Some Hecke algebras of GL_n (type A)

Finite(Iwahori-) Hecke algebra H_n^f

$$\begin{aligned} \text{Generators} \quad & T_1, \dots, T_{n-1} \\ \text{Relations} \quad & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ & T_i T_j = T_j T_i \qquad (|i-j| > 1) \\ & (T_i - q)(T_i + 1) = 0 \end{aligned}$$

• Non-degenerate means q is a root of 1; Degenerate means $q = 1$, $H_n^f \cong k\mathfrak{S}_n$
• Last relation $\leftrightarrow (T_i - Q)(T_i + Q^{-1})$ via transformation $T_i \mapsto \sqrt{q} T_i$
• $(T_i - Q)(T_i + Q^{-1})$-presentation renormalizes the natural inner product on H_n^f
• When $q \neq 0$, these are symmetric algebra via symmetrizing form $T_w \mapsto \delta_{1,w}$
• **nil**-version: replace last relation by $T_i^2 = 1$; symmetrizing form is δ_{w,w_0} where w_0 is longest element

Affine Hecke algebra H_n

	Non-degenerate	Degenerate	nil
Form	$\mathbb{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \otimes H_n^f$	$\mathbb{Z}[X_1, \dots, X_n] \otimes H_n^f$	$\mathbb{Z}[X_1, \dots, X_n] \otimes H_n^f$
Generators	$X_1^{\pm 1}, \dots, X_n^{\pm 1}; T_1, \dots, T_{n-1}$	$X_1, \dots, X_n; T_1, \dots, T_{n-1}$	$X_1, \dots, X_n; T_1, \dots, T_{n-1}$
Relations	relations for non-degen. H_n^f $X_i^{\pm 1} X_j^{\pm 1} = X_j^{\pm 1} X_i^{\pm 1}$ $T_i X_j = X_j T_i \quad j \neq i, i+1$ $T_i X_{i+1} - X_i T_i = (q-1) X_{i+1}$ $X_i X_i^{-1} = 1$	relations for degen. H_n^f $X_i X_j = X_j X_i$ $T_i X_j = X_j T_i \quad j \neq i, i+1$ $T_i X_{i+1} - X_i T_i = 1$	relations for degen. H_n^f $X_i X_j = X_j X_i$ $T_i X_j = X_j T_i \quad j \neq i, i+1$ $T_i X_{i+1} - X_i T_i = 1$ $T_i X_i - X_{i+1} T_i = -1$

• In the nil case, X_i is called divided difference operator ∂_i

$$\partial_i(f(x_1, \dots, x_n)) = \frac{f - s_i f}{x_i - x_{i+1}}$$

where s_i acts on $k[x_1, \dots, x_n]$ by sapping x_i and x_{i+1}
• Third new relation can be written as $T_i X_i T_i = q X_{i+1}$
• $q = 1$: $T_i X_{i+1} - X_i T_i = 1$ is the same as saying $T_i X_i - X_{i+1} T_i = -1$
• Think of H_n as $U_q(\mathfrak{sl}_n)$, then $\mathbb{Z}[X_1, \dots, X_n]$ is analogue of the Cartan subalgebra, with evals $\{1, q, q^2, \dots, q^{e-1}\}$ where e is as in the definition for H_n^Λ

Cyclotomic Hecke algebra/Ariki-Koike algebra H_n^Λ or $H_{l,n} = H(G(l,1,n))$
(1) Brundan-Kleshchev definition
I=vertex set of quiver $A_{e-1}^{(1)}$ (or A_∞)=$\mathbb{Z}/e\mathbb{Z}$ $\quad [\leftrightarrow \quad \mathfrak{g} = \widehat{\mathfrak{sl}}_e$ or $\mathfrak{sl}_\infty]$
quantum characteristic of $q := e =$
either: minimal +ve integer s.t. $1 + q + \ldots + q^{e-1} = 0$
or: 0 if no such e exists
$\Lambda \in P^+$ (a positive dominant weight)
Level of $\Lambda = l = \sum_{i \in I} (\Lambda, \alpha_i)$

(2) Ariki-Koike definition Take $Q_1, \dots, Q_l \in R$
This corresponds to taking values $\in \{0, q^i, i\}$ in Brundan-Kleshchev definition.

	Non-degenerate	Degenerate
Form	$H_n/\langle\prod_{i\in I}(X_1-q^i)^{(\Lambda,\alpha_i)}\rangle$	$H_n/\langle\prod_{i\in I}(X_1-i)^{(\Lambda,\alpha_i)}\rangle$
Generators	$T_0(\leftrightarrow X_1), T_1,\dots,T_{n-1}$	
Relations	Relations from H_n^f with second one replaced by: $T_iT_j=T_jT_i \qquad (\lvert i-j\rvert>1;\quad i,j=0,\dots,n-1)$ $(T_0-Q_1)\cdots(T_0-Q_l)=0$ $T_0T_1T_0T_1=T_1T_0T_1T_0$	

- isom to cyclotomic KLR-algebra for type A
- The extra relation translates to $y_1^{\delta_{i_1,0}}e(\boldsymbol{i})=0$ (or $y_1^{(\Lambda,\alpha_{i_1})}e(\boldsymbol{i})=0$) in KLR
- Renewed relation comes from first 2 relations of affine case
- T_0,T_1-relation comes from T_iX_{i+1}-relation of affine case

- The subalgebra gen. by $T_1,\dots,T_{n-1}$ is H_n^f
- In particular, with above parameters
  - $q=1, Q_k=\xi^k$ (ξ primitive $\sqrt[l]{1}$) $\Rightarrow$ $R(C_l\wr\mathfrak{S}_n)$
  - $l=1$ or Λ fundamental $\Rightarrow$ H_n^f
  - $l=2$ $\Rightarrow$ Hecke algebra of type B (can rewritten as $(T_0-Q)(T_0+1)=0$)
  - $l=2, q=1$ $\Rightarrow$ Morita to Brundan-Stroppel diagram algebra

- $\bigoplus_n H_n^\Lambda$-mod categorify irred. h.w. $\mathfrak{g}$-module $V(\Lambda)$ but not as $U_q(\mathfrak{g})$-module

$\rightsquigarrow$ AIM: make $U_q(\mathfrak{g})$ visible
$\rightsquigarrow$ we need q to correspond to some good grading of H_n^Λ
$\rightsquigarrow$ consider cyclotomic KLR-algebra (Brundan-Kleshchev's result showed this is possible)