Brundan-Stroppel's generalised Khovanov's arc algebras

Family of symmetric f.d. algebras: H_m^n
Family of quasi-hereditary f.d. algebras: K_m^n
K_m^n = q.h. cover of H_m^n
$H_m^n \cong H_n^m; \quad K_m^n \cong K_n^m \;\Rightarrow\;$ can assume $m \leq n$
Direct limits give $K_m^\infty, K_m^{\pm\infty}, K_\infty^\infty$, these are locally unital, and has highest weight structure.
Both families are positively graded cellular [BS1]
K_*^* are Koszul [BS2]
Khovanov categorification of Jones polynomial: H_n^n

Results of BS3:

$\mathfrak{g} = \mathfrak{gl}_{m+n}$
$\mathfrak{l} = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$
$\mathfrak{p} = \mathfrak{l} + \mathfrak{b}$
$\mathcal{O}$ = parabolic analogue of BGG category associated to Hermitian pair $(\mathfrak{g}, \mathfrak{l})$
$\mathcal{O}(m,n)$ = category of f.g. $\mathfrak{g}$-modules
locally finite over $\mathfrak{p}$
semisimple over $\mathfrak{h}$
with weight in $\mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_{m+n}$
$= \bigoplus$ integral blocks of $\mathcal{O}$

Previous works of Stroppel: K_m^n-fdmod $\cong$ PerverseSheaves(Gr$(m, m+n)$) $\cong \mathcal{O}_0$

Construct weights set = $\Lambda(m,n)$ (m ups, n downs)
Construct blocks set = $\Lambda(m,n)/\sim$ (equivalence given by permutation of arrows)
(Note: Γ is principal block $\Rightarrow$ $K_\Gamma \cong K_m^n$)
Define $K(m,n) = \bigoplus_{\Gamma \in \Lambda(m,n)/\sim} K_\Gamma$

(Main) Theorem

$$\begin{array}{rcl} \mathcal{O}(m,n) & \xrightarrow{\sim} & K(m,n)\text{ -fdmod} \\ \text{simples } \mathcal{L}(\lambda) & \mapsto & \text{simples } L(\lambda) \end{array}$$

restricting to principal block:

$$\mathcal{O} \xrightarrow{\sim} K_m^n\text{-fdmod}$$

The followings are some important theorems proved along the way:

Let I be indexing set $= \{o+1, o+2, \dots\}$
$I^+ = I \cup (I+1)$ (e.g. $o = 0, I = I^+$) need $0 \leq m, n \leq |I| + 1$
U = universal enveloping algebra of GL_{I^+}

Theorem $K_0(K(m,n)\text{-Mod}) \cong \bigwedge^m V \otimes \bigwedge^n V \cong K_0(\mathcal{O}(m,n))$ as $\dot{U}$-module

$P(m,n)$ = set of weights with non-zero multiplicity in $\bigwedge^m V \otimes \bigwedge^n V$
$\Lambda = \Lambda_{o+m} + \Lambda_{o+n}$ = unique max. elt. in $P(m,n)$ wrt dominance order
$\Rightarrow$ any weight has form $\Gamma = \Lambda - \alpha$

Fix $\Gamma = \Lambda - \alpha$
Define $T^\Lambda_\alpha \in K_\Gamma$-Mod and $\mathcal{T}^\Lambda_\alpha \in \mathcal{O}_\Gamma$

To prove equivalence of categories above, is equivalent to, showing $E^\Lambda_\alpha := \mathrm{End}_{K_\Gamma}(T^\Lambda_\alpha)^{op} \cong \mathrm{End}_{\mathfrak{g}}(\mathcal{T}^\Lambda_\alpha)$

Theorem Identify K^Λ_α to K_Γ or 0 (depends on situation)
E^Λ_α Morita to $H^\Lambda_\alpha = eK^\Lambda_\alpha e$ some idem. e
KLR-algebra $R^\Lambda_\alpha \xrightarrow{\sim} E^\Lambda_\alpha$ as graded algebras

Let $p \leq q$ integers (typically $p = o + m, q = o + n$)
H_d = degenerate affine Hecke algebra
$H^{p,q}_d = H_d / \langle (x_1 - p)(x_1 - q) \rangle$
= degenerate cyclotomic Hecke algebra of level 2
= degenerate cyclotomic Hecke algebra (Ariki-Koike algebra) of $G(2, 1, d)$

Theorem For some idem. e_α, there is a unique algebra isomorphism $\sigma : e_\alpha H^{p,q}_d \xrightarrow{\sim} R^\Lambda_\alpha$

Applications: Basis of Specht modules, Khovanov-Lauda conjecture on level 2

<u>Results of BS4</u>:

$G = GL(m|n)$ General linear supergroup
$\mathcal{F} \oplus \Pi\mathcal{F}$ = category of all f.d. G-modules
$M \in \mathcal{F} = \mathcal{F}(m|n)$ if $M = M_+$
$M \in \Pi\mathcal{F}$ if $M = M_-$

Construct weight set = $\Lambda(m|n)$
Construct block set = $\Lambda(m|n)/\sim$
Define $K(m|n) = \bigoplus_{\Gamma \in \Lambda(m|n)/\sim} K_\Gamma$

(Main) Theorem

$$\begin{array}{rcl} \mathcal{F}(m|n) & \xrightarrow{\sim} & K(m|n)\text{ -fdmod} \\ \text{simples } \mathcal{L}(\lambda) & \mapsto & \text{simples } L(\lambda) \\ \text{Kac (Verma) modules } \mathcal{V}(\lambda) & \mapsto & \text{cell modules } \Delta(\lambda) \\ \text{PIMs } \mathcal{P}(\lambda) & \mapsto & \text{PIMs } P(\lambda) \end{array}$$

Important theorems proved/used along the way of proving the main one:

$R^{p,q}_d$ = cyclotomic KLR-algebra, identified with $e_{d,q,p} H^{p,q}_d$ as in BS3

Theorem
If $d \leq \min(m, n)$, then $R^{p,q}_d = H^{p,q}_d$ and as algebras, we have

$$\Phi : H^{p,q}_d \xrightarrow{\sim} \mathrm{End}_G(\mathcal{V}(\lambda_{p,q}) \otimes V^{\otimes d})^{op}$$

Theorem (Super Schur-Weyl duality)
For any $d \geq 0$, Φ (above) is surjective

$R^{p,q} = \bigoplus_{d \geq 0} R^{p,q}_d$
Generalised Khovanov algebra $K^{p,q} := eK(m|n)e$ some (non-central) idem e

Theorem $K^{p,q}$ Morita to $R^{p,q}$

Results of BS5

$B_{r,s}(\delta)$ = walled Brauer algebra over $\mathbb{C}$, $\delta \in \mathbb{Z}$

Construct weight set Λ, using δ
($\Lambda \leftrightarrow$ set of all bipartitions)
Construct block set $\Lambda/\sim$
$\rightsquigarrow \quad K(\delta) := \bigoplus_{\Gamma \in \Lambda/\sim} K_\Gamma$

Define $\Lambda_{r,s} := \{\lambda \in \Lambda \mid |\lambda^{\mathrm{L}}| = r - t \text{ and } |\lambda^{\mathrm{R}}| = s - t \text{ for } 0 \le t \le \min(r,s)\}$
$$\dot{\Lambda}_{r,s} := \begin{cases} \Lambda_{r,s} & \text{if } \delta \neq 0 \text{ or } r \neq s \text{ or } r = s = 0, \\ \Lambda_{r,s} \setminus \{(\varnothing, \varnothing)\} & \text{if } \delta = 0 \text{ and } r = s > 0. \end{cases}$$
Idem. $e_{r,s} := \sum_{\lambda \in \dot{\Lambda}_{r,s}} e_\lambda$

(Main) Theorem $B_{r,s}(\delta)$ Morita to $e_{r,s}K(\delta)e_{r,s}$

Corollary: $B_{r,s}(\delta)$ is Koszul for $\delta \neq 0$