.e. $\lambda\mu=1$)
μ has to be central □

Lemma 5.8
Let B and C be finite dimensional k-algebras, $A=B\otimes_k C$

(1) If A is simple then B and C are
(2) $Z(A)=Z(B)\otimes Z(C)$
(3) If B and C are central then A is central simple
(4) B central simple $\Rightarrow$ B^{op} is central simple

Lemma 5.9
B central simple, then $B^e = B \otimes B^{op} \cong M_n(k)$

Proof
By Lemma 5.8, B central simple $\Rightarrow$ B^{op} central simple ;$\Rightarrow$ $B \otimes B^{op}$ central simple
But there is a ring hom. $B \otimes B^{op} \to M_n(k)$ where $n = \dim_k B$, given by multiplication μ as follows

$$\begin{array}{rcl} \mu : B \otimes B^{op} & \to & \mathrm{End}_k(B) \cong M_n(k) \\ a \otimes b & \mapsto & (u \mapsto aub) \end{array}$$

$B \otimes B^{op}$ is simple $\Rightarrow$ $\ker \mu = \{0\}$
Count dimension to see we have an isomorphism □

Remark. For any finite dimensional k-algebra B, the image of $B \otimes B^{op}$ under μ is often called multiplicative algebra $M(B)$ of B

Theorem 5.10 (Jacobson-Bourbaki)
Let A be a finite dimensional simple k-algebra
For a subalgebra B of $\mathrm{End}_k(A)^{op}$ and a subalgebra D of $\mathrm{End}_k(A)$. Define:

$$\begin{array}{rcl} \alpha(B) & = & \mathrm{End}_B(A_B) \\ \beta(D) & = & \mathrm{End}_D({}_D A) \end{array}$$

Then $\alpha(B)$ is a subalgebra of $\mathrm{End}_k(A)$ and $\beta(D)$ a subalgebra of $\mathrm{End}_k(A)^{op}$
and α, β give a (mutually inverse) one-to-one correspondence

$$\left\{ \begin{array}{c} \text{subalgebras of} \\ \mathrm{End}_k(A)^{op} \\ \text{containing } M(A) \end{array} \right\} \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow[\beta]{} \end{array} \left\{ \begin{array}{c} \text{subalgebras of} \\ Z(A) \\ \text{containing } k \end{array} \right\}$$

Note if $B_1 \subseteq B_2$, then $\alpha(B_1) \supseteq \alpha(B_2)$, and similarly for β

(Proof omitted)

Morita equivalence

Recall from Definition 2.23 that basic algebra of an Artinian algebra A is $\mathrm{End}_A(\bigoplus P_i)^{op}$, where the summation is over all the projective indecomposable A-modules.

For central simple algebras, we have the following

Lemma 5.11
For two central simple k-algebras $A \cong M_n(D_1)$,$B \cong M_m(D_2)$, (so $Z(A) = k = Z(B)$) TFAE:

(1) The basic algebra of A and B are isomorphic

(2) There is a division algebra D which is also central simple

$$A \cong M_n(D) \quad , \quad B \cong M_m(D)$$

(3) $\exists r, s \in \mathbb{Z}_{>0}$ s.t.

$$A \otimes M_r(k) \cong B \otimes M_s(k)$$

Proof

(1)⇒(2)

Take $D := D_1 \cong D_2$

D is central: $k = Z(A) \cong Z(D_1)$ and $k = Z(B) \cong Z(D_2)$

D is simple: By Wedderburn, $D_1^{op} \cong \mathrm{End}_A(A)$ and $D_2^{op} \cong \mathrm{End}_B(B)$, i.e. D_1, D_2 are the basic algebras of A and B

(2)⇒(3)

$A \otimes_k M_m(k) \cong M_{nm}(D) \cong B \otimes M_n(k)$

(3)⇒(1)

If $A \otimes M_r(k) \cong B \otimes M_s(k)$

$\Rightarrow \quad M_{rn}(D_1) \cong M_{sn}(D_2)$

But uniqueness from Wedderburn $\Rightarrow D_1 \cong D_2$ □

Remark. The central simple algebras are equivalent if these are satisfied. This is in fact equivalent notion of the Morita equivalence on central simple algebras. In general, Morita equivalence is about equivalence of module categories

Definition 5.12

(For general rings) A and B are Morita equivalent if there are

$$A - B \text{ bimodule } P$$
$$B - A \text{ bimodule } Q$$

with surjective maps

$$\phi : P \otimes_B Q \twoheadrightarrow A \qquad \text{of } A - A \text{ bimodules}$$
$$\psi : Q \otimes_A P \twoheadrightarrow B \qquad \text{of } B - B \text{ bimodules}$$

satisfying

$$x\psi(y \otimes z) = \phi(x \otimes y)z$$
$$y\phi(z \otimes w) = \psi(y \otimes z)w$$

for $x, z \in P, y, w \in Q$.

(ϕ, ψ are necessarily isomorphisms. See Benson 2.2.4)

If A and B are Morita equivalent with P, Q, ψ, ϕ as above. Then by associativity of the tensor product, and by the fact that ψ, ϕ are isomorphisms:

$$Q \otimes_A - : {}_A\mathbf{mod} \rightarrow {}_B\mathbf{mod}$$
$$P \otimes_B - : {}_B\mathbf{mod} \rightarrow {}_A\mathbf{mod}(\text{category of left } A\text{-modules})$$

provides an equivalence of categories.

Exercise 1: (Benson 2.2.6) Two module categories, ${}_A\mathbf{mod}, {}_B\mathbf{mod}$ are equivalent $\Leftrightarrow \mathrm{End}_A(P)^{op} \cong B$ for some A-module P s.t. P projective and every A-module is homomorphic image of a direct sum of copies of P

(such P is called progenerator of ${}_A\mathbf{mod}$)

Exercise 2: (Benson 2.2.7) If ${}_A\mathbf{mod}$ is equivalent to ${}_B\mathbf{mod}$, then $Z(A) \cong Z(B)$

Denote Morita equivalence classes by $[\cdot]$. Thus $[M_n(D)] = [D]$, and so every equivalence class of central simple algebras has a division algebra representative.

Definition 5.13

The Brauer group $B(k)$ is the group of equiv. classes of central simple k-algebras with $[A][B] = [A \otimes B]$

Remark. Lemma 5.8 and 5.9 ensures it is a group; with $[B]^{-1} = [B^{op}]$, and identity $[B^e] = [k]$

Example: $B(k) = \{1\}$ if k is algebraically closed
Exercise: $B(\mathbb{Q})$ is an infinite group

Lemma 5.14
If $\phi : k \to K$ field homomorphism, then ϕ induces

$$\begin{aligned} \phi_* : B(k) &\to B(K) \\ [A] &\mapsto [A \otimes_\phi K] \end{aligned}$$

($\otimes_\phi$: the subscript is to emphases how K is a k-vector space) Thus we can define a functor:

$$\begin{aligned} \text{Category of fields} &\to \text{Category of abelian groups} \\ k &\mapsto B(k) \\ \phi &\mapsto \phi_* \end{aligned}$$

Note: Different ϕ yield different homomorphisms $B(k) \to B(K)$

Theorem 5.15 (Noether-Skolem)
Let A be central simple algebra, B is a simple subalgebra of A
If $\chi : B \to A$ an algebra homomorphism, then $\exists a \in A^\times$ with $\chi(b) = aba^{-1} \ \forall b \in B$

(Proof omitted)

Theorem 5.16 (Double-centraliser)
If A is a central simple k-algebra, B is a simple subalgebra of A

(1) the centraliser $C_A(B)$ is simple

(2) $(\dim_k B)(\dim_k C_A(B)) = \dim_k A$

(3) $C_A(C_A(B)) = B$

(4) If B is central simple k-algebra, then $C_A(B)$ is central simple algebra, and $A \cong B \otimes_k C_A(B)$

(Proof omitted)

Lemma 5.17
Let D be a division k-algebra.
If $x \in D$, then there is a subfield K of D with $x \in K$
If $\dim_k D < \infty$, then the subalgebra $k[x]$ generated by k and x is a subfield of D containing x

Proof
$k[x]$ is an integral domain $\quad \Rightarrow \quad k[x] \cong k[X]/\mathfrak{p}$ for some prime ideal
If $\mathfrak{p} = 0$, we get $k[X] \hookrightarrow D$
If D finite dimensional, $\mathfrak{p}$ is maximal and so $k[x]$ is a subfield of D □

Definition 5.18
k is <u>n-closed</u> if $\nexists$ proper extensions K with $|K : k| \big| n$

e.g. every field is 1-closed.
k is n-closed $\forall n \Rightarrow k$ algebraically closed

Lemma 5.19
If A is a simple finite dimensional k-algebra with maximal subfield k
Then $A \cong M_n(k)$ and k is n-closed where $n = (\dim_k A)^{1/2}$

Proof
Wedderburn $\Rightarrow A \cong M_n(D)$ for a division k-algebra D
But $D = k$ as otherwise Lemma 5.17 contradicts the maximality of k
If k is not n-closed there is a proper extension K of k with $|K : k| \big| n$
But $A \cong M_n(k)$ contains a subfield isomorphic to K, by considering $\begin{array}{ccc} K & \to & M_n(k) \\ x & \mapsto & \text{multi. by } x \end{array}$
This contradicts maximality of k □

Lemma 5.20
Let A be a central simple k-algebra, K subfield of A with $|K : k| = m$, TFAE:

(1) K is maximal subfield

(2) $C_A(K) \cong M_n(K)$ and K is n-closed

If these conditions are satisfied, $\dim_k A = (mn)^2$

Proof omitted.

Corollary 5.21
$\dim_k A = r^2$ for some r. For a subfield K of A, $|K : k| \big| r$

Definition 5.22
The degree A is

$$\operatorname{Deg} A := r = (\dim_k A)^{1/2}$$

Note if K is a subfield of central simple A then $|K : k| \leq \operatorname{Deg} A$ by Corollary 5.21
So if $|K : k| = \operatorname{Deg} A$, then K is a maximal subfield. But converse is not necessarily true (Exercise)

Definition 5.23
K is <u>strictly maximal</u> in A if $|K : k| = \operatorname{Deg} A$

Lemma 5.24

(1) K is strictly maximal $\Leftrightarrow C_A(K) = K$

(2) If A is a division algebra, then every maximal subfield is strictly maximal

Proof

(1) <u>$\Rightarrow$</u>: Lemma 5.20
<u>$\Leftarrow$</u>: Follows from Double-centraliser theorem 5.16:
$C_A(K) = K \Rightarrow \dim_k A / \dim_k K = \dim_k C_A(K) = \dim_k K$

(2) If K is a maximal subfield of a division algebra A
$\Rightarrow$ $M_n(K) \cong C_A(K) \leq A$ (by Lemma 5.19)
$\Rightarrow$ $n = 1$ since A has no nilpotent elements.
$\Rightarrow$ $C_A(K) = K$ and K is strictly maximal

□

<u>Exercise</u>:
k field of characteristic $\neq 2$. A central simple k-algebra with $\operatorname{Deg} A = 2$. Then $A \cong$ quaternion algebra

<u>Exercise</u>:
The only finite dimensional non-commutative division $\mathbb{R}$-algebra is $\mathbb{H}$. Hence $B(\mathbb{R}) \cong \mathbb{Z}/2\,\mathbb{Z}$

Definition 5.25
A splitting field K of central simple algebra A is a field extension K of k where $A \otimes_k K \cong M_n(K)$, where $n = \operatorname{Deg} A$

Thus if $\phi_* : B(k) \to B(K)$ as in Lemma 5.14
K splitting field $\Leftrightarrow [A] \in \ker \phi_*$

Notation: $B(K/k) = \ker \phi_*$ relative Brauer group

Exercise: Every maximal subfield of central simple k-algebra A is a splitting field for A

Up to equivalence, the converse is true:

Lemma 5.26
A central simple k-algebra, $|K : k| < \infty$. TFAE:

(1) K is a splitting field for A

(2) $\exists B$ central simple s.t. $[B] = [A]$ and K is strictly maximal subfield of B

(3) $\exists B$ central simple s.t. $[B] = [A]$ and K is maximal subfield of B

Proof omitted

Lemma 5.27
If D is a central simple division k-algebra
If K is a subfield of D, maximal w.r.t. separable extension K/k
Then K is strictly maximal subfield of D

Proof of this is based on the fact if all subfield of D were purely inseparable extensions then $D = k$

Theorem 5.28
If A central simple k-algebra then $\exists B$ central simple and a strictly maximal subfield L of B s.t. $[B] = [A]$ and L is a (finite) Galois extension of k

Proof
$[A] = [D]$ for some division k-algebra D
By Lemma 5.27, D has strictly maximal subfield such that K/k is separable
Then let L be a Galois extension of k containing K
Since K splits D (by exercise), so does L
Theorem now follows from Lemma 5.26 $\square$

Corollary 5.29

$$B(k) = \bigcup_{\substack{L/k \text{ finite} \\ \text{Galois}}} B(L/k)$$

every element of $B(L/k)$ has form $[A]$ where A (unique up to isom.) contains L as strictly maximal subfield

Theorem 5.30
Let A be central simple k-algebra containing strictly maximal subfield L with L/k Galois extension with $\operatorname{Gal}(L/k) = G$. Then:

$$A \cong (L, G, \phi) \qquad \text{crossed product}$$

for some 2-cocycle $\phi : G \times G \to L^\times$

Sketch
$\exists$ a set $\{t_g | g \in G\} \subseteq A$ s.t. $\forall l \in L, g \in G$, $l^g = t_g^{-1} l t_g$ by Noether-Skolem Theorem 5.15

Then we prove $\{t_g | g \in G\}$ are linearly independent and note that the strict maximality of L in A implies $\dim_L A = |L : k| = |G|$
so that $\{t_g | g \in G\}$ form an L-vector space basis of A

Let $\phi(g, h) = t_g t_h (t_{gh})^{-1} \in L^\times$
One can check it is a 2-cocycle □

Corollary 5.31
The following mapping is surjective

$$\begin{array}{rcl} \{\text{2-cocycle}\} & \twoheadrightarrow & B(L/k) \\ \phi & \mapsto & [(L, G, \phi)] \end{array}$$

Theorem 5.32
If L/k is Galois extension with $\mathrm{Gal}(L/k) = G$, then the following map is an isomorphism of abelian groups

$$\begin{array}{rcl} H^2(G, L^\times) & \xrightarrow{\sim} & B(L/k) \\ [\phi] & \mapsto & [(L, G, \phi)] \end{array}$$

Proof
Check that the kernel of the map in Corollary 5.31 consist of a coboundary

$(L, G, \phi) \cong (L, G, \psi) \Leftrightarrow \phi(g, h)\psi(g, h)^{-1} = \theta(g)\theta(gh)^{-1}\theta(h)$ for some $\theta : G \to L$

Abelian group homomorphism: $[(L, G, \phi)] \otimes [(L, G, \psi)] = [(L, G, \phi\psi)]$ □

6 Supplementary Material

6.1 Example of Group Algebra

$G = S_3 = \langle g = (12), h = (132) \rangle$
For char k=0 (or $p \neq 2, 3$), kS_3 semisimple, there are 3 conjugacy classes in S_3
$\Rightarrow \quad \leq 3$ irreducible modules (equality if k algebraically closed)

$(S_3)_{ab} = S_3/S_3'$, order 2 and there are two 1-dimensional irreducible:

$$\begin{array}{rcl} U_1 & = & \text{trivial module} \\ U_2 & = & \text{1-dimensional module, } \begin{array}{rcl} g & \mapsto & -1 \\ h & \mapsto & 1 \end{array} \\ U_3 & = & \text{2-dimensional module} \\ & & g \text{ acts like } \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \\ & & h \text{ acts like } \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \end{array}$$

These are all with entries in prime subfield $\Rightarrow$ these three irreducibles are defined for all such field.

$$kG \cong \underbrace{M_1(k)}_{U_1} \oplus \underbrace{M_1(k)}_{U_2} \oplus \underbrace{M_2(k)}_{\substack{U_3 \\ \text{2 copies}}}$$

Now consider $k = \mathbb{F}_3$, number of p-regular ccl is 2 $\Rightarrow$ expect 2 irreducibles
Take $S_1 = \overline{U_1}$ arising from U_1 mod 3.
Take $S_2 = \overline{U_2}$ arising from U_2 mod 3.
Think about $\overline{U_3}$, 2-dimensional, $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow \quad \begin{aligned} g \sim T \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} T^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\ h \sim T \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} T^{-1} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$\Rightarrow$ get 1-dimensional subspace, eigenspace for both g, h

$$\Rightarrow \qquad 0 \to S_2 \to \overline{U_3} \to S_1 \to 0$$

This sequence is non-split since there is no copy of S_1 in $\overline{U_3}$
We have projective indecomposable associated to S_1, S_2:

$$\mathbb{F}_3\, S_3 / \underbrace{J(\mathbb{F}_3\, S_3)}_{\text{4-dimensional}} \cong \underbrace{M_1(\mathbb{F}_3) \oplus M_1(\mathbb{F}_3)}_{\text{2-dimensional}}$$

$\mathbb{F}_3\, S_3$ is splitting as a direct sum of these two P_1, P_2:

P_1 = projective indecomposable associated with S_1 has composition factor (from top): S_1, S_2, S_1
$\text{Rad}(P_1) = \overline{U_3}$
Dual of our non-split extension is reducible with S_2 on top.

P_2 = projective indecomposable associated with S_2 has composition factor (from top): S_2, S_1, S_2

Both P_1, P_2 are 3-dimensional, $\mathbb{F}_3\, S_3 = P_1 \oplus P_2$. There is only one block by Proposition 2.27

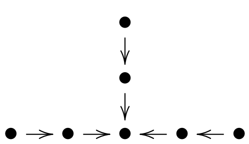
Vertex of indecomposable P_1 is trivial subgroup $\{1\}$ and source $\mathbb{F}_3$, so is P_2. Because P_i are direct summand of $\mathbb{F}_3 \uparrow_{\{1\}}^{S_3}$

Indecomposable $\overline{U_3} = \mathbb{F}_3 \uparrow_{\langle h \rangle}^{S_3}$ (as h acts trivially on $\overline{U_3}$)
$\Rightarrow$ vertex $= \langle h \rangle$, source = trivial module

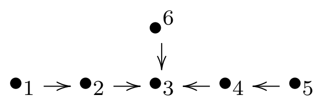
Note that $S_3 = N_{S_4}(\langle h \rangle)$
$\Rightarrow$ Green correspondence tells us that $\overline{U_3}$ correspond to an indecomposable of $\mathbb{F}_3\, S_4$ with vertex $\langle h \rangle$

6.2 Example on Quiver

Q' :



Q :



From Pierce: Representation (kQ-module) V of Q at vertex x is rigid, if any automorphism of the representation is scalar on V_x where x is a vertex on Q
(Recall V_x = subspace associated with vertex x, and $V = \bigoplus_x V_x$)

Lemma 6.1
If V is representation for Q, rigid at x, $\dim V_x > 1$, and we form a new quiver Q' with one extra vertex y, and extra arrow ${}_y\bullet \to \bullet_x$
Then Q' has infinite representation type

Proof
Form representation V'_ϕ of Q' by taking the same subspaces and maps as for V, and associate 1-dimensional space V_y with arrow ${}_y\bullet \to \bullet_x$ represented by $\phi : V'_y \to V_x$

Rigidity at x of $V \Rightarrow$ If $V'_\phi \cong V'_{\phi_1}$, then ϕ, ϕ_1 are scalar multiples of each other
$\Rightarrow$ have infinitely many isomorphism classes of such representations (assuming base field infinite)

But they all have same dimension (vector) $= d$
Krull-Schmidt decomposition $\Rightarrow \exists$ infinitely many indecomposable of dimension $\leq d$ $\square$

So, for our Q, Q' above, we want to show that there exists representation of Q with rigidity at 6.

Take W as the 3-dimensional space with basis e_1, e_2, e_3

$$\begin{aligned}
V_1 &= \langle e_1 \rangle \\
V_2 &= \langle e_1, e_2 \rangle \\
V_3 &= W \text{ with arrows represented by canonical embedding} \\
V_4 &= \langle e_2, e_3 \rangle \\
V_5 &= \langle e_3 \rangle \\
V_6 &= \langle e_1 + e_2, e_2 + e_3 \rangle
\end{aligned}$$

Now we get:

$$\begin{aligned}
V_2 \cap V_4 &= \langle e_2 \rangle \\
V_6 \cap V_2 &= \langle e_1 + e_2 \rangle \\
V_6 \cap V_4 &= \langle e_2 + e_3 \rangle
\end{aligned}$$

If θ is an automorphism of this V
$\Rightarrow \quad \theta_i = \psi|_{V_i}$ (the induced automorphism at vertex i), where $\psi =$ automorphism of W

Any automorphism of W leaving the subspaces invariant is a scalar multiple of identity
$\Rightarrow$ our representation is rigid at all vertex
$\Rightarrow$ by last lemma, Q' is of infinite representation type and we have infinitely many indecomposables arising as summands of our V'_ϕ as constructed in the Lemma

Try something similar for

$$Q \; : \quad \begin{array}{c} \bullet \\ \downarrow \\ \bullet \to \bullet \to \cdots \to \bullet \to \bullet \leftarrow \bullet \leftarrow \bullet \end{array}$$

(number of arrows on the left side of the junction ≥ 2)

Similar argument shows that why we get finite representation type from finite Dynkin diagram, but not for any extended Dynkin/Euclidean diagram.

7 Exercises

This section list the exercises that occurred in the notes. Some solution will be presented, correctness is not guaranteed. These uses the following standard references recommended by the lecturer:

(1) Alperin, *Local Representation Theory*, CUP CSAM Series

(2) Benson, *Representations and Cohomology vol. I*, CUP CSAM Series 30

(3) Pierce, *Associative Algebras*, Springer

(4) Robinson, *A Course in the Theory of Groups*, chapter 11, Springer

and the following reference used by the typesetter

(1) Auslander, Reiten, Smalø, *Representation Theory of Artin Algebras*, CUP CSAM Series 36

(2) Assem, Simson, Skowronski, *Elements of the Representation Theory of Associative Algebras, vol. 1*, CUP LMS Student Text 65

which will be abbreviated as ARS and ASS respectively.

Exercise 7.1
Let $A = \left\{ \begin{pmatrix} q & r \\ 0 & s \end{pmatrix} : q \in \mathbb{Q}, r, s \in \mathbb{R} \right\}$
Show that it is not left Artinian but is right Artinian, and similarly for Noetherian.

Solution
We will use the following lemma:

Claim: Let $R = \begin{pmatrix} S & B \\ 0 & T \end{pmatrix} = \left\{ \begin{pmatrix} s & b \\ 0 & t \end{pmatrix} : s \in S, b \in B, t \in T \right\}$.
Then R is (left) right Noetherian $\Leftrightarrow$ S, T are (left) right Noetherian and $({}_SB)$ B_T is finitely generated.

Proof of Claim:
$\underline{\Leftarrow}$:
S, T right Noetherian
$\Rightarrow \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \cong S \times T \Rightarrow$ right Noetherian.
If $b_1, \ldots, b_n$ generate B as right T-module

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix}$$

generate R as right $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$-module
$\Rightarrow$ R right Noetherian

$\underline{\Rightarrow}$
Suppose R is left Noetherian, consider the projection maps

$$\begin{array}{ccc} R & \twoheadrightarrow & S \\ \begin{pmatrix} s & b \\ 0 & t \end{pmatrix} & \mapsto & s \end{array} \qquad \begin{array}{ccc} R & \twoheadrightarrow & S \\ \begin{pmatrix} s & b \\ 0 & t \end{pmatrix} & \mapsto & t \end{array}$$

$\Rightarrow \quad S, T$ left Noetherian

$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ is left ideal of R and have list of generators

$$\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad b_1, \dots, b_n$ generate ${}_S B$. ■

Given the claim, apply with $S = \mathbb{Q}, T = \mathbb{R}, B = \mathbb{R}$. Now both S, T are left and right Noetherian. Also, $\mathbb{R}$ is finitely generated as right $\mathbb{R}$-module, but not finitely generated as left $\mathbb{Q}$-module. Hence the claim implies A is right Noetherian but not left Noetherian.

Similar result can be done for Artinian property. □

Exercise 7.2
M is f.g. module over Artinian ring $A \Rightarrow M$ satisfies DCC on submodules

Solution
See Commutative Algebra course notes, section Chain Condition. □

Exercise 7.3
A left Noetherian $\Rightarrow$ any left inverse is necessarily a right inverse

Exercise 7.4
Prove Schur's Lemma

Exercise 7.5
Show that $G = SL(2, p)$ has exactly p ccls of elements of order not divisible by p

Solution
We will use the following facts:

(1) $\forall x \in G, \exists! y, z \in G$ s.t. $x = yz = zy$ with $y = p$-regular element (i.e. p'-elements) and $z = p$-singular element
(Note this result is true for any group in general)

(2) For G a group of matrix, the p-singular elements are the matrices with eigenvalue being 1

(3) Matrix has order prime to $p \Leftrightarrow$ it is diagonalizable

Using the above facts, the following matrices are the one we are interested in:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad (\lambda \in \mathbb{F}_p^\times)$$

$$z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad d = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad b = \text{element of order } p+1 \text{ not diagonalisable over } \mathbb{F}_p^\times$$

The following are the ccls and their size:

ccl	1	z	a^i	b^i	c	d	zc	zd
size	1	1	$p(p+1)$	$p(p-1)$	$(p^2-1)/2$	$(p^2-1)/2$	$(p^2-1)/2$	$(p^2-1)/2$

For $a^i, i = 1, \dots (p-3)/2$
For $b^i, i = 1, \dots (p-1)/2$
First four ccls are p-regular, the last four are p-singular □

Exercise 7.6
Exercises under the subclaim in the $SL(2,p)$ section:

(1) Check that W_i is $k\langle g\rangle$-module

(2) Check W_i/W_{i-1} is 1-dimensional trivial $k\langle g\rangle$-module

(3) Complete the induction part of proof

Solution

(1) $g \cdot X^i Y^{r-i} = (X+Y)^i Y^{r-i} = (X^i + \cdots + Y^i)Y^{r-i} = X^i Y^{r-i} + \cdots + Y^r \in W_i$

(2) $g \cdot X^i Y^{r-i} \equiv X^i Y^{r-i} \mod W_i$, by above. Hence 1-dimensional and $k\langle g\rangle$-trivial.

(3) Let $w \in W_i \setminus W_{i-1}, w = \alpha X^i Y^{r-i} + w'$ some $w' \in W_{i-1}$
Apply induction on i, for $i = 1$, $W_1 = \langle Y^r\rangle$, so clear as any elements are of form αY^r
For $i > 1$, using the two facts from above,

$$(g-1)w = \underbrace{\alpha(g-1)X^i Y^{r-i}}_{W_{i-1}\setminus W_{i-2}} + \underbrace{(g-1)w'}_{\in W_{i-1}}$$

But $(g-1)w \in W_{i-2}$ as W_{i-1}/W_{i-2} is trivial module
$\Rightarrow \quad (g-1)w \in W_{i-1} \setminus W_{i-2}$, now invoke induction hypothesis.

□

Exercise 7.7
Show that for matrix algebra A, $[A, A]$ has codimension 1 and consist all the matrices of trace zero

Solution
$[A, A] \subseteq \{$ trace zero $\}$ is clear. To prove the other side, write E_{ij} as the standard basis for the matrix algebra, where the (i,j)-th entry is 1, and 0 elsewhere. We have the following,

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} = \begin{cases} E_{il} & j = k, i \neq l \\ -E_{kj} & i = l, k \neq j \\ E_{ii} - E_{jj} & j = k, i = l \end{cases}$$

Third line gives the required result. □

Exercise 7.8
Prove that $(a+b)^p \equiv a^p + b^p \mod [kG, kG]$, $\forall a, b \in kG$

Solution
Obvious as $p \equiv 0$ in k and $[kG, kG]$ is a k-algebra. □

Exercise 7.9
A is local $\Leftrightarrow$ the non-invertible element form a left ideal $\mathfrak{m}$

Solution
$\underline{\Leftarrow}$: Any non-trivial ideal must not contain unit, so is always contained in $\mathfrak{m}$
$\underline{\Rightarrow}$: For all non-units $x \in A$, have $(x) \subseteq \mathfrak{m}$ the unique maximal ideal of A, union of all such ideals is $\mathfrak{m}$ □

Exercise 7.10
A is local $\Rightarrow A/J(A)$ division ring

Solution
A local $\Rightarrow$ {non-units} $= J(A)$ =maximal ideal $\Rightarrow$ for all non-zero $a \in A \setminus J(A)$, $a \in A^\times \Rightarrow a + J(A)$ has inverse ☐

Exercise 7.11
Prove Proposition 2.9 on the equivalence notion of projective modules

Solution
See commutative algebra notes. ☐

Exercise 7.12
Show $A_1 = \mathrm{End}_A(\bigoplus P_i)^{op}$ is a basic algebra, where the direct sum is over all the distinct indecomposable projective A-modules.

Solution
Let $e = \sum_i e_i$ with each e_i being the orthogonal idempotents in A correspond to the projective A-modules, one each from distinct isomorphism classes. So Lemma 1.14 says $A_1 \cong eAe = \bigoplus_{i,j} e_i A e_j$ (direct sum as e_i, e_j primitive)

$$\Rightarrow \quad A_1/J(A_1) \cong \bigoplus_{i,j} e_i A e_j / e_i J(A) e_j \cong \bigoplus_{k=1}^{r} \Delta_k$$

where $\Delta_i \cong \mathrm{End}_A(S_i)^{op}$. Lift the result back to A_1. ☐

Remark. See Benson 2.2.1; ASS I.6; ARS II.2

Exercise 7.13
Let G be the linear affine group $\begin{pmatrix} k^\times & k \\ 0 & 1 \end{pmatrix}$.

Show that all the simple kG-modules are 1-dimensional and construct the Ext^1-quiver. Try the same for triangular group

Solution
Details are needed. The Ext^1-quiver is an oriented cycle with each vertex correspond to each of the 1-dimensional modules. ☐

Exercise 7.14
(Hard) $G = SL(2, p)$, k algebraically closed with characteristic p
For $p = 2$, decompose kG as direct sum of blocks
For p odd, show V_p is projective and the simples that associated to the blocks are (1) V_p, (2) $V_1, V_3, \ldots, V_{p-2}$, (3) $V_2, V_4, \ldots, V_{p-1}$

Remark. See Alperin for solution

Exercise 7.15
Let Q be a quiver with n vertices and oriented cycle. Show that, for k finite, kQ has infinite representation type.

Solution
To mimic what we did in the k infinite case, we use the fact that there are infinitely many finite extension of k. Let K be a finite extension of k, and $\lambda \in K$, then we do exactly what we did in the infinite case.

Let each vector space correspond to each vertex be a copy of K, notice since this is a finite field extension, so K is a finite dimensional vector space over k.
Then assign maps $e_i \mapsto e_{i+1}$ to ${}_i\bullet \to \bullet_{i+1}$ where e_i are basis the vector space at $i = 1, \dots, n-1$.
Assign the map $e_n \mapsto \lambda e_1$ to ${}_n\bullet \to \bullet_1$

This representation is indecomposable. We have infinitely many such representation over k as there are infinitely many finite extension of k. □

Exercise 7.16
Prove that $\bullet \rightrightarrows \bullet$ has infinite representation type.

Solution
Let the both vertex be k, one arrow be identity, the other be multiplication by $\lambda \in k^\times$. This is indecomposable, but for k infinite, this has infinitely many such representation. □

Remark. This is called the Kronecker quiver and is a very interesting object in the study of quiver. Any literature on quiver theory (e.g. ASS, ARS) has detailed exposition on the subject.

Exercise 7.17

(1) Let Q, Q' be two quivers with same tree as underlying graph, show that there exists some choice of $j_1, \dots, j_s$ s.t. $s_{j_1} \cdots s_{j_s} Q = Q'$

(2) Deduce that if the underlying graph of a quiver is a tree, then representation type is independent of orientation of arrows

Solution

(1) Combinatorics.

(2) Applying s_j correspond to applying functor S_j^- to the kQ-modules, its effect on the dimension is reflection by an element of the Weyl group, correspond to j. Hence representation type stays the same.

□

Exercise 7.18
Show that $\overline{I_N} = I_N \mathbb{Z}G = \mathbb{Z}GI_N$, where I_N is the augmentation ideal in $\mathbb{Z}N$

Solution
$\overline{I_N} = \ker(\mathbb{Z}G \to \mathbb{Z}(G/N))$, the mapping is:

$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g_i \in T} \left(\sum_{g \in g_i N} \alpha_g \right) g_i N$$

where T is transversal set.
So for $x \in \overline{I_N}$, $\sum_{g \in g_i N} \alpha_g = 0 \ \ \forall g_i \in T$ This is equivalent to $y \in I_N \cong I_{g_1 N} \cong I_{g_2 N} \cong \cdots$

I_N is generated by $\{h - 1 | h \in N\}$ over $\mathbb{Z}$
I_{gN} is generated by $\{(h-1)g | h \in N\}$ over $\mathbb{Z}$, for all $g \in T$
$\Rightarrow \quad \overline{I_N}$ generated by $\{(h-1)g | h \in N, g \in T\}$ □

Exercise 7.19
Complete the proof of Lemma 4.16

Remark. See Robinson

Exercise 7.20
Prove the analogous result to Proposition 4.20 and Corollary 4.21 for homology groups

Exercise 7.21
Check the induced map

$$\begin{array}{rcl} \theta : R_{ab} & \to & \overline{I_R}/I_F\overline{I_R} \\ rR' & \mapsto & (r-1) + I_F\overline{I_R} \end{array}$$

is a $\mathbb{Z}G$-homomorphism.

Solution
Note that G acts on R via conjugation in F, so the induced action on R_{ab} is $\pi(f)\cdot rR' = f^{-1}rfR' \ \ \forall f \in F, r \in R$
Let the standard presentation be $1 \to R \xrightarrow{\iota} F \xrightarrow{\pi} G \to 1$

$$\begin{array}{rcl} \theta(\pi(f)\cdot rR') & = & \theta(f^{-1}rfR') \ = \ (f^{-1}rf - 1) + I_F\overline{I_R} \\ & = & f^{-1}(r-1)f + I_F\overline{I_R} \ = \ \pi(f)\cdot(r-1) + I_F\overline{I_R} \\ & = & \pi(f)\cdot(\theta(rR')) \end{array}$$

□

Exercise 7.22
Show that if $f : G \times G \to M$ is normalised, then f defines a group structure on $M \times G$

(1) 2-cocycle condition gives associativity

(2) Find the 2-sided identity and inverse of non-split extension given by normalised 2-cocycle f

(3) Show that changing the choice of section s corresponds to modifying f by a coboundary

Solution

(1) Already know $(u,g)(v,h) = (u + gv + f(g,h), gh)$, so

$$\begin{array}{rcll} ((u,g)(v,h))(w,k) & = & (u + gv + f(g,h), gh)(w,k) & \\ & = & (u + gv + ghw + f(g,h) + f(gh,k), ghk) & \\ & = & (u + gv + ghw + f(g,hk) + gf(h,k), ghk) & \text{(2-cocycle condition)} \\ & = & (u + g(v + hw + f(h,k)) + f(g,hk), ghk) & \\ & = & (u,g)((v,h)(w,k)) & \end{array}$$

(2) Since we have write M additively, G multiplicatively, identity is (0,1). Using the formula $(u + gv + f(g,h), gh)$ again, we know the inverse of (u,g) is $(g^{-1}(-u), g^{-1})$

(3) Let s' be another section. Then $s'(g) = xs(g)$ for some $x \in \ker\pi = i(M)$
So we have a 1-cochain $\phi : G \to M$ sending g to $x \in M$ (identifying M and $i(M)$)
Apply the boundary map to see ϕ is a 2-coboundary: $\partial^2\phi(g_1,g_2) = g_1\phi(g_2) - \phi(g_1g_2) + \phi(g_1)$

□

Exercise 7.23
Let $1 \to R \to F \to G \to 1$ be a presentation of G, C be a trivial $\mathbb{Z}G$-module. Show that there exists exact sequence

$$\mathrm{Hom}_{\mathbb{Z}}(F_{ab}, C) \to \mathrm{Hom}_{\mathbb{Z}}(R/[R,F], C) \to H^2(G,C) \to 0$$

Solution
Let $1 \to C \to E \to G \to 1$ be the (central) extension corresponding to $\theta \in H^2(G,C)$
G acts trivially on $C \Rightarrow F$ acts trivially on C

$$\Rightarrow \quad H^1(F,C) = \mathrm{Der}(F,C) = \mathrm{Hom}_{\mathbb{Z}}(F,C) \cong \mathrm{Hom}_{\mathbb{Z}}(F_{ab},C)$$

The first equality is by the condition that F acts trivially on C
The second equality is because of trivial action of F implies $\phi(f_1 f_2) = \phi(f_1)f_2 + \phi(f_2) = \phi(f_1) + \phi(f_2)$, so the cocycle condition becomes the condition for group homomorphism.
The last isomorphism comes from the fact that every group homomorphism $F \to C$ has kernel containing $[F,F]$, so each $\phi \in \mathrm{Hom}_{\mathbb{Z}}(F,C)$ induces $\phi' \in \mathrm{Hom}_{\mathbb{Z}}(F/[F,F],C)$ and this is a isomorphism.

By a similar reason, i.e. $[R,F]/[R,R] \subseteq \ker \phi \ \ \forall \phi \in \mathrm{Hom}_{\mathbb{Z}G}(R_{ab},C)$,

$$\Rightarrow \quad \mathrm{Hom}_{\mathbb{Z}G}(R_{ab},C) \cong \mathrm{Hom}_{\mathbb{Z}}(R/[F,R],C)$$

Put the terms back into MacLane Theorem to get desired result. □

Remark. See Robinson.

Exercise 7.24
Let $1 \to R \to F \to G \to 1$ be a presentation of G, with G and C being abelian group, and C is a trivial $\mathbb{Z}G$-module. Show that there exists exact sequence

$$\mathrm{Hom}_{\mathbb{Z}}(F_{ab},C) \to \mathrm{Hom}_{\mathbb{Z}}(R/[F,F],C) \to \mathrm{Abext}(G,C) \to 0$$

Solution
$\mathrm{Abext}(G,C)$ is a subgroup of $H^2(G,C)$ in the previous exercise. So we need to find a condition which ensures the extension is abelian. For this we first describe explicitly what an extension is. By MacLane's Theorem, each $[\theta] \in H^2(G,C)$ arises from $\phi \in \mathrm{Hom}_{\mathbb{Z}G}(R_{ab},C)$, which correspond to $\phi' \in \mathrm{Hom}_{\mathbb{Z}F}(R,C)$
The extension section tells us that extension correspond to (equivalence class of 2-cocycle) θ is of the form

$$1 \to C \to F \times C/\phi'(R) \to G \to 1$$

Claim: $E = F \times C/\phi'(R)$ is abelian $\Leftrightarrow F' \leq \phi'(R)$

Proof of Claim:
<u>$\Leftarrow$</u>:
Let $f \in F', r \in R$. If $f = \phi'(r) = \phi(r)r$, then $\phi(r) = r^{-1}f \in F \cap C = 1$
$\Rightarrow f = r$ and $\phi(f) = \phi(r) = 1 \Rightarrow \ker \phi' \supseteq F'$

<u>$\Rightarrow$</u>:
$\ker \phi \supseteq F' \quad \Rightarrow \quad \phi'(f) = \phi(f)f = f \ \ \forall f \in F \quad \Rightarrow \quad F' \leq \phi'(R)$ ■

Applying this to the exact sequence in the previous exercise to get desired result. □

Remark. See Robinson

Exercise 7.25
Prove Lemma 4.36: If G abelian, then $H_2(G) \cong G \wedge G$.
Also deduce $H_n(G)$ for G being finitely generated abelian group

Solution
G abelian $\Rightarrow F' \leq R \Rightarrow H_2(G) \cong F'/[F,R]$ by Hopf's formula. Now consider the homomorphism:

$$\begin{array}{rcl} G \times G & \to & F'/[F,R] \\ (f_1R, f_2R) & \mapsto & [f_1,f_2][F,R] \end{array}$$

this is well-defined and bilinear, hence by universal property, induces a (unique) map $G \otimes G \to F'/[F,R]$, given by $fR \otimes fR \mapsto [f,f][F,R] = [F,R]$ (the identity in $F'/[F,R]$), hence inducing a map $G \wedge G \to F'/[F,R]$

Now we construct the inverse of this map. Choose a set of free generators $\{x_1, x_2, \ldots\}$ of F, we use the fact:

$$F'/[F',F] \text{ is free abelian on the set of } [x_i,x_j][F',F] \;\; i < j \in \{1,2,\ldots\}$$

This implies that there exists a map

$$\begin{aligned} \phi_0 : F'/[F',F] &\to F/R \wedge F/R \, (= G \wedge G) \\ [x_i,x_j][F',F] &\mapsto x_iR \wedge x_jR \end{aligned}$$

and extends bilinearly (as commutator is a bilinear operator).

Note that $\phi_0([F,R]/[F',F]) = 0$, so ϕ_0 induces $\phi : F'/[F,R] \to G \wedge G$ and this is an inverse of the previous map. □

Remark. See Robinson

Exercise 7.26
Prove Lemma 4.37

Exercise 7.27
Prove Lemma 4.40

Exercise 7.28
(Hard) Show that $\forall \alpha \in \mathrm{Ext}^n_{RG}(M,M'), n > 0, |G|\alpha = 0$
(Note that the special case $M = \mathbb{Z}$ is the Proposition 4.20)

Exercise 7.29
Show that ${}_A\,\mathbf{mod}, {}_B\,\mathbf{mod}$ are equivalent if and only if $\mathrm{End}_A(P)^{op} \cong B$ for some A-module P s.t. P is a progenerator for A

Solution
$\Leftarrow$: B acts on P on the right, so $P \in {}_A\,\mathbf{mod}_B$
Let $Q = \mathrm{Hom}_A(P,A) \in {}_B\,\mathbf{mod}_A$

$$\begin{aligned} \phi : P \otimes_B Q &\to A \\ x \otimes \theta &\mapsto \theta(x) \end{aligned}$$

is surjective as A is homomorphic image of finite copies of P (as P is a progenerator)

$$\begin{aligned} \psi : Q \otimes_A P &\to B \\ \theta \otimes x &\mapsto f \quad \text{s.t. } f^{op} : y \mapsto \theta(y)x \end{aligned}$$

This map is also surjective as P is a summand of free (left) A-module and so every endomorphism is sum of endomorphisms that factor through P.

Now check the required condition for Morita equivalence:

$$\begin{aligned} m\psi(\theta \otimes x) &= \theta(m)x &= \phi(m \otimes \theta)x \\ f\phi(x \otimes g) &= f(y)g(x) &= \psi(f \otimes x)g \end{aligned}$$

(Think carefully!)

$\Rightarrow$: Given ϕ, ψ, P, Q as in the definition of Morita equivalence We know that $\phi : P \otimes_B Q \xrightarrow{\sim} A$ and $\psi : Q \otimes_A P \xrightarrow{\sim} B$ are isomorphisms (this is not trivial but can be easily checked) □

Remark. See Benson 2.2

Exercise 7.30
Show that if ${}_A\,\mathbf{mod}$ and ${}_B\,\mathbf{mod}$ are equivalent, then $Z(A) \cong Z(B)$

Solution
Let $\lambda \in Z(A)$, then multiplication by λ is a natural transformation from the identity functor id: ${}_A\,\mathbf{mod} \to {}_A\,\mathbf{mod}$, i.e. for all $M, N \in {}_A\,\mathbf{mod}$, $f \in \mathrm{Mor}({}_A\,\mathbf{mod})$, , the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{id}(f)} & N \\ {\scriptstyle\lambda\cdot}\downarrow & & \downarrow{\scriptstyle\lambda\cdot} \\ M & \xrightarrow{\mathrm{id}(f)} & N \end{array}$$

Hence $\tau \in Z(B)$

Claim: All natural transformation of the identity functor are of this form

Proof of Claim:
Let τ be a natural transformation, set $M = {}_AA$ and $\lambda = \tau_{{}_AA}(1) \in A$, then $\forall N \in {}_A\,\mathbf{mod}, m \in N$, define

$$\begin{aligned} f : {}_AA &\to N \\ \lambda &\mapsto \lambda m \end{aligned}$$

As τ is natural transformation, we have

$$\tau_N(m) = \tau_N(f(1)) = f(\tau_{{}_AA}(1)) = f(\lambda) = \lambda m$$

$\Rightarrow \quad \tau$ is multiplication by λ ■

This claims say that for all $\tau \in Z(B)$, we get $\lambda \in Z(A)$ □

Remark. See Benson 2.2. Also c.f. Yoneda lemma.

Exercise 7.31
Show that $B(\mathbb{Q})$ is an infinite group

Exercise 7.32
Let K be maximal subfield of A, show that this does not necessary give $|K : k| = \operatorname{Deg} A$

Exercise 7.33
char $k \neq 2$. Let A be a central simple k-algebra. Show that if $\operatorname{Deg} A = 2$, then A is isomorphic to a quaternion algebra

Exercise 7.34
Show that the only finite dimensional non-commutative division $\mathbb{R}$-algebra is H. (i.e. $B(\mathbb{R}) \cong \mathbb{Z}/2\,\mathbb{Z}$)

Exercise 7.35
Show that every maximal subfield of the central simple k-algebra A is a splitting field of A

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