Lie Group, Lie Algebra and their Representations

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Recommended Books:
A. Kirillov - An introduction to Lie groups and Lie algebras
J-P. Serre - Complex semisimple Lie algebra
W. Fulton, J. Harris - Representation theory
Kirillov is the closest to what we will cover, Fulton-Harris is longer but with lots of example, which provides a good way to understand representation theory.
This course fit in especially well with Differential Geometry and Algebraic Topology.

Definition 1
A Lie group is a group which is also a smooth manifold

Example:
($\mathbb{R}$,+) is a Lie group of dimension 1
$S^1 = \{z \in \mathbb{C} : |z| = 1\}$ under multiplication

Definition 2
The n-sphere $S^n = \{(x_0, \dots, x_n)\} \in \mathbb{R}^{n+1} \,|x_0^2 + \dots + x_n^2 = 1\}$ is an n-manifold

Many interesting Lie groups act on S^2
Example:
$SO(3)$=group of rotation in $\mathbb{R}^3$ (this is non-abelian)
$PGL(2, \mathbb{C})$ acts on $S^2 = \mathbb{C} \cup \{\infty\}$ as Mobius transformation
Here $SO(3) \subseteq PGL(2, \mathbb{C}) = GL(2, \mathbb{C}) / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{C}^\times \right\}$
$z \mapsto 2z$, say, is in $PGL(2, \mathbb{C})$ acting on $S^2 = \mathbb{C} \cup \{\infty\}$

Examples of Lie groups

- $(\mathbb{R}^n, +)$ any $n \in \mathbb{N}$ (or any finite dimensional real vector space)
- $\mathbb{R}^\times = \{x \in \mathbb{R} \,|x \neq 0\}$ under multiplication
- $\mathbb{C}^\times = \{x \in \mathbb{C} \,|x \neq 0\}$ under multiplication
- $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) |\det A \neq 0\}$ under mulitiplication
- $GL(V)$ - General Linear group, where V is a finite dimensional vector space
- $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) |\det A = 1\} = \{f : \mathbb{R}^n \to \mathbb{R}^n$ linear and preserves volume$\}$ (Special Linear group)
- $O(n)$ (Orthogonal group)
- $Sp(2n, \mathbb{R})$ (Sympletic group)
- $U(n)$ (Unitary group)
- $SU(n)$ (Special Unitary group)

Remark: S^0, S^1, S^3 are the only spheres that are also Lie groups

Orthogonal group

$$
\begin{aligned}
O(n) &= \{A \in M_n(\mathbb{R}) | AA^\top = 1\} \\
&= \{f : \mathbb{R}^n \to \mathbb{R}^n \,| f \text{ linear and preserves distances } \} \\
&= \{f : \mathbb{R}^n \to \mathbb{R}^n \,| \langle f(x), f(y)\rangle = \langle x, y\rangle \;\forall x, y \in \mathbb{R}^n\}
\end{aligned}
$$

where the standard inner product on $\mathbb{R}^n$ is

$$
\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in \mathbb{R}
$$

Elements of $O(n)$ includes rotations and reflections

Note that det is a homomorphism $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$ and this restricts to $\det : O(n) \to \{\pm 1\}$ since, $A \in O(n) \Rightarrow 1 = \det(1) = \det(AA^\top) = \det(A)\det(A^\top) = \det(A)^2$

Definition 3
Special orthogonal group

$$
SO(n) = \{A \in M_n(\mathbb{R}) | AA^\top = 1, \det A = 1\}
$$

Elements include rotations but not reflections (on $\mathbb{R}^n$)
$SO(n)$ is a subgroup of index 2 in $O(n)$.
In fact, $O(n)$ has 2 connected component, the one containing 1 is $SO(n)$

Also note that $SO(2) \cong S^1$

Symplectic group

$$
Sp(2n, \mathbb{R}) = \{f : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \,| w(x, y) = w(f(x), f(y)) \forall x, y \in \mathbb{R}^{2n}\}
$$

where w is a non-degenerate alternating bilinear form on $\mathbb{R}^{2n}$:

$$
w\big((q_1, \ldots, q_n, p_1, \ldots, p_n), (q_1', \ldots, q_n', p_1', \ldots, p_n')\big) = \sum_{i=1}^{n} q_ip_i' - p_iq_i'
$$

for some choice of basis.

Remark: Any non-degenerate alternating bilinear form w on $\mathbb{R}^n$ must have n even, and after a change of basis, such a form is given by above formula

Example:
$Sp(2n, \mathbb{R}) \subset SL(2n, \mathbb{R})$
$Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$ = the group of area preserving linear maps

Unitary group

$$
U(n) = \{f : \mathbb{C}^n \to \mathbb{C}^n \text{ linear and preserves distance}\} \;\; (= GL(n, \mathbb{C}) \cap O(2n))
$$

Definition 4
The standard inner product on $\mathbb{C}^n$ is the nondegenerate positive definite Hermitian form

$$
\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n)\rangle = \sum_{i=1}^{n} z_1\overline{w_1}
$$

Note that the length of a vector $z \in \mathbb{C}^n (= \mathbb{R}^{2n})$ is $\|z\| = \sqrt{\langle z, z\rangle}$
(as $z = x_iy, z\overline{z} = |z|^2 = x^2 + y^2$)

So, we have

$$\begin{aligned} U(n) &= \{f : \mathbb{C}^n \to \mathbb{C}^n \text{ linear } | \langle f(x), f(y)\rangle = \langle x, y\rangle \ \forall x, y \in \mathbb{C}^n\} \\ &= \{A \in GL(n, \mathbb{C}) | AA^* = 1\} \qquad (A^* = \overline{A^\top}) \end{aligned}$$

<u>**Special Unitary group**</u>
The det of a unitary matrix gives a homomorphism $\det : U(n) \to S^1 \subset \mathbb{C}^\times$

$$SU(n) = \ker \det|_{U(n)}$$

<u>Example</u>
$U(1) = \{z \in M_1(\mathbb{C}) | z\overline{z} = 1\} = S^1$
$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \middle| a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$
Wee see that $SU(2)$ is diffeomorphic to $S^3 = \{(x_0, x_1, x_2, x_3) | x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$

<u>Remark</u>: S^0, S^1, S^3 are the only sphere that are also Lie groups

Some Basics of Smooth Manifold

Definition 5
A subset $M \subseteq \mathbb{R}^n$ is called <u>k-dimensional manifold</u> (in $\mathbb{R}^n$) if for every point $x \in M$, the following condition is satisfied:
(M) There is an oen set $U \ni x$, and open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ s.t.

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{x \in V | x_{k+1} = \cdots x_n = 0\}$$

Definition 6
Let $U \subseteq \mathbb{R}^n, n > 0$ be an open set, a <u>smooth function</u> $f : U \to \mathbb{R}$ (or C^∞) if all partial derivatives

$$\frac{\partial^r}{\partial_{x_{i_1}} \cdots \partial_{x_{i_r}}} f \qquad (r \geq 0)$$

are defined and continuous on U

For $U \subseteq \mathbb{R}^m$ open, a <u>smooth mapping</u> $f : U \to \mathbb{R}^n$ is a function s.t. $f = (f_1, \ldots, f_m)$, with $f_i : U \to \mathbb{R}$ smooth function

For $U, V \subseteq \mathbb{R}^n$, a <u>diffeomorphism</u> $f : U \to V$ of <u>degree</u> n is a smooth map (on $\mathbb{R}^n$) with a smooth inverse

The <u>derivative</u> $df|_x$ of a smooth map $f : U(\subseteq \mathbb{R}^m) \to \mathbb{R}^n$ at a point $x \in U$ is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ given by matrix $(\frac{\partial f_i}{\partial x_j})$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

composite of the smooth maps is smooth, and $d(g \circ f)|_x = dg|_{f(x)} \circ df|_x$

Theorem 7 (Inverse Function Theorem)
Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ be a smooth map. Suppose $df|_x$ is an isomorphism, for some $x \in U$
Then $\exists V \subset U$, $V \ni x$, s.t. $f(V)$ is open and f is a diffeomorphism from V to $f(V)$

Theorem 8 (Implicit Function Theorem)
Let $U \subseteq \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$ smooth map. Suppose $df|_x$ is surjective at a point $x \in U$ $(n \leq m)$
Then $\exists V \subseteq U, V \ni x$ and diffeomorphism $\phi : W \to V$ $(W \subseteq \mathbb{R}^n$ open), s.t.

$$f(\phi(x_1, \ldots, x_n)) = (x_1, \ldots, x_n)$$

Definition 9
A submersion is a smooth map which has derivatives being surjective everywhere

A smooth submanifold $X \subseteq \mathbb{R}^N$ of dimension n is a subset s.t. $\forall x \in X, \exists$ nbhd $U \ni x$, $U \subseteq \mathbb{R}^N$ and a submersion $F : U \to \mathbb{R}^{N-n}$ s.t. $X \cap U = F^{-1}(0) \subseteq U$

Example:
Claim: The sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a smooth n-dimensional submanifold

Proof
$S^n = F^{-1}(0)$, where

$$\begin{aligned} F : \mathbb{R}^{n+1} &\to \mathbb{R} \\ (x_0, \ldots, x_n) &\mapsto x_0^2 + \cdots + x_n^2 - 1 \end{aligned}$$

We have to check that F is a submersion at points $x \in S^n$:

$$dF = (2x_0, 2x_1, \ldots, 2x_n) \qquad \text{(row matrix)}$$

This is surjective whenever $(x_0, \ldots, x_n) \neq (0, \ldots, 0) \quad \Rightarrow \quad$ surjective everywhere on S^n □

Example:
$X = \{(x, y) \in \mathbb{R}^2 \,|\, xy = a\}$ is a smooth 1-dimensional submanifold for nonzero $a \in \mathbb{R}$, but not for $a = 0$, here we have:

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto xy - a \end{aligned}$$

Example:
$X = \{x \in \mathbb{R} \,|\, x^2 = 0\}$ is an 0-dimensional submanifold of $\mathbb{R}$, but $x^2 : \mathbb{R} \to \mathbb{R}$ is not a submersion at 0. To prove that X is a 0-dimensional submanifold, you have to notice that $X = \{x \in \mathbb{R} \,|\, x = 0\}$

Definition 10
The tangent space to a smooth n-dimensional submanifold $X \subseteq \mathbb{R}^N$ at a point $x \in X$ (if we describe X as $X = F^{-1}(V)$ for some submersion $F : V \to \mathbb{R}^{N-n}$) is defined as:

$$T_x X = \ker(dF|_x : \mathbb{R}^N \to \mathbb{R}^{N-n})$$

This is an n-dimensional linear subspace of $\mathbb{R}^N$

Let $X \subset \mathbb{R}^N$ be a smooth n-dimensional submanifold.
A function $f : X \to \mathbb{R}$ is smooth $\Leftrightarrow$ near each point $x \in X$, f is the restriction of a smooth function on an open nbhd of X in $\mathbb{R}^N$ (0-dimensional submanifold of $\mathbb{R}^N$ = discrete subset)

For submanifold $X \subseteq \mathbb{R}^M$, $Y \subseteq \mathbb{R}^N$ (dim=m, n resp.) a smooth map $f : X \to Y$ has derivative $df|_x : T_x X \to T_{f(x)} Y$ which is a linear map
A diffeomorphism between 2 submanifolds is a smooth map with smooth inverse.

Fact: (Hausdorff countable basis) Every smooth manifold is diffeomorphic to a submanifold of $\mathbb{R}^N$

For submanifold $X \subseteq \mathbb{R}^M, Y \subseteq \mathbb{R}^N$ (dim=m, n resp.), the product $X \times Y \subseteq \mathbb{R}^M \times \mathbb{R}^N = \mathbb{R}^{M+N}$ is a smooth submanifold. It has the product topology

Lie Group

Definition 11
A Lie group G is a smooth manifold which is also a group s.t.

$$\begin{aligned} \text{multiplication } : G \times G &\to G \qquad (g, h) \mapsto gh \\ \text{inverse } : G &\to G \qquad g \mapsto g^{-1} \end{aligned}$$

are smooth maps. We have a point $1 \in G$ (the identity)

Note that a Lie group need not be connected. (0-dimensional submanifold of $\mathbb{R}^N$=discrete subset)
In particular, we can view any group (say countable) as a 0-dimensional Lie group.

Lemma 12
Let G be a Lie group, G^0 be the connected component of G containing 1. Then $G^0 \trianglelefteq G$ and G/G^0 is discrete (with the quotient topology)

Proof
multiplication : $G \times G \to G$ is continuous, so it maps connected space $G^0 \times G^0$ onto connected subset of G, which contains 1.
$\Rightarrow G^0 \times G^0 \twoheadrightarrow G^0$
Likewise, inverse : $G^0 \twoheadrightarrow g^0$. Therefore, $G^0 \leq G$

To show $G^0 \trianglelefteq G$, need to show $\forall g \in G$ the map $\begin{array}{rcl} C_g : G & \to & G \\ x & \mapsto & gxg^{-1} \end{array}$ sends G^0 to G^0
Have C_g smooth⇒continuous, and 1↦1
$\Rightarrow \quad C_g : G^0 \twoheadrightarrow G^0$
$\Rightarrow \quad G^0 \trianglelefteq G$

We have, $\forall g \in G$ a diffeomorphism $\begin{array}{rcl} L_g : G & \to & G \\ x & \mapsto & gx \end{array}$
(Can check that $L_{g^{-1}}$ is an inverse map, using that G is associative)
Therefore, $L_g(G^0) = gG^0$ is the connected component of G containing g.
We know that G is the disjoint union of some of these left cosets gG and G/G^0 is the set of cosets.
To show that G/G^0 has discrete topology. I have to show that each component gG^0 is open in G. In fact, all connected component in any manifold are open subsets □

Lemma 13
Let G be a connected Lie group, Then G is generated by a neighbourhood of $1 \in G$

Proof
Let N be an neighbourhood of $1 \in G$
Let $H \leq G$, generated by N
$\Rightarrow H$ open in G because $\forall h \in H$ $hN \subseteq H$ and hN is an open subset of G containing h
In fact, H is also closed in G if$x \in G - H \Rightarrow xN \subseteq G - H$
(If $xn = h \in H$ for some $n \in N$, then $x = hn^{-1}$ #)
So H is open and closed and contains 1 $\Rightarrow H = G$ since G is connected □

ref.: Armstrong, Basic Topology

Definition 14
A homomorphism $f : G \to H$ of Lie groups is a group homomorphism which is also smooth

Lemma 15
Let $f : G \to H$ be a homomorphism of connected Lie groups. Suppose that

$$df|_1 : T_1G \twoheadrightarrow T_1H \tag{1}$$

Then $f : G \twoheadrightarrow H$

Proof
By the Implicit Function Theorem, f maps some neighbourhood of $1 \in G$ onto some neighbourhood of $1 \in H$, so $f(G)$ contains the subgroup of H generated by this neighbourhood which is all of H because H is connected □

Example:
$f : \mathbb{R} \to S^1 \subseteq \mathbb{C}^\times \quad i \mapsto e^{it}$ is a homomorphism of Lie groups (It's smooth, and it's a hom. because

$f(s+t) = f(s)f(t))$
Its derivative at 1 is

$$\left.\frac{d(e^{it})}{dt}\right|_{t=0} = ie^{it}|_{t=0} = i \tag{2}$$

which is an isomorphism $\mathbb{R} = T_0\,\mathbb{R} \cong T_1 S^1 = i\,\mathbb{R} \subset \mathbb{C}$
So lemma applies and indeed $\mathbb{R} \twoheadrightarrow S^1$
In fact, $S^1 \cong \mathbb{R}\,/\,\mathbb{Z}$ where $2\pi\,\mathbb{Z} = \mathbb{Z} = \ker f$

Definition 16
A closed Lie subgroup H of a Lie group G is a closed submanifold of G which is a subgroup of G

Note that such a subgroup H is a Lie group. Indeed, multi: $H \times H \to H$ is just the restriction of multi: $G \times G \to G$ so it is also smooth, likewise for inverses.

Use this to prove that the classical groups actually are Lie groups

Example:
$GL(n, \mathbb{R})$. This is an open subset of $M_n\,\mathbb{R} = \mathbb{R}^{n^2}$ so it is a smooth n^2-dimensional manifold. Multiplication of matrices is smooth (in fact, polynomial or mapping smooth)

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \ldots & \cdots \\ \vdots & \ddots \end{pmatrix}$$

is a smooth function. Inverse is a polynomial in entries of given matrix A and in $1/\det A$ which is a smooth function of $GL(n, \mathbb{R}) = \{A \mid \det A \neq 0\}$. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$SL(n, \mathbb{R}) = \{A \in GL(n\,\mathbb{R}) \mid \det A = 1\}$ This is a closed Lie subgroup of $GL(n, \mathbb{R})$. Clearly it is a closed subgroup
To show: $SL(n, \mathbb{R})$ is a smooth submanifold of dimension $n^2 - 1$. It suffices to check that $SL(n, \mathbb{R})$ is a smooth submanifold near $1 \in G(n, \mathbb{R})$ using left translation (see notes for pictures)

It suffices to show that $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$ is a submersion near 1.
To do this, we see how det changes as you move from $1 \in GL(n, \mathbb{R})$. So look at $A = 1 + \epsilon B,\ B \in M_n\,\mathbb{R}$.
We solve the equation

$$\det A = 1 (\mathrm{mod}\ \epsilon^2)$$

We compute:

$$\begin{aligned} \det(1 + \epsilon B) &= \det\left(1 + \epsilon\begin{pmatrix} b_{11} & \cdots \\ \vdots & \ddots \end{pmatrix}\right) \\ &= (1 + \epsilon b_{11}) \cdots (1 + \epsilon b_{nn}) \pmod{\epsilon^2} \\ &= 1 + \epsilon(b_{11} + \cdots + b_{nn}) \pmod{\epsilon^2} \\ \Rightarrow \ker(d(\det)|_1) &= \{B \in M_n\,\mathbb{R} \mid \mathrm{tr}(B) = 0\} \end{aligned}$$

This is a codimension 1 subspace of $M_n\,\mathbb{R}$ so det is a submersion at $1 \in GL(n, \mathbb{R})$, so $SL(n, \mathbb{R})$ is a closed Lie subgroup , and $\mathfrak{sl}(n) = T_1 SL(n, \mathbb{R}) = \{B \in M_n\,\mathbb{R} \mid \mathrm{tr}(B) = 0\}$
$\mathfrak{gl}(n) = M_n\,\mathbb{R} = T_1 GL(n, \mathbb{R})$

Example:
Orthogonal group $O(n)$. Again this is a closed subgroup of $GL(n,\mathbb{R})$. To show that it is a smooth submanifold it suffice to check that near $1 \in GL(n,\mathbb{R})$. So we differentiate these equations for $O(n) \subset GL(n,\mathbb{R})$
So, for $B \in \mathfrak{gl}(n)$ we compute where is :

$$(1+\epsilon B)(1+\epsilon B)^t = 1 \pmod{\epsilon^2} \tag{3}$$
$$(1+\epsilon B)(1+\epsilon B)^t = 1+\epsilon(B+B^t) \pmod{\epsilon^2} \tag{4}$$

$F : GL(n,\mathbb{R}) \to \mathbb{R}^{n^2}$
We have $O(n) = F^{-1}(1)$ for some smooth mapping
and we have computing $\ker(dF) = \{B \in \mathfrak{gl}(n) \mid B + B^t = 0\}$
$\Rightarrow \quad \dim_{\mathbb{R}}(\ker(dF)) = \dim(\text{zero diagonal matrix}) = \frac{n(n-1)}{2}$

So we would like to say that $O(n)$ is the fibre of a smooth map $GL(n,\mathbb{R}) \to \mathbb{R}^{n^2-(n(n-1)/2)} = \mathbb{R}^{n(n+1)/2}$
Can we define $O(n)$ using only $n(n+1)/2$ equations?
Yes, since $\forall A \in GL(n,\mathbb{R}), AA^t$ is symmetric
So $AA^t = 1$ reduces to $n(n+1)/2$ equations.
So $O(n)$ is a smooth submanifold of dimension $n(n-1)/2$ in $GL(n,\mathbb{R})$ and hence a closed Lie subgroup.
Also $\mathfrak{so}(n) = T_1O(n) = \{B \in \mathfrak{gl}(n) | B^t = -B\}$

Example:
Unitary group $U(n) \subset GL(n,\mathbb{C})$. We just show that it is a smooth (real) submanifold of $GL(n,\mathbb{C})$ near 1
Differentiate the equation for $U(n) \subset GL(n,\mathbb{C})$ at 1:
Write $A = 1 + \epsilon B,\ B \in \mathfrak{gl}(n,\mathbb{C}) = M_n\,\mathbb{C}$
Solve

$$(1+\epsilon B)(1+\epsilon B)^* = 1 \pmod{\epsilon^2}$$
$$(1+\epsilon B)(1+\epsilon B)^* = 1+\epsilon(B+B^*) \pmod{\epsilon^2}$$

So $U(n) = F^{-1}(1) \subset GL(n,\mathbb{C})$ where
$\ker(dF|_1) = \{B \in \mathfrak{gl}(n,\mathbb{C}) \mid B^* = -B\} = \{\text{skew hermitian matrices}\} = i\{\text{hermitian matrices}\}$ and $\mathfrak{gl}(n,\mathbb{C}) = \{\text{hermitian}\} + \{\text{skew-hermitian}\}$

$$\text{Skew-hermitian matrix is } \begin{pmatrix} ia & z \\ -\bar{z} & ib \end{pmatrix} \qquad a,b \in \mathbb{R}, z \in \mathbb{C}$$

So $\dim_{\mathbb{R}}(\ker(dF|_1)) = (1/2)\dim_{\mathbb{R}} \mathfrak{gl}(n,\mathbb{C}) = n^2$
So I would like to define $U(n) \subset GL(n,\mathbb{C})$ by exactly $2n^2 - n^2 = n^2$ real equations.

Indeed, for any $A \in GL(n,\mathbb{C}),\ AA^*$ is always hermitian (since $(AB)^* = B^*A^*$). So $AA^* = 1$ reduces to only n^2 real equations (say that the element of AA^* above diagonal are zeroes and the elements on diagonal, which are real =1)
So $U(n)$ =fibre of submersion $GL(n,\mathbb{C}) \to \mathbb{R}^{n^2}$ so it is a closed Lie subgroup of $GL(n,\mathbb{C})$

Definition 17
For $A \in M_n(K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, define the exponential of A by:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} \in M_nK \tag{5}$$

To check that this series converges, define the norm:

$$\|A\| := \sup_{\|x\|=1, x\in\mathbb{R}^n} \|Ax\| \tag{6}$$

Clearly $\|AB\| \leq \|A\|.\|B\|$
$\Rightarrow \quad \forall A \in M_n(K)$, $\|\frac{A^n}{n!}\| \leq \frac{\|A\|^n}{n!}$ and this series converges in $\mathbb{R} \quad \forall \|A\|$. So the series of matrices converges absolutely.

Easy that $\exp : M_n \mathbb{R} \to M_n \mathbb{R}$ is smooth and $\exp : M_n \mathbb{C} \to M_n \mathbb{C}$ is complex analytic.

Also, for $\|A\| < 1$ define the logarithm

$$\log(1 + A) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A^n}{n} \tag{7}$$

This converges for $\|A\| < 1$.
$\Rightarrow \quad$ log is defined on the open ball of radius 1 around $1 \in M_n(K)$

Theorem 18 (1) For x in some neighbourhood of $0 \in M_n K$, $\log(\exp(x)) = x$.
For X with $\|X - 1\| < 1$, $\exp(\log(X)) = X$

(2) $\exp(x) = 1 + x + \cdots$. That is $\exp(0) = 1$ and $d\exp|_0 = \mathrm{id}_{M_n K}$

(3) If $xy = yx$ in $M_n K$, then $\exp(x + y) = \exp(x)\exp(y)$ In particular, $\exp(x)\exp(-x) = 1$ for any $x \in M_n K$. So $\exp(x) \in GL(n, K)$

(4) For a fixed $x \in M_n K$, define a smooth map $\mathbb{R} \to GL(n, K)$ by $t \mapsto \exp(tx)$. Then $\exp((s+t)x) = \exp(sx)\exp(tx)$ $\forall s, t \in K$. In other wrods, $t \mapsto \exp(tx)$ is a homomorphism of Lie groups.

(5) The exponential map commutes with conjugation and transpose. That is $\exp(A \times A^{-1}) = A\exp(x)A^{-1}$ and $\exp(x)^t = \exp(x^t)$

Proof

(1) follows from the fact that $\log(\exp(x)) = x$ for $x \in \mathbb{R}$, so that is true as an identity of formal power series. So it works for a matrix X

(2) -

(3) Try to compute $\exp(x)\exp(y)$ for any $x, y \in M_n(K)$

$$\begin{aligned} \exp(x)\exp(y) &= (1 + x + x^2/2 + \cdots)(1 + y + y^2/2 + \cdots) && (8)\\ &= 1 + (x + y) + (x^2/2 + xy + y^2/2) + \cdots && (9)\\ \text{and } \exp(x + y) &= 1 + (x + y) + (x + y)^2/2 + \cdots && (10)\\ &= 1 + (x + y) + (x^2 + xy + yx + y^2)/2 + \cdots && (11) \end{aligned}$$

If $yx = yx$, then $\exp(x + y) = \exp(x)\exp(y)$ is an identity of power series in commmuting variables, say because it's true for $x, y \in \mathbb{R}$

(4) follows from (3) because for any $x \in M_n K$, and any $s, t \in K$ sx and tx commute. So $\exp(sx + tx) = \exp(sx)\exp(tx)$

(5) These follow from the power series for exp, using that $(AxA^{-1})^n = Ax^nA^{-1}$, and likelwise $(x^t)^n = (x^n)^t$

□

Definition 19
A one-parameter subgroup of a Lie group G is a homomorphism $\mathbb{R} \to G$ of Lie groups

The theorem gives, for any $x \in \mathfrak{gl}(n, \mathbb{R})$, a one-parameter subbgroup of $GL(n, \mathbb{R})$, $\mathbb{R} \to GL(n, \mathbb{R})$ with tangent vector at 0 is $x \in \mathfrak{gl}(n, \mathbb{R}) = T_1 GL(n, \mathbb{R})$

Theorem 20
For every classical group $G \subseteq GL(n,K)$ (to be listed), G is a closed Lie subgroup of $GL(n,K)$. In fact, if we let $\mathfrak{g} = T_1 G$, then exp gives diffeomorphism, for some neighbourhood U of 1 in $GL(n,K)$ and $\mathfrak{u}$ of 0 in $\mathfrak{gl}(n,K)$, $U \cap G \overset{\log}{\underset{\exp}{\rightleftarrows}} \mathfrak{u} \cap \mathfrak{g}$

The classical groups:

(1) Compact (real) groups: $SO(n), U(n), SU(n), Sp(n)$

(2) $GL(n,K)$, $SL(n,K)$, $SO(n,K)$, $O(n,K)$; for $K = \mathbb{R}$ or $\mathbb{C}$

(3) Real Lie group: $Sp(2n,\mathbb{R})$

(4) Complex Lie gorup: $Sp(2n,\mathbb{C})$

Example:
$O(n,\mathbb{C})$ = subgroup of $GL(n,\mathbb{C})$ preserving the symmetric $\mathbb{C}$-bilinear form:

$$< (z_1,\ldots,z_n),(w_1,\ldots,w_n) >= \sum z_i w_i$$

$Sp(2n,\mathbb{C})$ = subgroup of $GL(2n,\mathbb{C})$ preserving the standard $\mathbb{C}$-symplectic (i.e. alternating nondegenerate) form:

$$w((z_1,\ldots,z_{2n}),(w_1,\ldots,w_{2n})) = (z_1 w_{n+1} - z_{n+1} w_1) + (z_2 w_{n+2} - z_{n+2} w_2) + \cdots$$

Compact symplectic group
$Sp(n)$:=subgroup of $GL(n,H)$ preserving distance on $H^n = \mathbb{R}^{4n}$.
Here the quaternions $H = \mathbb{R}\,1 \oplus \mathbb{R}\,i \oplus \mathbb{R}\,j \oplus \mathbb{R}\,k$ determined by $i^2 = k^2 = j^2 = -1$ and $ij = k$ (etc.).

Say we define an H-vector space V to be a right H-module, $va \in V$ for $a \in H$.
For example $H^n = \{(z_1,\ldots,z_n)^\top \mid z_i \in H\}$ is an H-vector space.
$GL(n,H) := \{$invertible H-linear maps $H^n \to H^m\} \subseteq M_n(H)$

Warning: det only defined for matrices uses a commutative ring.

Why are $O(n), U(n), Sp(n)$ compact?
$O(n) = \{$ matrix with column $i = A(e_i) \mid A(e_1),\ldots,A(e_n) \in \mathbb{R}^n$ orthonormal$\} \subseteq M_n\,\mathbb{R} = \mathbb{R}^{n^2}$
is a closed bounded subset and hence compact

$U(n) = GL(n,\mathbb{C}) \cap O(2n)$ which is closed subset of $O(2n)$ hence compact.

$Sp(n) = GL(n,H) \cap O(4n) \subset GL(4n,\mathbb{R})$ a closed subset of $O(4n)$, so $Sp(n)$ is compact

Proof of Theorem 20 in a few cases:
$SL(n,\mathbb{R})$: Claim that: for $x \in \mathfrak{gl}(n,\mathbb{R})$, near 0, $\exp(x) \in SL(n,\mathbb{R}) \Leftrightarrow x \in \mathfrak{sl}(n,\mathbb{R}) := \{x \in \mathfrak{gl}(n) \mid \operatorname{tr}(x) = 0\}$.

Use Jordan canonical form: For any $x \in M_n\,\mathbb{C}$, x is conjugate (over $\mathbb{C}$) to an upper-triangular matrix.
So $\exp(x)$ is conjugate (over $\mathbb{C}$) to $\begin{pmatrix} e^{a_1} & & * \\ & \ddots & \\ 0 & & e^{a_n} \end{pmatrix}$.
In particular,

$$\begin{aligned} \det\exp(x) &= e^{a_1}\cdots e^{a_n} && (12)\\ &= e^{a_1+\cdots a_n} && (13)\\ &= \exp(\operatorname{tr}(x)) && (14) \end{aligned}$$

So,

$$\exp(x) \in SL(n,\mathbb{R}) \quad \Leftrightarrow \quad \det \exp(x) = 1 \tag{15}$$

$$\Leftrightarrow \quad \exp(\mathrm{tr}(x)) = 1 \Leftrightarrow \mathrm{tr}(x) \in 2\pi i\,\mathbb{Z} \tag{16}$$

For x near 0, this happens $\Leftrightarrow \mathrm{tr}(x) = 0$

Definition 21
vector field V on a smooth manifold M assigns to every point $p \in M$ a tangent vector $v_p \in T_pM$ s.t. in any coordinate chart, it has the form

$$v = \sum_{i=1}^{n} f_i(p) \frac{\partial}{\partial x_i} \tag{17}$$

where $f_1, \ldots, f_n$ are smooth functions $M \to \mathbb{R}$ (see picture)

Here we write $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ for the standard basis to $T_p\,\mathbb{R}^n$ for every $p \in \mathbb{R}^n$

Two ways to think of tangent vectors at $p \in M$:

(1) A smooth curve $c : \mathbb{R} \to M$ has a tangent vector $c'(t) \in T_{c(t)}M$

(2) Differentiate a smooth function F on M in the direction of tangent vector $X \in T_pM$ at point p (one definition: pick a curve c with $c'(0) = X$ and then define $X(f) = \frac{d}{dt}|_{t=0} f(c(t))$)

We can identify T_pM with the space of "derivation at p", $X : C^\infty(M) \to \mathbb{R}$, $\mathbb{R}$-linear, s.t. $X(fg) = f(p)X(g) + X(f)g(p) \in \mathbb{R}$
In particular, in some coordinates, $\frac{\partial}{\partial x_1}\big|_p, \cdots \frac{\partial}{\partial x_n}\big|_p$ are derivation at p

Theorem 22 (Existence and Uniqueness for ODEs)
Let M be a smooth manifold, X a vector field on M, $p \in M$.
Then $\forall a < 0, b > 0$, $\exists$at most one curve $c : (a,b) \to M$ s.t. $c(0) = p$ and $c'(t) = X_{c(t)} \in T_{c(t)}M$
Also, $c(t)$ exists on some open interval around 0, the maximal interval might or might not be $\mathbb{R}$. If M is compact then $c(t)$ is defined $\forall t \in \mathbb{R}$

Theorem 23
Let G be a Lie group, $x \in T_1G$. Then $\exists!$ one parameter subgroup $f : \mathbb{R} \to G$ s.t. $f'(0) = x$

Proof
(see picture)

Suppose we have such a f. We know that $\forall t, t_0 \in \mathbb{R}$, $f(t + t_0) = f(f_0)f(t) \in G$
For $t_0 \in \mathbb{R}$, and think if t near 0. Then $f(t + t_0) = L_{f(t_0)}f(t) \in G$
Differentiate this w.r.t. t at $t = 0$ gives:
$f'(t_0) = dL_{f(t_0)}(x) \in T_{f(x_0)}G$, since $f'(0) = x \in T_1G$ so define a left-invariant vector field X on G by: $\forall g \in G$, take the tangent vector $X_g := (dL_g)(x) \in T_gG$
So $f(t)$ must be the unique solution to the ODE: $f(0) = 1 \in G$ and $f'(t) = X_{f(t)} \in T_{f(t)}G$ $\forall t \in (a,b) \subseteq \mathbb{R}$
One checks that a solution to the ODE is a one-parameter subgroup.
Suppose we have defined $f : [0,T] \to G$ with $f(s+t) = f(s)f(t)$ for $s, t, s+t \in [0,T]$. Then we can define f on $[T, 2T]$ by $f(T+t) = f(T)f(t)$ for $t \in [0,T]$. (see picture) Repeat process. $\square$

Definition 24
Let G be a Lie group. Then the exponential map $\exp : \mathfrak{g} \to G$ (where $\mathfrak{g} = T_1G$) is defined by

$$\exp(x) = f(1) \tag{18}$$

where $f : \mathbb{R} \to G$ is the unique one-parameter subgroup with $f'(0) = x \in \mathfrak{g}$
(This is smooth, by theorems on ODEs)

Notice that for $t \in \mathbb{R}$, $\exp(tx) = f(t)$. That is, $t \mapsto \exp(tx)$ is the unqiue one-parameter subgroup $\mathbb{R} \to G$ with tangent vector x at time 0.

For $G = GL(n, \mathbb{R})$ it follows that this map is the same as the matrix exponential

$$\exp : \mathfrak{gl}(n, \mathbb{R}) \to GL(n, \mathbb{R}) \tag{19}$$

More generally, let $H < G$ be a closed Lie subgroup of Lie group G. For x near 0 in $\mathfrak{g}$, $\exp(x) \in H \Leftrightarrow x \in \mathfrak{h}$ (see picture)
Remark: $d\exp|_0 : \mathfrak{g} \to T_1G = \mathfrak{g}$ is the identity map, so exp gives a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of 1 in G for any Lie group.

Lemma 25
For any conencted Lie group G, G is generated by the subset $\exp(\mathfrak{g}) \subset G$

Proof
By Inverse Function Theorem, since $d\exp|_0$ =identity on $\mathfrak{g}$, $\exp(\mathfrak{g})$ contain a neighbourhood of 1 in G. Since G is connected, this generates G as a group. □

Corollary 26
Let G, H be Lie groups, G connected. Let $\alpha, \beta : G \to H$ be homomorphisms s.t. $d\alpha|_1 = d\beta|_1 : \mathfrak{g} \to \mathfrak{h}$. Then $\alpha = \beta$

Proof
For any $x \in \mathfrak{g}$, then $t \mapsto f(\exp(tx))$ is a one-parameter subgroup $f : \mathbb{R} \to H$. The tangent vector to this one-parameter subgroup in H is $df|_1(x) \in \mathfrak{h}$, so $f(\exp(tx)) = \exp(tdf|_1(x))$.
Since α and β have the same derivative at 1, we have $\alpha(\exp(tx)) = \beta(\exp(tx))$ $\forall t \in \mathbb{R}, x \in \mathfrak{g}$. So, $\alpha = \beta$ on $\exp(\mathfrak{g}) \subset G$. So $\alpha = \beta$ on all of G. □

Example

- For G abelian Lie group, exp is "globally well-behave". It is a group homomorphism $\exp : \mathfrak{g} \to G$, it is surjective, and it is a covering map (See Armstrong, Basic Topology)
- $G = S^1$, then $\exp : \mathfrak{g} \to S^1$ is the map $\mathbb{R} \to S^1 \quad t \mapsto e^{it}$

For G nonabelian, exp need not be a covering map even if it is surjective.

Example $G = Sp(1) = \{z \in H \cong \mathbb{R}^4 \mid |z| = 1\} \cong S^3$ group under multiplication.
$\mathfrak{g} \to G = S^3$ sends all vectors of length π to the point -1; all vectors of length 2π to 1 etc.

Let G be a Lie group. We have a smooth map $f : U \times U \to V$ where $0 \in U \subseteq V \subseteq \mathfrak{g}$ are open subsets of $\mathfrak{g} = T_1G$ s.t. $\exp(f(x, y)) = \exp(x)\exp(y) \in G$
This satisfies $f(0, y) = y$ and $f(x, 0) = x \quad \forall x, y \in \mathfrak{g}$. So the Taylor series for f at $(0, 0) \in \mathfrak{g} \times \mathfrak{g}$ begins:

$$f(x, y) = x + y + f_2(x, y) + f_3(x, y) + \cdots \tag{20}$$

In general $f_2(x, y) = \sum a_{ij}x_ix_j + \sum b_{ij}x_iy_j + \sum c_{ij}y_iy_j = \sum b_{ij}x_iy_j$
In this case, $f_2(x, y)$ is a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

Definition 27
The Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined by $f_2(x, y) = \frac{1}{2}[x, y]$.
We have

$$f(x, x) = \exp^{-1}(\exp(x)\exp(x)) \tag{21}$$
$$= \exp^{-1}(\exp(2x)) = 2x \tag{22}$$

(More generally, $\exp(sx)\exp(tx) = \exp((s + t)x) \quad \forall s, t \in \mathbb{R}, x \in \mathfrak{g}$)
Therefore, $[x, x] = 0 \quad \forall x \in \mathfrak{g}$. This defines $[\ ,\ ]$ is alternating. As a result, $[x, y] = -[y, x] \quad \forall x, y \in \mathfrak{g}$
Proof: $0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y]$ □

The Lie bracket measures the non-commutativity of G, in a nbhd of $1 \in G$. In particular, if G is abelian, then $[\ ,\ ]$ is 0

Example: Compute the Lie bracket for $G = GL(n, \mathbb{R})$
Here $[\ ,\ ] : \mathfrak{gl}(n) \times \mathfrak{gl}(n) \to \mathfrak{gl}(n)$, we have

$$\begin{aligned} f(x,y) &= \log(\exp(x)\exp(y)) && (23)\\ &= \log((1 + x + \frac{x^2}{2} + \cdots)(1 + y + \frac{y^2}{2} + \cdots)) && (24)\\ &= \log(1 + \left[(x+y) + (\frac{x^2}{2} + xy + \frac{y^2}{2}) + \cdots\right]) && (25)\\ &= \left[(x+y) + (\frac{x^2}{2} + xy + \frac{y^2}{2}) + \cdots\right] - \left[\frac{(x+y)^2}{2} + \cdots\right] + \cdots && (26)\\ &= x + y + \frac{1}{2}(xy - yx) + \cdots && (27) \end{aligned}$$

So the Lie bracket on $\mathfrak{gl}(n)$ is

$$[x, y] = xy - yx \tag{28}$$

We can use this formula to compute the Lie bracket for closed Lie subgroups $G \subseteq GL(n)$. For $x, y \in \mathfrak{g} \subseteq \mathfrak{gl}(n)$, xy need not be in $\mathfrak{g}$, but $xy - yx$ will be in $\mathfrak{g}$, and that is $[x, y] \in \mathfrak{g}$

Remark: If G is a complex Lie group, then $\mathfrak{g}$ is a complex vector space, and $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is $\mathbb{C}$ −bilinear and alternating.

Other ways to think of the Lie bracket:

$$\exp(sx)\exp(ty)\exp(sx)^{-1}\exp(ty)^{-1} = \exp\{st[x, y] + \cdots\}$$

(for $s, t \in \mathbb{R}$ near 0, $x, y \in \mathfrak{g}$).
Can check from definition on $[\ ,\ ]$. Yet another way,

$$\exp(sx)\exp(ty)\exp(sx)^{-1} = \exp\{ty + st[x, y] + \cdots\}$$

Lemma 28
For any homomorphism $f : G \to H$ of Lie groups, $df|_1 : \mathfrak{g} \to \mathfrak{h}$ is compatible (commute) with Lie brackets:

$$[df|_1(x), df|_1(y)] = df|_1[x, y] \quad \forall x, y \in \mathfrak{g} \tag{29}$$

Proof
Easy, using that $f(\exp(tx)) = \exp(t \cdot df|_1(x))$ □

Definition 29
A representation V of a Lie group G is a vector space over $K = \mathbb{R}$ or $\mathbb{C}$ with a smoth map $G \times V \to V$ s.t.:

(1) $(gh)(x) = g(h(x)) \quad \forall g, h \in G, x \in V$ (definition of group action on a set)

(2) $1(x) = x \quad \forall x \in V$

(3) $\forall g \in G, x \mapsto gx$ is a linear map $V \to V$

Note: these maps $x \mapsto gx$ are in $GL(V)$, so we can think of a representation as a homomorphism of Lie groups $G \to GL(V)$

Example:
We could have every $g \in G$ act as identity on V, a trivial representation of G. In particular, $V = \mathbb{C}$ is the trivial complex representation of G

Example:
$GL(n, \mathbb{R})$ has an obvious representation on $\mathbb{R}^n$ the standard representation. So any subgroup of $GL(n, \mathbb{R})$ say $O(n)$, has a standard representation on $\mathbb{R}^n$

Example:
For any Lie group G and any $g \in G$, conjugation: $\begin{array}{rcl} C_g : G & \to & G \\ h & \mapsto & ghg^{-1} \end{array}$ is an isomorphism of Lie groups. The derivative of C_g is a linear map

$$\mathrm{Ad}(g) := dC_g|_1 : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g} \tag{30}$$

Lemma 30
$\mathrm{Ad} : G \to GL(\mathfrak{g})$ is a linear representation, called adjoint, of G

Proof
$C_{gh} = C_g C_h \quad \forall g, h \in G$
Taking derivatives shows that $\mathrm{Ad}(gh) = \mathrm{Ad}(g)\,\mathrm{Ad}(h)$ □

Since C_g is a group homomorphism $G \to G$, by Lemma 28, we have:

$$\mathrm{Ad}(g)[x, y] = [\mathrm{Ad}(g)(x), \mathrm{Ad}(g)(y)] \in \mathfrak{g} \quad \forall g \in G, x, y \in \mathfrak{g} \tag{31}$$

Example:
For $G = GL(n, \mathbb{R})$ the adjoint representation of $GL(n)$ of n-dimensional is:

$$g \in GL(n, \mathbb{R}), x \in \mathfrak{gl}(n) \quad \mathrm{Ad}(g)(x) = gxg^{-1} \in \mathfrak{gl}(n) \tag{32}$$

The formula (31) can be checked by hand (exercise) in this case that
For $g \in GL(n), x, y \in \mathfrak{gl}(n) \quad g[x, y]g^{-1} = [gxg^{-1}, gyg^{-1}]$

Note that the adjoint representation measures the non-commutativity of G. If G is abelian, then the adjoint representation is trivial.

Lemma 31
Let G be any Lie group
Let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$ be the derivative at 1 of the adjoint representation of G. Then $\forall x, y \in \mathfrak{g}$

$$\mathrm{ad}(x)(y) = [x, y] \in \mathfrak{g} \tag{33}$$

Proof
For $g \in G, y \in \mathfrak{g}$, we have

$$\mathrm{Ad}(g)(y) = \frac{d}{dt}\bigg|_{t=0} g \exp(ty) g^{-1} \tag{34}$$

Therefore, $\forall x, y \in \mathfrak{g}$,

$$\begin{aligned} \mathrm{ad}(x)(y) &= \frac{d}{ds}\bigg|_{s=0} \mathrm{Ad}(\exp(sx))(y) && (35) \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \exp(sx)\exp(ty)\exp(sx)^{-1} && (36) \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \exp(ty + st[x, y] + \cdots) && (37) \\ &= [x, y] && (38) \end{aligned}$$

□

I know that $\mathrm{Ad}: G \to GL(\mathfrak{g})$ is a Lie group homomorphism.
Therefore, by Lemma 28, the linear map $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ must preserve Lie brackets: $\mathrm{ad}[x,y] = [\mathrm{ad}\, x, \mathrm{ad}\, y] \in \mathfrak{gl}(\mathfrak{g})$
But I know how to compute the Lie bracket in $\mathfrak{gl}(V)$.

$$\begin{aligned} \mathrm{ad}[x,y] &= [\mathrm{ad}\, x, \mathrm{ad}\, y] = (\mathrm{ad}\, x)(\mathrm{ad}\, y) - (\mathrm{ad}\, y)(\mathrm{ad}\, x) \in \mathfrak{gl}(\mathfrak{g}) && (39)\\ \Rightarrow \quad \forall x,y,z \in \mathfrak{g} \qquad \mathrm{ad}[x,y](z) &= (\mathrm{ad}\, x)(\mathrm{ad}\, y)(z) - (\mathrm{ad}\, y)(\mathrm{ad}\, x)(z) && (40) \end{aligned}$$

That is,

$$\begin{aligned} [[x,y],z] &= [x,[y,z]] - [y,[x,z]] && (41)\\ &= -[[y,z],x] + [y,[z,x]] && (42)\\ &= -[[y,z],x] - [[z,x],y] && (43) \end{aligned}$$

Theorem 32 (The Jacobi identity)
For any lie group G, any $x,y,z \in \mathfrak{g} := T_1G$, we have a Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.t.

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

Proof
This can be proved directly from the power series $f(x,y)$ s.t.:

$$\begin{aligned} \exp(f(x,y)) &= \exp(x)\exp(y)\\ f(x,y) &= x + y + \frac{1}{2}[x,y] + f_3(x,y) + \cdots \end{aligned}$$

Associativity of this operation, i.e. $f(f(x,y),z) = f(x, f(y,z))$ implies the Jacobi identity
$(xy)z = x(yz)$ in $G \quad \Rightarrow \quad$ Jacobi identity in $\mathfrak{g}$
Failure of $xy = yx$ in $G \quad \Rightarrow \quad$ failure of $[\ ,\ ]$ in $\mathfrak{g}$
$xyx^{-1}y^{-1} = (yxy^{-1}x^{-1})^{-1} \quad \Rightarrow \quad [x,y] = -[y,x]$ in $\mathfrak{g}$ □

Definition 33
Let k be any field. A Lie algebra over k is a k-vector space $\mathfrak{g}$ with an alternating k-bilinear form $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the Jacobi identity

For any Lie group G, $\mathfrak{g} := T_1G$ is a Lie algebra over $\mathbb{R}$
For any complex Lie group G, $\mathfrak{g} := T_1G$ is a complex Lie algebra.

Note that, if we pick a basis $e_1, \dots, e_n$ for a Lie algebra $\mathfrak{g}$ over k, $\mathfrak{g}$ is determined by the n^3 different numbers $a_{ijk} \in k$, the structure constants:

$$[e_i, e_j] = \sum_{k=1}^{n} a_{ijk} e_k \qquad 1 \le i,j,k \le n \tag{44}$$

These numbers satisfy some simple conditions, alternating and Jacobi identity.

Definition 34
A homomorphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$ over k is a k-linear map s.t.

$$f[x,y] = [f(x), f(y)] \in \mathfrak{h} \qquad \forall x,y \in \mathfrak{g} \tag{45}$$

If $f : G \to H$ is a homomorphism of Lie groups, then $df|_1 : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebra over $\mathbb{R}$

Definition 35
A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a k-linear subspace $\mathfrak{h} \subset \mathfrak{g}$ s.t.

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \tag{46}$$

(that is, $[x, y] \in \mathfrak{h} \quad \forall x, y \in \mathfrak{h}$) If $H \leq G$ is a closed Lie subgroup, then $T_1 H$ is a Lie subalgebra of $T_1 G$

Definition 36
An ideal $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$ is a k-linear subspace $\mathfrak{h} \subset \mathfrak{g}$ s.t.

$$[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h} \tag{47}$$

If $H \trianglelefteq G$ is a normal closed Lie subgroup of a Lie group, then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ (Adjoint representation, $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ preserves the linear subspace $\mathfrak{h} \subset \mathfrak{g}$,i.e. $\mathfrak{h}$ is an ideal of $\mathfrak{g}$)

Lemma 37
Let $f : \mathfrak{g} \to \mathfrak{h}$ be any homomorphism of Lie algebra over a field k. Then $\ker f$ is an ideal in $\mathfrak{g}$, and $\mathfrak{g} / \ker(f) \subset \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{h}$
Conversely, if $\mathfrak{a} \subset \mathfrak{g}$ is any ideal, then $\mathfrak{g} / \mathfrak{a}$ is a Lie algebra in a natural way.

Proof
f is a k-linear map, so $\ker(f) = \{x \in \mathfrak{g} \,|\, f(x) = 0 \in \mathfrak{h}\} \subset \mathfrak{g}$ is a k-linear subspace. If $x \in \ker(f)$ and $y \in \mathfrak{g}$, then

$$f[x, y] = [f(x), f(y)] = [0, f(y)] = 0$$

So $[x, y] \in \ker(f)$. That is, $\ker(f)$ is an ideal

If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, let $x, y \in \mathfrak{g} / \mathfrak{a}$. Let $\tilde{x}, \tilde{y} \in \mathfrak{g}$ s.t. they maps to x, y under $\mathfrak{g} \to \mathfrak{g} / \mathfrak{a}$. Then,

$$[x, y] := [\tilde{x}, \tilde{y}] \in \mathfrak{g} \mod \mathfrak{a}$$

This is well-defined in $\mathfrak{g} / \mathfrak{a}$ because $\mathfrak{a}$ is an ideal
Alternating and Jacobi identity on $\mathfrak{g} / \mathfrak{a}$ are immediate from $\mathfrak{g}$ $\square$

Theorem 38
Let G be any Lie group, and let $\mathfrak{h} \subset \mathfrak{g}$ be any Lie subalgebra. Then $\exists! H$ connected Lie group with a homomorphism $H \to G$ which is an injective immersion and with $T_1 H = \mathfrak{h} \subset \mathfrak{g}$

Definition 39
A smooth map of manifolds, $f : M \to N$ is an immersion if $df|_x : T_x M \to T_{f(x)} N$ is injective $\forall x \in M$

Example:
There is an immersion $\mathbb{R} \to \mathbb{R}^2$ with image: (see notes for pictures)

Even if an immersion is injective, it needs not be a homeomorphism onto its image $f(M) \subset N$ (with the subspace topology)

Example 2(see notes)

Example 3

There is a homeomorphism of Lie groups $f : \mathbb{R} \to (S^1)^2 = \mathbb{R}^2 / \mathbb{Z}^2$ which is an injective immersion but with $f(\mathbb{R}) \subset (S^1)^2$ not closed and $f : \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism

Some one-parameter subgroup $\mathbb{R} \to \mathbb{R}^2 / \mathbb{Z}^2$ is given by $f(t) = (t, at)$ where $a \in \mathbb{R}$
If $a \in \mathbb{Q}$, then $f(\mathbb{R}) \cong S^1$ and it's a closed Lie subgroup of $(S^1)^2$
If $a \notin \mathbb{Q}$, then $f : \mathbb{R} \to (S^1)^2$ is an injective immersion, but $f(\mathbb{R})$ not closed in $(S^1)^2$

$f(\mathbb{R})$ looks like: (see notes for picture)
$f(\mathbb{R})$ is dense in $(S^1)^2$ (not closed)

Sketch Proof of theorem
This is proved in:
M. Spuak, A comprehensive introduction to differential geometry
F. Warner, Foundations of differential manifolds and Lie groups
Part III Differential Geometry course later this term

Given a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ this determines what T_xH should be for any $x \in H$. We must have $T_xH = (dL_x|_1)(\mathfrak{h}) \subseteq T_x(G)$. So H is tangent to a "smooth distribution" $S_x \subset T_xG \quad \forall x \in G$. The assumption that $\mathfrak{h}$ is a Lie subalgebra is exactly the hypothesis for "Frobenius Theorem", which ensures the existence of an immersed connected "submanifold" with the given tangent space everywhere.
This manifold H (through 1) is unique if you take it to be maximal. One checks that it is a subgroup. □

Theorem 40
Let G be a simply connected Lie group G, H any Lie group. Then there is a one-to-one correspondence between Lie group homomorphism $G \to H$ and Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$. i.e. (and more explicitly)

$$\{f : G \to H \text{ Lie group hom.}\} \leftrightarrow \{df|_1 : \mathfrak{g} \to \mathfrak{h} \text{ Lie algebra hom.}\}$$

Proof
Roughly:

We know that a homomorphism $f : G \to H$ determines a Lie algebra homomorphism $df|_1 : \mathfrak{g} \to \mathfrak{h}$. We have shown that any homomorphism $\alpha : \mathfrak{g} \to \mathfrak{h}$ of Lie algebra comes from a homomorphism of Lie groups using that G is simply connected

Idea: α gives a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$, namely the graph of α, $\Gamma_\alpha = \{(x, \alpha(x)) \in \mathfrak{g} \times \mathfrak{h} \,|x \in \mathfrak{g}\}$. So this correspond to some connected Lie group K with an injective immersion $K \hookrightarrow G \times H$

One checks that $K \cong G$ and $G \to G \times H$ is the graph of a homomorphism $G \to H$

In details: Given $f : \mathfrak{g} \to \mathfrak{h}$ a Lie group homomorphism.
The graph of f, $\Gamma_f := \{(x, f(x)) : x \in \mathfrak{g}\}$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$, $[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [h_1, h_2])$ (Note that $[(g, 0), (0, h)] = 0$).

So there is a connected Lie group K with an injective immersion and with Lie algebra$=\mathfrak{k} \subseteq \mathfrak{g} \times \mathfrak{h}$, $\mathfrak{k} \cong \mathfrak{g}$
So the composition $K \hookrightarrow G \times H \to G$ induces an isomorphism on tangent space at 1.
Therefore, (by Example Sheet 1), $K \to G$ is a covering map. But G is simply connected, so $K \cong G$.
So we get our homomorphism $G \to H$ □

Corollary 41
Two simply connected Lie groups are isomorphism iff their Lie algebras are isomorphic

Proof
$\Leftarrow$: If $f : \mathfrak{g} \xrightarrow{\sim} \mathfrak{h}$ isomorphic as Lie algebra, then both f and f^{-1} come from homomorphism $G \to H$

and $H \to G$ (Here, G, H are simply connected Lie groups with those Lie algebras). You can check that both compositions $G \to H \to G$ and $H \to G \to H$ are the identity $\square$

Theorem 42 (Ado's Theorem)
Every finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ can be embedded as a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some $n < \infty$
(c.f.: Fulton-Harris, Appendix C)

Theorem 43
Every finite dimensional real Lie algebra $\mathfrak{g}$ is the Lie algebra of a unique simply connected Lie group G. Also, every finite dimensional complex Lie algebra is the Lie algebra of a unique simply connected complex Lie group.

Proof
Use Ado's Theorem.
Given that, we have $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$
Therefore, there is a connected Lie group G with Lie algebra $\mathfrak{g}$ and an injective immersion $G \hookrightarrow GL(n, \mathbb{R})$
Therefore the universal cover G is the simply connected Lie group we want. $\square$

Can we describe all the connected Lie groups with a given Lie algebra $\mathfrak{g}$?
Let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then any connected Lie group with Lie algebra $\mathfrak{g}$ has the form $G = \tilde{G}/Z$ for some discrete central subgroup $Z \subseteq \tilde{G}$

Example:
Describe all n-dimensional connected abelian Lie groups. Here $\mathfrak{g} \cong \mathbb{R}^n$ with $[\ ,\ ] = 0$
Here $\tilde{G} = (\mathbb{R}^n, +)$. What are the discrete subgroups $Z \subseteq \tilde{G}$? (see picture)

We have $Z \cong \mathbb{Z}^a$ for some $0 \leq a \leq n$. Then $\tilde{G}/Z \cong (S^1)^a \times \mathbb{R}^{n-a}$ as a Lie group $G = \tilde{G}/Z$

Example:
What are all the connected Lie groups with Lie algebra $\mathfrak{su}(2)$?
One is $SU(2) \cong S^3 \cong Sp(1)$, hence is simply connected.
$Z(SU(2)) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in SU(2)\} = \{\pm 1\} \subseteq SU(2)$
So the possible connected groups with Lie algebra $\mathfrak{su}(2)$ are $SU(2)$ and $SU(2)/\{\pm 1\} = PSU(2) = SO(3)$
The isomorphism $SU(2)/\{\pm 1\} \xrightarrow{\sim} SO(3)$ is given by the adjoint representation $SU(2) \to GL(\mathfrak{su}(2)) \cong GL(\mathbb{R}^3)$
Image=$SO(3)$, kernel=$Z(SU(2)) = \{\pm 1\}$

More generally, for any connected Lie group G,

$$\ker(\mathrm{Ad} : G \to GL(\mathfrak{g})) = Z(G) = \mathrm{centre}(G) = \{g \in G | gh = hg \quad \forall h \in G\}$$

Definition 44
A (finite dimensional) representation V of a Lie algebra $\mathfrak{g}$ over a field k, also called a $\mathfrak{g}$-module, is a k-vector space together with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}_k(V)$
Equivalently, ρ gives a bilinear map $\mathfrak{g} \times V \to V$ which satisfies

$$[u, v](x) = u(v(x)) - v(u(x)) \quad \forall u, v \in \mathfrak{g}, x \in V$$

Remark:
In this sense, a representation of Lie group is finite dimensional by definition. But the definition of a representation of a Lie algebra makes sense even for V infinite dimensional

Example: ad:$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation of $\mathfrak{g}$, the adjoint representation

Given a real representation V of a Lie group G. V is also a representation of the Lie algebra $\mathfrak{g}$. Explicitly, this representation of $\mathfrak{g}$ is given by:

$$u(x) = \frac{d}{dt}\bigg|_{t=0} \underbrace{\exp(tu)}_{G}(x) \in V \qquad u \in \mathfrak{g}, x \in V$$

Conversely, let G be a simply connected Lie group. Then a finite dimensional representation V of $\mathfrak{g}$ comes from a unique representation of G

$$\{\text{f.d. real repn } \rho : \mathfrak{g} \to \mathfrak{gl}(V)\} \leftrightarrow \{\text{f.d. real repn } \rho : G \to GL(V)\}$$

By the commutativity of exponential map and the representation of $\mathfrak{g}$. Explicitly,

$$\begin{aligned} \rho(\underbrace{\exp(u)}_{G})(x) &= x + \rho(u)(x) + \frac{\rho(u)^2}{2!}(x) + \cdots \qquad u \in \mathfrak{g}, x \in V \\ &= \underbrace{\exp(\rho(u))} \in GL(V)(x) \end{aligned}$$

Also, for a complex Lie group G, complex analytic representation of G give representations of the Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, and this is an equivalence for finite dimensional representations if G is simply connected.

$$\begin{aligned} \{\text{f.d. } \mathbb{C} \text{ analytic repn } \rho : G \to GL(V)\} &\leftrightarrow \{\text{f.d. } \mathbb{C}\text{-repn } \rho : \mathfrak{g} \to \mathfrak{gl}(V)\} \quad G \text{ simply connected} \\ \{\mathbb{C} \text{ analytic repn } \rho : G \to GL(V)\} &\rightsquigarrow \{\mathbb{C}\text{-repn } \rho : \mathfrak{g} \to \mathfrak{gl}(V)\} \end{aligned}$$

Example:
Complex analytic representations of $SL(2, \mathbb{C})$ are equivalent to finite dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$
Indeed, $SL(2, \mathbb{C})$ is simply connected, , because $S^3 = SU(2) \hookrightarrow SL(2, \mathbb{C})$ ($SL(2, \mathbb{C})$ is dimension 3 over $\mathbb{C}$) is a homotopy equivalence.

Let V be a complex representation of a real Lie group G. Then we have a homomorphism of real Lie groups $G \to GL(V) \quad \Rightarrow \quad$ gives a homomorphism of real Lie algebras $\mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$
But this is equivalent to a representation of the complex Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$
We can describe this as $\mathfrak{g} \oplus i\,\mathfrak{g}$, with $\mathbb{C}$ acting in the obvious way. It is a complex Lie algebra (define $[\ ,\ ]$ to be $\mathbb{C}$-bilinear). If $\dim_{\mathbb{R}} \mathfrak{g} = n$, then $\dim_{\mathbb{C}}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}) = n$

$$\boxed{\mathbb{C}\text{-repn of } \underline{\text{real}}\ G} \rightsquigarrow \boxed{\mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})} \leftrightarrow \boxed{\mathbb{C}\text{-repn of } \underline{\text{complex}}\ \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\,\mathfrak{g}}$$

Example:
Complex representations of the compact Lie group $SU(2)$
$\leftrightarrow$ complex representation of the real Lie algebra $\mathfrak{su}(2)$
$\leftrightarrow$ representations of the complex Lie algebra $\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$

Proof
$\mathfrak{su}(2) = \{A \in M_2\,\mathbb{C} \,|\, \mathrm{tr}(A) = 0, A + A^* = 0\}$
$i\mathfrak{su}(2) = \{A \in M_2\,\mathbb{C} \,|\, \mathrm{tr}(A) = 0, A^* = A\}$
$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$ □

Representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$

$$\mathfrak{sl}(2,\mathbb{C}) = \{A \in M_2\,\mathbb{C} \,|\, \mathrm{tr}(A) = 0\}$$

A basis for $\mathfrak{sl}(2)$ as a $\mathbb{C}$ vector space is:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We compute the Lie brackets:

$$\begin{aligned} [h,e] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ \Rightarrow \qquad [h,e] &= 2e \end{aligned}$$

We compute that $[h,f] = -2f$ and $[e,f] = h$

Let V be any finite dimensional representation of $\mathfrak{sl}(2\,\mathbb{C})$.
That is: we have $e, f, h \in \mathrm{End}(V)$ which satisfy

$$\begin{aligned} [h,e] &= he - eh = 2e \\ [h,f] &= -2f \\ [e,f] &= h \end{aligned}$$

Idea: Divide up V according to eigenspaces with respect to h.
If $V \neq 0$, then (since we are over $\mathbb{C}$) h has some eigenvector

that is, $\exists x \in V, x \neq 0$ and $hx = \lambda x$ for some $\lambda \in \mathbb{C}$.
What can we say about ex and $fx \in V$?
We know that, for example:
$hex - ehx = 2ex$ $hex - ehx = hex - e(\lambda x) = h(ex) - \lambda(ex)$
Thus

$$h(ex) = (\lambda + 2)ex$$

That is, ex is the $(\lambda+2)$-eigenspace for h

Likewise, using that $[h,f] = -2f$, we find that

$$h(fx) = (\lambda - 2)fx$$

i.e. f maps the λ-eigenspace for h into the $(\lambda-2)$-eigenspace for h

Notice that for any $x \neq 0$ an h-eigenvector with weight λ (=eigenvalue for h), then
ex has weight $\lambda + 2$
e^2x has weight $\lambda + 4$, etc.
But since V is finite dimensional, h has only finitely many eigenvalues on V. Therefore, $e^r x$ must be 0 for some $r \geq 1$

Likewise, $f^r x$ must be 0 for some $r \geq 1$

Definition 45
A highest weight vector x in a representation of $\mathfrak{sl}(2,\mathbb{C})$ is a vector $x \neq 0$ in V which is an h-eigenvector (so $hx = \lambda x$ for some $\lambda \in \mathbb{C}$) and $ex = 0$

If $V \neq 0$ is a finite dimensional representation of $\mathfrak{sl}(2)$, then V contains a highest weight vector, we have shown.

Let x be a highest weight vector in a finite dimensional representation of $\mathfrak{sl}(2)$, with weight $\lambda \in \mathbb{C}$.
We know that $hx = \lambda x$ and $ex = 0$
What can we say about fx?
It is weight $\lambda - 2$ that is : $h(fx) = (\lambda - 2)fx$ What is efx?
We know that $[e, f] = h \in \mathfrak{sl}(2)$,hence in End$(V)$
Therefore, $efx - fex = hx$. But $fex = 0$ as $ex = 0$ and $hx = \lambda x$
So $e(fx) = hx$
Next, what can we say about f^2x?
We know that $h(f^2x) = (\lambda - 4)(f^2x)$
What is $e(f^2x)$? It is some vector of weight $\lambda - 2$.
We use that $[e, f] = h$ again:

$$\begin{aligned} ef^2x &= fe(fx) + h(fx) \\ &= f(\lambda x) + (\lambda - 2)fx \\ &= (2\lambda - 2)fx \end{aligned}$$

One more step: What is $e(f^3x)$?
Again, use $[e, f] = h$

$$\begin{aligned} ef^3x &= fef^2x - hf^2x \\ &= f((2\lambda - 2)fx) + (\lambda - 4)f^2x \\ &= (3\lambda - 6)f^2x \end{aligned}$$

Summary:
f^rx has weight $\lambda - 2r$ for some $r \geq 0$
$e(fx) = \lambda x$
$e(f^2x) = (2\lambda - 2)fx$
$e(f^3x) = (3\lambda - 6)f^2x$
etc. By induction, we show that for $r \geq 1$,

$$\begin{aligned} e(f^rx) &= (r\lambda - 2(1 + 2 + \cdots + (r-1))) - f^{r-1}x \\ &= (r\lambda - r(r-1))f^{r-1}x \\ &= r(\lambda - r + 1)f^{r-1}x \end{aligned}$$

Say $f^{r+1}x$ is the first element that becomes 0. Then, $x, fx, f^2x, \ldots, f^rx$ are all nonzero in V. They are all h-eigenvectors with different eigenvalues, namely, $\lambda, \lambda - 2, \ldots, \lambda - 2r \in \mathbb{C}$
Therefore, $x, fx, \ldots, f^rx$ are linearly independent in V. Let $S \subset V$ be the $\mathbb{C}$-linear subspace they span.
Then $S \subseteq V$ is a <u>subrepresentation</u> of V for $\mathfrak{sl}(2)$

Definition 46
Let V be a representation of a Lie algebra $\mathfrak{g}$ over k. Then a <u>subrepresentation</u> $S \subseteq V$ (or <u>$\mathfrak{g}$-submodule</u>) is a k-linear subspace s.t. $ux \in S \quad \forall u \in \mathfrak{g}, x \in s$.

Definition 47
An <u>irreducible representation</u> V of a Lie algebra $\mathfrak{g}$ is a representation s.t. $V \neq 0$ and V contains no $\mathfrak{g}$-submodules $0 \subsetneq S \subsetneq V$

Suppose that V is a finite dimensional irreducible representation of $\mathfrak{sl}(2)$. Let x be a highest weight vector in V. Then the subspace $S = \mathbb{C}\{x, fx, \ldots, f^rx\} \subset V$ is equal to V

What can we say about the weight $\lambda \in \mathbb{C}$ of the highest weight vector x?

Theorem 48
The weight of a highest weight vector for a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ is a natural number

Proof
We use that $e(f^{r+1}x) = (r+1)(\lambda - r)f^r x$, but $f^{r+1}x = 0$
$\Rightarrow \quad 0 = (r+1)(\lambda - r)f^r x \in V$ where $f^r x \neq 0 \in V$
$\Rightarrow \quad$ must have $(r+1)(\lambda - r) = 0 \in \mathbb{C}$
Here $r \in \{0, 1, 2, \dots\} = \mathbb{N}_0$
$\Rightarrow \quad r+1 \neq 0 \in \mathbb{C} \qquad \Rightarrow \qquad \lambda = r$ □

Notice that the representation of $\mathfrak{sl}(2)$ given by the above formulae, for any $r \in \mathbb{N}$, are completely determined by the number $\lambda (= r$ in later formulae)
That is, this representation has basis:

$$x, fx, f^2x, \dots, f^\lambda x$$

The formula we wrote describe how $\mathfrak{sl}(2)$ acts in this basis

Theorem 49
The finite dimensional irreducible representation V of $\mathfrak{sl}(2, \mathbb{C})$ are classified up to isomorphism by one number $\lambda \in \mathbb{N}$, the weight of a highest weight vector (unique up to scalars in V) in V.
(Here $\dim_{\mathbb{C}} V_\lambda = \lambda + 1$)

How do these irreducible representations of $\mathfrak{sl}(2)$ arises in nature?
There is the standard representation $V \cong \mathbb{C}^2$ of the group $SL(2, \mathbb{C})$.
Therefore, any $\lambda \in \mathbb{N}$, $S^\lambda V$ (the λth symmetric power) is also a representation of $SL(2, \mathbb{C})$
Here, if V has $\mathbb{C}$-basis e_1, e_2, $S^\lambda V$ means the $\mathbb{C}$-vector space of homogeneous polynomials of degree λ in e_1, e_2. That is:

$$S^\lambda V = \{a_0 e_1^\lambda + a_1 e_1^{\lambda-1} e_2 + \cdots a_\lambda e_2^\lambda\}$$

If $f \in SL(2, \mathbb{C})$

$$f(e_1^a e_2^{\lambda-a}) = f(e_1)^a f(e_2)^{\lambda-a} \in S^\lambda V$$

This representation, as a representation of $\mathfrak{sl}(2, \mathbb{C})$ is the irreducible representation we described

Tensor Product

Theorem 50
For any vector spaces V, W over a field k, there is a vector space $V \otimes_k W$ (the tensor product) with a k-bilinear map $f : V \times W \to A$, $\exists!$ linear map $g : V \otimes_k W \to A$ with $f = (V \times W \to V \otimes_k W \xrightarrow{g} A)$

Proof
See commutative algebra (Part III)/representation theory (Part II) □

Example:
If V has a basis $e_1, \dots, e_m$ and W has a basis $f_1, \dots f_n$, then $V \otimes_k W$ has a baiss $e_i \otimes f_j$, $1 \leq i \leq m, 1 \leq j \leq n$. So $\dim_k(V \otimes_k W) = (\dim_k V)(\dim_k W)$
So every element of $V \otimes W$ can be written as $\sum a_{ij} e_i \otimes f_j,\ a_{ij} \in k$
Note: Some element can be written $v \otimes w$ for a simple $v \in V$, $w \in W$
Not-so-related-notes: Compare the direct sum:

$$V \oplus W = \{(v, w) | v \in V, w \in W\}$$

here, $\dim_k(V \oplus W) = \dim_k V + \dim_k W$

Example:
$V^* \otimes_k W = \{k \text{ linear maps } V \to W\}$
if V is finite dimensional.
A linear map correspond to a "symbol" $f \otimes w$ where $f \in V^* = \mathrm{Hom}_k(V,k) \Leftrightarrow$ it has rank ≤ 1
(Here $f \otimes w \in V^* \otimes W$ corresponds to the linear map $V \to W, x \mapsto f(x)w \in W$ $(f(x) \in k)$)

Symmetric Products and Exterior Products

Definition 51
Let V be a k-vector space, $a \in \mathbb{N}$, Then the a-th symmetric power $S^aV = \mathrm{Sym}^aV$ is the quotient space

$$V \otimes_k \cdots \otimes_k V / \left(v_1 \otimes \cdots \otimes v_a = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(a)}\right), \sigma \in S_a$$

which is a k-vector space
Write $v_1 v_2 \cdots v_a$ for the image of $v_1 \otimes \cdots \otimes v_a$ in S^aV. If V has a k-basis $e_1, \ldots e_n$, then S^aV is the space of homogeneous polynomial of degree a in $e_1, \ldots, e_n$.
We compute that $\dim_k S^aV = \binom{n+a-1}{a}$

Definition 52
For a k-vector space V, $a \geq 0$, the a-th exterior power of V is

$$\bigwedge^a V := V \otimes_k \cdots \otimes_k V / \left(v_1 \otimes \cdots \otimes v_a = \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(a)}\right), \sigma \in S_a$$

which is a k-vector space

Here, for example: $v \wedge v = 0 \;\; \forall v \in V$ and $v \wedge w = - \wedge v \;\; \forall v, u \in V$
If V has a basis $e_1, \ldots, e_n$, then $\wedge^a V$ has a k-basis $e_{i_1} \wedge \cdots e_{i_a}$ if $1 \leq i_1 < \cdots < i_a \leq n$
So $\dim_k \bigwedge^a V = \binom{n}{a}$

If V, W are representations of any Lie group G, then $S^aV, \wedge^a V$ and $V \otimes_k W$ are also representations of G (G acts by $g(v \otimes w) = gv \otimes gw$,etc)

Let V, W be representations of a Lie group G.
Then $V \otimes_k W$ is a representation of G, hence a representation of the Lie algebra $\mathfrak{g}$. How does $u \in \mathfrak{g}$ act on $V \otimes W$? We have

$$\begin{aligned} u(v \otimes w) &= \left.\frac{d}{dt}\right|_{t=0} \exp(tu)(v \otimes w) \\ &= \left.\frac{d}{dt}\right|_{t=0} (1 + tu + \cdots)(v \otimes w) \\ &= (1 + tu + \cdots)(v) \otimes (1 + tu = \cdots)(w) \\ &= v \otimes + t(uv \otimes w + v \otimes uw) + O(t^2) \end{aligned}$$

So $u \in \mathfrak{g}$ acts on $V \otimes W$ by the Leibniz rule:

$$u(v \otimes w) = uv \otimes w + v \otimes uw$$

If V, W are any representation of a Lie algebra $\mathfrak{g}$ over a field (representation could be infinite dimensional) then the Leibniz rule defines a representation of $\mathfrak{g}$ on $V \otimes_k W$

Likewise, for a representation V of a Lie algebra $\mathfrak{g}$ over a field, S^aV is a representation of $\mathfrak{g}$ given by

$$u(v_1 \cdots v_a) = (uv_1)(v_2 \cdots v_a) + v_1(uv_2) \cdots v_a + \cdots + v_1 \cdots v_{a-1}(uv_a)$$

for $u \in \mathfrak{g}, v_1, \cdots, v_a \in V$.

Likewise, action of $\mathfrak{g}$ on $\bigwedge^n V$ is given by,

$$u(v_1 \wedge \cdots \wedge v_a) = (uv_1) \wedge v_2 \wedge \cdots \wedge v_a + \cdots + v_1 \wedge \cdots \wedge v_{a-1} \wedge (uv_a)$$

Example:
How does the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ act on S^aV, where $V \cong \mathbb{C}^2$ is the standard representation?
We can write how e, f, h act on the basis $e_1^a, e_1^{a-1}e_2, \cdots, e_1e_2^{a-1}, e_2^a$
We have

$$e(e_1) = 0, e(e_2) = e_1$$
$$f(e_1) = e_2, f(e_2) = 0$$
$$h(e_1) = e_1, h(e_2) = -e_2$$

$$\begin{aligned} \Rightarrow \qquad h(e_1^ie_2^{a-i}) &= i(he_1)e_1^{i-1}e_2^{a-i} + (a-i)e_1^ih(e_2)e_2^{a-i-1} \\ &= (i-(a-i))e_1^ie_2^{a-i} \\ &= (2i-a)e_1^ie_2^{a-i} \qquad (0 \le i \le a) \end{aligned}$$

$$\Rightarrow \begin{cases} e_1^a & \text{is in weight } a \\ e_1^{a-1}e_2 & \text{is in weight } a-2 \\ \vdots & \\ e_2^a & \text{is in weight } -a \end{cases}$$

Example Sheet 2: compute action of e and f, you use that e_1^a is the highest weight vector, up to scalars, so $S^aV \cong$ the irreducible representation of $\mathfrak{sl}(2)$ of the highest weight a, $a \in \mathbb{N}$

Definition 53
If $S \subseteq V$ is a $\mathfrak{g}$-submodule then V/S is also a representation of $\mathfrak{g}$, the quotient representation.

Definition 54
Let $\mathfrak{g}$ be a Lie algebra over a field, and let V, W be two $\mathfrak{g}$-modules. Then a $\mathfrak{g}$-linear map $f : V \to W$ (or a homomorphism of representation of $\mathfrak{g}$) is a k-linear map such that $f(ux) = uf(x) \in W, u \in \mathfrak{g}, x \in V$
We say $V \cong W$ if there is a $\mathfrak{g}$-linear map $V \to W$ which is bijective

Lemma 55 (Schur's Lemma) (1) Let $\mathfrak{g}$ be a Lie algera $\mathfrak{g}$ over a field k, V, W irreducible representation of $\mathfrak{g}$. If $V \ncong W$, then $\mathrm{Hom}_{\mathfrak{g}}(V,W) = \{\mathfrak{g}\text{-linear maps } V \to W\} = 0$

(2) Let $k = \mathbb{C}$, let V be a finite dimensional irreducible representation of $\mathfrak{g}$ over $\mathbb{C}$.
Then $\mathrm{Hom}_{\mathfrak{g}}(V,V) \cong \mathbb{C} \cdot 1_V$

Proof

(1) Let $f : V \to W$ be a $\mathfrak{g}$-linear map. Suppose $f \neq 0$. Then $f(V) \subseteq W$ is a $\mathfrak{g}$-submodule and non-zero. So $f(V) = W$ since W is irreducible. Likewise, $\ker(f) \subseteq V$ is a $\mathfrak{g}$-submodule, and it is not equal to V. So $\ker(f) = 0$
So $f : V \to W$ is a $\mathfrak{g}$-linear isomorphism #

(2) What can we say about $\mathrm{Hom}_{\mathfrak{g}}(V,V)$ for an irreducible representation V of $\mathfrak{g}$?
One shows that $\mathrm{Hom}_{\mathfrak{g}}(V,V)$ is a division algebra over k (that is every $f \neq 0$ has an inverse)
Suppose that $k = \mathbb{C}$, and V is irreducible and finite dimensional.
Let $f : V \to V$ be a nonzero $\mathfrak{g}$-linear map. Know that $\exists x \in V, x \neq 0$ s.t. $f(x) = \lambda x$ some $\lambda \in \mathbb{C}$
Look at $f - \lambda 1_V \in \mathrm{Hom}_{\mathfrak{g}}(V,V)$
We know that this $\mathfrak{g}$-linear map sends $x \neq 0$ in V to 0. So $f - \lambda 1_V$ is not isomorphism, it must be 0, so $f = \lambda \cdot 1_V$

$\square$

Corollary 56
Let $\mathfrak{g}$ be an abelian Lie algebra over $\mathbb{C}$. Then every finite dimensional irreducible representation of $\mathfrak{g}$ is 1-dimensional. The 1-dimensional representation of $\mathfrak{g}$ are corresponding to the linear maps $\mathfrak{g} \to \mathbb{C}$

Proof
Let V be a finite dimensional irreducible representation. Then $\mathrm{Hom}_{\mathfrak{g}}(V,V) = \mathbb{C}\, 1_V$ by Schur's Lemma.
But for any $u \in \mathfrak{g}$, we have for any $v \in \mathfrak{g}, x \in V$ $uv(x) - vu(x) = [u,v](x) = 0(x) = 0$
So $u \in \mathrm{Hom}_{\mathfrak{g}}(V,V)$ so every element of $\mathfrak{g}$ acts by scalars on V
So every k-linear subspace of V is $\mathfrak{g}$-invariant. Since V is irreducible, $\dim_{\mathbb{C}} V = 1$ ✓

1-dimensional representation of $\mathfrak{g} \leftrightarrow$ homomorphism of Lie algebra
$\mathfrak{g} \to \mathfrak{gl}(1,\mathbb{C}) = \mathbb{C} \leftrightarrow$ a $\mathbb{C}$-linear map $\mathfrak{g} \to \mathbb{C}$, because $\mathfrak{g}$ is abelian $\square$

Definition 57
A finite dimensional representation of a Lie algebra $\mathfrak{g}$ is completely reducible if $V \cong V_1 \oplus \cdots \oplus V_r$ with V_i irreducible representations of $\mathfrak{g}$, for some $r \geq 0$

For any finite dimensional representation V of $\mathfrak{g}$, we can always find sub-$\mathfrak{g}$-moddules

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r = V$$

s.t. V_i/V_{i-1} are irreducible.
This need NOT imply that $V \cong \bigoplus V_i/V_{i-1}$

Example: Let $\mathfrak{g}$ be the 1-dimensional Lie algebra over $\mathbb{C}$, $\mathfrak{g} = \mathbb{C}\, e$. Then a representation of $\mathfrak{g}$ is exactly a vector space V with an endomorphism $e : V \to V$. We know how to classify such representations (Jordan Normal Form) in some basis for V

$$e = \left(\begin{array}{ccc|cc|c|c} a & 1 & & & & & \\ & a & 1 & & & & \\ & & a & & & & \\ \hline & & & b & 1 & & \\ & & & & b & & \\ \hline & & & & & c & \\ & & & & & & \ddots \end{array}\right)$$

Look at $S = \mathbb{C}\{e_1, e_2\}$. That is an e-invariant subspace of V and two such matrices are conjugate $\Leftrightarrow$ they are the same up to reordering the Jordan block

So a representation of the Lie algebra $\mathbb{C}\, e$ is completely reducible $\Leftrightarrow e \in \mathrm{End}(V)$ is diagonalizable

More generally, if a representation V of a Lie algebra $\mathfrak{g}$ has $\mathfrak{g}$-invariant subspace S, then (in a suitable basis for V) $\mathfrak{g} \to \mathrm{End}(V) = M_n\, \mathbb{C}$ maps into

$$\left(\begin{array}{c|c} A & * \\ \hline 0 & * \end{array}\right)$$

with A a $\dim S \times \dim S$ matrix

If $V = V_1 \oplus V_2$ as a representation of $\mathfrak{g}$, then (in some basis for V) the representation $\mathfrak{g} \to \mathrm{End}(V)$ maps into

$$\left(\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right)$$

with A a $\dim V_1 \times \dim V_1$ matrix, B a $\dim V_2 \times \dim V_2$ matrix

Theorem 58
Every finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible. Therefore representations of the groups $SL(2,\mathbb{C})$ and complex representations of $SU(2)$ are completely reducible.

Proof
We will show that the complex representations of $SU(2)$ are completely reducible, that implies the statement on $\mathfrak{sl}(2,\mathbb{C})$. More generally, we have the following theorem. $\square$

Theorem 59
For any compact Lie group G all its real or complex representations are completely reducible

Proof
(We will consider the $\mathbb{C}$-case, proof for the real is similar)
Let V be a complex representation of a compact Lie group G. We will show that V is unitary, that is: $\exists$ a positive definite hermitian form $\langle\ ,\ \rangle$ on V s.t.

$$\langle gx, gy\rangle = \langle x, y\rangle \qquad \forall x, y \in V, g \in G$$

(Recall properties of hermitian form:
(1) $\langle x, y\rangle : V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in x and conjugate-linear in y
(2) $\langle x, y\rangle = \overline{\langle y, x\rangle} \in \mathbb{C}$ (3) Positive definite) If V is a unitary representation of G, let $S \subseteq V$ be a G-invariant subspace. Then $S^\perp \subseteq V$, $S^\perp = \{x \in V | \langle x, y\rangle = 0\ \forall y \in S\}$ is also a G-subspace of V. Because $\langle\ ,\ \rangle$ is positive definite, $V = S \oplus S^\perp$. Repeating the process we see that V is a direct sum of irreducible representation

To prove that every $\mathbb{C}$-representation of a compact Lie group G is unitary, we average
For an oriented n-manifold M, let $w \in \Omega^n(M)$ be a smooth n-form. (So, at every $p \in M, w \in \bigwedge^n(T_p^* M)$)
In local coordinates,

$$w = f(x_1, \ldots, x_n) dx_1 \wedge \cdots \wedge dx_n$$

If g is compactly supported smooth for $g : M \to \mathbb{R}$, then we can define

$$\int_M gw \in \mathbb{R}$$

In local coordinates, this is

$$\int gf dx_1 \cdots dx_n$$

On a compact Lie group G let w be any non-zero element of $\bigwedge^{n'}(\mathfrak{g}^*) \cong \mathbb{R}$
This extends uniquely to a right-invariant n-form w on G
Use this to integrate all smooth functions on G, because G is compact
Because w is right-invariant, we have

$$\int_{g\in G} f(g) w = \int_{g \in G} f(gh) w \qquad \forall h \in G$$

Let V be a complex representation of a compact Lie group G
Let $\langle\ ,\ \rangle_0$ be a positive definite hermitain form on V
Define a hermitian form on V by

$$\langle x, y\rangle = \int_G \langle gx, gy\rangle_0 w(g) \quad \forall x, y \in V$$

THis is a hermitian form on V. It is positive definite because the integral of a positive form is positive.
Finally,

$$\langle hx, hy\rangle = \int_G \langle ghx, ghy\rangle_0 w(g) = \int_G \langle gx, gy\rangle_0 w(g) = \langle x, y\rangle$$

$\square$

Therefore, for any finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, h is always diagonalizable (=semisimple) in End(V)
Also e, f are always nilpotent on V (That is, $e^N = 0$ and $f^N = 0$ for some $N > 0$)
This is somehow related to the fact that
$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is daigonalizable in $M_2\,\mathbb{C}$
$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ nilpotent ($N = 2$)

Definition 60
The character of a representation V of $\mathfrak{sl}(2,\mathbb{C})$ is

$$\chi(V) = \sum_{j\in\mathbb{Z}} (\dim V_j) t^j \in \mathbb{Z}[t, t^{-1}]$$

where

$$\begin{aligned} V_j &= \text{weight-}j \text{ subspace of } V \\ &= \{x \in V | hx = jx\} \end{aligned}$$

(We know the eigenvalues of h on V are in $\mathbb{Z}$)

Easy Fact: The character of a representation of $\mathfrak{sl}(2,\mathbb{C})$ determine the representation up to isomorphism

Example: If V is a representation of $\mathfrak{sl}(2)$ with $\chi(V) = t^{-2} + 3 + t^2$. What is V?
Let A be the 2-dimensional standard representation of $\mathfrak{sl}(2)$, then

$$\chi(S^m A) = t^{-m} + t^{-m+2} + \cdots + t^{m-2} + t^m$$

and $\chi(V \oplus W) = \chi(V) + \chi(W)$ and $\chi(V \otimes_{\mathbb{C}} W) = \chi(V)\chi(W)$
Answer to question:
$V = S^2 A \oplus$(some representation with character 2)
$= S^2 A \oplus \mathbb{C} \oplus \mathbb{C}$ (where $\mathbb{C} = S^0 A$)

Theorem 61 (Clebsch-Gordon)
For any $a, b \in \mathbb{N}$, $a \leq b$, we have

$$S^a V \otimes S^b V \cong S^{a+b} V \oplus S^{a+b-2} V \oplus \cdots \oplus S^{a-b} V \tag{48}$$

as representation of $\mathfrak{sl}(2,\mathbb{C})$ (or the group $SL(2,\mathbb{C})$ or $SU(2)$)

Proof
Compute the character of the left side

$$\chi_{S^a V}(t) = t^{-a} + t^{-a+2} + \cdots + t^a$$

Want to know what does

$$(t^{-a} + t^{-a+2} + \cdots + t^a)(t^{-b} + t^{-b+2} + \cdots + t^b)$$

equals to.
Note that all weights in $S^a V \otimes S^b V$ are $\cong a + b \mod 2$
(see pictures in handwritten notes) □

Nilpotent and Solvable Lie Algebras

Definition 62
An abelian Lie algebra $\mathfrak{g}$ over a field k is a Lie algebra with $[\ ,\ ] = 0$

Definition 63
For a Lie algebra $\mathfrak{g}$ over k. The commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ (or derived algebra of $\mathfrak{g}$) is the k-linear subspace spanned by $[x, y]$, $x \in \mathfrak{g}, y \in \mathfrak{g}$
Then $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of the quotient Lie algebra $\mathfrak{g}^{\text{ab}} := \mathfrak{g} / [\mathfrak{g}, \mathfrak{g}]$ is abelian the abelianization of $\mathfrak{g}$

Definition 64
Let $\mathfrak{g}$ be a Lie algebra over a field k. The derived series of $\mathfrak{g}$ is defined by $Z^0 \mathfrak{g} = \mathfrak{g}$ and

$$Z^{j+1} \mathfrak{g} = [Z^j \mathfrak{g}, Z^j \mathfrak{g}]$$

for $j \geq 0$. Clearly,

$$\mathfrak{g} = Z^0 \mathfrak{g} \supset Z^1 \mathfrak{g} \supset Z^2 \mathfrak{g} \supset \cdots$$

Definition 65
$\mathfrak{g}$ is solvable if $Z^j \mathfrak{g} = 0$ for some $j \geq 0$

Lemma 66
A Lie algebra is solvable $\Leftrightarrow$ there is a sequence of Lie subalgebras

$$0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_r = \mathfrak{g}$$

s.t. $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i+1}$ and $\mathfrak{g}_{i+1} / \mathfrak{g}_i$ is abelian

Also any Lie subalgebra and any quotient Lie algebra of a solvable Lie algebra is solvable

Example 67
The set of upper triangular matrices $\mathfrak{b} \subset \mathfrak{gl}(n)$ form a solvable Lie algebra

Proof
Let $x, y \in \mathfrak{b}$. Then $[x, y] = xy - yx \in \mathfrak{u} = \{\text{strictly upper triangular matrices}\} \subset \mathfrak{gl}(n)$
Let e_{ij} = the matrix with 1 in row i and column j and 0 otherwise for $1 \leq i, j \leq n$
We have $e_{ij} e_{kl} = \delta_{jk} e_{il}$ where

$$\delta_{jk} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}$$

So $\qquad [e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$
$e_{ij} \in \mathfrak{b} \Leftrightarrow i \leq j$ and $e_{ij} \in \mathfrak{u} \Leftrightarrow i < j$
For $r \geq 0$, let $\mathfrak{u}_r$ =span of the matrices e_{ij} with $i + r \leq j$
So $\mathfrak{u}_0 = \mathfrak{b}, \mathfrak{u}_1 = \mathfrak{u}$, etc. Then we compute that $[\mathfrak{u}_i, \mathfrak{u}_j] \subset \mathfrak{u}_{i+j}$
So $[\mathfrak{u}_1, \mathfrak{u}_1] \subset \mathfrak{u}_2$ $\quad [\mathfrak{u}_2, \mathfrak{u}_2] \subset \mathfrak{u}_4$ etc. and so $\mathfrak{b}$ (and $\mathfrak{u}$) are solvable

Here $\mathfrak{b}(\mathbb{C})$ is the Lie algebra of the complex Lie group B = {upper triangular matrix} = { upper traingular matrix with diagonal entries in $\mathbb{C}^\times$} $\subset GL(n, \mathbb{C})$
Also $\mathfrak{u}(\mathbb{C})$ is the Lie algebra of the complex Lie group U = {upper triangular matrix with diagonal entries being 1}

□

Remark. B =a Borel subalgebra in $GL(n, \mathbb{C})$
U = a group of nilpotent matrices

Example:
$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is NOT solvable since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (since $[e, f] = h, [h, e] = 2e, [h, f] = -2f$)

Definition 68
The lower central series of a Lie algebra $\mathfrak{g}$ over a field k is $Z_0\,\mathfrak{g} = \mathfrak{g}$ and

$$Z_{j+1}\,\mathfrak{g} = [\mathfrak{g}, Z_j\,\mathfrak{g}]$$

for $j \geq 0$.
$\mathfrak{g}$ is nilpotent if $Z^j\,\mathfrak{g} = 0$ for some $j \geq 0$

Lemma 69
A Lie algebra $\mathfrak{g}$ is nilpotent $\Leftrightarrow$ there is sequence of ideals in $\mathfrak{g}$

$$0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

s.t. $\mathfrak{g}_{j+1}$ is central in $\mathfrak{g}\,/\,\mathfrak{g}_j \ \forall j$
Equivalently, $\mathfrak{g}\,/\,\mathfrak{g}_j$ is a central extension of $\mathfrak{g}\,/\,\mathfrak{g}_{j+1}$
(We saw $\mathfrak{g}$ is a central extension of $\mathfrak{h}$ if there is a central ideal $\mathfrak{z} \subset \mathfrak{g}$ such that $\mathfrak{h} \cong \mathfrak{g}\,/\mathfrak{z}$)

Definition 70
The centre of a Lie algebra $\mathfrak{g}$ is

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g}\,|\,[x, y] = 0\ \forall y \in \mathfrak{g}\} \tag{49}$$

Remark. If G is a Lie group then $Z(\mathfrak{g})$ is the Lie algebra of $Z(G)$
An ideal $\mathfrak{z} \subset \mathfrak{g}$ is central if $\mathfrak{z} \subset Z(\mathfrak{g})$ (also, $\mathfrak{z}$ central $\Rightarrow$ $[\mathfrak{g}, \mathfrak{z}] = 0$)

Example:
The Lie algebra $\mathfrak{u}$ of strictly upper triangular matrices in $\mathfrak{gl}(n)$ is nilpotent, because $\mathfrak{u} = \mathfrak{u}_1$, $[\mathfrak{u}_1, \mathfrak{u}_1] \subset \mathfrak{u}_2$, $[\mathfrak{u}_1, \mathfrak{u}_2] \subset \mathfrak{u}_3$ and so on (see previous example) whereas $\mathfrak{b} \subset \mathfrak{gl}(n)$ is NOT nilpotent for $n \geq 2$

Lemma 71
Any Lie subalgebra and any quotient Lie algebra of a nilpotent Lie algebra is nilpotent

Example 72
Classify all Lie algebras $\mathfrak{g}$ over $\mathbb{C}$ of dimension ≤ 2 up to isomorphism

$\dim_{\mathbb{C}} \mathfrak{g} = 1$: Let e_1 be a basis for $\mathfrak{g}$ as a $\mathbb{C}$-vector space. We have $[e_1, e_1] = 0$
So there is only one 1-dimensional Lie algebra over $\mathbb{C}$ up to isomorphism

$$\mathfrak{g} \cong \mathbb{C} = \mathfrak{u} \subset \mathfrak{gl}(2)$$

($\mathfrak{u}$ is the set of 2x2 strictly upper triangular matrices in $\mathbb{C}$)

$\dim_{\mathbb{C}} \mathfrak{g} = 2$: Let e_1, e_2 be a basis for $\mathfrak{g}$ as a $\mathbb{C}$ vector space.
Then $[e_1, e_1] = 0,\ [e_1, e_2] = a_1e_1 + a_2e_2$ $(a_1, a_2 \in \mathbb{C}), [e_2, e_2] = 0$ (and $[e_2, e_1] = -a_1e_1 - a_2e_2$)

Case 1:
If $a_1 = a_2 = 0$, then $\mathfrak{g}$ is the 2-dimensional abelian Lie algebra,

$$\mathfrak{g} = \mathbb{C}^2 \cong \mathbb{C} \times \mathbb{C}$$

(it iss the Lie algebra of the complex Lie group $(\mathbb{C}^2, +)$ or $(\mathbb{C}^\times)^2$ for example)

Case 2:
Suppose $\mathfrak{g}$ not abelian. Then $\dim_{\mathbb{C}}[\mathfrak{g}, \mathfrak{g}] = 1$
Let e_1 be a basis for $[\mathfrak{g}, \mathfrak{g}]$ and let e_2 be any other basis element for $\mathfrak{g}$
Then $[e_1, e_2] = ae_1$ where $0 \neq a \in \mathbb{C}$
By changing e_2 to a nonzero multiple, we can arrange to have $[e_1, e_2] = e_1$
So there is at most one non-abelian Lie algebra over $\mathbb{C}$ up to isomorphism

This IS a Lie algebra since it is the Lie algebra of matrices $\left\{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\right\} \subset \mathfrak{gl}(2)$ which is the Lie algebra of the group of

$$G = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C}\right\}$$

We compute $\mathfrak{g} = \mathbb{C}\{e_{11}, e_{12}\}$ and $[e_{11}, e_{12}] = \delta_{11}e_{12} - \delta_{21}e_{11} = e_{12}$
This Lie algebra $\mathfrak{g}$ is solvable because $D^1\,\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{C}\,e_{12}$
and $Z^2\,\mathfrak{g} = [\mathbb{C}\,e_{12}, \mathbb{C}\,e_{12}] = 0$
But $\mathfrak{g}$ is NOT nilpotent because:
$Z_1\,\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{C}\,e_{12}$
$Z_2\,\mathfrak{g} = [\mathfrak{g}, \mathbb{C}\,e_{12}] = \mathbb{C}\,e_{12}$
So $Z_j\,\mathfrak{g} \neq 0 \quad \forall j$ i.e. $\mathfrak{g}$ is not nilpotent

Example:
$\mathfrak{u}$ =set of strictly upper triangular 3×3 matrices, called the Heinsenberg Lie algebra is the smallest nilpotent but not abelian Lie algebra
Here $\mathfrak{u} = \mathbb{C}\{e_{12}, e_{23}, e_{13}\}$ with $[e_{12}, e_{23}] = e_{13}$
$[e_{12}, e_{13}] = 0$
$[e_{23}, e_{13}] = 0$

Lemma 73
Let $\mathfrak{g}$ be a Lie algebra over a field k. Let $\mathfrak{a}, \mathfrak{b}$ be solvable ideals in $\mathfrak{g}$. Then $\mathfrak{a} + \mathfrak{b} = \{x + y | x \in \mathfrak{a}, y \in \mathfrak{b}\}$ is a solvable ideal in $\mathfrak{g}$

Proof
Clearly $\mathfrak{a} + \mathfrak{b}$ is an ideal, have an isomorphism of Lie algebras:

$$\mathfrak{a} \,/\, \mathfrak{a} \cap \mathfrak{b} \xrightarrow{\sim} (\mathfrak{a} + \mathfrak{b}) / \,\mathfrak{b}$$

LHS is solvable since $\mathfrak{a}$ is solvable, and RHS is a Lie algebra
Since $\mathfrak{b}$ is solvable, $\mathfrak{a} + \mathfrak{b}$ is solvable □

Definition 74
The radical of a Lie algebra $\mathfrak{g}$ (finite dimensional over k), denote $\mathrm{rad}(\mathfrak{g})$ is the maximal solvable ideal in $\mathfrak{g}$

Definition 75
A Lie algebra $\mathfrak{g}$ is semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$

Definition 76
A Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ is not abelian and the only ideal in $\mathfrak{g}$ are 0 and $\mathfrak{g}$

Lemma 77
A simple Lie algebra $\mathfrak{g}$ is semisimple

Proof
If $\mathrm{rad}(\mathfrak{g}) \neq 0$, then $\mathfrak{g} = \mathrm{rad}(\mathfrak{g})$. So $\mathfrak{g}$ is solvable. We have $[\mathfrak{g}, \mathfrak{g}] = 0$ or $\mathfrak{g}$.
We have $[\mathfrak{g}, \mathfrak{g}] \neq 0$ because $\mathfrak{g}$ is not abelian. And $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ because we assumed $\mathfrak{g}$ was solvable (and we knew $\mathfrak{g} \neq 0$) □

Examples of simple Lie algebra: $\mathfrak{sl}(n, \mathbb{C})$ for $n \geq 2$
$\mathfrak{sp}(2n, \mathbb{C})$ for $n \geq 1$
$\mathfrak{so}(n, \mathbb{C})$ for $n = 3$ or $n \geq 5$

To check that a Lie algebra $\mathfrak{g}$ is simple, it is equivalent to check that $\mathfrak{g}$ not abelian and the adjoint representation of $\mathfrak{g}$ is irreducible
(Recall that $\mathrm{ad}(x)(y) = [x, y]$, for $x, y \in \mathfrak{g}$)

For example, for $\mathfrak{sl}(2)$, the adjoint representation $\mathfrak{sl}(2) \cong S^2(V)$, $V = \mathbb{C}^2$, which is irreducible
The exceptional cases:
$\mathfrak{sl}(1,\mathbb{C}) = 0$, $\mathfrak{so}(2,\mathbb{C}) \cong \mathbb{C}$, which is abelian (so not simple)
This is due to

$$\begin{array}{ccc} SO(2,\mathbb{C}) & \cong & \mathbb{C}^\times \\ \cup| & & \cup| \\ SO(2) & = & S^1 \end{array}$$

and $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$
(since $SO(4,\mathbb{C}) \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\{(1,1),(-1,-1)\}$)

Note that for any Lie algebra $\mathfrak{g}$, $\mathfrak{g}\,/\,\mathrm{rad}(\mathfrak{g})$ is semisimple

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{g}\,/\,\mathrm{rad}(\mathfrak{g}) \\ \cup| & & \cup| \\ \pi^{-1}(I) & & I \end{array} \quad \Rightarrow I = 0$$

$\pi^{-1}(I)$ (Solvable ideal in $\mathfrak{g}$) $\to I$ solvable ideal
$\Rightarrow I = 0$

Dual Representation
Let V be a representation of a group G. Is V^* a representation of G?

Given $g \in G$, we have a linear map $g : V \to V$, hence a linear map $g^* : V^* \to V^*$, $g^*(f)(x) = f(g(x)) \quad \forall x \in V$
We have $(gh)^* = h^* g^*$
We define a representation of G on V^* by $g \mapsto (g^*)^{-1} \in GL(V^*)$

in terms of a basis for V, the dual representation to a representation $G \xrightarrow{\rho} GL(n,k)$ is $G \to GL(n,k) \to GL(n,k)$, $A \mapsto (A^t)^{-1}$ because the matrix for f^* is f^t, $((AB)^t)^{-1} = (A^t)^{-1}(B^t)^{-1}$ so this is a representation

By taking derivatives, you find if V is a representation of a Lie algebra $\mathfrak{g}$ over k, V^* is a representation of $\mathfrak{g}$, by :

$$(uf)(x) = -f(ux) \subset k \qquad u \in \mathfrak{g}, f \in V^*, x \in V$$

Example:
If V and W are representations of a Lie algebra $\mathfrak{g}$ then $\mathrm{Hom}(V,W)$ is a representation of $\mathfrak{g}$, by $\mathrm{Hom}(V,W) = V^* \otimes W$ namely,

$$v \otimes w \mapsto (\phi : v \mapsto \alpha(v)w)$$

for a linear map $f \in \mathrm{Hom}(V,W)$ and $u \in \mathfrak{g}$

$$(uf)(v) = -f(uv) + uf(v) \in W \qquad v \in V$$

We see that the subspace $\mathrm{Hom}(V,W)^{\mathfrak{g}}$ is exactly the space $\mathrm{Hom}_{\mathfrak{g}}(V,W)$ of $\mathfrak{g}$-linear maps $V \to W$

Definition 78
For any representation V of a group G, the space V^G of invariant is $\{x \in V | gx = x \; \forall g \in G\}$
For a representation V of a Lie algebra $\mathfrak{g}$ the space of $\mathfrak{g}$-invariant is

$$V^{\mathfrak{g}} := \{x \in V | ux = 0 \; \forall u \in \mathfrak{g}\}$$

Likewise, given a representation V of a Lie algebra $\mathfrak{g}$ we can write the action of $\mathfrak{g}$ on the space $(V \otimes V)^* = V^* \otimes V^*$ of bilinear forms on V. The result is a bilinear form $B(\cdot, \cdot)$ on V is $\mathfrak{g}$-invariant $\Leftrightarrow B(ux, y) + B(x, uy) = 0\ \forall x, y \in V$

Example: Let $\langle\ ,\ \rangle$ be the standard representation bilinear form on $\mathbb{C}^n$. Then a element $A \in \mathfrak{gl}(n\,\mathbb{C})$ preserves $\langle\ ,\ \rangle$

$$\begin{aligned} &\Leftrightarrow \quad \underbrace{\langle Ax, y\rangle}_{\langle x, A^t y\rangle} + \langle x, Ay\rangle = 0 \quad x, y \in \mathbb{C}^n \\ &\Leftrightarrow \quad A + A^t = 0 \\ &\Leftrightarrow \quad A \in \mathfrak{so}(n, \mathbb{C}) \end{aligned}$$

Definition 79
Let V be a representation of a Lie algebra $\mathfrak{g}$ over a field k. The trace form associated to V is the symmetric bilinear form on $\mathfrak{g}$ defined by

$$B_V(x, y) = \operatorname{tr}(\rho(x)\rho(y)) \in k \quad x, y \in \mathfrak{g}$$

where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is the given bilinear maps $V \to V$, $GL(V) = \{\text{linear isom } V \to V\}$
This is symmetric because $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A, B : V \to V$ linear

Definition 80
The Killing form of a Lie algebra $\mathfrak{g}$ over k is the trace form $B_{\mathfrak{g}} = K$ associated to the adjoint representation, i.e.

$$K(x, y) = \operatorname{tr}(\underbrace{\operatorname{ad}(x)\operatorname{ad}(y)}_{\in \mathfrak{gl}(\mathfrak{g})})$$

Lemma 81
Let V be a finite dimensional representation of a Lie algebra. Then the trace form B_V on $\mathfrak{g}$ is (ad-)invariant, i.e.

$$B_V(\operatorname{ad}(u)\cdot(x), y) + B_V(x, \operatorname{ad}(u)\cdot(y)) = 0$$

Proof
We have to show that for any $x, y \in \mathfrak{g}$

$$B_V(u(x), y) + B_V(x, u(y)) = 0\ \forall u \in \mathfrak{g}$$

$(u(x) = (\operatorname{ad} u)(x))$ i.e. we want to show $B_V([u, x], y) + B_V(x, [u, y]) = 0$
i.e. want to show that:

$$\operatorname{tr}(\rho([u, x])\rho(y)) + \operatorname{tr}(\rho(x)\rho([u, y])) = 0$$

We know that $\rho([u, x]) = \rho(u)\rho(x) - \rho(x)\rho(u)$ because ρ is a representation of $\mathfrak{g}$ on V, so LHS is

$$\begin{aligned} & \operatorname{tr}(\rho(u)\rho(x)\rho(y) - \rho(x)\rho(u)\rho(y) + \rho(x)\rho(u)\rho(y) - \rho(x)\rho(y)\rho(u)) \\ = \ & \operatorname{tr}(\rho(u)\rho(x)\rho(y) - \rho(x)\rho(y)\rho(u)) \\ = \ & 0 \end{aligned}$$

□

In particular, the Killing form on any Lie algebra $\mathfrak{g}$ is ad-invariant

Lemma 82
Let $\mathfrak{a}$ be any ideal in a Lie algebra $\mathfrak{g}$. Then the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{a}$ is a Killing form of $\mathfrak{a}$

Proof
We have to show that for any $x, y \in \mathfrak{a}$

$$\operatorname{tr}_{\mathfrak{g}}((\operatorname{ad} x)(\operatorname{ad} y)) = \operatorname{tr}_{\mathfrak{a}}((\operatorname{ad} x)(\operatorname{ad} y))$$

Choose a basis for $\mathfrak{g}$ as a vector space starting with a basis for $\mathfrak{a}$
Then for $x \in \mathfrak{a}$, $\operatorname{ad} x \in \operatorname{Hom}_k(\mathfrak{g}, \mathfrak{g})$ has the form (see notes)

(since $\mathfrak{a}$ is an ideal)
Therefore, for $x, y \in \mathfrak{a}$, $(\operatorname{ad} x)(\operatorname{ad} y)$ acting on $\mathfrak{g}$ should be

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & 0 \end{array}\right)$$

which is easily observed $\operatorname{tr}_{\mathfrak{a}}(\operatorname{ad} x)(\operatorname{ad} y) = \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} x)(\operatorname{ad} y)$ □

Remark. For any Lie algebra $\mathfrak{g}$ over a field k, the $\ker(\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})) = Z(\mathfrak{g})$ (the map is $x \mapsto (y \mapsto [x, y])$)
So $\mathfrak{g}/Z(\mathfrak{g}) \hookrightarrow \mathfrak{gl}(n)$, where $n = \dim \mathfrak{g}$
So Ado's Theorem is obvious for $\mathfrak{g}$ with $Z(\mathfrak{g}) = 0$ such as semisimple Lie algebras.
This also applies to some non-semisimple Lie algebra, such as the 2-dimensional nonabelian Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \quad [e_{11}, e_{12}] = e_{12}$$

and we compute that $Z(\mathfrak{g}) = 0$

Recall: the Killing form on a Lie algebra is $K(x, y) = \operatorname{tr}_{\mathfrak{g}}((\operatorname{ad} x)(\operatorname{ad} y)) \in k$
This is an ad-invariant symmetric bilinear, form on $\mathfrak{g}$

Example: For $\mathfrak{g}$ abelian, the Killing form is 0.
More generally, if $\mathfrak{g}$ is nilpotent, then the Killing form is 0: we have the lower central series:

$$\mathfrak{g} = Z_0\,\mathfrak{g} \supset Z_1\,\mathfrak{g} \supset \cdots \supset Z_r\,\mathfrak{g} = 0$$

where $Z_{j+1}\,\mathfrak{g} = [\mathfrak{g}, Z_j\,\mathfrak{g}]$
So for any $x \in \mathfrak{g}$, $(\operatorname{ad} x)(Z_j\,\mathfrak{g}) \subset Z_{j+1}\,\mathfrak{g}$
So $(\operatorname{ad} x)(\operatorname{ad} y)$ is nilpotent: $\mathfrak{g} \to \mathfrak{g}$, so $K(x, y) = 0$ on $\mathfrak{g}$ nilpotent.

For $\mathfrak{g}$ solvable, $K(x, y)$ can be non-zero
Example:

$$\mathfrak{g} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \quad [e_{11}, e_{12}] = e_{12}$$

Here, $(\operatorname{ad} e_{11})(e_{11}) = b$ $\quad (\operatorname{ad} e_{11})(e_{12}) = e_{12}$
$(\operatorname{ad} e_{12})(e_{11}) = -e_{12}$ $\quad (\operatorname{ad} e_{12})(e_{12}) = 0$
We compute that $K(e_{11}, e_{11}) = 1, K(e_{11}, e_{12}) = 0, K(e_{12}, e_{12}) = 0$

Theorem 83 (Cartan's criterion for solvable Lie algebras)
A Lie algebra $\mathfrak{g}$ over a field k with char $k = 0$ is solvable $\Leftrightarrow$ $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$

(Proof ommitted)

Lemma 84
Let $\mathfrak{g}$ be a Lie algebra over a field k, $\mathfrak{a} \trianglelefteq \mathfrak{g}$ an ideal. Then $\mathfrak{a}^\perp$ (with respect to the Killing form) is an ideal

Proof
Use that the Killing form is ad-invariant

$$K(\underbrace{[x,y]}_{\in\mathfrak{a}},z) + K(y,[x,z]) = 0 \quad \forall x,y,z \in \mathfrak{g}$$

Let $x \in \mathfrak{g}, y \in \mathfrak{a}, z \in \mathfrak{a}^\perp$
Then first term is 0 so we have $0 = K(y,[x,z])$
So $[x,z] \in \mathfrak{a}^\perp$ (since $y \in \mathfrak{a}$ is arbitrary)
So $\mathfrak{a}^\perp$ is an ideal in $\mathfrak{g}$ □

Corollary 85 (Cartan's criterion for semisimple Lie algebra)
Let char k=0. A Lie algebra $\mathfrak{g}$ over k is semisimple $\Leftrightarrow$ K nondegenerate on $\mathfrak{g}$

Proof
K nondegenerate means that $K(x,y) = 0 \ \forall y \quad \Rightarrow \quad x = 0$
$\Rightarrow \qquad (K \text{ nondegenerate } \Leftrightarrow \mathfrak{g}^\perp = 0)$

$\underline{\Rightarrow}$:
First suppose $\mathfrak{g}$ is semisimple. By lemma, $\mathfrak{g}^\perp$ is an ideal in $\mathfrak{g}$
Also, the Killing form of $\mathfrak{g}$ restricts to 0 on $\mathfrak{g}^\perp$
So the Killing form of $\mathfrak{g}^\perp$ is 0 (by a previous lemma).
By Cartan's criterion for solvable Lie algebra, $\mathfrak{g}^\perp$ is solvable. Since $\mathfrak{g}$ is semisimple, $\mathfrak{g}^\perp = 0$
That is, the Killing form on $\mathfrak{g}$ is nondegenerate

Conversely, suppose $\mathfrak{g}$ is <u>not</u> semisimple, so rad($\mathfrak{g}$)$\neq 0$ □

Lemma 86
If $\mathfrak{a}$ is an ideal in a Lie algebra $\mathfrak{g}$, then $[\mathfrak{a},\mathfrak{a}]$ is also an ideal in $\mathfrak{g}$

Proof
For any $x \in \mathfrak{g}, y, z \in \mathfrak{a}$, we have

$$[x,[y,z]] = -[y,\underbrace{[z,x]}_{\in\mathfrak{a}}] - [z,\underbrace{[x,y]}_{\in\mathfrak{a}}] \in [\mathfrak{a},\mathfrak{a}]$$

$\Rightarrow \quad$ [rad($\mathfrak{g}$),rad($\mathfrak{g}$)] $= Z^1$rad($\mathfrak{g}$) is an ideal in $\mathfrak{g}$
as is $Z^2\,\mathfrak{g}, Z^3\,\mathfrak{g}, \ldots$ (these are terms in derived series)
$\Rightarrow$ $\mathfrak{g}$ contains a nonzero <u>abelian</u> ideal $\mathfrak{a}$
We will show that $\mathfrak{a} \subset \mathfrak{g}^\perp$ so Killing form is degenerate.

Pick a basis for $\mathfrak{g}$ over k that starts with a basis for abelian ideal $\mathfrak{a}$. Then for any $x \in \mathfrak{a}$

$$\operatorname{ad} x = \left(\begin{array}{c|c} 0 & * \\ \hline \underbrace{0}_{\dim\mathfrak{a}} & \underbrace{0}_{\dim\mathfrak{g}-\dim\mathfrak{a}} \end{array}\right) \qquad \operatorname{ad} y = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$$

$$\Rightarrow \quad (\operatorname{ad} x)(\operatorname{ad} y) = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad K(x,y) = \operatorname{tr}(\operatorname{ad} x)(\operatorname{ad} y) = 0$$

□

Corollary 87
Every semisimple Lie algebra $\mathfrak{g}$ over a field k of characteristic 0 is a product of simple Lie algebra $\mathfrak{g} \cong \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$

Proof
Let $\mathfrak{a} \trianglelefteq \mathfrak{g}$ be an ideal. We know that $\mathfrak{a}^\perp$ is also an ideal in $\mathfrak{g}$ so $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal in $\mathfrak{g}$
The Killing form of $\mathfrak{g}$ is 0 on $\mathfrak{a} \cap \mathfrak{a}^\perp$

By a previous lemma, the Killing form of the Lie algebra $\mathfrak{a} \cap \mathfrak{a}^\perp$ is 0
By Cartan's criterion, $\mathfrak{a} \cap \mathfrak{a}^\perp$ is solvable.
Since $\mathfrak{g}$ is semisimple, we have $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$
(By conunting dimensions) $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as a vector space
Notice that $[\mathfrak{a}, \mathfrak{a}^\perp] = 0$ because $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are both ideals in $\mathfrak{g}$
So $\mathfrak{g} \cong \mathfrak{a} \times \mathfrak{a}^\perp$ as a Lie algebra
By induction on dimension of $\mathfrak{g}$, $\mathfrak{g}$ is product of simple Lie algebras □

Example:
$\mathfrak{gl}(n, \mathbb{C})$ is not semisimple.
In fact, $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C}) \times \mathbb{C} \cdot 1$
Clearly $\mathfrak{gl}(n, \mathbb{C})$ is a direct sum as a vector space as above, and $[\mathfrak{sl}(n, \mathbb{C}), \mathbb{C} \cdot 1] = 0$
So rad($\mathfrak{gl}(n, \mathbb{C})$)=$\mathbb{C} \cdot 1 = Z(\mathfrak{gl}(n, \mathbb{C}))$

Semisimple and Nilpotent elements

Definition 88
Let $\mathfrak{g}$ be a Lie algebra over a field k
An element $x \in \mathfrak{g}$ is called semisimple if the linear map $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable (=semisimple)
An element $x \in \mathfrak{g}$ is called nilpotent if $\operatorname{ad} x$ is nilpotent

Example:
For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ $\quad$ (*or* $\mathfrak{sl}(n, \mathbb{C})$) $\quad x \in \mathfrak{gl}(n, \mathbb{C})$ is semisimple or nilpotent in this sense if and only if $x : \mathbb{C}^n \to \mathbb{C}^n$ is diagonalizable or nilpotent.

Definition 89
A Lie subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is toral if it is abelian and consists of semisimple elements

Example:
$$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \quad \mathfrak{t} = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \Bigg| a_i \in \mathbb{C} \right\} \text{ is a toral subalgebra}$$
Here $\mathfrak{t}$ is the Lie algebra of the complex (multiplicative) Lie group

$$T = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \Bigg| a_i \in \mathbb{C}^\times \right\} \cong (\mathbb{C}^\times)^n$$
$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix} = \begin{pmatrix} a_1 b_1 & & 0 \\ & \ddots & \\ 0 & & a_n b_n \end{pmatrix} \in T$$

In dealing with complex subgroups of $GL(n, \mathbb{C})$, a torus means a group $\cong (\mathbb{C}^\times)^a$ some $a \geq 0$

Lemma 90
Let V be a finite dimensional vector space. Let $S \subset \operatorname{End}(V)$ be a set of commuting semisimple linear

maps $V \to V$. Then we can simultaneously diagonalize all the maps in S. Equivalently,

$$V = \bigoplus_{\lambda : S \to \mathbb{C}} V(\lambda)$$

where $V(\lambda) = \{x \in V | s(x) = \lambda(s)x \ \forall s \in S\}$

Proof

Say $S = \{s_1, s_2, \ldots\}$

We know that s_1 is diagonalizable, so $V = \bigoplus_{\lambda_1 \in \mathbb{C}} V(\lambda_1)$, $V(\lambda_1) = \{x \in V | s_1(x) = \lambda_1 x\}$

Since s_2 commutes with s_1, s_2 maps each s_1-eigenspace $V(\lambda_1)$ into itself

So $s_2 : V(\lambda_1) \to V(\lambda_1)$ is diagonalizable for $\lambda_1 \in \mathbb{C}$

so $V = \bigoplus_{\lambda_1, \lambda_2 \in \mathbb{C}} V(\lambda_1, \lambda_2)$

where $V(\lambda_1, \lambda_2) = \{x \in V | s_1(x) = \lambda_1 x \quad s_2(x) = \lambda_2 x\}$

etc. □

Remark. The trace form on $\mathfrak{gl}(n, \mathbb{C})$ associate to the standard representation $\langle x, y \rangle = \mathrm{tr}(xy)$, is a nondegenerate symmetric bilinear form on $\mathfrak{gl}(n, \mathbb{C})$ It has

$$\begin{aligned} \langle e_{ij}, e_{kl} \rangle &= \begin{cases} 1 & (k,l) = (i,j) \\ 0 & \text{otherwise} \end{cases} \\ &= \mathrm{tr}(e_{ij} e_{kl}) \\ &= \mathrm{tr}(\delta_{jk} e_{il}) = \delta_{jk} \delta_{il} \end{aligned}$$

The Killing form on $\mathfrak{sl}(n, \mathbb{C})$ is equal to $2n \, \mathrm{tr}(xy)$

Theorem 91

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a toral subalgebra. Let $\langle \ , \ \rangle$ be a nondegnerate ad-invariant symmetric bilinear form on $\mathfrak{g}$ (e.g. the Killing form). Then

(1) $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$
where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \, | \forall y \in \mathfrak{t} \ [y, x] = \alpha(y)x\}$ (the "α-eigenspace" for $\mathfrak{t}$ acting on $\mathfrak{g}$).
In particular, $\mathfrak{t} \subset \mathfrak{g}_0$ (will soon show they are in fact equal)

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \quad \forall \alpha, \beta \in \mathfrak{t}^*$

(3) If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to $\langle \ , \ \rangle$.

(4) $\forall \alpha \in \mathfrak{t}^*$, the bilinear form restricts to a nondegenerate $\mathfrak{g}_\alpha \times \mathfrak{g}_\alpha \to \mathbb{C}$

Proof

(1) For each $y \in \mathfrak{t}$, $\mathrm{ad}\, y$ is a semisimple linear map $\mathfrak{g} \to \mathfrak{g}$ and all these linear maps commute (because $0 = \mathrm{ad}[x, y] = [\mathrm{ad}\, x, \mathrm{ad}\, y]$ for $x, y \in \mathfrak{t}$ abelian)
So we can simultaneously diagonalize $\mathfrak{g}$ with respect to all of $\mathfrak{t}$
Easy to see that the eigenvalues $\alpha : \mathfrak{t} \to \mathbb{C}$ of any basis element of $\mathfrak{g}$ must be linear, that is, $\alpha \in \mathfrak{t}^*$

(2) Let $\alpha, \beta \in \mathfrak{t}^*, y \in \mathfrak{g}_\alpha, z \in \mathfrak{g}_\beta, x \in \mathfrak{t}$. Then

$$\begin{aligned} [x, [y, z]] &= -[y, [z, x]] - [z, [x, y]] \\ &= [y, [x, z]] - [z, [x, y]] \\ &+ [y, \beta(x)z] - [z, \alpha(x)y] \\ &= (\beta(x) + \alpha(x))[y, z] \\ &= (\alpha + \beta)(x)[y, z] \end{aligned}$$

so $[y, z] \in \mathfrak{g}_{\alpha+\beta}$

(3) Use that $\langle\ ,\ \rangle$ is ad-invariant

$$\langle [x,y],z\rangle + \langle y,[x,z]\rangle = 0 \quad \forall x,y,z \in \mathfrak{g}$$

Let $x \in \mathfrak{t}, y \in \mathfrak{g}_\alpha, z \in \mathfrak{g}_\beta$. Then

$$\begin{aligned} 0 &= \langle \alpha(x)y,z\rangle + \langle y,\beta(x)z\rangle \\ &= (\alpha(x)+\beta(x))\langle y,z\rangle \end{aligned}$$

so if $\langle y,z\rangle \neq 0$, then we must have $(\alpha+\beta)(x) = 0\ \forall x \in \mathfrak{t}$. That is, $\alpha+\beta = 0 \in \mathfrak{t}^*$

(4) This follows from $\langle\ ,\ \rangle$ being nondegenerate on $\mathfrak{g}$ together with (3)

□

Definition 92
A Cartan subalgebra in a complex semisimple Lie algebra is a maximal toral subalgebra

Lemma 93
Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra. Then $\mathfrak{t}$ is equal to its own centralizer in $\mathfrak{g}$ (hence $\mathfrak{g}_0$)

$$\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{t}) = \{x \in \mathfrak{g}\,|\,[x,y] = 0\ \forall y \in \mathfrak{t}\}$$

(Proof omitted)

Thus, for any Cartan subalgebra $\mathfrak{t}$ in a $\mathbb{C}$-semisimple Lie algebra $\mathfrak{g}$, we have

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{0\neq\alpha\in\mathfrak{t}^*} \mathfrak{g}_\alpha$$

because $\mathfrak{g}_0 = \mathfrak{t}$. Remind again:

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}\,|\,[h,x] = \alpha(h)x\ \forall h \in \mathfrak{t}\}$$

This is called the root space of decomposition of $\mathfrak{g}$. The eigenspaces $\mathfrak{g}_\alpha \neq 0$ with $\alpha \neq 0 \in \mathfrak{t}^*$ are called the root spaces. The $0 \neq \alpha \in \mathfrak{t}^*$ with $\mathfrak{g}_\alpha \neq 0$ are called the roots of $\mathfrak{g}$. Write $R \subset \mathfrak{t}^*$ be the set of roots of $\mathfrak{g}$

Example:
Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and let $\mathfrak{t}$ =the space of diagonal matrices in $\mathfrak{g}$=$\{(a_1,\ldots,a_n) \in \mathbb{C}^n\,|a_1+\cdots+a_n = 0\}$
This is a toral subalgebra. Claim that this is a Cartan subalgebra
To see that, conjugate the eigenspace decomposition of $\mathfrak{g}$ with respect to $\mathfrak{t}$
We use that for $i \neq j, [e_{ii}, e_{ij}] = e_{ij}$
Therefore, for any diagonal matrix $y = (y_1,\ldots,y_n)$

$$[y, e_{ij}] = (y_i - y_j)e_{ij}$$

Define a linear function $\epsilon_1,\ldots,\epsilon_n \in \mathfrak{t}^*$ by

$$\epsilon_i(y_1,\ldots,y_n) = y_i$$

Then the above calculation shows that for $i \neq j$, $e_{ij} \in \mathfrak{g}_{\epsilon_i-\epsilon_j}$. Thus

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{i\neq j} \underbrace{\mathfrak{g}_{\epsilon_i-\epsilon_j}}_{\mathbb{C}\cdot e_{ij}}$$

$\epsilon_i - \epsilon_j \neq 0 \in \mathfrak{t}^*\ \ \forall i \neq j \quad \Rightarrow \quad t = \mathfrak{g}_0$
$\Rightarrow \quad Z_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$, i.e. $\mathfrak{t}$ is a Cartan subalgebra.
We have found the root-space decomposition of $\mathfrak{sl}(n,\mathbb{C})$

See what this is for $\mathfrak{sl}(2,\mathbb{C})$ Here $\mathfrak{t} = \mathbb{C}\cdot\begin{pmatrix}1 & \\ & -1\end{pmatrix}$

$$\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}1 & \\ & -1\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

where $\mathbb{C}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \mathfrak{g}_{\epsilon_2-\epsilon_1} = \mathfrak{g}_{-2\epsilon_1}$
$\mathbb{C}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \mathfrak{t}$
$\mathbb{C}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \mathfrak{g}_{\epsilon_1-\epsilon_2} = \mathfrak{g}_{2\epsilon_1}$
(Note: On $\mathfrak{t} \subset \mathfrak{sl}(n), \epsilon_1 + \ldots + \epsilon_n = 0 \in \mathfrak{t}^*$, so in $\mathfrak{sl}(2,\mathbb{C})$, $\epsilon_2 = -\epsilon_1 \in \mathfrak{t}^* = (\mathbb{C}\cdot h)^*$)
These just the same as formulae:

$$[h,f] = -2f \quad [h,h] = 0 \quad [h,e] = 2e$$

Lemma 94
$\mathfrak{sl}(n,\mathbb{C})$ is simple for $n \geq 2$

Proof
It is not abelian, because $[e_{12}, e_{21}] = e_{12}e_{21} - e_{21}e_{12} = e_{11} - e_{22} \neq 0 \in \mathfrak{sl}(n,\mathbb{C})$
We have to show that any nonzero ideal $\mathfrak{a} \trianglelefteq \mathfrak{sl}(n,\mathbb{C})$ must equal $\mathfrak{sl}(n,\mathbb{C})$
We know,in particular, that $[\mathfrak{t},\mathfrak{a}] \subset \mathfrak{a}$
That implies $\mathfrak{a} = (a_n\,\mathfrak{t})\oplus$(the subspace spanned by some set of e_{ij}'s, $i \neq j$)
Claim that $\mathfrak{a} \cap \mathfrak{t} \neq 0$. If not, $\mathfrak{a} \supset e_{ij}$ some $i \neq j$, So $\mathfrak{a}$ contains

$$[e_{ij}, e_{ji}] = e_{ii} - e_{jj}$$

So $\mathfrak{a} \cap \mathfrak{t} \neq 0$
Next for any $k \notin \{i,j\}$, we have

$$[e_{ii} - e_{jj}, e_{ik}] = e_{ik}$$

So $\mathfrak{a}$ contains e_{ik} and hence $\mathfrak{a}$ contains

$$[e_{ik}, e_{ki}] = e_{ii} - e_{kk}$$

Therefore $\mathfrak{a}$ contains $\mathfrak{t}$ ($\mathfrak{t}$ is spanned by $e_{11} - e_{22}, e_{11} - e_{33}, \ldots, e_{11} - e_{nn}$)
Therefore, for any $i \neq j$, $\mathfrak{a}$ contains

$$[\underbrace{e_{ii} - e_{jj}}_{\in\mathfrak{t}\subset\mathfrak{sl}(n)}, e_{ij}] = e_{ij} - [e_{jj}, e_{ij}] = 2e_{ij}$$

So $\mathfrak{a} = \mathfrak{sl}(n,\mathbb{C})$. That is, $\mathfrak{sl}(n,\mathbb{C})$ is simple □

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra. Let $\langle\ ,\ \rangle$ be an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$
We know that $\langle\ ,\ \rangle : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is nondegenerate for all $\alpha \in \mathfrak{t}^*$
For $\alpha = 0$, this gives the $\langle\ ,\ \rangle$ is nondegenerate on $\mathfrak{t}$
We can use this form to identify

$$\begin{array}{rcl}\mathfrak{t} & \cong & \mathfrak{t}^* \\ x & \mapsto & (y \mapsto \langle x,y\rangle \in \mathbb{C})\end{array}$$

For $\alpha \in \mathfrak{t}^*$, write the corresponding element of $\mathfrak{t}$ as $H_\alpha \in \mathfrak{t}$, i.e.

$$\alpha(x) = \langle H_\alpha, x\rangle \qquad \forall x \in \mathfrak{t}$$

Lemma 95
Let $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$. Then $[e,f] = \langle e,f\rangle H_\alpha$

Proof
It will suffice to show that $\forall x \in \mathfrak{t}$, $\langle x, [e,f]\rangle = \langle x, \langle e,f\rangle H_\alpha\rangle$. The right side here is $\langle e,f\rangle \alpha(x)$
Use that $\langle\ ,\ \rangle$ on $\mathfrak{g}$ is ad-invariant

$$\begin{aligned} \langle x, [e,f]\rangle &= -\langle [e,x], f\rangle \qquad \forall x \in \mathfrak{t} \\ &= \langle [x,e], f\rangle \\ &= \langle \alpha(x)e, f\rangle \\ &= \alpha(x)\langle e,f\rangle \end{aligned}$$

□

Lemma 96
Let $\alpha \in R \subset \mathfrak{t}^*$. Then

(1) $\langle\alpha,\alpha\rangle \neq 0$
(Equivalently, $\langle H_\alpha, H_\alpha\rangle \neq 0$)

(2) Let $\alpha \in R$. Let $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ s.t. $\langle e,f\rangle = \frac{2}{\langle\alpha,\alpha\rangle}$. Also, let $h_\alpha = \frac{2H_\alpha}{\langle\alpha,\alpha\rangle} \in \mathfrak{t}$.
Then $\alpha(h_\alpha) = 2$ and the elements $e, f, h_\alpha \in \mathfrak{g}$ satisfy the relation defining $\mathfrak{sl}(2,\mathbb{C})$. Denote this Lie subalgebra $\mathfrak{sl}(2,\mathbb{C})_\alpha \subset \mathfrak{g}$.

Proof

(1) Assume that $\langle\alpha,\alpha\rangle = 0 \in \mathbb{C}$. THen $\alpha(H_\alpha) = 0$. We know that $\langle\ ,\ \rangle : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is nondegenerate and $\mathfrak{g}_\alpha \neq 0$, so there are elements $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ with $\langle e,f\rangle \neq 0$. Let $h = [e,f] (\in \mathfrak{g}_0 = \mathfrak{t}) = \langle e,f\rangle H_\alpha (\neq 0)$.
Claim that h, e, f span a Lie subalgebra of $\mathfrak{g}$. Indeed, we have

$$\begin{aligned} [h,e] &= \alpha(h)e = 0 \\ [h,f] &= \alpha(h)f = 0 \end{aligned}$$

Look at the action of $\operatorname{ad} h$ on $\mathfrak{g}$, it is diagonalizable, so

$$\mathfrak{g} = \bigoplus_{c\in\mathbb{C}} \mathfrak{g}_c$$

where $\mathfrak{g}_c = \{x \in \mathfrak{g} \,|\, [h,x] = cx\}$
How do $\operatorname{ad} e$ and $\operatorname{ad} f$ act on $\mathfrak{g}$?
Because e and f commute with h, e and f map each subspace $\mathfrak{g}_c$ into itself for all $c \in \mathbb{C}$
We have $h = [e,f]$ as endomorphism on $\mathfrak{g}_c$ for each $c \in \mathbb{C}$
Therefore $\operatorname{tr}(h|_{\mathfrak{g}_c}) = 0$
But h acts by multiplication by c on $\mathfrak{g}_c$
So, if $\mathfrak{g}_c \neq 0$, then we must have $c = 0$
That means that $h \in Z(\mathfrak{g})$. But $\mathfrak{g}$ is semisimple, so $h = 0$ #

(2) $\alpha(h_\alpha) = \frac{2\alpha(H_\alpha)}{\langle\alpha,\alpha\rangle} = \frac{2\langle\alpha,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 2$
$(\langle\alpha,\alpha\rangle = \alpha(H_\alpha) = \langle H_\alpha, H_\alpha\rangle)$
We know that

$$\begin{aligned} [e,f] &= \langle e,f\rangle H_\alpha \\ &= \frac{2H_\alpha}{\langle\alpha,\alpha\rangle} \\ &= h_\alpha \\ \text{and} \quad [h_\alpha, e] &= \alpha(h_\alpha)e = 2e \\ \text{and} \quad [h_\alpha, f] &= \alpha(h_\alpha)f = -2f \end{aligned}$$

□

Lemma 97
Let α be a root, and let $\mathfrak{sl}(2,\mathbb{C})_\alpha \subset \mathfrak{g}$ be the Lie subalgebra spanned by $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ and h_α as above.
Consider the linear subspace of $\mathfrak{g}$

$$V = \mathbb{C} \cdot h_\alpha \oplus \left(\bigoplus_{0 \neq k \in \mathbb{Z}} \mathfrak{g}_{k\alpha} \right) \subset \mathfrak{g}$$

Then V is an irreducible representation of $\mathfrak{sl}(2\,\mathbb{C})_\alpha$, and $\dim_\mathbb{C} \mathfrak{g}_\alpha = 1$

Proof
Here $\mathfrak{sl}(2,\mathbb{C})_\alpha \subset \mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation. We have to show that $\operatorname{ad} e, \operatorname{ad} f$ and $\operatorname{ad} h_\alpha$ map V to itself. We have

$$\begin{aligned} (\operatorname{ad} e)(\mathfrak{g}_{k\alpha}) &\in \mathfrak{g}_{(k+1)\alpha} \\ (\operatorname{ad} e)(\mathfrak{g}_{-\alpha}) &= \langle e, f \rangle H_\alpha \in \mathbb{C} \cdot h_\alpha \end{aligned}$$

by Lemma 95.
Same argument show that $(\operatorname{ad} f)(V) \subset V$
Because $h_\alpha = [e, f]$, h_α also maps V into itself
So $V \subset \mathfrak{g}$ is a representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$
What are its weight? The weight of a vector $x \in \mathfrak{g}_{k\alpha}$ (w.r.t. $h_\alpha \in \mathfrak{t}$) is $k\alpha(h_\alpha) = 2k$
(So $V \cong (S^0 A)^{\oplus a_0} \oplus (S^2 A)^{\oplus a_2} \oplus \cdots$ where $A \cong \mathbb{C}^2$ is the standard representation of $\mathfrak{sl}(2,\mathbb{C})$) And the 0-th weight space of V is 1-dimensional

$$\begin{aligned} \operatorname{ch}(S^0 V) &= && && \bullet^0 \\ \operatorname{ch}(S^2 V) &= && \bullet^{-2} && \bullet^0 && \bullet^2 \\ \operatorname{ch}(S^4 V) &= \bullet^{-4} && \bullet^{-2} && \bullet^0 && \bullet^2 && \bullet^4 \end{aligned}$$

So V is irreducible, as a representation of $\mathfrak{sl}(2,\mathbb{C})$. So all (nonzero) weight spaces of V are 1-dimensional. Since $\mathfrak{g}_\alpha \neq 0$, $\dim_\mathbb{C} \mathfrak{g}_\alpha = 1$ □

Detour: Semidirect Products
Let $N \trianglelefteq G$ be a normal subgroup of a group
We say that G is a semidirect product $G = H \ltimes N$ if there is a subgroup $H \leq G$ that maps isomorphically to G/N

$$1 \to N \to G \to G/N \to 1$$

Conversely, given groups H and N, what do we need to define a group $G = H \ltimes N$?
Given a semidirect product group, we get a homomorphism

$$\begin{aligned} H &\to \operatorname{Aut}(N) \\ h &\mapsto (n \mapsto hnh^{-1}) \end{aligned}$$

Conversely, given H, N a homomorphism $\phi : H \to \operatorname{Aut}(N)$, define a semidirect product group $G = H \ltimes N$

$$(h_1 n_1) \cdot (h_2 n_2) = \underbrace{(h_1 h_2)}_{\in H} \underbrace{(h_2^{-1} n_1 h_2 n_2)}_{= \phi(h_2)(n_1) n_2 \in N}$$

Example:
The group of isometries of $\mathbb{R}^n$ is $O(n) \ltimes \mathbb{R}^n$ (isometries that fixes 0 $\ltimes$ translation)
The group of affine translations of $\mathbb{R}^n$ is $GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$
The homomorphism $GL(n,\mathbb{R}) \to \operatorname{Aut}(\mathbb{R}^n)$ is the obvious one

Example:
The group

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C}\right\}$$

is a semidirect product $\mathbb{C}^\times \ltimes \mathbb{C}$

Lemma 98
Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra, $\alpha \in R$ a root (i.e, $\alpha \in \mathfrak{t}^*, \mathfrak{g}_\alpha \neq 0, \alpha \neq 0$)
Then the Lie subalgebra $\mathfrak{sl}(2,\mathbb{C})_\alpha \subset \mathfrak{g}$ and the element $\alpha^\vee = h_\alpha \in \mathfrak{t}$ are independent of the choice of nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$

Proof
We know that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are 1-dimensional so $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}$ has dimension ≤ 1 and in fact it has dimension being 1 as we showed. The $\mathfrak{sl}(2,\mathbb{C})_\alpha$ is

$$\mathfrak{sl}(2,\mathbb{C})_\alpha = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$$

which clearly does not depend on choice of $\langle \ , \ \rangle$.
The element $\alpha^\vee$ is the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ s.t. $\alpha(\alpha^\vee) = 2$ □

Remark. The equation $\alpha(\alpha^\vee) = 2$ means that $\operatorname{ad}\alpha^\vee$acts on $\mathfrak{g}_\alpha$ by multiplication by 2, i.e.

$$[\alpha^\vee, x] = 2x \quad \forall x \in \mathfrak{g}_\alpha$$

The element $\alpha^\vee \in \mathfrak{t}$ associate to a root $\alpha \in \mathfrak{t}^*$ is called the coroot associated to α

Example:
Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}), \mathfrak{t} =$ diagonal matrices in $\mathfrak{sl}(n,\mathbb{C})$ Any element of $\mathfrak{t}$ can be written by $(y_1,\dots,y_n) \in \mathbb{C}^n$ with $y_1 + \cdots + y_n = 0$. The roots are $\epsilon_i - \epsilon_j \in \mathfrak{t}^*$ for $i \neq j$ where $\epsilon_n(y_1,\dots,y_n) = y_i$ (we have $\epsilon_1 + \ldots + \epsilon_n = 0$ in $\mathfrak{t}^*$)

That means $[(y_1,\dots,y_n), e_{ij}] = (y_i - y_j)e_{ij}$ $(i \neq j)$ The coroot $\alpha_{ij}^\vee = e_{ii} - e_{jj}$ because that is the unique element of $[\mathbb{C}\, e_{ij}, \mathbb{C}\, e_{ji}]$ s.t. $(\epsilon_i - \epsilon_j)(e_{ii} - e_{jj}) = 2$ That means that $(f =)e_{ji}, (h =)e_{ii} - e_{jj}, (e =)e_{ij}$ satisfy the relations in $\mathfrak{sl}(2,\mathbb{C})$

Theorem 99 (Structure of complex semisimple Lie algebra)
Let $\mathfrak{g}$ be a $\mathbb{C}$ semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra. Let $\mathfrak{g} = \mathfrak{t} \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha)$ be the root space decomposition . Let $\langle \ , \ \rangle$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Then

(1) R spans $\mathfrak{t}^*$ as a $\mathbb{C}$-vector space

(2) For each root α, $\mathfrak{g}_\alpha$ is 1-dimensional

(3) For any two roots α, β, the number

$$n_{\alpha\beta} = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$$

is an integer

(4) For any $\alpha \in R$, the reflection s_α on $\mathfrak{t}^*$ is defined by

$$s_\alpha(x) = x - \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} x$$

(this makes sense since $\langle \alpha, \alpha \rangle \neq 0$) For any root $\beta \in R, s_\alpha(\beta)$is a root

(5) For any root α, if $c\alpha$ is also a root ($c \in \mathbb{C}$), then $c = \pm 1$

(6) For any roots $\alpha, \beta \neq \pm\alpha$, then subspace

$$V = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$$

is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha \subset \mathfrak{g}$

(7) If α, β are root s.t. $\alpha+\beta$ is also a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ (Of course, if $\alpha+\beta \neq 0$ and not a root then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$)

Proof

(1) Suppose there was an element $h \in \mathfrak{t}$ which $\alpha(h) = 0$ for all roots α; we want to show that $h = 0$. The assumption means that $\operatorname{ad} h$ acts by 0 on $\mathfrak{g}_\alpha$ $\forall \alpha \in R$. It also acts by 0 on $\mathfrak{t}$, so $\operatorname{ad} h = 0$ that is $h \in Z(\mathfrak{g})$. But $\mathfrak{g}$ semisimple, so $Z(\mathfrak{g}) = 0$, so $h = 0$

(2) Proved

(3) Consider $\mathfrak{g}$ as a representation of $\mathfrak{sl}(2,\mathbb{C})_\beta \subset \mathfrak{g}$. The weight for this $\mathfrak{sl}(2)$, i.e. for for $\beta^\vee \in \mathfrak{t}$, of $\mathfrak{g}_\alpha$ is $\alpha(\beta^\vee)$. But we know the weights of any finite dimensional representation of $\mathfrak{sl}(2)$ are in $\mathbb{Z}$, so $\alpha(\beta^\vee) \in \mathbb{Z}$. We define

$$\beta^\vee = \frac{2H_\beta}{\langle \beta, \beta \rangle}$$

where $\alpha(H_\beta) = \langle \alpha, \beta \rangle$. So $n_{\alpha\beta} \in \mathbb{Z}$.

(4) Consider the subspace of $\mathfrak{g}$ defined by

$$V = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$$

This is a representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ (Clear, since e in this $\mathfrak{sl}(2)$ lives in $\mathfrak{g}_\alpha$, f lives in $\mathfrak{g}_{-\alpha}$)
For any finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, its weights are symmetric but 0.
(x weight $\Rightarrow$ $-x$ weight)
We know that $\mathfrak{g}_\beta \neq 0$ and the weight of $\alpha^\vee (= \text{“}h_\beta\text{”})$ on $\mathfrak{g}_\beta$ is $\beta(\alpha^\vee)$.
More generally, the weight of $\mathfrak{g}_{\beta+n\alpha}$ wrt $\alpha^\vee$ is $\beta(\alpha^\vee) + 2n$
$\Rightarrow$ $-\beta(\alpha^\vee)$ must also be weight in this representation V of $\mathfrak{sl}(2,\mathbb{C})_\alpha$
$\Rightarrow$ $\mathfrak{g}_{\beta-\beta(\alpha^\vee)\alpha} \neq 0$
$\Rightarrow$ $\beta - \beta(\alpha^\vee)\alpha$ is a root

(5) Consider the subspace of $\mathfrak{g}$

$$V = \bigoplus_{0\neq n\in\mathbb{Z}} \mathfrak{g}_{n\alpha} \oplus \mathbb{C}\cdot\alpha^\vee$$

We showed that this is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ But $\mathfrak{sl}(2,\mathbb{C})_\alpha \subset V$
So $\mathfrak{sl}(2,\mathbb{C})_\alpha = V$ So if α a root and $n\alpha$ is a root with $n \in \mathbb{Z}$, then $n = \pm 1$
Suppose α and $c\alpha$ are roots ($c \in \mathbb{C}^*$) Then $n_{\alpha,c\alpha} \in \mathbb{Z}$
That is $c \in (1/2)\,\mathbb{Z}$ and $1/c \in (1/2)\,\mathbb{Z}$
So $c \in \{\pm 1, \pm\frac{1}{2}, \pm 2\}$ We have excluded ± 2 and that also excludes $\pm\frac{1}{2}$

(6) Look at the weights of V as a representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ i.e. the eigenvalues wrt $\alpha^\vee \in \mathfrak{t}$. These weights are $\beta(\alpha^\vee) + 2n$.
So the weights of V as a representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ are all $\equiv \beta(\alpha^\vee) \mod 2$
Also, all the weight spaces have dimensional ≤ 1. These implies that V is irreducible

(7) Know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. (LHS is subspace of 1-dimensional space, RHS is 1-dimensional space)
Want to show that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ Look at the subspace of $\mathfrak{g}$:

$$V = \bigoplus_{0\neq n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$$

This is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_\alpha$. We are given that $\mathfrak{g}_\beta \neq 0$ and $\mathfrak{g}_{\beta+\alpha} \neq 0$. So V has non-zero weight spaces with the weights $\beta(\alpha^\vee)$ and $\beta(\alpha^\vee) = 2$
In particular, if the weight spaces V_k and V_{k+2} are not 0 ($k \in \mathbb{Z}$), then $e : V_k \to V_{k+2}$ is not the zero map. That means that $e \in \mathfrak{g}_\alpha$ has $\operatorname{ad} e : \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+\alpha}$ NOT the zero map. That is, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$

□

Lemma 100
Let $\mathfrak{g}$ be a complex semisimple Lie algebra

(1) $\mathfrak{t} \subseteq \mathfrak{g}$ a Cartan subalgebra. Let $\mathfrak{t}_\mathbb{R} \subseteq \mathfrak{t}$ be the real vector space spanned by the coroots $\alpha^\vee, \alpha \in \mathbb{R}$ Then $\mathfrak{t} = \mathfrak{t}_\mathbb{R} \oplus i\, \mathfrak{t}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, and the Killing form of $\mathfrak{g}$ is real and positive definite on $\mathfrak{t}_\mathbb{R}$

(2) Let $\mathfrak{t}^*_\mathbb{R}$ = $\mathbb{R}$-vector space spanned by root in $\mathfrak{t}^*$. Then $\mathfrak{t}^* = \mathfrak{t}^*_\mathbb{R} \oplus i\, \mathfrak{t}^*_\mathbb{R}$ and the form on $\mathfrak{t}^*$ corresponds to the Killing form of $\mathfrak{g}$ on $\mathfrak{t}$ is positive definite on $\mathfrak{t}^*_\mathbb{R}$

Proof

(1) Let $h \in \mathfrak{t}_\mathbb{R}$ so $h = \sum_{\alpha \in \mathbb{R}} c_\alpha \alpha^\vee$ $c_\alpha \in \mathbb{R}$. Then using the killing form

$$\langle h, h \rangle = \operatorname{tr}_\mathfrak{g}((\operatorname{ad} h)(\operatorname{ad} h))$$

Here

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathbb{R}} \mathfrak{g}_\alpha$$

So $\langle h, h \rangle = \sum_{\alpha \in \mathbb{R}} \alpha(h)^2$
But $\alpha(h) \in \mathbb{R}$ because $\alpha(\beta^\vee) \in \mathbb{Z}$ for all roots α, β
($\alpha(\beta^\vee) = n_{\alpha\beta} = 2\frac{<\alpha,\beta>}{<\beta,\beta>}$) so $\langle h, h \rangle$ is real and ≥ 0
and if it is 0 then $\alpha(h) = 0$ $\forall$roots α. That implies $h = 0$ i.e. the Killing form is positive definite on $\mathfrak{t}_\mathbb{R}$
So the Killing form of $\mathfrak{g}$ is negative definite on $i\, \mathfrak{t}_\mathbb{R}$. So $\mathfrak{t}_\mathbb{R} \cap i\, \mathfrak{t}_\mathbb{R} = 0$. But the coroots span $\mathfrak{t}$ as a complex vector space, so $\mathfrak{t} = \mathfrak{t}_\mathbb{R} + i\, \mathfrak{t}_\mathbb{R}$. That is $\mathfrak{t} = \mathfrak{t}_\mathbb{R} \oplus i\, \mathfrak{t}_\mathbb{R}$

(2) Follows from (1)

□

Example:
$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \mathfrak{t}$ =diagonal matrices $\subseteq \mathfrak{g}$
Then $\mathfrak{t}_\mathbb{R}$ =real diagonal matrices of trace 0 = Lie algebra of $(\mathbb{R}^*)^{n-1}$

Definition 101
A root system R is a finite set of nonzero element in a real vector space E with inner product s.t.

(1) R spans E as a real vector space

(2) $\forall \alpha, \beta \in R$,

$$n_{\alpha\beta} := \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$$

(3) For every root $\alpha \in R$, the reflection

$$s_\alpha(x) = x - \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

($\alpha : E \to E$) maps the set R of roots into itself

A root system is reduced if when α is a root , $c\alpha$ is a root for some $c \in \mathbb{R}$, then $c = \pm 1$

Important Example:
For $\mathfrak{g}$ a $\mathbb{C}$-semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra, let $E = \mathfrak{t}_{\mathbb{R}}^*$ with the (dual of) the Killing form of $\mathfrak{g}$. Then the set of roots $R \subset E$ is a root system

Definition 102
Given a root system $R \subset E$, the coroot $\alpha^\vee$ corresponds to a root $\alpha \in E$ is

$$\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

Then $n_{\alpha\beta} = \langle \alpha, \beta^\vee \rangle$, and the reflection s_α on E is

$$s_\alpha(x) = x - \langle \alpha^\vee, x \rangle \alpha$$

There is a geometric way to understand $n_{\alpha\beta}$:
Let $p_\alpha : E \to E$ be the orthogonal projection onto $\mathbb{R} \cdot \alpha \subset E$, then $p_\alpha(\beta) = (n_{\alpha\beta}/2) \cdot \alpha$. So $n_{\alpha\beta} \in \mathbb{Z}$ means that for all roots α, β, if project β orthogonally to $\mathbb{R} \cdot \alpha$, then have $\beta \in \mathbb{Z}\, \frac{\alpha}{2}$

Example:
The root system of $\mathfrak{sl}(n, \mathbb{C})$ is the A_{n-1} root system
Look at the A_2 root system (corresponds to $\mathfrak{sl}(3, \mathbb{C})$)

Definition 103
The Weyl group W of a root system is the subgroup of $GL(E)$ generated by the reflections s_α, $\alpha \in R$

Lemma 104

(1) The Weyl group W is a finite subgroup of $O(E)$, and $R \subset E$ is invariant under the action of W

(2) For $w \in W, \alpha \in R$

$$w s_\alpha w^{-1} = s_{w(\alpha)} \in W$$

Proof

(1) Clearly $W \subset O(E)$, because any reflection s_α is in $O(E)$. Clearly, $W(R) = R$. Any element $w \in W$ acts on R by some permutation, and there are only finitely many permutations of R. But if $w \in W$ acts as the identity on R, then $w = 1 \in GL(E)$, because R spans E

(2) Clearly $w s_\alpha w^{-1}$ is a reflection in $O(E)$. And $w s_\alpha w^{-1}(w(\alpha)) = w s_\alpha(\alpha) = -w(\alpha)$. So $w s_\alpha w^{-1}$ must equal $s_{w(\alpha)}$

□

Pairs of roots and rank-2 root systems

Lemma 105
Let $R \subset E$ be a root system and let $\alpha, \beta \in R$ s.t. $\alpha \notin \mathbb{R}\cdot\beta$. After switching α and β if necessary, can assume $|\alpha| \leq |\beta|$, ($|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$)
By changing β to $-\beta$ if necessary, can assume that $\langle \alpha, \beta \rangle \leq 0$
Then one of the following holds

(1) $\langle \alpha, \beta \rangle = 0$. Thus α, β are a angle $\pi/2$

(2) $n_{\alpha\beta} = -1$ and $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$. Here α, β are at angle $2\pi/3$

(3) $n_{\alpha\beta} = -2$ and $\langle \alpha, \alpha \rangle = \frac{1}{2}\langle \beta, \beta \rangle$. Here α, β are at angle $3\pi/4$

(4) $n_{\alpha\beta} = -3$ and $\langle \alpha, \alpha \rangle = \frac{1}{3}\langle \beta, \beta \rangle$. Here α, β are at angle $5\pi/6$

Proof
Since $\langle \alpha, \beta \rangle = 0$, $n_{\alpha\beta}$ are integers ≤ 0.
But

$$\begin{aligned} n_{\alpha\beta} n_{\beta\alpha} &= \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \\ &= \frac{4\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} \leq 4 \quad \text{(by Cauchy Schwarz)} \end{aligned}$$

One possibility is (1) $\langle \alpha, \beta \rangle = 0$
So we can assume $n_{\alpha\beta}$, $n_{\beta\alpha}$ are integers ≤ 0. Also, since $\beta \notin \mathbb{R}\cdot\alpha$. We have strict map in Cauchy-Schwarz, so $n_{\alpha\beta} n_{\beta\alpha} \leq 3$
Also $|n_{\alpha\beta}| \leq |n_{\beta\beta}|$ because α is the shorter root
So $n_{\alpha\beta} = -1$ and $n_{\beta\alpha} \in \{-1, -2, -3\}$ □

Definition 106
Rank of a root system $R \subset E$ is $\dim_{\mathbb{R}} E$

Theorem 107
Any reduced rank-2 root system is isomorphic to $A_1 \times A_1$, A_2, C_2 or G_2

Proof
Let R be a rank-2 root system. Choose roots $\alpha, \beta \notin \mathbb{R}\cdot\alpha$ s.t. angle(α, β) is as small as possible.
Easy to see that $\langle \alpha, \beta \rangle \geq 0$.
Apply lemma to α, $\gamma := -\beta$
We find the possible lengths and angles between α, γ
By applying reflections in α, γ, we find that $R \supset (A_1 \times A_1, A_2, C_2$ or $G_2)$ root system in cases (1)-(4) in previous lemma
$\nexists$ other roots in R, otherwise we would have two roots at a smaller angle than between α and β □

Remark. (1) For any roots α, β in a root system R,

$$n_{\alpha\beta} n_{\beta\alpha} = \frac{4\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = 4\cos^2\theta$$

where θ = angle between α and β
Suppose $\langle \alpha, \beta \rangle \leq 0, \alpha \notin \mathbb{R}\cdot\beta$, $|\alpha| \leq |\beta|$, we showed that $n_{\alpha\beta} n_{\beta\alpha} = 0, 1, 2$ or 3 (see picture)
so $\cos\theta = 0$, $\frac{-1}{2}$, $\frac{-\sqrt{2}}{2}$ or $\frac{-\sqrt{3}}{2}$.

(2) For these rank-2 root systems, the Weyl group is the dihedral group of order $4(\cong \mathbb{Z}/2 \times \mathbb{Z}/2)$, $6(\cong S_3)$, 8 or 12
In general, the dihedral group of order $2n$ is a semidirect product $\mathbb{Z}/2 \ltimes \mathbb{Z}/n$ ($\mathbb{Z}/n$ the normal subgroup)

Positive roots and simple roots

Definition 108
Let R be a root system in a Euclidean space E. Pick an element $v \in E$ with $\langle v, \alpha \rangle \neq 0$ for all roots α. Then we call the set of positive roots $R_+ \subset R$

$$R_+ = \{\alpha \in R | \langle \alpha, v \rangle > 0\}$$

Otherwise, negative roots.
Clearly, $R = R_+ \sqcup R_-$ and $R_- = -R_+$
Fix a set of positive roots R_+

Definition 109
A root $\alpha \in R$ is simple if it is positive and it is not a sum of two positive roots.
Write $\Pi \subset R_+$ for the set of simple roots.
Clearly, every positive root can be written

$$\alpha = \sum_{i=1}^{l} n_i \alpha_i \quad n \in \mathbb{N}, \quad \alpha_1, \dots, \alpha_l \text{ simple roots}$$

$\langle v, \alpha + \beta \rangle = \underbrace{\langle v, \alpha \rangle}_{>0} + \underbrace{\langle v, \beta \rangle}_{>0}$

Lemma 110
For any two simple roots $\alpha \neq \beta$, $\langle \alpha, \beta \rangle \leq 0$

Proof
Suppose $\langle \alpha, \beta \rangle > 0$
Ten α and $-\beta$ must be positioned as in one of the rank-2 root system with possibilities (see pictures)

In all these cases, $\beta - \alpha$ is again a root
So either $\beta - \alpha$ is a positive root or a negative root.
If $\beta - \alpha \in R_+$, then β is not simple, if $\alpha - \beta \in R_+$, then α not simple □

Theorem 111
Let R be a root system, R_+ a set of positive roots. Then the corresponding simple roots form a basis for E, as a $\mathbb{R}$-vector space

Proof
Clearly, the simple roots span E, because every positive root in R can be written $\sum n_i \alpha_i$, $n_i \in \mathbb{N}$ where $\alpha_1, \dots, \alpha_l$ are the simple roots; so the negative roots can be written $\sum n_i \alpha_i$, $n_i \in \mathbb{Z}, n_i \leq 0$, so $\alpha_1, \dots, \alpha_l$ span E.
We show that $\alpha_1, \dots, \alpha_l$ are $\mathbb{R}$-linear independent. If not, we can write

$$\sum_{i \in S} c_i \alpha_i = \sum_{i \in T} d_i \alpha_i$$

where $S \cap T = \emptyset$, $c_i > 0, d_i > 0$, and at least one of S, T is nonempty
First notice that $w \in E$ is not 0, because $\langle v, w \rangle > 0$
So we know that $\langle w, w \rangle > 0$
But we have $\langle w, w \rangle = \langle \sum_{i \in S} c_i \alpha_i, \sum_{i \in T} d_i \alpha_i \rangle \leq 0$ because $c_i, d_i > 0$
and $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$. Contradiction
So the simple roots form a basis for E $\square$

Definition 112
The rank of a root system $R \subset E$ is $\dim_{\mathbb{R}} E$. So number of simple roots $=\text{rk}(R)$

The rank of a $\mathbb{C}$-semisimple Lie algebra $\mathfrak{g}$ is the $\mathbb{C}$-dimension of a Cartan subalgebra

In fact, let G be a semisimple complex Lie group. Then any two Cartan subalgebras $\mathbb{C}^l \subset \mathfrak{g}$ are conjugate by some element of G
So G (or $\mathfrak{g}$) has a well-defined root system (up to isomorphism).

Remark. Any two sets of positive roots in a a root system R are equivalent by some element of the Weyl group W

Dynkin Diagram:
Let R be a root system $\subset E$. Let R^+ be a set of positive roots. The Dynkin diagram of R is a graph with one vertex for each simple roots and with edges:

Remark. Let $\mathfrak{g}_1, \mathfrak{g}_2$ are $\mathbb{C}$-semisimple Lie algebras. Let $\mathfrak{t}_1, \mathfrak{t}_2$ be Cartans in $\mathfrak{g}_1, \mathfrak{g}_2$, then $\mathfrak{t}_1 \times \mathfrak{t}_2$ is a Cartan in $\mathfrak{g}_1 \times \mathfrak{g}_2$.
($\mathfrak{g}_1 \times \mathfrak{g}_2 = \begin{pmatrix} \mathfrak{g}_1 & 0 \\ 0 & \mathfrak{g}_2 \end{pmatrix}$) The root system of $\mathfrak{g}_1 \times \mathfrak{g}_2$ is $R = R_1 \sqcup R_2 \subset E_1 \oplus E_2$, where $\langle E_1, E_2 \rangle$

In general, the Dynkin diagram of the product of two root systems is the disjoint union of the two Dynkin diagrams

Exercise:
$\mathfrak{sl}(2, \mathbb{C}) \leftrightarrow A_1 \leftrightarrow$ Dynkin diagram with 1 vertex
$\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \leftrightarrow$ Dynkin diagram with 2 vertices
And show the Dynkin diagram of other rank-2 root system

Exercise:
$\mathfrak{sl}(n \mathbb{C}) \leftrightarrow$ root system $A_{n-1} = \{\epsilon_i - \epsilon_j | i \neq j\} \subset \mathbb{R}^{n-1} = \{a_1\epsilon_1 + \cdots + a_n\epsilon_n | a_1 + \cdots + a_n = 0\}$
because: use the restriction of $\mathbb{R}^{n-1}$ of the standard inner product. Let $v = a_1\epsilon_1 + \cdots + a_n\epsilon_n$ where $a_1 > a_2 > \cdots > a_n$. Then the positive roots are $\epsilon_i - \epsilon_j$, $1 \leq i \leq j \leq n$

The simple roots are $\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n$
With our inner product $\langle \alpha_i, \alpha_j \rangle = 2$ for $i = 1, \ldots, n-1$, and

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 0 & |i-j| \geq 2 \\ -1 & |i-j| = 1 \\ 2 & i = j \end{cases}$$

$$\Rightarrow \qquad n_{\alpha_i \alpha_j} = \begin{cases} 0 & |i-j| \geq 2 \\ -1 & |i-j| = 1 \\ 2 & i = j \end{cases}$$

$\mathfrak{sl}(n, \mathbb{C})$ has root system of type A_{n-1}, and the Weyl group of A_{n-1} is the symmetric group S_n.
Simple roots $= \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n\} \subset \mathbb{R}^{n-1}$
The reflection $s_{\epsilon_i - \epsilon_j}$ for $i \neq j$, switches i and j coordinates in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$

Root system C_n of $\mathfrak{sp}(2n, \mathbb{C}), n \geq 1$
$Sp(2n, \mathbb{C}) = \{A \in GL(2n, \mathbb{C}) | AJA^T = J\}$ where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$
The Lie algebra $\mathfrak{sp}(2n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n, \mathbb{C}) | AJ + JA^T = 0\}$
We compute that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}(2n, \mathbb{C}) \Leftrightarrow B$ and C are symmetric and $D = -A^T$
A Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$ consists of the diagonal matrices in $\mathfrak{sp}(2n, \mathbb{C})$, that is :

$$y = \operatorname{diag}(y_1, \ldots, y_n, -y_1, \ldots, -y_n) \quad y_i \in \mathbb{C}$$

One computes how such a diagonal matrix acts on $\mathfrak{sp}(2n, \mathbb{C})$. You find that the roots are $\pm y_i \pm y_j$ $\forall i \neq j$, $1 \leq i, j \leq n$ and $\pm 2y_i$ for $1 \leq i \leq n$
(Excellent Exercise: Check this)
Can work out the coroots.
Killing form on $\mathfrak{g}$ restricts to a nonzero multiple of the standard symmetric bilinear form on $\mathfrak{t}_{\mathbb{R}}^* \cong \mathbb{R}^n$
So the reflection $s_{y_i - y_j}$ for $i \neq j$ switches coordinates i and j on a point in $\mathbb{R}^n = \mathfrak{t}_{\mathbb{R}}^*$
The reflection $s_{y_i + y_j}$ switches coordinates i and j and changes their signs:

$$s(y_1, \ldots, s_n) = (y_1, \ldots, -y_j, \ldots, -y_i, \ldots, y_n)$$

The reflection $s_{\pm 2y_i}$ changes the sign of the i-th coordinate in $\mathbb{R}^n$
So the Weyl group of the C_n root system is the semidirect product $S_n \ltimes (\mathbb{Z}/2)^n \subset O(n)$

The positive roots are $y_i \pm y_j$ for $1 \leq i < j \leq n$ and $2y_i$ for $1 \leq i \leq n$
The standard choice of simple roots are $y_1 - y_2, y_2 - y_3, \ldots, y_{n-1} - y_n, 2y_n$
So the C_n Dynkin diagram is:

<u>Example</u>: C_2=

$W = S_2 \ltimes (\mathbb{Z}/2)^2 \cong$ dihedral group of order 8

Root system B_n of $\mathfrak{so}(2n+1, \mathbb{C})$

It is easiest to describe $SO(2n+1, \mathbb{C})$ as the subgroup of $GL(2n+1, \mathbb{C})$ preserving the symmetric bilinear form defined by C with entry $(i, 2n+1-i)$ as 1 and everywhere else 0.
The corresponding bilinear form $\mathbb{C}^{2n+1}$ is

$$\langle (x_1, \dots, x_{2n+1}), (y_1, \dots, y_{2n+1}) \rangle = x_1 y_{2n+1} + x_2 y_{2n} + \cdots + x_{2n+1} y_1$$

Here, the Cartan subalgebra of $\mathfrak{so}(2n+1, \mathbb{C})$ is the diagonal matrices in $\mathfrak{so}(2n+1, \mathbb{C})$

Here $\mathfrak{so}(2n+1, \mathbb{C}) = \{A \in \mathfrak{gl}(2n+1, \mathbb{C}) | AC + CA^T = 0\}$
Cartan $\mathfrak{t} = \{(y_1, \dots, y_n, 0, -y_n, \dots, -y_1)\} | y_1, \dots, y_n \in \mathbb{C}\}$
The roots are $\{\pm y_i \pm y_j | i \neq j, 1 \leq i, j \leq n\} \cup \{y_i | 1 \leq i \leq n\}$
So the Weyl group $W(B_n) = S_n \ltimes (\mathbb{Z}/2)^n \subset O(n)$
A standard set of simple roots is

$$\{y_1 - y_2, y_2 - y_3, \dots, y_{n-1} - y_n, y_n\}$$

So the Dynkin diagram for $B_n \leftrightarrow \mathfrak{so}(2n+1, \mathbb{C})$

Root system D_n of $\mathfrak{so}(2n, \mathbb{C})$

Again, easier to denote this using the symmetric bilinear form C
A Cartan subalgebra $\mathfrak{t}$ =diagonal matrices in $\mathfrak{g}$
$= \{\mathrm{diag}(y_1, \dots, y_n, -y_n, \dots, -y_1)\}$
The roots are: $\{\pm y_i \pm y_j, i \neq j, 1 \leq i, j \leq n\}$
So Weyl group $W = S_n \ltimes (\mathbb{Z}/2)^{n-1}$ A standard choice of simple roots is

$$\{y_1 - y_2, y_2 - y_3, \dots, y_{n-1} - y_n, y_{n-1} + y_n\}$$

So the D_n Dynkin diagram:

Theorem 113
The following classification are equivalent

(1) Complex semisimple Lie algebras up to isomorphism

(2) Reduced root systems

(3) Dynkin diagrams of root system

In this correspondence
simple Lie algebra $\leftrightarrow$ Irreducible reduced root systems $\leftrightarrow$ connected Dynkin diagram.

The possible Dynkin diagrams are:

$A_n,\ n \geq 1$:

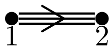
$B_n,\ n \geq 2$:

$C_n,\ n \geq 2$:

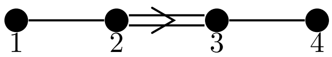
$D_n,\ n \geq 2$:

(The followings are exceptional simple Lie algebras)

G_2 :



F_4 :



E_6:

E_7:

E_8:

Sketch proof:
One part is pure Euclidean geometry:
Show that Dynkin diagram of a reduced irreducible root system is one of these graphs.
Indeed, consider the unit vertices $v_1, \ldots, v_n \in E \cong \mathbb{R}^n$ in the directions of the simple roots.
Then $v_1, \ldots, v_n$ are linearly independent, and the different ones are at angle $\pi/2, 2\pi/3, 3\pi/4$ or $5\pi/6$
That alone implies that the corresponding Coxeter diagram (Dynkin diagram without arrows) is one of these listed
Then the Dynkin diagram of R must be given by some choice of directions on or .
So the Dynkin diagram of R must be one listed.

We know that A_n, B_n, C_n, D_n correspond to complex semisimple Lie algebras.

One can write down root system correspond to the G_2, F_4, E_6, E_7, E_8 Dynkin diagrams (see Example Sheet 3). But why do they come from simple Lie algebras?

The complex simple group G_2 =group of automorphisms of the octonions $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathbb{O}$ = the real 8-dimensional non-associative division algebra, defined by Cayley.
That implies that $G_2(\mathbb{C}) \subset GL(7, \mathbb{C})$ (because $1 \in \mathbb{O}$ is fixed)

It is harder to describe the 5 exceptional Lie algebras, because they do not have low-dimensional representations.

G	$\dim(G)$	$\dim_{\mathbb{C}}$(smallest nontrivial repn of G)
G_2	14	7
F_4	53	26
E_6	78	27
E_7	133	56
E_8	248	248

(Classical Lie algebra of dimension N has a nontrivial representation of dimension $\sim \sqrt{N}$) □

Existence and Uniqueness of semisimple Lie algebra with given root system or Dynkin diagram
Serre's relations
(defining the semisimple Lie algebra with a given Dynkin diagram)

Given a Dynkin diagram with l vertices define a complex Lie algebra $\mathfrak{g}$ as the quotient of the free Lie algebra on generators

$$H_1, \ldots, H_l, E_1, \ldots, E_l, F_1, \ldots, F_l$$

modulo the relations to be shown later

Given a number n, the free Lie algebra F_n has the property:

$$\mathrm{Hom}_{\text{Lie alg.}}(F_n, \mathfrak{g}) = \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{n \text{ times}}$$

That is, F_n is generated by n elements $x_1, \ldots, x_n$. You can define it as the k-vector space spanned by all possible irreducibles

$$[[x_1, [x_2, x_3]], x_4]$$

The free Lie algebra is graded:

$$F_n = k^n \oplus \bigwedge{}^2(k^n) \oplus (\ )_{\deg 3} \oplus \cdots$$

($k^n = k\{x_1, \ldots, x_n\}$ and $[x_i, x_j] \in \bigwedge^2(k^n)$ $i < j$)

We will have simple roots $\alpha_1, \ldots, \alpha_l$ in our semisimple Lie algebra $\mathfrak{g}$;

$$\begin{array}{lll} H_i = \alpha_i^\vee & \in & \mathfrak{t} \subset \mathfrak{g} \\ E_i \in \mathfrak{g}_{\alpha_i} & & F_i \in \mathfrak{g}_{-\alpha_i} \\ & \text{s.t.} & [E_i, F_i] = \alpha_i^\vee (= H_i) \end{array}$$

So the modulo relation (Serre's relation) required above is:

$$\begin{aligned} [H_i, H_j] &= 0 \quad \forall i, j \\ [E_i, F_i] &= H_i \\ [E_i, F_j] &= 0 \quad \text{for } i \neq j \\ [H_i, E_j] &= n_{ji} E_j \end{aligned}$$

where $n_{ji} = n_{\alpha_j \alpha_i} = \alpha_j(\alpha_i^\vee) \in \mathbb{Z}$
and further:

$$\begin{aligned} [H_i, F_j] &= -n_{ji} F_j \\ (\mathrm{ad}\, E_i)^{1-n_{ji}}(E_j) &= 0 \quad \forall i \neq j \\ (\mathrm{ad}\, F_i)^{1-n_{ji}}(F_j) &= 0 \quad \forall i \neq j \end{aligned}$$

Here $n_{ji} \in \{0, -1, -2, -3\}$ as shown by the Dynkin diagram, so $(1 - n_{ji}) \in \{1, 2, 3, 4\}$

Why are these relations true in semisimple Lie algebra $\mathfrak{g}$?
It will suffice to show that $\alpha_j + (1 - n_{ji})\alpha_i$ is not a root in $\mathfrak{t}^*$ (it is clearly not 0)
This is because this expression is the reflection $s_{\alpha_i}(\alpha_j - \alpha_i)$; we know $\alpha_j - \alpha_i$ is not a root, and we know that the set of roots is preserved by the Weyl group

Example:
G_2

The picture shows that $(\operatorname{ad} E_1)^4(E_2) = 0$, because $\alpha_2 + 4\alpha_1 \notin R$, i.e. $\mathfrak{g}_{\alpha_2+4\alpha_1} = 0$

Compact Lie groups and complex semisimple groups

Definition 114
Let G be a connected compact Lie group. We say that a connected complex Lie group $G_{\mathbb{C}}$ is the complexification of G if $\exists$ inclusion $G \hookrightarrow G_{\mathbb{C}}$ s.t. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $\pi_1(G) \cong \pi_1(G_{\mathbb{C}})$

Example:
$\mathbb{C}^*$ is the complexification of S^1 (here $\pi_1 \cong \mathbb{Z}$, because $S^1 \cong \mathbb{R}/\mathbb{Z}$ and $\mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$)
More generally $GL(n, \mathbb{C})$ is the complexification of $U(n)$ since $\mathfrak{gl}(n, \mathbb{C})$ =(skew-hermitian matrices)$\oplus i\{$skew-hermitian matrices$\}$ and $\pi_1 GL(n, \mathbb{C}) \cong \pi_1 U(n) \cong \mathbb{Z}$

Example:
$SU(n)_{\mathbb{C}} \cong SL(n, \mathbb{C})$ $(\pi_1 = 1)$
$SO(n)_{\mathbb{C}} \cong SO(n, \mathbb{C})$ $(\pi_1 \cong \mathbb{Z}/2)$
$Sp(n)_{\mathbb{C}} \cong Sp(2n, \mathbb{C})$ $(\pi_1 = 1)$
$(Sp(n) = O(4n) \cap GL(n, \mathbb{H}) \subset GL(4n, \mathbb{R})$
$Sp(n) = U(2n) \cap Sp(2n, \mathbb{C}) \subset GL(2n, \mathbb{C}))$

Definition 115
A connected complex Lie group is reductive if it is the complexification of some compact Lie group

Example:
$\mathbb{C}$ is not reducitive because any compact subgroup of $\mathbb{C}$ is $\{0\}$

Corollary 116 (Weyl's Unitary Trick)
The $\mathbb{C}$ analytic representations of any complex reductive group are completely reducible

Proof
We are given a compact Lie group G with complexification $= G_{\mathbb{C}}$
Complex representations of the real Lie algebra $\mathfrak{g}$ are equivalent to representations of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$
So complex representation of the universal cover $\widetilde{G}$ are equivalent to the complex analytic representations of $\widetilde{G_{\mathbb{C}}}$. I know that $G = \widetilde{G}/Z$ and $G_{\mathbb{C}} = \widetilde{G_{\mathbb{C}}}/Z$, because $Z = \pi_1 G = \pi_1 G_{\mathbb{C}}$. So complex representations of G are equivalent to complex analytic representations of $G_{\mathbb{C}}$. The first ones are completely reducible, so are the second ones $\square$

Theorem 117

(1) Every connected complex semisimple group is reductive

(2) A compact connected Lie group is determined up to isomorphism by its complexification

(Proof omitted)

Corollary 118
The finite dimensional representations of a complex semisimple Lie algebra $\mathfrak{g}$ are completely reducible

Proof
Let $G_{\mathbb{C}}$ be the corresponding simply connected complex Lie group
By above theorem (1), $G_{\mathbb{C}}$ is reductive, so its representations are completely reducible. They are equivalent to finite dimensional representations of $\mathfrak{g}$ $\square$

In particular, $\exists!$ simply connected compact Lie group with given Dynkin diagram
Most we have seen:
$A_n : SU(n+1)$
$B_n : Spin(2n+1) =$ simply connected double cover of $SO(2n+1)$
$C_n : Sp(n)$
$D_n : Spin(2n)$
But there are also simply connected compact Lie group of type G_2, F_4, E_6, E_7, E_8

Example:
The compact Lie group G_2 is $\mathrm{Aut}(\mathbb{O})$ (recall $\mathbb{O}$ is octonions over $\mathbb{R}$)

Example:
(Complex analytic) representations of $\mathbb{C}^*$ are direct sums of 1-dimensional representations, by completely reducibility + Schur's Lemma.
$\mathbb{C}^* = \mathbb{C}/2\pi i\,\mathbb{Z}$
A 1-dimensional representation of $\mathbb{C}^*$ is a homomorphism $\mathbb{C}^* \to GL(1,\mathbb{C}) = \mathbb{C}^*$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}^* = \mathbb{C}/2\pi i\,\mathbb{Z} & \dashrightarrow & \mathbb{C}/2\pi i\,\mathbb{Z} = \mathbb{C}^* \end{array}$$

Here $a \in \mathbb{C}$ gives a homomophism $\mathbb{C}^* \to \mathbb{C}^* \Leftrightarrow a \in \mathbb{Z}$
So the 1-dimensional representations of $\mathbb{C}^*$ are $\mathbb{C}^* \to \mathbb{C}^* \quad z \mapsto z^a$ for some $a \in \mathbb{Z}$

Remark. For a compact connected Lie group G, the inclusion $G \hookrightarrow G_{\mathbb{C}}$ is a homotopy equivalence.

Example:
$S^1 \hookrightarrow \mathbb{C}^\times, \quad U(n) \hookrightarrow GL(n,\mathbb{C})$
Prove that $GL(n,\mathbb{C})$ deformation retracts onto $U(n)$, using Gram-Schmidt

Low-dimensional isomorphism of classical groups
Example:
$SL(2,\mathbb{C}) \cong Sp(2,\mathbb{C}), SO(3,\mathbb{C}) \cong SL(2,\mathbb{C})/\{\pm\}$ because all have Dynkin Diagram (one vertex no edge)

$SL(n,\mathbb{C})$= subgroup of $GL(n,\mathbb{C})$ preserving a nonzero element of $\bigwedge^n V^*$ where $V \cong \mathbb{C}^n$
$Sp(n,\mathbb{C})$= subgroup of $GL(2n,\mathbb{C})$ preserving a nondegenerate element of $\bigwedge^{2n} V^*$ where $V \cong \mathbb{C}^{2n}$
The homomorphism $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$ given by the representation S^2V, $V \cong \mathbb{C}^2$.

If you have symplectic forms on V and W, you get a nondegenerate symmetric form on $V \otimes W$, hence on S^2V and $\bigwedge^2 V$
$D_2 \cong A_1 \times A_1$: $\quad \bullet \quad \bullet \quad$ (2 vertices, no edge)

$\Rightarrow \quad SO(4,\mathbb{C}) = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\{(1,1),(-1,-1)\}$
Proof: If V_1 and V_2 are the standard representations of two copies of $SL(2,\mathbb{C})$, then $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ acts on $V_1 \otimes V_2$ preserving a symmetric form $\square$

$C_2 = B_2 \cong$

$\Rightarrow \quad Sp(4,\mathbb{C})/\{\pm 1\} \cong SO(5,\mathbb{C})$
Proof: Let V=the standard representation of $Sp(4,\mathbb{C})$, $V \cong \mathbb{C}^4$.
$\Rightarrow \quad Sp(4,\mathbb{C})$ acts on $\bigwedge^2 V \cong \mathbb{C}^6$, and $\bigwedge^2 V \cong \mathbb{C} \oplus M_5$
$\Rightarrow \quad$ get homomorphism $Sp(4,\mathbb{C}) \to SO(5,\mathbb{C})$.
By counting dimensions, this is surjective. □

Finally, $D_3 \cong A_3$:

So $SL(4,\mathbb{C})/\{\pm 1\} \cong SO(6,\mathbb{C})$
$(\pi_1(SO(6,\mathbb{C})) = \mathbb{Z}/2)$
Exercise: A proof in terms of linear algebra

Representation Theory of complex semisimple Lie algebra

Let V be a finite dimensional representation of $\mathfrak{g}$. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra. For each root $\alpha \in R$, we have a copy of $\mathfrak{sl}(2,\mathbb{C}) \subset \mathfrak{g}$ (denoted $\mathfrak{sl}(2,\mathbb{C})_\alpha$ before) and we can view V as a representation of this $\mathfrak{sl}(2,\mathbb{C})$. We know that the coroot $\alpha^\vee \in \mathfrak{sl}(2,\mathbb{C})_\alpha$ acts diagonalizably on V
Moreover, all coroot $\alpha^\vee$ are in $\mathfrak{t}$ an abelian Lie algebra so they all commute in their action on V. So we can simultaneously diagonalise V wrt all $\alpha^\vee \in \mathfrak{t}$
So all of $\mathfrak{t}$ acts diagonalizably on V

$$V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda \qquad (\lambda\text{-weight spaces})$$

where $V_\lambda = \{x \in V | h(x) = \lambda(h)x \; \forall h \in \mathfrak{t}\}$
Moreover, for each $\alpha \in R$, the weights of V wrt $\alpha^\vee \in \mathfrak{sl}(2,\mathbb{C})_\alpha$ must be integers (Theorem 48)
That means that if $V_\lambda \neq 0$, then $\lambda(\alpha^\vee) \in \mathbb{Z}$

Definition 119
The weight lattice P of $\mathfrak{g}$ is $\{\lambda \in \mathfrak{t}^* \,|\lambda(\alpha^\vee) \in \mathbb{Z} \;\forall \alpha \in R\}$
The root lattice Q of $\mathfrak{g}$ is the $\mathbb{Z}$-submodule of $\mathfrak{t}^*$ spanned by the roots α

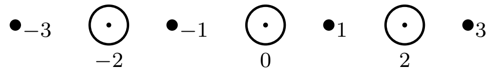
We know that $Q \subset P$ because $\alpha(\beta^\vee) \in \mathbb{Z} \;\forall \alpha, \beta \in R$
Let l =rank $\mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{t}$
Then $Q \cong \mathbb{Z}^l$ because the roots span $\mathfrak{t} \cong \mathbb{C}^l$ as a complex vector space
and $P \cong \mathbb{Z}^l$ and it contains Q as a subgroup of finite index. We can describe P as

$$P = \{\lambda \in \mathfrak{t}^* \,|\lambda(\alpha_1^\vee) \in \mathbb{Z}, \ldots, \lambda(\alpha_l^\vee) \in \mathbb{Z}\}$$

where $\alpha_1, \ldots, \alpha_l$ are the simple roots
So Q must have finite index in P, because the weights of a finite dimensional representation of $\mathfrak{sl}(2)$ are in $\mathbb{Z}$, we have

$$V = \bigoplus_{\lambda \in P} V_\lambda$$

Example:
For $\mathfrak{sl}(2,\mathbb{C})$, $P \cong \mathbb{Z}, Q \cong 2\mathbb{Z} \subset \mathbb{Z}$



$P = \{\bullet\} \cup Q = \{\bullet\} \cup \{\odot\}$
For $\mathfrak{sl}(n,\mathbb{C}), Q = \{\sum_{i=1}^n a_i\epsilon_i | a_i \in \mathbb{Z}, \sum a_i = 0\}$

The weight lattice is

$$P = \mathbb{Z}^n \,/\, \mathbb{Z}(1,1,\dots,1) = \left\{ \sum_{i=1}^{n} a_i\epsilon_i \middle| a_i \in \mathbb{Z} \right\} /(\epsilon_1 + \cdots + \epsilon_n = 0)$$

One sees that $P/Q \cong \mathbb{Z}/n$

In terms of the <u>group</u> $G = SL(n,\mathbb{C})$
Let $T \subset SL(n,\mathbb{C})$ be the maximal torus $\cong (\mathbb{C}^\times)^{n-1}$ with Lie algebra $\mathfrak{t}$. Then

$$P = \mathrm{Hom}(T,\mathbb{C}^\times) \cong \mathbb{Z}^{n-1}$$
$$Q \cong \mathrm{Hom}(T/Z(G),\mathbb{C}^\times)$$

where $Z(SL(n,\mathbb{C})) \cong \mu_n(\mathbb{C})$ (group of n-th roots of unity)

Notice that any representation of $SL(n,\mathbb{C})$ has weights in P. But the adjoint representation of $SL(n,\mathbb{C})$ has weights in $Q \subset P$

Definition 120
The <u>adjoint group</u> with a given semisimple Lie algebra $\mathfrak{g}$ is $G/Z(G)$ for any group G with Lie algebra $\mathfrak{g}$

We have $V = \bigoplus_{\lambda\in P} V_\lambda$
Easy to check that for $e \in \mathfrak{g}_\alpha$, $e(V_\lambda) \subset V_{\lambda+\alpha}$
Therefore, for any $\lambda \in P$ and any root α,

$$\bigoplus_{n\in\mathbb{Z}} V_{\lambda+n\alpha}$$

is an $\mathfrak{sl}(2,\mathbb{C})_\alpha$-subrepresentation of V
Use that the character of a finite dimensional representation of $\mathfrak{sl}(2)$ are invariant under sign change, $\mathfrak{t} \mapsto \mathfrak{t}^{-1}$
That means that $\dim V_\lambda = \dim V_{s_\alpha(\lambda)}$ for every root α, because $s_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha$

Definition 121
The <u>character</u> of any finite dimensional representation of $\mathfrak{g}$ is

$$\mathrm{ch}(V) = \sum_{\lambda\in P} n_\lambda e^\lambda \in \text{ the group ring } \mathbb{Z}[P] \cong \mathbb{Z}[\mathbb{Z}^l]$$

Here, we say that $e^\lambda e^\mu = e^{\lambda+\mu}$ for $\lambda,\mu \in P \cong \mathbb{Z}^l$, $e^0 = 1$, $n_\lambda = \dim_{\mathbb{C}} V_\lambda$

Corollary 122
The character of any finite dimensional representation of $\mathfrak{g}$ is invariant under the Weyl group.

Definition 123
The <u>fundamental weights</u> $w_1, \dots w_l$ of $\mathfrak{g}$ are the elements of $\mathfrak{t}^*$ s.t. $w_i(\alpha_j^\vee) = \delta_{ij}$
Easy to see that $P = \mathbb{Z}\,w_1 \oplus \cdots \oplus \mathbb{Z}\,w_l$
Then $\mathbb{Z}[P] \cong \mathbb{Z}[e^{w_1}, (e^{w_1})^{-1}, \dots, e^{w_l}, (e^{w_l})^{-1}]$

e.g. character of a representation of $\mathfrak{sl}(3,\mathbb{C})$ correspond to A_2

Definition 124
A highest weight vector $x \in V$ is a nonzero element $x \in V_\lambda$ for some $\lambda \in P$ s.t. $e(x) = 0\ \forall e \in \mathfrak{g}_\alpha$ with α a positive root

Clearly, every nonzero finite dimensional representation V of $\mathfrak{g}$ contains a highest weight vector. Start with $x \in V_\lambda, x \neq 0$. If $e(x) \neq 0$ for some $e \in \mathfrak{g}_\alpha$,$\alpha \in R^+$, then look at $e(x) \in V_{\lambda+\alpha}$ Repeat

Lemma 125
Let V be a finite dimensional representation of a complex semisimple Lie algebra (Fix $\mathfrak{t} \subset \mathfrak{g}, R^+ < R$)
Let $x \in V$ be a highest weight vector. Then

$$M := \sum_{\substack{f_i \in \mathfrak{g}_\alpha \\ \alpha \in R_-, r \geq 0}} ef_1 \cdots f_r(x) \subset V$$

is an irreducible subrepresentation of V

Proof
First show that M is a sub-$\mathfrak{g}$-module of V. Clear that $f(M) \subset M$ if $f \in \mathfrak{g}_\alpha, \alpha \in R_-$
If $x \in V_\lambda$ then $f_1 \cdots f_r(x) \in V_{\lambda+\alpha_1+\cdots\alpha_r}$ ($\alpha_1, \ldots \alpha_r$ negative roots)
We show, for $e \in \mathfrak{g}_\alpha$ with α^+ that $e(f_1 \cdots f_r(x)) \in M$
by induction on r, true for $r = 0$ since $e(x) = 0$
If true for $r - 1$, then

$$ef_1 \cdots f_r(x) = [e, f_1]f_2 \cdots f_r(x) + f_1 \underbrace{ef_2 \cdots f_r(x)}_{\in M \text{ by induction}}$$

Here $[e, f_1] \in \mathfrak{g}_\alpha$ for $\alpha \in R_+$ or R_- or $\alpha = 0$, and we are done in all cases by induction. So M is a sub-$\mathfrak{g}$-module of V

If M is not irreducible then $M = M_1 \oplus M_2$ for some non-zero $\mathfrak{g}$-modules. We would have $\mathbb{C}\, x = M_\lambda = (M_1)_\lambda \oplus (M_2)_\lambda$
So x is in one of M_1 or M_2, say M_1 WLOG. So $M = M_1 \Rightarrow\ M$ is irreducible □

Define a partial order on the weight lattice $P = \mathbb{Z}\,\omega_1 \oplus \cdots \mathbb{Z}\,\omega_l \cong \mathbb{Z}^l$ by $\lambda \leq \mu$ if

$$\mu = \lambda + \sum_{i=1}^{l} n_i \alpha_i$$

where $\alpha_1, \ldots, \alpha_l$ are the simple roots, $n_i \in \mathbb{N}$

In the module M in Lemma, all weight that occur are $\leq \lambda$ (the weight of x). For example, if V is an irreducible $\mathfrak{g}$-module, then $M = V$, so all weights in V are $\leq \lambda$, the weight of a highest weight vector. So an irreducible $\mathfrak{g}$-module has a unique highest weight vector, up to nonzero scalar. Also the weight λ of this highest weight vector is uniquely determined by V.
Moreover, let V be an irreducible $\mathfrak{g}$-module with highest weight $\lambda \in P \cong \mathbb{Z}^l \subset \mathfrak{t}^* \subset \mathbb{C}^l$
For each positive root α, think of V as a representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha \subset \mathfrak{g}$
Then a highest weight $x \in V$ for $\mathfrak{g}$ is also a highest weight vector for $\mathfrak{sl}(2, \mathbb{C})_\alpha$. Therefore, the weight of X wrt $\alpha^\vee$ is a nonnegative integer.
So $\lambda(\alpha_i^\vee) \geq 0$ for $i = 1, \ldots, l$

Definition 126
The dominant weights $P^+ \subset P$ are

$$\begin{aligned} \{\lambda \in P | \lambda(\alpha_i^\vee) > 0\} &= \{\lambda \in \mathfrak{t}^* | \lambda(\alpha_i^\vee) \in \mathbb{N}\} \\ &= \mathbb{N}\omega_1 \oplus \cdots \oplus \mathbb{N}\omega_l \end{aligned}$$

where $\omega_1, \ldots, \omega_l$ are the fundamental weights

Remark. In literature, these are the weights that lies in the closure of Weyl chamber

.

Lemma 127
A finite dimensional irreducible representation of $\mathfrak{g}$ is uniquely determined by its highest weight

Proof
Let V, W be finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight vectors $x \in V, y \in W$ with the same weight $\lambda \in P \subset \mathfrak{t}^*$. Then $V \oplus W$ is a representation of $\mathfrak{g}$ and $x + y \in V \oplus W$ is a highest weight vector, with the same weight λ. As in previous lemma, let M=sub-$\mathfrak{g}$-module of $V \oplus W$ spanned by $x + y$; thus an irreducible subrepresentation of $V \oplus W$. We have $\mathfrak{g}$-linear projections

$$M \hookrightarrow V \oplus W \twoheadrightarrow V$$
$$M \hookrightarrow V \oplus W \twoheadrightarrow W$$

These are nonzero $\mathfrak{g}$-linear maps of irreducible representations of $\mathfrak{g}$, so they are isomorphic by Schur's Lemma. So $V \cong M \cong W$ □

Theorem 128
There is a finite dimensional irreducible representation of $\mathfrak{g}$ with any given dominant weight as its highest weight

Sketch Proof
It suffices to find irreducible representations of $\mathfrak{g}$ with highest weight the fundamental weights $\omega_1, \ldots, \omega_l$
Indeed, if V and W are irreducible representations of $\mathfrak{g}$ with highest weights λ and μ; then $V \otimes_{\mathbb{C}} W$ contains a highest weight vector with weight $\lambda + \mu$
(Take $x \otimes y \in V \otimes W$, for $x \in V, y \in W$ highest weight vectors) $e(x \otimes y) = ex \otimes y + x \otimes ey = 0$ for $e \in \mathfrak{g}_\alpha, \alpha \in R_+$
So $V \otimes W$ contains an irreducible $\mathfrak{g}$-module with highest weight $\lambda + \mu$
So the irreducible representation with the highest weight $d_1\omega_1 + \cdots + d_l\omega_l, \quad d_i \in \mathbb{N}$ occurs inside $V_1^{\otimes d_1} \otimes \cdots V_l^{\otimes d_l}$ where V_i is irreducible with highest weight ω_i □

A slight improvement: if V has highest weight λ, then S^dV contains the irreducible representation with highest weight $d\lambda$, so $V_{d_1\omega_1+\cdots d_l\omega_l} \subset S^{d_1}V_1 \otimes \cdots \otimes S^{d_l}V_l$

Corollary 129
A finite dimensional representation of $\mathfrak{g}$ is uniquely determined by its character in $\mathbb{Z}[P]^W$

Proof
Subtract off one irreducible character at a time □

Example:
The representation of S^dV of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is irreducible,for any $d \geq 0$.
Here, $V \cong \mathbb{C}^n$ is the standard representation of $\mathfrak{g}$

We compute the character of S^dV
let $e_1, \ldots, e_n$ be the usual basis for V
Then S^dV has a basis $e_1^{i_1} \cdots e_n^{i_n}$ for $i_1, \ldots, i_n \geq 0, i_1 + \cdots + i_n = d$
The element $e_1^{i_1} \cdots e_n^{i_n}$ has weight $i_1\epsilon_1 + \cdots + i_n\epsilon_n \in P = \mathbb{Z}\,\epsilon_1 \oplus \cdots \mathbb{Z}\,\epsilon_n$
can see that all these weights are different, i.e. all "weight multiplicities" for S^dV are isomorphic to $\mathbb{Z}^n$ or 0

Look for highest weight vectors in S^dV. That is we try to solve

$$e_{ab}(e_1^{i_1} \cdots e_n^{i_n}) = 0 \qquad \forall 1 \leq a < b \leq n$$

Here

$$e_{ab}(e_j) = \begin{cases} e_a & \text{if } j = b \\ 0 & \text{otherwise} \end{cases}$$

The only highest weight vector, therefore, is e_1^d up to scalars
So S^dV is an irreducible representation of $\mathfrak{sl}(n, \mathbb{C})$

Weyl Character Formula
One proof: write down the Bernstein-Gelfand-Gelfand resolution of finite dimensional irreducible $\mathfrak{g}$-modules ($\mathfrak{g} = \mathbb{C}$ −semisimple Lie algebra) by (infinite dimension) Verma modules

For any $\lambda \in \mathfrak{t}^*$, the Verma module M_λ with highest weight λ is the "universal" highest weight module with a highest weight vector with weight λ. That means: choose a set of positive roots $R^+ \subset R$. Then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$ and we write $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}_+$ a Borel subalgebra. λ determines a linear map $\mathfrak{b} \to \mathbb{C}$ by $\lambda : \mathfrak{t} \to \mathbb{C}$ and sending $\mathfrak{n}_+$ to 0

$$\begin{array}{rcll} \lambda : \mathfrak{b} & \to & \mathbb{C} & \\ h & \mapsto & \lambda(h) & h \in \mathfrak{t} \\ x & \mapsto & 0 & x \in \mathfrak{n} \end{array}$$

A Verma module has, let $\mathbb{C}_\lambda$ be the 1-dimensional representation $\mathfrak{b}$ given by this linear map. Then

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, V)$$

(first V is any $\mathfrak{g}$-module, second V is considered as an representation of $\mathfrak{b}$)
(More concretely, $M_\lambda = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda) = U\,\mathfrak{g} \otimes_{U\,\mathfrak{b}} \mathbb{C}_\lambda$)
More concretely, let $f_1, \ldots, f_N$ be a basis for $\mathfrak{n}_-$. Then a basis B for M_λ as a $\mathbb{C}$-vector space:

$$M_\lambda = \bigoplus_{i_1, \ldots, i_n \in \mathbb{N}} \mathbb{C} \cdot f_1^{i_1} \cdots f_N^{i_N}(x)$$

here x is the highest weight vector in M_λ with highest weight λ. (Convince yourself that M_λ IS a representation of $\mathfrak{g}$)
We have an obvious surjection $M_\lambda \twoheadrightarrow L_\lambda$ for $\lambda \in P_+$, where L_λ= the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ, and is unique up to isomorphism

Example:
For $\mathfrak{sl}(2)$, $\lambda \in \mathfrak{t}^* \cong \mathbb{C}$ the Verma module M_λ has characters

we worked out how $e \in \mathfrak{sl}(2)$ acts on $f^r x$ for any $r \in \mathbb{N}$:

$$e(f^r x) = r(\lambda + 1 - r) f^{r-1} x$$

(prove by induction on r)

Exercise: Show that, for $\mathfrak{g} = \mathfrak{sl}(2)$, $\lambda \in \mathbb{C}, \lambda \notin \mathbb{N}$

By contrast, if $\lambda \in \mathbb{N}$, then we have $M_{-\lambda-2} \subset M_\lambda$

In fact:

$$0 \to M_{-\lambda-2} \to M_\lambda \to L_\lambda \to 0$$

Notice that M_λ is NOT completely reducible

Definition 130
The Weyl group W of $\mathfrak{g}$ is generated by simple reflections $s_i = s_{\alpha_i} \quad 1 \leq i \leq l$.
Define the length of $w \in W$ be

$$\begin{aligned} l(w) &:= \min\{d \geq 0 | w = s_{i_1} \cdots s_{i_d} \in W\} \\ &= |\{\alpha \in R_+ | w(\alpha) \in R_-\}| \end{aligned}$$

Define

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \in P$$

Note that $s_i \rho = \rho - \alpha_i$ for all simple reflection s_i Define the dot action or shifted action of W on P by $w \cdot \lambda = w(\lambda + \rho) - \rho$

Theorem 131 (Bernstein-Gelfand-Gelfand Resolution)
Let $\mathfrak{g}$ be a $\mathbb{C}$-semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ Cartan, $R_+ \subset R$ a set of positive roots. For $\lambda \in P_+$ let L_λ =the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ. Then there is an exact sequence of representation of $\mathfrak{g}$

$$0 \to \bigoplus_{\substack{w \in W \\ l(w) = l_0}} M_{w\lambda} \to \cdots \to \bigoplus_{\substack{w \in W \\ l(w) = 1}} M_{w\lambda} \to M_\lambda \to L_\lambda \to 0$$

where l_0 is the maximal length

We can use the dot action to get a formula for the character of L_λ, $\lambda \in P_+$, then M_λ has the same "size" as a polynomial in N variables, $N = |\dim_{\mathbb{C}} \mathfrak{n}_-| = |R_+|$, so

$$\operatorname{ch}(M_\lambda) = \frac{e^\lambda}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})}$$

Corollary 132 (Weyl Character Formula)
For any $\lambda \in P_+$

$$\operatorname{ch}(L_\lambda) = \frac{\sum_{w \in W}(-1)^{l(w)} e^{w\lambda}}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})} = \frac{\sum_{w \in W}(-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})} = \frac{\sum_{w \in W}(-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W}(-1)^{l(w)} e^{w(\rho)}}$$

(the last equality comes from setting $\lambda = 0$ in the first/second equality)
Remark. $(-1)^{l(w)} = \det(w$ acting on $\mathfrak{t}^*_{\mathbb{R}}) = \{\pm 1\}$

Example:
For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), W = S_{n-1}$, $(-1)^{l(w)} = \operatorname{sgn}(w) \in \{\pm 1\}$
(the simple reflections here are $s_i = (i, i+1)$)

Using l'Hopitals's rule, get:

Corollary 133
For any $\lambda \in P_+$

$$\dim_{\mathbb{C}} L_\lambda = \frac{\prod_{\alpha \in R_+}\langle \lambda + \rho, \alpha \rangle}{\prod_{\alpha \in R_+}\langle \rho, \alpha \rangle} = \frac{\prod_{\alpha \in R_+}(\lambda + \rho)(\alpha^\vee)}{\prod_{\alpha \in R_+}\rho(\alpha^\vee)}$$

Some examples:
Seen that for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $S^d V$ is an irreducible $\mathfrak{g}$-module for $d \geq 0$ where $V \cong \mathbb{C}^n$ is the standard representation. This has highest weight dw_1, where w_1=highest weight of V=1st fundamental weight

Now consider, for $1 \leq d \leq n-1$, $\bigwedge^V$
(Note $\bigwedge^0 V \cong \bigwedge^n V \cong \mathbb{C}$ as representations of $\mathfrak{sl}(n, \mathbb{C})$)

What are the fundamental weights for $\mathfrak{sl}(n, \mathbb{C})$? First way:

$\alpha_i = e_i - e_{i+1}$
We have, for example $\alpha_1(\alpha_1^\vee) = 2, \alpha_1(\alpha_2^\vee) = -1$, $\alpha_1(\alpha_j^\vee) = 0$ for $j \geq 3$ so $\alpha_1 = 2w_1 - w_2$ etc.

Second way:
We know that the simple coroots are $\alpha_i^\vee = e_{ii} - e_{i+1,i+1} \in \mathfrak{t}$
So the fundamental weighs are $w_1 = \epsilon_1$, $w_2 = \epsilon_1 + \epsilon_2, \dots, w_{n-1} = \epsilon_1 + \cdots + \epsilon_{n-1}$

The weights of $\bigwedge^d V$ are: the basis element $e_{i_1} \wedge \cdots \wedge e_{i_d} \in \bigwedge^d V$ has weight $\epsilon_{i_1} + \cdots \epsilon_{i_d} \in P = (\mathbb{Z}\,\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\,\epsilon_n)/\,\mathbb{Z}(\epsilon_1 + \cdots + \epsilon_n) \cong \mathbb{Z}^{n-1}$
These are all different so all weight multiplicities for $\bigwedge^d V$ are 1 (or 0), $\mathfrak{n}_+$ is spanned by the elements $e_{ab} \in \mathfrak{sl}(n, \mathbb{C})$ with $1 \leq a < b \leq n$
So there is only one highest weight vector in $\bigwedge^d V$, up to scalars, $e_1 \wedge e_2 \cdots \wedge e_d$
So $\bigwedge^d V$ is an irreducible representation of $\mathfrak{sl}(n, \mathbb{C})$ for $1 \leq d \leq n-1$ and its highest weight is $\epsilon_1 + \cdots + \epsilon_d = w_d$
So $V, \bigwedge^2 V, \dots, \bigwedge^{n-1} V$ are the fundamental representations of $\mathfrak{sl}(n)$, the irreducible representations correspond to the fundamental weight.
Using the character formulas, you can work out how to decompose tensor product of $L_{\lambda_1} \otimes L_{\lambda_2}$ as a sum of irreducible representations.

Example:
As a representation of $\mathfrak{sl}(n, \mathbb{C})$, $V^* \cong \bigwedge^{n-1} V$ (and more generally $(\bigwedge^i V)^* \cong \bigwedge^{n-i} V$)

Proof

We can "multiply" $\bigwedge^a V \otimes \bigwedge^b V \to \bigwedge^{a+b} V$, this is $\mathfrak{sl}(n)$-linear.
For $n = \dim V$, this is a dual pairing, $\bigwedge^i V \otimes \bigwedge^{n-i} V \to \bigwedge^n V = \mathbb{C}$ □

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