Elliptic Curves

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Chapter 1

Informal Introduction

Number Theory
—Diophantis equations: Algebraic NT, Arithmetic geometry, Birch-Swinnerton-Dyer Conjecture
—Primes: Analytic NT, Riemann Hypothesis

This course is a an introduction of arithmetic geometry

$$V : \begin{cases} f_1(x_1, \ldots, x_m) = 0 \\ \quad \vdots \\ f_n(x_1, \ldots, x_m) = 0 \end{cases}$$

System of polynomial equations with $\mathbb{Z}$-coefficient (algebraic variety over $\mathbb{Q}$)

Main question Describe:
$V(\mathbb{Q})$ =set of rational solutions ($x_i \in \mathbb{Q}$)
$V(\mathbb{Z})$ =set of integer solutions ($x_i \in \mathbb{Z}$)

Example: Is $V(\mathbb{Q})$ infinite (or empty)?
Exercise (Fermat's Last Theorem):
$x^n + y^n = z^n$ has no $\mathbb{Z}$-solutions with $xyz \neq 0, x, y, z \in \mathbb{Z}$ for $n > 2$
$\Leftrightarrow \quad V : x^n + y^n = 1$ has $V(\mathbb{Q}) \subseteq \{(\pm 1, 0), (0, \pm 1)\}$ for $n > 2$
Generally, simplest case is 1 equation in 2 variables

$C : f(x, y) = 0 \qquad \deg f = d$
Plane curve
If C is non-singular projective, then $C(\mathbb{C})$=compact Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$

When can $C(\mathbb{Q})$ be infinite?

$g = 0$: Either $C(\mathbb{Q}) = \emptyset$ or $C(\mathbb{Q})$ infinite. $\exists$ algorithm to determine which
$g = 1$: Unsolved problem (BSD conjecture)
$g \geq 2$: Falting's Theorem (= Mordell Conjecture) (very hard) $C(\mathbb{Q})$ always finite

$\underline{g = 0}$
C line, $ax + by = c$, $C(\mathbb{Q})$ infinite
or
C conic, $f(x, y) = 0$, $\deg f = 2$, (circle, parabola, hyperbola)

E.g.: $C : x^2 + y^2 = 1$
What is $C(\mathbb{Q})$?
Take $Q = (-1, 0)$ and line l_t through Q of slope $t \in \mathbb{Q}$

Claim: 2nd point of intersection P_t is in $C(\mathbb{Q})$

Proof

$$\begin{cases} x^2 + y^2 = 1 \\ y = t(x+1) \end{cases}$$

$\Leftrightarrow \quad x^2 + t^2(x+1)^2 - 1 = 0$ quadratic equation on x with $\mathbb{Q}$-coeff., 1st root $x = -1$ rational
$\Rightarrow$ 2nd root rational

Explicitly,

$$(t^2 + 1)x^2 + 2t^2x + (t^2 - 1) = 0$$

has roots

$$\begin{aligned} x &= -1 \qquad & y &= 0 \\ x &= \frac{1-t^2}{1+t^2} \qquad & y &= t\left(\frac{1-t^2}{1+t^2} + 1\right) = \frac{2t}{1+t^2} \end{aligned}$$

i.e.

$$P_t = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

Conversely, $P \in C(\mathbb{Q}) \Rightarrow$ line PQ has slope $\in \mathbb{Q}$
$\Rightarrow \quad P = P_t$ for some $t \in \mathbb{Q}$

$$\begin{array}{ccc} \mathbb{Q} \cup \{\infty\} & \leftarrow 1:1 \rightarrow & C(\mathbb{Q}) \\ t & \mapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \\ \frac{y}{x+1} & \leftarrow\!\shortmid & (x, y) \end{array}$$

(in fact, $C \cong \mathbb{P}^1_{\mathbb{Q}}$) □

Corollary
Every Pythagorean triples $a^2 + b^2 = c^2$, $a, b, c \in \mathbb{N}$, are of the form

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

(Put $t = m/n$)
Remark:

$C : f(x, y)$ any conic
Either $C(\mathbb{Q}) \neq \emptyset \Rightarrow C(\mathbb{Q})$ infinite, $C \cong \mathbb{P}^1_{\mathbb{Q}}$ (same proof)
or $C(\mathbb{Q}) = \emptyset$ can happen
E.g. $x^2 + y^2 = -1, C(\mathbb{R}) = \emptyset$
E.g.2 $x^2 + y^2 = 3$ no solution mod 3 ($C(\mathbb{Q}_3) = \emptyset$)

Theorem 1.0.1 (Hasse-Minkowski)
C conic, then

$$C(\mathbb{Q}) \neq \emptyset \Leftrightarrow C(\mathbb{R}) \neq \emptyset, C(\mathbb{Q}_p) \neq \emptyset \; \forall p$$

In fact, write $C : ax^2 + by^2 = c$ (easy), $a, b, c \in \mathbb{Z}$ Then enough to check $\mathbb{R}, \mathbb{Q}_p$ for $p|2abc$
Solves $g = 0$ completely

$g = 1$: Elliptic curves - can be represented as a plane cubic

$$E : y^2 = x^3 + Ax + B \quad (A, B \in \mathbb{Q})$$

use Riemann-Roch Theorem:
If $P, Q \in E(\mathbb{Q})$, then line PQ intersect E in third point $R \in E(\mathbb{Q})$

Theorem 1.0.2
Define operation $+$ as follows:
$P + Q = R' = R$ reflected in x-axis
This makes $E(\mathbb{Q})$ into an abelian group

This gives elliptic curves a very rich structure

Theorem 1.0.3 (Mordell-Weil)
$E(\mathbb{Q})$ is a finitely generated abelian group

Our course:

- Geometry of ECs, group law
- Structure of $E(\mathbb{C})$, $E(\mathbb{F}_q)$, $E(\mathbb{Q}_p)$
- Mordell-Weil Theorem
- State Birch-Swinnerton-Dyer Conjecture and related bits

Chapter 2

Curves

2.1 Background

k algebraically closed (e.g. $k = \mathbb{C}$)

Definition 2.1.1
Affine space $\mathbb{A}^n = \mathbb{A}^n_k = \{(a_1, \dots, a_n) | a_i \in k\}$
Projective space $\mathbb{P}^n = \mathbb{P}^n_k = \{(a_0 : a_1 : \cdots : a_n) | a_i \in k$, not all $0\} / \sim$
where $(a_0 : \cdots : a_n) \sim (\alpha a_0 : \cdots : \alpha a_n)$ $\forall \alpha \in k^\times$

$\mathbb{P}^n$ covered by $\mathbb{A}^n$'s:

$$\begin{array}{rcl} \mathbb{A}^n & \hookrightarrow & \mathbb{P}^n \\ (a_1, \dots, a_n) & \mapsto & [1 : a_1 : \cdots : a_n] \end{array}$$

This gives a copy of $\mathbb{A}^n$ in $\mathbb{P}^n$, say $\mathbb{A}^n_0$.
Similarly, get $\mathbb{A}^n_0, \mathbb{A}^n_1, \dots, \mathbb{A}^n_n \hookrightarrow \mathbb{P}^n$
by $(a_1, \dots, a_n) \mapsto [a_0 : \cdots : 1 : \cdots : a_n]$ (1 at j-th place)
If $P \in \mathbb{P}^n$, say $P = (a_0 : \cdots : a_j : \cdots : a_n)$ with not all $a_n = 0$, say $a_j \neq 0$, then
$P = (a_0 : \dots : a_n) = (\frac{a_0}{a_j} : \cdots : \frac{a_j}{a_j} : \cdots : \frac{a_n}{a_j}) \in \mathbb{A}^n_j$
So $\mathbb{P}^n = \mathbb{A}^n_0 \cup \cdots \cup \mathbb{A}^n_n$ (this is called affine charts)

Example 2.1.2
Projective line $\mathbb{P}^1$
$\mathbb{P}^1 = \{(x : 1)\} \cup \{(1 : 0)\} = \mathbb{A}^1_1 \cup \{\infty$ point at infinity$\}$
$= \{(0 : 1)\} \cup \{(1 : y)\} = \{0\} \cup \mathbb{A}^1_0$
Algebraic subsets are $\emptyset, \mathbb{P}^1$, finite subsets $\{b_1, \dots, b_k\}$ zero set of $f(x, y) = \prod (x - b_i y)$

Definition 2.1.3
An (affine) algebraic set $V \subseteq \mathbb{A}^n$ is the set of all solutions to a system of polynomial equations in $x_1, \dots, x_n$

$$V : \begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \qquad \vdots \\ f_m(x_1, \dots, x_n) = 0 \end{cases}$$

A (projective) algebraic set $V \subseteq \mathbb{P}^n$ is the set of all solutions to a system of homogeneous polynomial equations in $x_0, \dots, x_n$
Exercise: Equivalent to $V \cap \mathbb{A}^n_j$ affine algebraic set $\forall j$

Definition 2.1.4
A (projecitve) curve is an infinite algebraic set $C \subseteq \mathbb{P}^n$ s.t. $Y \subsetneq C$ algebraic $\Rightarrow Y$ finite
(irreducible projective variety of dimension 1)

E.g. $\mathbb{P}^1$ is a curve

A curve $C \subseteq \mathbb{P}^2$ is plane curve. These are given by $C : f(x,y,z) = 0$, $f \in k[x,y,z]$ homog. irred.

E.g. $xy - z^2 = 0$
$xy = 1$ in $z = 1$ chart
$x = z^2$ in $y = 1$ chart
$y = z^2$ in $x = 1$ chart

We often write e.g. $C : xy = 1 \subseteq \mathbb{P}^2$ meaning associated projective curve $xy = z^2$

Algebraic sets in $\mathbb{P}^2$ are $\emptyset, \mathbb{P}^2$ finite unions of points and plane curves

2.1.1 Rational functions

Definition 2.1.5
A rational function on $\mathbb{A}^n$ is $f \in k(x_1, \ldots, x_n) =: k(\mathbb{A}^n)$
A rational function on $\mathbb{P}^n$ is $f = 0$ or

$$f = \frac{g(x_0, \ldots, x_n)}{h(x_0, \ldots, x_n)}$$

where g, h homog. polynomials of the same degree.

They form a field $k(\mathbb{P}^n)$; and in fact, $k(\mathbb{P}^n) = k(\mathbb{A}^n_j)$ $\forall$ chart

Example $k(\mathbb{P}^1) \ni \frac{y}{x+y} \leftrightarrow \frac{1}{x+1} \in k(\mathbb{A}^1)$ via,
from left to right, $y \mapsto 1$, and from right to left, homogenize.

Definition 2.1.6
$C \subseteq \mathbb{P}^n$ curve, $f = g/h \in k(\mathbb{P}^n), h \neq 0$ on C
The restriction of f to C

$$f : C \setminus \{\text{finite set}\} \to k$$

(not defined where $h = 0$) is a rational function on C. They form a field $k(C)$

Example 2.1.7

- $C \subseteq \mathbb{P}^2$ plane curve $f(x,y) = 0$ Then $k(C) = ff(k[x,y]/(f))$
- $C = \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$
  Then $k(C) = ff(k[x,y]/(y)) = ff(k[x]) = k(x)$
- $C : y^2 = x^3 + 1$. $k(C) = ff(k[x,y]/y^2 - x^3 - 1) \cong k(x, \sqrt{x^3+1})$

Fact: $k(C)$ is a finitely generated field of transcendence degree 1 over k; so $\forall f \in k(C) \setminus k$

$$k \overset{\text{transc.}}{\hookrightarrow} k(f) \cong k(t) \overset{\text{finite}}{\hookrightarrow} k(C)$$

Fact: (Not hard) Conversely, K f.g. field of tr.deg. 1 over $k \Rightarrow \exists C$ s.t. $k(C) \cong K$

Definition 2.1.8
$C \subseteq \mathbb{P}^n, D \subseteq \mathbb{P}^m$ curves. A rational map $\phi : C \dashrightarrow D$ is one given by rational functions

$$\phi(P) = (f_0(P) : \cdots : f_m(P))$$

where $f_i \in k(C)$, not all 0.
Note: This may not be defined on finitely many points.

Definition 2.1.9
We say ϕ is defined at $P \in C$ if $f_0 g, \ldots, f_m g$ defined at P for some $g \in k(C)^\times$
If ϕ is defined everywhere, ϕ is a morphism

A non-constant $\phi : C \to D$ induces

$$\begin{aligned} \phi^* : k(D) &\hookrightarrow k(C) \\ f &\mapsto \phi^*(f) := f \circ \phi \end{aligned}$$

injective (since fields) of finite index (tr.deg 1)

Definition 2.1.10
Degree of morphism is $\deg \phi = [k(C) : \phi^* k(D)]$

Conversely, any injection $k(D) \hookrightarrow k(C)$ comes from a unique rational map $C \to D$

Example 2.1.11
$C : x^2 + y^2 = 1$, $D : y = 0$, $\phi(x, y) := (x, 0)$
$k(C) \cong k(x, \sqrt{1 - x^2})$, $k(D) \cong k(x)$
So induces $\phi^* x = x$
$\deg \phi = [k(x, \sqrt{1 - x^2}) : k(x)] = 2$

Exercise: $\{$ Rational maps $C \to \mathbb{P}^1$ $\} = k(C)$

2.1.2 Smoothness

Definition 2.1.12
Affine curve C (defined by $f_1, \ldots, f_m$) is non-singular at $P = (a_1, \ldots, a_n) \in C$ if the matrix $A = \left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i,j}$ has rank $n - 1$ (note the rank is always $\leq n - 1$)

Formal derivative

$$\frac{\partial(cx^i y^j \cdots)}{\partial x} := cix^{i-1} y^j \cdots + \text{linearity}$$

with usual rules, product rule, chain rule, etc.

Definition 2.1.13
Projective curve $C \subseteq \mathbb{P}^n$ is non-singular at P if $C \cap \mathbb{A}^n_j$ non-singular at P for some (equivalently, for any) chart containing P

Example 2.1.14
Plane curve $C : f(x, y) = 0$, f irreducible, singular at $P = (a, b) \quad \Leftrightarrow \quad \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0$

We can think in terms of picture:

Example 2.1.15
$f = y^2 - x^3 = 0$
$\frac{\partial f}{\partial x} = -3x^2$
$\frac{\partial f}{\partial y} = 2y$
Both 0 at $(0,0)$ and not both 0 otherwise, so f has unique singular point $(0,0)$

Definition 2.1.16
C non-singular (or smooth) if it is non-singular at every point

Exercise: (char $k \neq 2$) Affine plane curve $y^2 = f(x)$ is non-singular $\Leftrightarrow$ $f(x)$ has non multiple roots

Fact: Non-singular $P \in C$ defines a discrete valuation ("order of vanishing at P")

$$\begin{aligned} v_p : k(C)^\times &\twoheadrightarrow \mathbb{Z} \\ f &\mapsto v_p(f) = \begin{cases} n > 0 & f \text{ has zero of order } n \text{ at } P \\ -n < 0 & f \text{ has pole of order } n \text{ at } P \\ 0 & f(P) \in k^\times \\ \infty & f \equiv 0 \end{cases} \end{aligned}$$

$$v_p(fg) = v_p(f) + v_p(g) \qquad v_p(f/g) = v_p(f) - v_p(g)$$
$$v_p(f \pm g) \geq \min(v_p(f), v_p(g))$$

Example 2.1.17
$C = \mathbb{P}^1, k(C) = k(X) \ni f = \frac{g}{h} = \frac{\prod (x-a_i)^{n_i}}{\prod (x-b_i)^{m_i}}$

$$v_{a_i} f = n_i, \quad v_{b_i} f = -m_i, \quad v_\infty f = \deg h - \deg g, \quad v_P f = 0 \text{ otherwise}$$

Definition 2.1.18
f is a uniformiser at P if $v_P f = 1$
One of coordinate functions $x_j - a_j$ is always a uniformiser at $P = (a_1, \ldots, a_n)$

Example 2.1.19
$C : x^2 + y^2 = 1, P = (a,b) \in C$
$P \neq (\pm 1, 0) \qquad x - a$ uniformiser
$P = (1,0) \qquad y$ uniformiser, $x - 1 = \frac{y^2}{x+1}$ (has valuation 2)

Lemma 2.1.20
If $\phi : C \to C'$ rational map, C non-singular, then ϕ is a morphism

Proof
$\phi = (f_0 : \cdots : f_n), \quad P \in C$
Say $v_P f_0 < v_P f_j, \ j \neq 0$
Then
$$\phi = \left(1 : \underbrace{\frac{f_1}{f_0} : \cdots : \frac{f_n}{f_0}}_{v_P \geq 0}\right)$$
defined at P $\square$

Corollary 2.1.21
If $\phi : C \to C'$ has degree 1, C, C' non-singular, then ϕ is an isomorphism

Proof
ϕ induces $\phi^* : k(C') \xrightarrow{\sim} k(C)$
$\exists\psi$ rational map $C' \to C$ s.t. $\phi\psi = \mathrm{id} = \psi\phi \Rightarrow \phi, \psi$ morphism by the lemma. □

Summary: There is an equivalence of categories

$$\begin{array}{ccc} \text{non-singular curves}/k & \to & \text{f.g. fields } K/k \text{ of tr.deg. } 1 \\ \text{(rational maps =) morphisms } \phi & \to & \text{fields inclusions} \\ C & \mapsto & k(C) \\ \left\{ \begin{array}{c} \text{discrete valuations on } K \\ v : K^\times \twoheadrightarrow \mathbb{Z} \text{ s.t.} \\ v(k^\times) = 0 \end{array} \right\} & \leftarrow\!\shortmid & K \end{array}$$

2.1.3 Divisor

All curves non-singular over $k = \overline{k}$

Definition 2.1.22
A divisor D on C is a formal finite linear combination of points

$$D = \sum_i n_i P_i \qquad n_i \in \mathbb{Z}, P_i \in C$$

$$\mathrm{Div}(C) = \{\text{divisors of } C\}$$

this is an abelian group.

$$\underline{\text{degree of divisor}} : \deg(D) = \sum_i n_i \in \mathbb{Z}$$

Divisor of degree zero forms $\mathrm{Div}^0 C$ a subgroup.

Non-constant $\phi : C \to C'$ induces homomorphisms

$$\begin{array}{rcll} \phi_* : \mathrm{Div}\, C & \to & \mathrm{Div}\, C' & \underline{\text{pushforward}} \\ (Q) & \mapsto & (P), P = \phi(Q) & \\ \phi^* : \mathrm{Div}\, C' & \to & \mathrm{Div}\, C & \underline{\text{pullback}} \\ (P) & \mapsto & \sum_{\phi(Q)=P} e_Q(Q) & \end{array}$$

where

$$e_Q = \underline{\text{ramification index}} := v_Q(\phi^* t_P) \geq 1$$

t_P is uniformiser at P

Fact: $\deg \phi^* P = \deg \phi$ always (in particular, ϕ surjective)

Example 2.1.23
(see picture)
$\phi^*(a) = (a, \sqrt{1-a}) + (a, -\sqrt{1-a}) \quad a \neq \pm 1$
$\phi^*(1) = 2(1, 0) \qquad (\phi^*(x-1) = x - 1$ has valuation 2)
$\phi^*(-1) = 2(-1, 0)$
We say that (1,0), (-1,0) are ramified (i.e. $e_Q > 1$)

Remark.

$$\begin{array}{c} k(C) \\ d \big\uparrow \\ k(C') \end{array} \qquad \begin{array}{ccc} Q_1 & Q_2 & \cdots \\ \diagdown & | & \diagup \\ & P & \end{array} \qquad \sum e_Q = d$$

(Note residue field $k = \overline{k} \implies f = 1$ always) Compare with algebraic number theory

$$\begin{array}{c} K \\ d \big\uparrow \\ \mathbb{Q} \end{array} \qquad \begin{array}{ccc} Q_1 & Q_2 & \cdots \\ \diagdown & | & \diagup \\ & \mathfrak{p} & \end{array} \qquad \sum e_i f_i = d$$

2.1.4 Frobenius map

If char $k = p$ then $a \mapsto a^p$ is a bijection (in fact, isomorphism) $k \to k$
$0 = x^p - b = (x - \sqrt[p]{b})^p$ has one solution in k
So

$$\begin{aligned} \phi : \mathbb{P}^1 &\to \mathbb{P}^1 \\ (x : y) &\mapsto (x^p : y^p) \end{aligned}$$

is a bijection on points
But $k(x^p) \hookrightarrow k(x)$ has index p, so $\deg\phi = p$.
Every point is ramified, $e_Q = p \ \ \forall Q \in \mathbb{P}^1$

Can do this for every curve:

Definition 2.1.24

$$C : \quad \begin{cases} f_1 = 0 \\ \vdots \\ f_m = 0 \end{cases} \qquad \subseteq \mathbb{P}^n \text{ curve}$$

$$C^{(p)} : \quad \begin{cases} f_1^{(p)} = 0 \\ \vdots \\ f_m^{(p)} = 0 \end{cases}$$

$f^{(p)} := f$ with all coefficients raised to p-th powers
The p-th power Frobenius map is:

$$\begin{aligned} \mathrm{Frob}_p : C &\to C^{(p)} \\ (x_0 : \cdots : x_n) &\mapsto (x_0^p : \cdots : x_n^p) \end{aligned}$$

Example 2.1.25
$C : y^2 = x^3 + Ax + B, A, B \in k$
$(y^2)^p = (x^3 + Ax + B)^p$
$(y^p)^2 = (x^p)^3 + A^p(x^p) + B^p \quad \Rightarrow \quad (x^p, y^p) \in C^{(p)}$

It is a bijection on points, $e_Q = p \quad \forall Q \in C$ (some uniformiser computation)

Alternatively, by definition of $e_Q : Q = a \in \mathbb{A}^1, P = a^p$,
$\phi^*(x - a^p) = x^p - a^p = (x - a)^p$ has valuation p at Q

So deg $\mathrm{Frob}_p = p$

Remark. $k \supseteq \mathbb{F}_p$
Say $f_i \in \mathbb{F}_p[x_1, \dots, x_n]$, i.e. C is defined over $\mathbb{F}_p$. Then

(1) $C = C^{(p)} \qquad (a \in \mathbb{F}_p \Leftrightarrow a^p = a)$

(2) $C(\mathbb{F}_p) := \{(a_1, \dots, a_n) \in C | a_i \in \mathbb{F}_p\}$
$=$ fixed points of $\mathrm{Frob}_p : C \to C$
$=$ fixed points of $(\mathrm{Frob}_p)^n$

This leads to Lefschetz trace formula, etale cohomology, Weil conjecture

Lemma 2.1.26
K f.g. field of tr.deg.1 over k, char $k = p$, $K' := K(\{\sqrt[p]{f}\}_{f \in K})$. Then

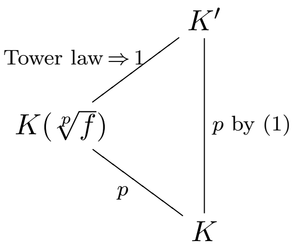
(1) $[K' : K] = p$

(2) $K' = K(\sqrt[p]{f})$ for any $f \in K$ with $\sqrt[p]{f} \notin K$

Proof

(1) $K = k(C), K' = k(C^{(1/p)}), C^{(1/p)} \xrightarrow{\mathrm{Frob}_p} C$ has degree p, $[K' : K] = p$

(2)



□

Definition 2.1.27
Finite field extension K'/K is <u>separable</u> if $\forall \alpha \in K'$ is a simple root of an irreducible polynomial $f(x) \in K[x]$
<u>Inseparable</u> otherwise.

Fact:

(1) char $K = 0 \quad \Rightarrow \quad$ every K'/K is separable

(2) char $K = p \quad \Rightarrow \quad K(\sqrt[p]{\alpha}), \alpha \in K, \sqrt[p]{\alpha} \notin K$ is inseparable.
Every F/K factors
$$K \underset{\text{separabe}}{\subseteq} K' \underset{\text{purely inseparable}}{\subseteq} F$$
<u>purely inseparable</u> means that the (inseparable) extension is obtained by successively adjoining p-th roots
<u>separable degree</u> $\deg_s F/K := [K' : K]$

(3) $F/M/K$ finite. Then $\deg_s F/K = \deg_s F/M \deg_s M/K$

(4) F/K separable $\Leftrightarrow$ $F = K(\alpha), \alpha$ root of some irred. polyn. $f(x) \in K[x]$ with $f'(\alpha) = 0 (\Leftrightarrow f' \not\equiv 0)$

For $\phi : C \to C'$ non-constant, we say ϕ is separable if $k(C)/\phi^* k(C')$ is.

Corollary 2.1.28

(1) Every ϕ factors $C \xrightarrow{(\mathrm{Frob}_p)^n} C^{(p^n)} \xrightarrow{\phi_0} C'$ with ϕ_0 separable

(2) Every C admits separable $\phi : C \twoheadrightarrow \mathbb{P}^1$ (i.e. $k(C) \supseteq k(t)$ separable extension)

(3) If $\phi : C \twoheadrightarrow C'$ separable, only finitely many points are ramified ($\Rightarrow$ In general, If $\phi : C \twoheadrightarrow C'$ arbitrary, all but finitely many $P \in C'$ have exactly $\deg_s \phi$)

Proof

(1) Fact (2) + Lemma

(2) Let $f = t_p \in k(C)$ be a unit at (some) $P \in C$; check that $f : C \to \mathbb{P}^1$ is separable

(3) May assume $C' = \mathbb{P}^1$ by Fact (2).
Write $k(C) = k(t)(\alpha)$, α root of irred. polyn. $f \in k[t]$
Then {ramified points} $\subseteq$ {those where $f'(\alpha) = 0, l \neq 0$}

$\square$

2.1.5 Divisors of functions

Definition 2.1.29
For $f \in k(C)^\times$ define divisor of f

$$\begin{aligned}\mathrm{div}(f) = (f) &:= \sum_{P \in C} v_P(f) \cdot (P) \\ &= f^*((0)) - f^*((\infty))\end{aligned}$$

Remark. Has degree $\deg f - \deg f = 0$

Definition 2.1.30
$D, D' \in \mathrm{Div}(C)$ are linearly equivalent, write $D' \sim D$, if $D - D' = \mathrm{div}(f)$ for some $f \in k(C)^\times$
$D \sim 0$ are called principal divisors

Definition 2.1.31

$$\begin{aligned}\mathrm{Pic}^0(C) &:= \mathrm{Div}^0(C)/\sim \\ \mathrm{Pic}(C) &:= \mathrm{Div}(C)/\sim \;\cong \mathrm{Pic}^0(C) \times \mathbb{Z}\end{aligned}$$

In algebraic number theory

points	$\leftrightarrow$	prime ideals
Div(C)	$\leftrightarrow$	group of fractional ideals
Principal	$\leftrightarrow$	Principal ideals
Pic(C)	$\leftrightarrow$	Class group

Example 2.1.32
$C = \mathbb{P}^1$. For $P, Q \in \mathbb{A}^1 \subseteq \mathbb{P}^1$
$(P) \sim (Q)$ $\quad$ $[\ (P) - (Q) = \operatorname{div} \frac{x-P}{x-Q}\ ]$
$\Rightarrow \quad \mathrm{Pic}^0(\mathbb{P}^1) = \{0\}$ and deg define isomorphism $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$

Conversely, if C is a curve on which $(P) \sim (Q)$ for some $P, Q \in C$ then $C \cong \mathbb{P}^1$
Proof: Take $f \in k(C)^\times$ s.t. $\operatorname{div}(f) = (P) - (Q)$. Then $f : C \to \mathbb{P}^1$ has only one pole at Q and so has degree $1 \Rightarrow C \cong \mathbb{P}^1$

2.1.6 Differentials

Definition 2.1.33
A (rational) differential on a non-singular curve C is a formal finite sum

$$\omega = \sum_i f_i \, dg_i, \quad f_i, g_i \in k(C)$$

subject to relations

$$\begin{aligned} d(g_1 g_2) &= g_1 dg_2 + g_2 dg_1 \\ d(g_1 + g_2) &= dg_1 + dg_2 \\ da &= 0 \ \forall a \in k \subseteq k(C) \end{aligned}$$

Example 2.1.34
(char $k \neq 2$) $C : x^2 + y^2 = 1$
we have, for example, $d(x^2 y) = x^2\, dy + 2xy\, dx$
Generally, any $f\, dg = f \cdot {g_x}' \cdot dx + f \cdot {g_y}' \cdot dy$
$\Rightarrow$ can express any w as $f_1\, dx + f_2\, dy$

Also $x^2 + y^2 = 1$
$\Rightarrow \quad 2x\, dx + 2y\, dy = 0$
$\Rightarrow \quad dy = -(x/y) dx$
$\Rightarrow \quad \forall \omega \ \ \exists! f \in k(C)$ s.t. $\omega = f\, dx$

$$\Rightarrow \qquad \{\text{differentials on } C\} = k(C) \cdot dx$$

Similarly, for any C, we have the 1-dimensional $k(C)$-vector space

$$\{\text{differentials on } C\} \quad = \quad k(C) \cdot df$$

For any f s.t. $K(C)/k(f)$ is separable, let ω be a differential on C. For $P \in C$, write

$$\omega = f \cdot dt_P, \qquad t_P \text{ uniformiser at } P$$

and define

$$\begin{aligned} v_P(\omega) &:= v_P(f) && \text{(independent of the choice of } t_P) \\ \operatorname{div}(\omega) &:= \sum_P v_P(\omega)\,(P) && \text{(finite sum)} \end{aligned}$$

Because any w, w' differ by a function,

$$\omega = f \cdot \omega' \quad \Rightarrow \quad \operatorname{div}(\omega) = \operatorname{div}(\omega') + \operatorname{div}(f) \sim \operatorname{div}(\omega')$$

So divisors of differential forms span a class $\mathbb{K} \in \mathrm{Pic}(C)$, the canonical class

ω regular at P if $v_P(w) \geq 0$
ω regular if all $v_P(w) \geq 0$ (i.e. $\operatorname{div}(w) \geq 0$)

2.1.7 Riemann-Roch

Definition 2.1.35
The <u>complete linear system</u> of a divisor D

$$\begin{array}{rcll} \mathcal{L}(D) & := & \{f \in k(C) \mid \operatorname{div}(f) + D \geq 0\} & k\text{-vector space} \\ & \leftrightarrow & \{D' \in \operatorname{Div}(C) \mid D' \geq 0 \text{ and } D \sim D'\} & \end{array}$$

(via $f \mapsto D' = D + \operatorname{div}(f)$)

Remark. $D \sim D' \Rightarrow \mathcal{L}(D) \cong \mathcal{L}(D')$

Example 2.1.36
$\mathcal{L}(0) = \{f \in k(C) \mid \operatorname{div}(f) \geq 0\}$ functions with no poles
$= k$ (f non-constant $\Rightarrow f : C \twoheadrightarrow \mathbb{P}^1$ hits ∞)

Example 2.1.37
$\mathcal{L}(3(P)) = \{f \in k(C) \mid \operatorname{div}(f) \geq -3(P)\}$ functions with pole of order ≤ 3 at P and no other poles

(Generally, "$\mathcal{L}(D)$ = functions with a pole at most at D")

<u>Exercise</u>:

(1) $\mathcal{L}(D) = 0$ when $\deg D < 0$ (equivalently, when $\deg D = 0$ and $D \nsim 0$)
(2) $\dim_k \mathcal{L}(D + P) \leq \dim_k \mathcal{L}(D) + 1$ ($\Rightarrow \dim_k \mathcal{L}(D) < \infty \ \forall D$)

Definition 2.1.38
The <u>genus of C</u> is

$$g(C) := \dim_k \mathcal{L}(\mathbb{K}) = \dim_k \mathcal{L}(\operatorname{div}(\omega)) \text{ for any } \omega \neq 0$$

<u>Fact</u> Non-constant $\phi : C \to C'$ induces pullback map on differential forms:

$$\omega = f dg \rightsquigarrow \phi^*\omega := (\phi^* f) d(\phi^* g)$$

and therefore

$$\phi^* : \mathcal{L}(\mathbb{K}_{C'}) \to \mathcal{L}(\mathbb{K}_C)$$

Not hard to see that ϕ^* injective $\Leftrightarrow \phi$ separable
and $\phi^* = 0 \Leftrightarrow \phi$ inseparable

Corollary 2.1.39
$g(C) \geq g(C')$ always (i.e. genus goes down under non-constant maps)

Remark. A non-singular plane curve $C \subseteq \mathbb{P}^2$, $C : f(x, y) = 0$ has genus

$$\begin{array}{rcll} g & = & \frac{(d-1)(d-2)}{2} & d = \deg f \\ & = & 0 & \text{for linear and conics} \\ & = & 1 & \text{for cubics} \\ & = & 3 & \text{for quartics} \end{array}$$

In particular, genus 2 curves (they exist) cannot embedded in $\mathbb{P}^2$

Theorem 2.1.40 (Riemann-Roch)
C non-singular curve. For every $D \in \operatorname{Div}(C)$

$$\dim \mathcal{L}(D) - \dim \mathcal{L}(\mathbb{K} - D) = \deg D - g + 1$$

Corollary 2.1.41

- $\deg \mathbb{K} = 2g - 2$ (Proof: Take $D = \mathbb{K}$)
- If $\deg D > 2g - 2$, then $\dim \mathcal{L}(D) = \deg D - g + 1$ (Proof: Since $\deg(\mathbb{K} - D) < 0$)

Lemma 2.1.42 (Classification of Curve of Genus 0)
A non-singular curve C has genus 0 $\Leftrightarrow$ $C \cong \mathbb{P}^1$

Proof

$\underline{\Leftarrow}$: $\mathbb{P}^1$ genus 0: uniformisers

$$\begin{aligned} t_a &= x - a, \quad a \in \mathbb{A}^1 \\ t_\infty &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} dx &= d(x-a) && \text{valuation 0 at } a \in \mathbb{A}^1 \\ dx &= d\left(\frac{1}{t_\infty}\right) = -\frac{1}{t_\infty^2} \cdot dt_\infty && \text{valuation } -2 \text{ at } \infty \end{aligned}$$

$\Rightarrow$ $\operatorname{div}(dx) = -2(\infty), \deg = -2 = 2g(\mathbb{P}^1) - 2$ (by Corollary)
$\Rightarrow$ $g(\mathbb{P}^1) = 0$

$\underline{\Rightarrow}$: Suppose a curve C has genus 0. Take $P \in C, D = (P)$
$\deg D > 2g - 2 = -2$ $\Rightarrow$ $\dim \mathcal{L}((P)) = 1 - 0 + 1 = 2$
$\Rightarrow$ $\mathcal{L}((P)) \supsetneq \mathcal{L}(0) = k$ $\Rightarrow$ $\exists f \in k(C)$ with a simple pole at P and no other poles
$\operatorname{div}(f) = -(P) + (Q)$ for some $Q \in C$
$\Rightarrow$ $f : C \xrightarrow{\sim} \mathbb{P}^1$ is an isom.

□

Corollary 2.1.43
k algebraically closed, every conic is isomorphic to $\mathbb{P}^1$

2.2 Cubics

Suppose char $k \neq 2, 3$, $C \subseteq \mathbb{P}^2$ non-singular of the form

$$\begin{aligned} C : y^2 &= x^3 + ax + b \qquad a, b \in k \\ &= (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \qquad \alpha_i \in k \end{aligned}$$

$C \cap \mathbb{A}^2$ non-singular $\Leftrightarrow$ α_i are distinct

(see picture for the 3 different cases)

Exercise: When α_i not distinct, C is singular, $k(C) \cong k(\mathbb{P}^1)$ (C has "geometric genus 0")

To get a morphism $\begin{array}{ccc} \mathbb{P}^1 & \to & C \\ t & \mapsto & P \end{array}$ of degree 1 (see picture)

Recall

$$\underset{(x:y:z)}{\mathbb{P}^2} = \underset{(x:y:1)}{\mathbb{A}^2_{z=1}} \cup \underset{(x:y:0)}{\mathbb{P}^1_{z=0}} \quad \leftarrow \text{line at } \infty$$

$$\begin{array}{ccc} C \subseteq \mathbb{P}^2 & : & y^2 z = x^3 + axz^2 + bz^3 \\ \underbrace{C \cap \mathbb{P}^1_{z=0}}_{\substack{\text{has unqiue pt.} \\ \mathcal{O} = (0:1:0) \\ \text{point at infinity}}} & : & 0 = x^3 + 0 + 0 \Rightarrow \begin{cases} x = 0 \\ z = 0 \\ y = 1 \end{cases} \end{array}$$

In the $y = 1$ chart
$C : z = x^3 + axz^2 + bz^3$
$\mathcal{O} = (0, 0)$ (see picture)
$g(x, z) = z - x^3 - axz^2 - bz^3$
$\frac{dg}{dz}\Big|_{(0,0)} = 1 \neq 0$
$\Rightarrow$ C non-singular at 0

So, $C \subseteq \mathbb{P}^2$ non-singular $\Leftrightarrow$ $C \cap \mathbb{A}^2_{z=1}$ non singular $\Leftrightarrow$ α_i distinct

Differentials:
e.g. $\text{div}(dx) = (P_1) + (P_2) + (P_3) - 3(0)$ (exercise: check)
this has degree $0 = 2g - 2$ (by Corollary of Riemann-Roch)
$\Rightarrow$ C has genus 1 ($= \frac{(3-1)(3-2)}{2}$ as expected)

$\text{div}(y) = (P_1) + (P_2) + (P_3) + \lambda(0)$ some λ
this has degree 0 $\Rightarrow \lambda = -3$
$\Rightarrow$ $\text{div}(\frac{dx}{y}) = 0$ since $w = \frac{dx}{y}$ has no zeroes, no poles
In fact, $\mathbb{K} = \langle \frac{dx}{y} \rangle$ as it is 1-dimensional over k by definition of genus.

Definition 2.2.1
An elliptic curve , $(E, \mathcal{O})$, is a non-singular projective curve E of genus 1 with a marked point $\mathcal{O}$

Example 2.2.2
(char $k \neq 2, 3$)

$$y^2 = x^3 + ax + b \qquad \mathcal{O} = (0 : 1 : 0)$$

is an elliptic curve in (simplified) Weierstrass form (if $\Delta_E = 16\Delta_{\text{RHS}} = -16(4a^3 + 27b^2) \neq 0$)

In any characteristic, have (generalised) Weierstrass form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

(char $k \neq 2, 3 \Rightarrow$ complete the square in LHS, complete the cube in RHS, then we get simplified form)

Theorem 2.2.3
Every elliptic curve is isomorphic to one in Weierstrass form

Proof
$(E, \mathcal{O})$ elliptic curve.

$$\begin{array}{rcl}
\dim \mathcal{L}(n(0)) & = & n-1+1 \ = \ n \ \text{ for } n \geq 1 \\
\mathcal{L}(1(0)) & = & k \ = \ \langle 1 \rangle \text{ constant} \\
\mathcal{L}(2(0)) & = & \langle 1, x \rangle \ \text{ where } x \in k(C) \text{ with double pole at } 0 \\
\mathcal{L}(3(0)) & = & \langle 1, x, y \rangle \ \text{ where } y \in k(C) \text{with triple pole at } 0 \\
 & & \qquad\qquad \text{Note that } y \notin k(x) \quad \text{(elements of } k(x) \text{ has even order)} \\
\mathcal{L}(4(0)) & = & \langle 1, x, y, x^2 \rangle \\
\mathcal{L}(5(0)) & = & \langle 1, x, y, x^2, xy \rangle \\
\underbrace{\mathcal{L}(6(0))}_{\text{dim=6}} & \ni & \underbrace{1, x, y, x^2, xy, \qquad \overbrace{x^3, y^2}^{\text{pole of order 6 at } \mathcal{O}}}_{\text{7 functions}}
\end{array}$$

$\Rightarrow$ must have a linear relation, involving both x^3, y^2

$$\underbrace{\alpha y^2}_{\neq 0} + \underbrace{\beta x^3}_{\neq 0} + \cdots = 0$$

Rescaling x, y may make $\alpha = 1, \beta = -1$

$$\begin{array}{rc}
 & y^2 - x^3 + \cdots = 0 \\
y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 & \text{for some } a_i \in k
\end{array}$$

Let $C \subseteq \mathbb{P}^2_{x,y,z}$ by a curve given by this equation

$$\begin{array}{rcl}
k(C) = ff(\dfrac{k[x,y]}{y^2 + \cdots = x^3 + \cdots}) & \hookrightarrow & k(E) \\
x & \mapsto & x \\
y & \mapsto & y
\end{array}$$

$[k(x,y) : k(x)] = 2$
This defines a map $E \to C$
$x : E \to \mathbb{P}^1$ has $x^*((\infty)) = 2(\mathcal{O})$ (as $\mathcal{O} \mapsto \infty$), so this map has degree 2

$$\begin{array}{ccc}
k(x) & \xrightarrow{\quad 2 \quad} & k(E) \\
 {\scriptstyle 2} \searrow & & \nearrow {\scriptstyle \cong} \\
 & k(x,y) = k(C) &
\end{array}$$

the lower left map is non-trivial, $y \in k(x)$ and its degree ≤ 2 by equation $y^2 + \cdots = x^3 + \cdots$
$\Rightarrow$ $k(C) \hookrightarrow k(E)$ is isomorphism, i.e. $E \to C$ has degree 1

If C is singular, then $k(C) \cong k(\mathbb{P}^1)$, and then $E \cong \mathbb{P}^1$ #
So C is non-singular □

Corollary 2.2.4
Every elliptic curve admits a degree 2 map to $\mathbb{P}^1$, namely $E \xrightarrow{x} \mathbb{P}^1$
Such curves (of any genus) are called hyperelliptic

$g = 1$ $\Rightarrow$ hyperelliptic
$g = 2$ $\Rightarrow$ hyperelliptic (exercise)
$g = 3$ $\Rightarrow$ Either a plane quartic or hyperelliptic, but not both

Remark. If E, E' in Weierstrass form and $E \cong E'$ then

$$\begin{aligned}\mathcal{L}_E(2(\mathcal{O})) &\cong \mathcal{L}_{E'}(2(\mathcal{O}))\\ \mathcal{L}_E(3(\mathcal{O})) &\cong \mathcal{L}_{E'}(3(\mathcal{O}))\end{aligned}$$

(these are k-vector spaces), so

$$\begin{aligned}x_{E'} &= u^2x + r \qquad && u \in k^\times\\ y_{E'} &= u^3y + sx + t \qquad && r, s, t \in k\end{aligned}$$

i.e. Weierstrass form is unique up to such transformations

Suppose char $k \neq 2, 3$:
• Simplified Weierstrass form unique up to

$$\begin{aligned}x_{E'} &= u^2x \qquad && u \in k^\times\\ y_{E'} &= u^3y\end{aligned}$$

and

$$E : y^2 = x^3 + ax + b \quad \cong \quad E' : (y')^2 = (x')^3 + a'x' + b' \Leftrightarrow \begin{cases} a' = u^4a\\ b' = u^6b\end{cases}$$

$(\Delta_{E'} = -16(4a'^3 + 27b'^2) = u^{12}\Delta_E)$

Definition 2.2.5
<u>j-invariant</u> $j(E) := 1728\frac{(-4a)^3}{\Delta}$

Example 2.2.6
- $y^2 = x^3 + ax$ has $j = 1728$
- $y^2 = x^3 + b$ has $j = 0$

Proposition 2.2.7

(1) $E \cong E' \Leftrightarrow j(E) = j(E')$

(2) For any $j \in k$ $\exists E$ s.t. $j(E) = j$, hence

$$\{\text{elliptic curves (up to isom.)}/k\} \overset{\text{1:1 map}}{\underset{}{\overset{j(E)}{\longleftrightarrow}}} k$$

Proof

(1)

$$\begin{aligned}\begin{matrix}a' = u^4a\\ b' = u^6b\end{matrix} &\Leftrightarrow \sqrt[4]{\frac{a'}{a}} = \sqrt[6]{\frac{b'}{b}}\\ &\Leftrightarrow \left(\frac{b'}{b}\right)^2 = \left(\frac{a'}{a}\right)^3\\ &\Leftrightarrow \frac{4a^3 + 27b^2}{a^3} = \frac{4(a')^3 + 27(b')^2}{(a')^3}\\ &\Leftrightarrow j(E) = j(E')\end{aligned}$$

Do $d = 0, b = 0$ separately $(j = 0, 1728)$

(2) $y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$ works for $j \neq 0, 1728$

$\square$

Corollary 2.2.8
The automorphism group $\mathrm{Aut}(E) = \{$ morphisms $\phi : E \to E$ s.t. $\phi(\mathcal{O}) = \mathcal{O}\}$ is

- $\mathbb{Z}/2\mathbb{Z}$ for $y^2 = x^3 + ax + b, a, b \neq 0$ $(j \neq 0, 1728)$
- $\mathbb{Z}/4\mathbb{Z}$ for $y^2 = x^3 + ax$ $(j = 1728)$
- $\mathbb{Z}/6\mathbb{Z}$ for $y^2 = x^3 + b$ $(j = 0)$

Proof

$$\begin{aligned}\mathrm{Aut}(E) &= \left\{ u \in k^\times \middle| \begin{array}{l} u^4 a = a \\ u^6 b = b \end{array} \right\} \\ &= \begin{cases} \{\pm 1\} & ab \neq 0 \\ \langle i \rangle & b = 0 \\ \langle \zeta_6 \rangle & a = 0 \end{cases}\end{aligned}$$

For most elliptic curves, $(x, y) \to (x, -y)$ is the only automorphism. $\square$

Remark. If char $k = 2, 3$

- Δ, j complicated polynomial, rational function of $a_1, \ldots, a_6$
- a_i change $a_i{}' = u^i a_i + \cdots$
- Proposition still holds
- $|\mathrm{Aut}(E)| \leq 24$

2.2.1 Group Law

Over $\mathbb{C}$: $E(\mathbb{C}) \cong \mathbb{C}/\text{lattice}$, group law = addition

Recall $\mathrm{Pic}^0(E) = \dfrac{\text{divisors of deg } 0}{\text{divisors of functions}}$, e.g. $\mathrm{Pic}^0\,\mathbb{P}^1 = \{0\}$
E elliptic curve

Theorem 2.2.9
The following map is a bijection

$$\begin{aligned} E &\to \mathrm{Pic}^0(E) \\ P &\mapsto (P) - (\mathcal{O}) \end{aligned}$$

Proof
Injective $\hookrightarrow$:
If $(P) - (\mathcal{O}) \sim (Q) - (\mathcal{O})$, then $(P) \sim (Q) \Rightarrow E \cong \mathbb{P}^1$ $\quad \#$ unless $P = Q$

Surjective $\twoheadrightarrow$:
Take $D \in \mathrm{Div}^0(E)$. By Riemann-Roch,
$\dim \mathcal{L}(\underbrace{D + (\mathcal{O})}_{\deg=1}) = 1$
$\Rightarrow \quad \exists f$ s.t. $\mathrm{div}(f) \geq \underbrace{-D - (\mathcal{O})}_{\deg=-1}$
$\Rightarrow \quad \mathrm{div}(f) = -D - (\mathcal{O}) + (P)$ for some $P \in E$
$\Rightarrow \quad D \sim (P) - (\mathcal{O})$ $\square$

Corollary 2.2.10
E has a structure of an abelian group

Proof
$\mathrm{Pic}^0(E)$ has structure of abelian group, apply theorem. $\square$

$E: \quad y^2 = x^3 + ax + b$

Identity $= \mathcal{O}$ because $(\mathcal{O}) - (\mathcal{O}) = 0 \in \mathrm{Pic}^0(E)$

Inverse of $P = (x_1, y_1)$ is $P' = (x_1, -y_1)$
$\mathrm{div}(x - x_1) = (P) + (P') - 2(\mathcal{O}) \Rightarrow (P) - (\mathcal{O}) \sim -[(P') - (\mathcal{O})]$

Addition $P = (x_1, y_1), Q = (x_2, y_2), P \neq -Q; P, Q \neq 0$
$P + Q + R = 0$ in $E \quad \Leftrightarrow \quad P + Q = (-R)$

Need function with

$$\begin{aligned} \mathrm{div}(f) &= (P) - (\mathcal{O}) + (Q) - (\mathcal{O}) + (R) - (\mathcal{O}) \\ &= (P) + (Q) + (R) - 3(\mathcal{O}) \end{aligned}$$

$(\Rightarrow \quad f \in \mathcal{L}(3\,\mathcal{O}) = \langle 1, x, y \rangle)$
So $f = \alpha y + \beta x + \gamma, \qquad \alpha \neq 0$
So $f = 0$ is an equation of a line passing through P and Q (tangent to P if $P = Q$) with $R =$ third point of intersection

Explicitly, solve

$$\begin{cases} y^2 = x^3 + ax + b & \text{elliptic curve} \\ y = \kappa(x - x_1) + y_1 & \text{line} \end{cases}$$

with

$$\kappa = \text{ slope } = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1} & P \neq Q \\ \frac{3x_1^2 + a}{2y_1} & P = Q \end{cases}$$

$(\kappa x + \cdots)^2 = x^3 + ax + b$
$x^3 - \kappa^2 x^2 + \cdots = 0$ and $\sum$roots $= \kappa^2$
$\Rightarrow \quad$ 3rd root x, y defining $R = (x, y)$ is

$$\begin{cases} x = \kappa^2 - x_1 - x_2 \\ y = \kappa(x - x_1) + y_1 \end{cases}$$

Hence,

$$(x_1, y_1) + (x_2, y_2) = (\kappa^2 - x_1 - x_2, -\kappa(x - x_1) - y_1)$$

Important: This shows (+some extra work when $P = \pm Q$ see Silverman Theorem III 3.6) that

$$\begin{aligned} \text{inverse} : E &\xrightarrow{i} E \\ \text{addition} : E \times E &\xrightarrow{\mu} E \end{aligned}$$

are morphisms, i.e. given by rational functions that are defined everywhere
That is, E is algebraic group (= group variety = group object in the category of varieties)

In particular, translation maps

$$\begin{aligned} \tau_Q : E &\to E \\ P &\mapsto P + Q \end{aligned}$$

are morphisms. (Proof: This is just composition $\mu \circ (\mathrm{id}, Q)$)

Corollary 2.2.11

$$\left\{ \begin{array}{c} \text{isomorphisms} \\ E \to E \text{ as a curve} \end{array} \right\} \cong \{\text{translations}\} \rtimes \mathrm{Aut}(E)$$
$$\cong E \rtimes \text{ finite groups}$$
$$\mathrm{Iso}(C,C) \cong \begin{cases} PGL_2(k) & g=0 \\ E \rtimes \text{finite group} & g=1 \\ \text{finite group} & g \geq 2 \end{cases}$$

Exercise: The only (affine or projective) curves that are algebraic groups are

- Additive group $\mathbb{G}_a = \mathbb{P}^1 \setminus \{\infty\} = (k, +)$
- Multiplicative group $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\} = (k^\times, \times)$
- Elliptic curves (the only projective algebraic groups in dimension 1)

Remark. genus(C)=g $\Rightarrow$ $\mathrm{Pic}^0(C)$ has a structure of a g-dimensional abelian variety (i.e. projective algebraic group, by definition) the Jacobian of C, denoted $\mathrm{Jac}(C)$

Fixing $P_0 \in C$, define the Abel-Jacobi map

$$\begin{array}{rcl} C & \to & \mathrm{Jac}(C) \\ P & \mapsto & (P) - (P_0) \end{array}$$

injective when $g > 0$, $\cong$ when $g = 1$. Every $D \in \mathrm{Pic}^0(C)$ is $\sim (P_1) + \cdots (P_g) - g(P_0)$, usually unique such.

2.2.2 Isogenies

Definition 2.2.12
An isogeny between elliptic curves is a morphism $\phi : E \to E'$ s.t. $\phi(\mathcal{O}) = \mathcal{O}$

Example 2.2.13

$$\begin{array}{rcl} [0] : E & \to & E \\ P & \mapsto & \mathcal{O} \end{array} \qquad \text{zero isogeny}$$

we let $\deg[0] := 0$, so $\deg(\phi \circ \psi) = \deg\phi \deg\psi$ for all isogenies

Example 2.2.14
Elements of $\mathrm{Aut}(E)$ are isogenies (of degree 1), e.g.

$$\begin{array}{rcl} [1] : E & \to & E \\ P & \mapsto & P \\ [-1] : E & \to & E \\ P & \mapsto & -P \end{array}$$

Example 2.2.15
Multiplication-by-m maps

$$\begin{array}{rcll} [m] : E & \to & E & \\ P & \mapsto & \underbrace{P + \cdots + P}_{m \text{ times}} & (m > 0) \\ P & \mapsto & \underbrace{(-P) + \cdots (-P)}_{m \text{ times}} & (m < 0) \end{array}$$

Example 2.2.16
([2] when char $k \neq 2, 3$)
$E: \; y^2 = x^3 + ax + b$

$$\begin{array}{rcll} [2] : E & \to & E & \\ P = (x, y) & \mapsto & P + P = (\kappa^2 - 2x, -\kappa(\kappa^2 - 2x - x) - y) & (\kappa = \dfrac{3x^2 + a}{2y}) \\ & & = \left(\underbrace{\dfrac{\frac{1}{2}(x^2 - a)^2 - 2bx}{x^3 + ax + b}}_{\psi(x)}, \cdots \right) & \end{array}$$

This has degree 4:

$$\begin{array}{ccc} E & \xrightarrow{[2]} & E \\ {\scriptstyle x,\deg=2}\downarrow & & \downarrow{\scriptstyle x,\deg=2} \\ \mathbb{P}^1 & \xrightarrow{\psi(x)} & \mathbb{P}^1 \end{array}$$

$\Rightarrow \quad \deg[2] = \deg(\psi : \mathbb{P}^1 \to \mathbb{P}^1) = \max(\deg(\text{numerator}), \deg(\text{denominator})) = 4$

E.g.: $[2]^*(\mathcal{O}) = (\mathcal{O}) + (T_1) + (T_2) + (T_3)$

Corollary 2.2.17
$[m] \neq [0] \quad 0 \neq m \in \mathbb{Z}$

Proof
In char $k \neq 2, 3$:
$[2] \neq [0]$
$[n] \neq [0]$ for n odd since $[n]T_1 = T_1$
$[mn] = [m] \circ [n]$ □

Theorem 2.2.18
An isogeny $\phi : E \to E'$ is a group homomorphism.

Proof
$\phi = [0]$ is a homomorphism, so assume ϕ is non-constant

Then recall: ϕ induces

$$\begin{array}{rcl} \phi_* : \mathrm{Div}(E) & \to & \mathrm{Div}(E') \\ (Q) & \mapsto & (\phi(Q)) \\ \phi^* : \mathrm{Div}(E') & \to & \mathrm{Div}(E) \\ (P) & \mapsto & \sum\limits_{\phi(Q)=P} e_Q(Q) \end{array}$$

<u>Fact</u>: (For all curves) Both map principal divisors to principal divisors:

$$\begin{array}{rcl} \phi^*(\mathrm{div}(f)) & = & \mathrm{div}(\phi^* f) \\ \phi_*(\mathrm{div}(f)) & = & \mathrm{div}(N(f)), \quad N(f) := \mathrm{Norm}_{k(E)/\phi^* k(E_2)}(f) \end{array}$$

Now $P + Q = R$ on $E \quad \Rightarrow \quad (P) - (\mathcal{O}) + (Q) - (\mathcal{O}) \sim (R) - (\mathcal{O})$
$\Rightarrow$ (by fact above) $(\phi(P)) - (\mathcal{O}) + (\phi(Q)) - (\mathcal{O}) \sim (\phi(R)) - (\mathcal{O})$
$\Rightarrow \quad \phi(P) + \phi(Q) = \phi(R) \quad$ in E' □

Corollary 2.2.19

(1)

$$\mathrm{Hom}(E_1, E_2) := \{ \text{ isogenies } E_1 \to E_2\}$$

is a torsion-free abelian group (will see later that $\cong \mathbb{Z}^r$, some $r \leq 4$)

(2) $\mathrm{End}(E) := \mathrm{Hom}(E, E)$ is a (not necessarily commutative) integral domain of characteristic 0, $\mathrm{Aut}(E) = \mathrm{End}(E)^\times$ its units

Proof

(1) $\phi + \psi :=$ composition $\begin{array}{ccccccc} E & \xrightarrow{\Delta} & E \times E & \xrightarrow{(\phi,\psi)} & E \times E & \xrightarrow{\mu} & E \\ P & \mapsto & (P, P) & \mapsto & (\phi(P), \psi(P)) & & \end{array}$

$\phi + \psi$ =morphism

Homomorphisms between abelian groups are abelian groups:

$$m\phi = 0 \Rightarrow [m] \circ \phi = [0] \Rightarrow [m] = [0] \text{ or } \phi = 0$$

(2) $\begin{array}{ccc} \mathbb{Z} & \hookrightarrow & \mathrm{End}(E) \\ m & \mapsto & [m] \end{array}$ injective ring hom $\Rightarrow$ char. 0

$\phi\psi = [0] \Rightarrow \phi = [0]$ or $\psi = [0]$

□

Most of the time $\mathrm{End}(E) = \mathbb{Z}$ (only $[m]$'s)

Definition 2.2.20
We say E has <u>complex multiplication</u> if $\mathrm{End}(E) \supsetneq \mathbb{Z}$ (this is very special)

Example 2.2.21
$E: \ y^2 = x^3 + x$ over $\mathbb{C}$ has $\mathrm{End}(E) \cong \mathbb{Z}[i]$

$$\begin{array}{rcl} [1] : (x, y) & \mapsto & (x, y) \\ [i] : (x, y) & \mapsto & (-x, iy) \end{array}$$

$[i]^2 = [-1] \Rightarrow \quad \mathrm{End}(E) \supseteq \mathbb{Z}[i]$
(for $\subseteq$, we will get from $\mathbb{C}$)

Example 2.2.22
$E: \ y^2 + y = x^3$ over $\overline{\mathbb{F}_2}$ has $\mathrm{End}(E) \cong \mathbb{Z} + \mathbb{Z}\, i + \mathbb{Z}\, j + \mathbb{Z}\, \frac{1+i+j+k}{2}$
where $i^2 = j^2 = k^2 = 1, ij = k, jk = i, ki = j$

$$[i]: \begin{array}{ccc} x & \mapsto & x+1 \\ y & \mapsto & y+x+\zeta \end{array} \qquad [j]: \begin{array}{ccc} x & \mapsto & x+\zeta^2 \\ y & \mapsto & y+\zeta x+\zeta \end{array} \qquad [k]: \begin{array}{ccc} x & \mapsto & x+\zeta \\ y & \mapsto & y+\zeta^2 x+\zeta \end{array}$$

$$[-1]: \begin{array}{ccc} x & \mapsto & x \\ y & \mapsto & y+1 \end{array}$$

$\mathrm{Frob}_2 : (x \mapsto x^2, y \mapsto y^2) = [j] + [k], (\mathrm{Frob}_c)^2 = [-2] \quad \Rightarrow \quad [2]$ inseparable

2.2.3 Invariant Differential

Definition 2.2.23
A differential $w \neq 0$ on E is an invariant differential if $\mathrm{div}(\omega) = 0$

Recall: $g(C) = \dim \mathcal{L}(\mathbb{K}) = 1$
$\Rightarrow \ \exists \omega$ with no poles, $\deg \mathbb{K} = 2g - 2 = 0$
$\Rightarrow$ has no zeroes either
$\Rightarrow$ such ω exist up to $\omega \mapsto \alpha\omega$ $(\alpha \in k^\times)$

Example 2.2.24
$E: \ y^2 = x^3 + ax + b \qquad \omega = \frac{dx}{y}$
For E in generalised Weierstrass form, $\omega = \frac{dx}{2y+a_1x+a_3}$

Theorem 2.2.25

(1) $\tau_P^*\omega = \omega \quad \forall P \in E$ and ω invariant differential on E (invariant differential invariant under translation)
(2) $(\phi + \psi)^*\omega = \phi^*\omega + \psi^*\omega \ \ \forall \phi, \psi : E \to E'$ isogenies, ω on E'
(3) $(\phi\chi)^*\omega = \chi^*(\phi^*\omega)$

Proof
Omitted (see Silvermann III 5.1, 5.2)
Idea: (1) uses brute force, (2) can get from formal groups (see later), (3) is easy given (1),(2) □

Remark. Recall: for $\phi : E \to E'$, non-zero isogeny
$\phi^*\omega \neq 0 \ \Leftrightarrow \ \phi^* : \mathcal{L}(\mathbb{K}_{E'}) \to \mathcal{L}(\mathbb{K}_E) \ \Leftrightarrow \ \phi$ separable
So, in particular,

$$\begin{array}{rcl} \mathrm{End}(E) & \to & k \\ \phi & \mapsto & \alpha = \dfrac{\phi^*\omega}{\omega} \qquad (\alpha \in k \text{ s.t. } \phi^*\omega = \alpha\omega) \end{array}$$

is a ring homomorphism, kernel = inseparable isogenies (but kernel=0 in char k=0)

Corollary 2.2.26
char $k = 0 \quad \Rightarrow \quad \mathrm{End}(E)$ is commutative

Corollary 2.2.27
$[m]^*\omega = m\omega$

(Check $m = 0, 1$. Then done by induction, using $(\phi + \psi)^*\omega$)

Corollary 2.2.28
For $m \neq 0$, $[m]$ separable $\quad \Leftrightarrow \quad$ char $k \nmid m$

Example 2.2.29
$E: \ y^2 = x^3 + x$ (over $\mathbb{C}$)

$$\begin{array}{rrcll} & [i]:(x,y) & \mapsto & (-x, iy) & \\ \Rightarrow & [i]^*\dfrac{dx}{y} & = & \dfrac{d(-x)}{iy} & = \ i\dfrac{dx}{y} \\ \Rightarrow & \mathrm{End}(E) & \hookrightarrow & \mathbb{C} & \\ & \bigcup\vert & & \bigcup\vert & \\ & \mathbb{Z}[i] & = & \mathbb{Z}[i] & \end{array}$$

<u>Exercise</u>: Describe $\mathrm{End}(E) = \mathbb{Z}\langle 1, i, j, \frac{1+i+j+k}{2}\rangle \to \overline{\mathbb{F}_2}$ for $E: \ y^2 + y = x^3$ over $\overline{\mathbb{F}_2}$

2.2.4 Galois Theory for Isogenies

If $\phi : E_1 \to E_2$ non-zero isogeny, then $\ker\phi = \phi^{-1}(\mathcal{O})$ is a finite subgroup

Example 2.2.30
$E: \ y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$
$\ker[2] = \{\mathcal{O}, T_1, T_2, T_3\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
Conversely, every finite subgroup $\Phi \subseteq E$ arises like this:

Theorem 2.2.31
$\phi : E_1 \to E_2$ separable isogeny, $\deg\phi = n \neq 0$

(1) ϕ is unramified, i.e. $|\phi^{-1}(P)| = n \ \ \forall P \in E_2$

(2) $K_1 = k(E_1)/\phi^* k(E_2) = K_2$ is Galois of degree n, and

$$\begin{array}{rcl} \ker(\phi^*) & \cong & \mathrm{Gal}(K_1/K_2) \\ f & \mapsto & \tau_P^* \end{array}$$

(this implies $\mathrm{Gal}(K_1/K_2)$ abelian)

(3) If $\psi : E_1 \to E_3$ another isogeny (may be inseparable) and $\ker\psi \supseteq \ker\phi$ then $\exists!\chi$ s.t. $\psi = \chi \circ \phi$

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ & \searrow^{\psi} & \downarrow {\scriptstyle \exists!\chi} \\ & & E_3 \end{array}$$

(4) Conversely, given any finite subgroup $\Phi \in E_1$, $\exists$ separable $\phi : E_1 \to$ some elliptic curve (denoted E_1/Φ) s.t. $\ker\phi = \Phi$

Proof

(1) By separability, $\exists\widetilde{P} \in E_2$ with n preimages $\widetilde{Q_1}, \dots, \widetilde{Q_n}$ by separability
If $\phi(Q) = P$ arbitrary, then

$$\underbrace{Q + (\widetilde{Q_1} - \widetilde{Q_1})}_{T_1}, \underbrace{Q + (\widetilde{Q_2} - \widetilde{Q_1})}_{T_2}, \dots, \underbrace{Q + (\widetilde{Q_n} - \widetilde{Q_1})}_{T_n}$$

are n direct preimages of P

(2) $\Phi := \ker\phi = \{T_1, \dots, T_n\}$ and

Claim: $\tau_{T_i}^* : k(E_2) \hookrightarrow k(E_1)$ preserves $\phi^*(k(E_2))$

Proof of Claim:

$\tau_{T_i}^*(\phi^* f) = \phi^* f(\cdot + T_i) = f(\phi(\cdot + T_i)) = f(\phi(\cdot) + \phi(T_i)) = f(\phi(\cdot) + \mathcal{O}) = f(\phi(\cdot)) = \phi^* f$ ■

$\Rightarrow \quad |\operatorname{Aut}(K_1/K_2)| \geq n$ ($\tau_{T_i}^* \in \operatorname{Aut}(K_1/K_2)\ \forall i$)
also $[K_1 : K_2] = n$, so by Galois theory, K_1/K_2 Galois and $|\operatorname{Gal}| = n$

(3) $K_3 = \psi^* k(E_3) \hookrightarrow K_1$
K_3 is fixed by $\{\tau_P^* | P \in \ker\psi\} \supseteq \{\tau_P^* | P \in \ker\phi\} = \operatorname{Gal}(K_1/K_2)$
$\Rightarrow \quad K_3 \subseteq K_2 \quad \Rightarrow \quad \exists!\chi : E_2 \to E_3$ inducing this inclusion
and $\psi = \chi \circ \phi$, $\chi(\mathcal{O}) = \psi(\mathcal{O}) = \mathcal{O}$
$\Rightarrow \quad \chi$ isogeny

(4) $\tau_P^* : k(E/\Phi) \hookrightarrow K_1 = k(E_1)$, where $P \in \Phi$
Let $K := K_1^\Phi$. By Galois theory, K_1/K is Galois of degree $|\Phi|$
In particular, tr.deg$K = 1 \Rightarrow K = k(C)$ for some (unique up to isom) non-singular curve C, get non-constant map

$$\phi : E_1 \to C \qquad \text{(this map is unramified, same argument as in (1))}$$

Recall $g(C) \leq g(E_1) = 1$
If $g(C) = 1 \Rightarrow$ done (define $\mathcal{O}_C = \phi(\mathcal{O}_{E_1})$)
If $g(C) = 0, C \cong \mathbb{P}^1$, check the following:

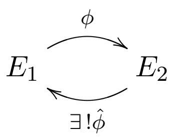
$$\operatorname{div}(\phi^* dx) = \sum_{\phi(Q)=\infty} e_Q(Q)$$

(Note dx has divisor $-2(\infty)$) all $a_Q < 0$, and this divisor has degree < 0 #

□

2.2.5 Dual Isogeny

Definition 2.2.32



We say E_1, E_2 are <u>isogeneous</u> if $\exists$ isogeny $\phi \neq 0 : E_1 \to E_2$

Proposition 2.2.33
$\phi : E_1 \to E_2$ isogeny of degree $m \neq 0$
Then $\exists!\,\hat{\phi} : E_2 \to E_1$ (the <u>dual</u> isogeny) s.t. $\hat{\phi}\phi = [m]$
(This proposition implies being isogeneous is an equivalence relation)

Proof
<u>Uniqueness:</u>
If $\hat{\phi}\phi = \psi\phi = [m] \Rightarrow (\hat{\phi} - \psi)\phi = [0] \Rightarrow$ (by $\phi \neq 0$) $\hat{\phi} = \psi$

<u>Existence</u>:
Suffice to show for ϕ separable and Frob_q

(1) <u>ϕ separable</u>: This implies $\#\ker\phi = m$, hence, $\forall P \in \ker\phi, mP = \mathcal{O} \Rightarrow \ker\phi \subseteq \ker[m]$
$\Rightarrow \quad$ done by previous Theorem 2.2.31(3).

(2) $\phi = \mathrm{Frob}_p, \quad p = \mathrm{char} k > 0, m = \deg \mathrm{Frob}_p = p$
w invariant differential on E
$[p]^* w = pw = 0 \Rightarrow [p]$ inseparable $\Rightarrow [p] = \mathrm{Frob}_p \circ \psi$ for some ψ

□

Theorem 2.2.34

(1) $\widehat{\phi}\phi = [m]$ on E_1, $\phi\widehat{\phi} = [m]$ on E_2
(2) $\widehat{\chi \circ \phi} = \widehat{\phi} \circ \widehat{\chi} \;\; \forall \chi : E_2 \to E_3$
(3) $\widehat{\phi + \psi} = \widehat{\phi} + \widehat{\psi} \;\; \forall \psi : E_1 \to E_2$
(4) $\widehat{[m]} = [m]$ and $\deg[m] = m^2 \;\; \forall m \in \mathbb{Z}$
(5) $\deg \widehat{\phi} = \deg \phi$
(6) $\widehat{\widehat{\phi}} = \phi$

Proof
May assume all isogenies $\neq [0]$

(1) $\widehat{\phi}\phi = [m]$ by definition
$\phi\widehat{\phi}\phi = \phi[m] = [m] \circ \phi \Rightarrow \phi\widehat{\phi} = [m]$ (as $\phi \neq 0$)

(2) $\chi\phi\widehat{\phi}\widehat{\chi} = \chi[\deg \phi]\widehat{\chi} = [\deg \phi][\deg \chi] = [\deg(\chi\phi)] = \chi\phi\widehat{\chi\phi}$
$\Rightarrow \quad \widehat{\phi}\widehat{\chi} = \widehat{\chi\phi}$

(3) Omitted (Silverman III 6.2)

(4) By induction: Clearly true for $m = -1, 0, 1$
$\widehat{[m+1]} = \widehat{[m]} + \widehat{[1]}$ by (3)
$= [m] + [1] = [m+1] = [\deg[m]] = [m]\widehat{[m]} = [m^2]$
$\Rightarrow \quad \deg[m] = m^2$

(5) $\widehat{\phi}\phi = [m]$ Take degrees

(6) $\widehat{\phi}\phi = [\deg \phi] = [\deg \widehat{\phi}] = \widehat{\widehat{\phi}}\widehat{\phi}$
$\Rightarrow \quad \phi = \widehat{\widehat{\phi}}$

□

Definition 2.2.35
A an abelian group. A quadratic form is a function $d : A \to \mathbb{R}$ s.t.

(1) $d(-x) = dx \;\; \forall x \in A$
(2) The pairing

$$\begin{aligned} \langle \; , \; \rangle : A \times A &\to \mathbb{R} \\ (\phi, \psi) &\mapsto d(\phi + \psi) - d\phi - d\psi \end{aligned}$$

is $\mathbb{Z}$-bilinear

Say d is positive-definite if $d(x) \geq 0$, and $d(x) = 0 \Leftrightarrow x = 0$

Corollary 2.2.36
$\deg : \mathrm{Hom}(E_1, E_2) \to \mathbb{Z}$ is a positive definite quadratic form

Proof
All clear except bilinearity. Using $[\cdot] : \mathbb{Z} \hookrightarrow \mathrm{End}(E_1)$:

$$\begin{aligned} \langle \phi, \psi \rangle &= [\deg(\phi + \psi)] - [\deg \phi] - [\deg \psi] \\ &= (\widehat{\phi + \psi}) \cdot (\phi + \psi) - \phi\widehat{\phi} - \psi\widehat{\psi} \\ &= \widehat{\phi}\psi + \widehat{\psi}\phi \qquad \text{bilinear} \end{aligned}$$

□

2.2.6 Torsion

Definition 2.2.37
The <u>m-torsion group</u> (or group of m-torsion points)

$$E[m] := \ker[m] = \{P | mP = 0\} \qquad (m \geq 1)$$

Corollary 2.2.38
If char $k \nmid m$ then $E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$

Proof
$[m]$ separable $\Rightarrow |E[m]| = m^2$ (because $\deg[m] = m^2$)

$$\left.\begin{array}{l} |E[m]| = m^2 \\ |E[d]| = d^2 \ \forall d | m \end{array}\right\} \Rightarrow E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$$

(Exercise: check this) □

Remark. $E[m] \cong E[p_1^{n_1}] \times \cdots E[p_k^{n_k}]$ if $m = p_1^{n_1} \cdots p_k^{n_k}$ prime decomposition

2.2.7 Tate module

l prime, $l \nmid$ char k

$$\cdots \xrightarrow{[l]} E[l^3] \xrightarrow{[l]} E[l^2] \xrightarrow{[l]} E[l] \overset{[l]}{\twoheadrightarrow} 0$$
$$E[l^n] = \mathbb{Z}/l^n\mathbb{Z} \times \mathbb{Z}/l^n\mathbb{Z}$$

Definition 2.2.39
The <u>l-adic Tate module</u> is

$$\begin{aligned} T_l E &:= \varprojlim_{n \geq 1} E[l^n] \\ &= \{(P_n)_{n \geq 1} | P_n \in E[l^n], [l]P_n = P_{n-1}\} \quad \text{(by defn)} \\ &= \mathbb{Z}_l \oplus \mathbb{Z}_l \quad \text{as an abelian group or } \mathbb{Z}_l\text{-module} \end{aligned}$$

Recall: The <u>l-adic integer</u> $\mathbb{Z}_l := \{(\cdots, a_2, a_1) | a_n \in \mathbb{Z}/l^n\mathbb{Z}, a_{n+1} \equiv a_n \mod l^n\}$
This a ring (component-wise) and $\supseteq \mathbb{Z} = \{(\cdots, a, a) | a \in \mathbb{Z}\}$

An isogeny $\phi : E_1 \to E_2$ induces linear maps

$$E_1[l^n] \to E_2[l^n]$$

so a $\mathbb{Z}_l$-linear map $\phi_l : T_l E_1 \to T_l E_2$ (think this as element of $M_2(\mathbb{Z}_l)$)

Theorem 2.2.40
E_1, E_2 elliptic curves. Then

$$\underbrace{\mathrm{Hom}(E_1,E_2)}_{\text{torsion free}\,\mathbb{Z}\text{-modules}} \otimes \mathbb{Z}_l \quad \hookrightarrow \quad \mathrm{Hom}(T_lE_1, T_lE_2)$$

Proof
Let $H = \mathrm{Hom}(E_1,E_2)$ torsion-free abelian group.
Now suppose $\phi \in H \otimes \mathbb{Z}_l$ s.t. $\phi_l = 0$

$$\phi = a_1\psi_i + \cdots a_t\psi_t \quad a_i \in \mathbb{Z}_l, \psi_i \in H$$

$M := \langle \psi_1, \ldots, \psi_t \rangle$. Use the following Lemma 2.2.41, replace ψ_i by a basis of M^{div}, may assume $M = M^{\mathrm{div}}$

$$\phi = a_1\psi_1 + \cdots a_t\psi_t \qquad \phi_l = 0$$

for all $n \neq 1$,
Since $(a_1 \mod l^n) \in \mathbb{Z}$ s.t. its class in $\mathbb{Z}/l^n\mathbb{Z}$ is the same as that of a_1

$$[a_1 \mod l^n]\psi_1 + \cdots [a_t \mod l^n]\psi_t \text{ kills } E[l^n]$$

$\Rightarrow$ factoring isogenies theorem

$$\begin{aligned} &= l^n \times (\text{some elts of } M^{\mathrm{div}} = M) \\ &= [l^nb_1]\psi_1 + \cdots + [l^nb_t]\psi_t \quad \text{for some } b_i \in \mathbb{Z} \end{aligned}$$

ψ_i basis of $M \Rightarrow a_i = l^nb_i \equiv 0 \mod l^n$
True for all $n \Rightarrow$ all $a_i = 0 \Rightarrow \phi = 0$ □

Lemma 2.2.41
If $M \subseteq H = \mathrm{Hom}(E_1,E_2)$ finitely generated subgroup, then

$$M^{\mathrm{div}} = \{\phi \in H | m\phi \in M \text{ for some } m \geq 1\}$$

is finitely generated

Proof
Note $M \otimes \mathbb{R}$ is a finite dimensional vector space, degree as quadratic form

$$\begin{aligned} M^{\mathrm{div}} &\hookrightarrow M \otimes \mathbb{R} \\ \text{open nbhd of 0:} \quad U = \{\phi \in M \otimes \mathbb{R} \,|\, \deg\phi < 1\} &\hookrightarrow M \otimes \mathbb{R} \end{aligned}$$

$M^{\mathrm{div}} \cap U = \{0\}$ (deg ≥ 1 for non-zero isogenies)
$\Rightarrow$ M^{div} discrete $\Rightarrow$ finitely generated □

Corollary 2.2.42

$$\begin{aligned} \mathrm{rk}_{\mathbb{Z}}\,\mathrm{Hom}(E_1,E_2) &\leq \mathrm{rk}_{\mathbb{Z}_l}\,\mathrm{Hom}(\mathbb{Z}_l^2, \mathbb{Z}_l^2) = 4 \\ \mathrm{rk}_{\mathbb{Z}}\,\mathrm{End}(E) &\leq 4 \end{aligned}$$

Easy algebra:
Any integral domain R of char 0 which has $\mathrm{rk}_{\mathbb{Z}} \leq 4$ and has a positive-definite quadratic form

$$d : R \to \mathbb{Z}$$
$$\text{s.t. } d(ab) = d(a)d(b)$$

then

(1) $R \cong \mathbb{Z} \qquad (d(x) = x^2)$ or

(2) $R \cong \mathcal{O}_K$ order in imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D}) \qquad (d(x) = |x|^2)$

(3) R rank 4 order in a quaternion algebra $\qquad (d(x) = a^2 + b^2 + c^2 + d^2)$

Corollary 2.2.43
End(E) is one of the 4 cases above, in character 0 (commutative) either (1) or (2)

2.3 Elliptic Curves over $\mathbb{C}$

2.3.1 Aside

A non-singular projective curve C over $\mathbb{C}$ with its usual complex topology is a compact (i.e. $\mathbb{P}^n_{\mathbb{C}}$ compact) complex manifold (i.e. non-singular) of dimension 1 (i.e. curve)
$\Rightarrow$ a complex Riemann surface.
Conversely, by Riemann Existence Theorem:
Every complex Riemann surface X comes from a C over $\mathbb{C}$

$$\begin{array}{ccc} C & \longrightarrow & X \\ \text{rational function} & & \text{meromorphic function} \\ \mathbb{C}(C) & = & \mathbb{C}(X) \end{array}$$

(This is an equivalence of categories)
(Note: This is very hard, the main step is to prove $\mathbb{C}(X) \neq \mathbb{C}$)

Universal curve $\widetilde{X}$ has a complex structure (easy),

$$X = \widetilde{X}/\pi_1(X)$$

($\pi_1(X)$ is a discrete group acting freely, the fundamental group)

Complex Uniformization Theorem (also hard). As a $\mathbb{C}$-manifold,

$$\begin{array}{ll} \widetilde{X} = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}} & \text{if } g(X) = 0 \\ \widetilde{X} = \mathbb{C} & \text{if } g(X) = 1 \\ \widetilde{X} = \{z : |z| < 1\} & \text{if } g(X) \geq 2 \end{array}$$

If $g = 1$, then $\mathrm{Aut}_{\mathbb{C}-\mathrm{inf}} \mathbb{C} = \{z \mapsto az + b | a, b \in \mathbb{C}\}$
fixed-point free ones $= \{z \mapsto z + w | w \in \mathbb{C}\}$
$\pi_1(X) = \cong \mathbb{Z} \oplus \mathbb{Z} \Rightarrow X \cong \mathbb{C}/\Lambda$ (Λ lattice)
$\Rightarrow \{\mathbb{C}/\Lambda\} =$ elliptic curves over $\mathbb{C}$

Our Goal: Do this explicitly

2.3.2 Theory

Recall function on $\mathbb{C}$ is meromorphic $\Leftrightarrow \forall a \in \mathbb{C}$ it has Laurent expansion at a:

$$f(z) = \sum_{n=n_0}^{\infty} c_n (z-a)^n \qquad c_{n_0} \neq 0 \text{ unless } f \equiv 0$$

Notation:

$$\begin{aligned} \operatorname{ord}_a f &:= n_0 \in \mathbb{Z} \quad \text{for order of vanishing at } a \text{ (discrete valuation)} \\ \operatorname{res}_a f &:= c_{-1} \quad \text{residue at } a \end{aligned}$$

Definition 2.3.1
A lattice $\Lambda \subseteq \mathbb{C}$ is a discrete subgroup of rank 2

$$\Lambda = \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2$$

(Note: Basis w_1,w_2 not unique, up to $GL_2(\mathbb{Z})$) We use π to denote the fundamental domain of Λ (i.e. the parallelogram spanned by w_1 an w_2)

An elliptic function (w.r.t to Λ) is a meromorphic function s.t.

$$f(z+w) = f(z) \qquad\qquad \forall z \in \mathbb{C}, w \in \Lambda$$

(These are precisely meromorphic functions on $X = \mathbb{C}/\Lambda$, they form a field $\mathbb{C}(X) \supseteq \mathbb{C}$)

Lemma 2.3.2
$f \not\equiv 0$ elliptic function

(1) f analytic (all $\operatorname{ord}_a f \geq 0$) $\Rightarrow$ f constant

(2) $\sum_{w \in \mathbb{C}/\Lambda} \operatorname{res}_w f = 0$

(3) $\sum_{w \in \mathbb{C}/\Lambda} \operatorname{ord}_w f = 0$

(4) $\sum_{w \in \mathbb{C}/\Lambda} \operatorname{ord}_w f \cdot w \in \Lambda$ (i.e. =0 in $\mathbb{C}/\Lambda$)

(Note: (2),(3),(4) are finite sums (π compact), well-defined)

Proof

(1) f analytic $\Rightarrow$ bounded on π $\Rightarrow$ bounded on $\mathbb{C}$ $\Rightarrow$ constant by Liouville's Theorem

(2) $\sum \text{res} = \frac{1}{2\pi i} \int_{\partial \pi} f(z) dz = \int + \int + \int + \int = 0$
(f elliptic)

(3) $\sum \text{ord} = \frac{1}{2\pi i} \int_{\partial \pi} \frac{f'}{f} dz = 0$ as above

(4) Use $z \frac{f'}{f}$ (Exercise)

□

Notation: $\mathcal{L}(n(0)) = \{\text{elliptic functions w.r.t. } \Lambda \text{ s.t. } f \text{ analytic for } z \notin \Lambda, \operatorname{ord}_z f \geq -n \text{ for } z \in \Lambda\}$

Lemma 2.3.2(1) $\Rightarrow$ $\mathcal{L}(0) = \mathbb{C}$ constants
Lemma 2.3.2(2) $\Rightarrow$ $\mathcal{L}(1(0)) = \mathbb{C}$ (by LHS=$\operatorname{res}_0 f$, RHS=0 $\Rightarrow$ analytic at 0 as well)

Definition 2.3.3
Eisenstein series of weight $2k$

$$G_{2k} = G_{2k}(\Lambda) := \sum_{\substack{w\in\Lambda \\ w\neq 0}} w^{-2k} \qquad k \geq 2$$

(Exercise: $\sum_{0\neq w\in\Lambda} \frac{1}{|w|^\alpha} < \infty \Leftrightarrow \alpha > 2$)

Example 2.3.4
$\Lambda = \mathbb{Z} + \sqrt{2}i\,\mathbb{Z}$
$G_4 = 2.23661\ldots$
$G_6 = 1.89217\ldots$

Theorem 2.3.5

$$\mathcal{L}(2(0)) = \langle 1, \wp(z)\rangle$$

where $\wp(z)$ unique elliptic function (Weierstrass $\wp$-function) s.t.

$$\wp(z) = \frac{1}{z^2} + O(z) \qquad \text{at } z = 0$$

($O(z)$ means, at $z = 0$, Laurent series has $c_{-1} = 0, c_0 = 0$)

Proof
Uniqueness:
$\dim \mathcal{L}(2(0)) \leq 2$, clear:
$\wp_1 - \wp_2 \in \mathcal{L}(0) \Rightarrow$ constant, zero at $z = 0$, as cannot have pole of order 1 by previous lemma $\Rightarrow 0$

Existence:
Define the Weierstrass $\wp$-function as follows

$$\wp(z) := \wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\substack{w\in\Lambda \\ w\neq 0}} \frac{1}{(z-w)^2} - \frac{1}{w^2}$$

If $|w| > 2|z|$

$$\left|\frac{1}{(z-w)^2} - \frac{1}{w^2}\right| = \left|\frac{z(2w-z)}{w^2(w-z)^2}\right| \leq \frac{10\cdot|z|}{|w|^3}$$

note $\sum_{|w|\geq 2|z|} \leq 10\cdot|z| \left(\sum \frac{1}{|w|^3}\right) < \infty$
So this converges uniformly on compact $\subseteq \mathbb{C}\setminus\Lambda$
$\Rightarrow$ analytic on $\mathbb{C}\setminus\Lambda$, double pole at $w \in \Lambda$

$\wp(z)$ elliptic:
$\wp(z)$ clearly even; in particular $\wp(\frac{w}{2}) = \wp(-\frac{w}{2})$ for $w \in \Lambda$

$$\wp'(z) = -2\sum_{w\in\Lambda} \frac{1}{(z-w)^3}$$

this clearly is elliptic

$$\Rightarrow \qquad \wp(z+w) - \wp(z) = c(w) \quad \text{constant } (w \in \Lambda)$$

$z = -\frac{w}{2} \Rightarrow c(w) = 0 \Rightarrow \wp(z)$ elliptic □

Remark. For $|z| < |w|$

$$\begin{aligned} \frac{1}{(z-w)^2} - \frac{1}{w^2} &= w^{-2}\left(\frac{1}{\left(1-\frac{z}{w}\right)^2} - 1\right) \\ &= \sum_{k=1}^{\infty} \frac{n+1}{w^{n+2}} z^n \end{aligned}$$

Sum over $w \in \Lambda$, interchange order

$$\Rightarrow \qquad \wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}$$

Theorem 2.3.6
Writing

$$\begin{aligned} g_2 &= g_2(\Lambda) &:=& \; 60G_4(\Lambda) \\ g_3 &= g_3(\Lambda) &:=& \; 140G_6(\Lambda) \end{aligned}$$

We get

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

Proof

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots \\ \wp(z)^3 &= \frac{1}{z^6} + 9G_4\frac{1}{z^2} + 15G_6 + \cdots \\ \wp'(z)^2 &= \frac{1}{4z^6} - 24G_4\frac{1}{z^2} - 80G_6 + \cdots \end{aligned}$$

LHS − RHS in Theorem is elliptic, holomorphic (i.e. analytic, i.e. no poles as all negative power of z cancel)
LHS − RHS $\equiv 0$ by Lemma 2.3.2 (1) □

Remark. (see picture)
$\wp(z)$ even, $\wp'(z)$ odd $\Rightarrow \wp'(T_i) = \wp'(-T_i) \Rightarrow \wp'(T_i) = 0$
$\frac{d}{dz}(\frac{1}{z^2}) = \frac{-2}{z^3} \Rightarrow -3(\mathcal{O})$

$$\operatorname{div}\wp'(z) = -3(\mathcal{O}) + (T_1) + (T_2) + (T_3)$$

and $\forall\, a \in C$

$$\begin{aligned} \operatorname{div}(\wp(z) - a) &= -2(\mathcal{O}) + (w) + (-w) \quad \text{for some } w \in \mathbb{C}/\Lambda \\ \text{and} \quad \operatorname{div}(\wp(z) - \wp(T_i)) &= -2(\mathcal{O}) + 2(T_i) \end{aligned}$$

in particular, $\wp(T_i)$ distinct

Theorem 2.3.7
$\Lambda \subseteq \mathbb{C}$ lattice, $X = \mathbb{C}/\Lambda$. Then

$$\mathbb{C}(X) = \mathbb{C}(\wp(z), \wp'(z))$$

Proof
Take $f \in \mathbb{C}(X)$. May assume f is even

$$\text{general } f = \underbrace{\frac{1}{2}(f(z) + f(-z))}_{\text{even elliptic}} + \underbrace{\frac{1}{2}(f(z) - f(-z))}_{\text{odd elliptic}} \quad \Rightarrow \quad \text{odd} = \wp' \times \text{even}$$

Now $\mathrm{div}(f) = n_1[(z_1) + (-z_1)] + \cdots + n_k[(z_k) + (-z_k)]$ for some $n_k \in \mathbb{Z}, z_k \in \mathbb{C}/\Lambda$
(check T_i carefully using f' odd)

Define

$$\widetilde{f} := \prod_i [\wp(z) - \wp(z_i)]^{n_i}$$

$\Rightarrow \quad \mathrm{div}(f) = \mathrm{div}(\widetilde{f}) + ?(0)$
both deg $\mathrm{div}(f)$, div $(\widetilde{f})$ =0 $\Rightarrow$ $\frac{\widetilde{f}}{f}$ has no zero or poles
$\Rightarrow$ holomorphic elliptic $\Rightarrow$ constant □

Write

$$E_\Lambda: \quad y^2 = 4x^3 - g_2 x - g_3$$

where $g_2 = g_2(\Lambda)$, $g_3 = g_3(\Lambda)$
$E_\Lambda \cong y^2 = (x - \wp(T_1))(x - \wp(T_2))(x - \wp(T_3))$
In particular, this is non-singular
Actually, $(\wp(z) : \wp'(z) : 1) \in \mathbb{P}^2$ and $\Lambda \mapsto (0:1:0) = \mathcal{O}$

Theorem 2.3.8
ϕ as follows is an analytic isomorphism of complex Lie groups

$$\begin{aligned} \phi : \mathbb{C}/\Lambda &\to E_\Lambda \\ z &\mapsto (\wp(z), \wp'(z)) \end{aligned}$$

Proof
Surjectivity:
$\mathcal{O}, (\alpha_i, 0)$ (where α_i is root of RHS) in the image
Take $(x, y) \in E_\Lambda$ where $y \neq 0, \infty$

$$\mathrm{div}(\wp(z) - x) = -2(\mathcal{O}) + (w_1) + (-w_1) \quad \text{for some } w_1 \in \mathbb{C}/\Lambda$$

$\Rightarrow \quad \wp(w_1) = x$

$$(\wp'(w))^2 = f(\wp(w)) = f(x) = y^2$$

$\Rightarrow \quad y = \wp'(w_1)$ or $y = -\wp'(w1) = \wp'(-w_1)$
$\Rightarrow$ either w_1 or $-w_1$ maps to (x, y)

Injectivity: Check T_i; otherwise follows from the proof of surjectivity

locally analytic isom:
$\frac{dx}{y}$ differential on E with no zeros/poles

$$\phi^* \frac{dx}{y} = \frac{d\wp(z)}{\wp'(z)} = \frac{\wp'(z) dz}{\wp'(z)} = dz$$

$\Rightarrow \quad \phi^*$ isomorphism on cotangent spaces

ϕ^{-1} group homomorphism
If $P_1 + P_2 + P_3 = \mathcal{O}$ on E_Λ
Take $f \in \mathbb{C}(E_\Lambda)$ s.t.

$$\mathrm{div}(f) = (P_1) + (P_2) + (P_3) - 3(\mathcal{O})$$

say $\phi : z_i \mapsto P_i$. Then

$$\mathrm{div}(\phi^* f) = (z_1) + (z_2) + (z_3) - 3(\mathcal{O})$$

(Note: $\phi^* f = f(\wp(z), \wp'(z))$ which is meromorphic)

previous Lemma 2.3.2 (4) $\Rightarrow z_1 + z_2 + z_3 = 0 \mod \Lambda$ □

Corollary 2.3.9
A divisor $D = \sum_{z_i \, \mathbb{C}/\Lambda} n_i(z_i)$ is a divisor of some elliptic function
$\Leftrightarrow \quad \sum n_i = 0$ and $\sum n_i z_i = 0 \mod \Lambda$

Proof
True on E_Λ □

2.3.3 Constructing Λ from E, and $\phi^{(-1)} : E \to \mathbb{C}/\Lambda$

(see picture)

If $\phi(z_0) = P_0$, then

$$z_0 = \int_0^{z_0} dz = \int_0^{z_0} \frac{d\wp(z)}{\wp'(z)} = \int_{\mathcal{O}}^{P_0} \frac{dx}{y} = \underbrace{\int_\infty^{x(P_0)} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}}_{\underline{\text{elliptic integral}}}$$

$(x(P_0) = x$-coordinate of $P_0)$

$$\Rightarrow \qquad P \mapsto \int_\infty^{x(P)} \frac{dx}{\sqrt{4f(x)}} \text{ is } \phi^{-1} : E \to \mathbb{C}$$

$\int_{\mathcal{O}}^{P_0} \frac{dx}{y}$ depends on the choice of a path from $\mathcal{O}$ to P (see picture)

The integral is well-defined up to $\mathbb{Z}$-multiples of $\int_{\gamma_1} \frac{dx}{y}, \; \int_{\gamma_2} \frac{dx}{y}$ with γ_1, γ_2 basis of $H_1(E, \mathbb{Z}) = \Lambda$ (see picture)

The lattice Λ is recovered as $\mathbb{Z} \cdot \int_{\gamma_1} \frac{dx}{y} + \mathbb{Z} \cdot \int_{\gamma_2} \frac{dx}{y} \subseteq \mathbb{C}$

Choose (picture)

$\Rightarrow \quad \int_{\gamma_1} = w_1, \int_{\gamma_2} w_2$

Example 2.3.10
$E: \quad y^2 = x(x-1)(x-3)$. Two well-defined choices of $\sqrt{x(x-1)(x-3)}$ on $\mathbb{C}$ with $(0,1)$ and $(3,\infty)$ removed, call them "$+\sqrt{\cdot}$" and "$-\sqrt{\cdot}$"

$E =$ $\cup$ $=$ $\cup$

$=$

Deform it $\Rightarrow$

$$\begin{aligned} w_1 &= \not{2}\int_0^1 \frac{dx}{\sqrt{\not{4}x(x-1)(x-3)}} = 0.620131\ldots \\ w_2 &= \not{2}\int_1^3 \frac{dx}{\sqrt{\not{4}x(x-1)(x-3)}} = 2.20335\ldots \cdot i \end{aligned}$$

This proves this E comes from a Λ (!!)
(namely, this $\Lambda = \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2$)

2.3.4 Conclusion

Let $E: \quad y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$

- If $\alpha_i \in \mathbb{R}$, E comes from a lattice $\Lambda = \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2$, $w_1 \in \mathbb{R}, w_2 \in i \cdot \mathbb{R}$
- If $\alpha_1 \in \mathbb{R}, \alpha_2 = \overline{\alpha_3}$ similar argument $\Rightarrow$ E comes from $\Lambda = \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2$, $w_1 \in \mathbb{R}$, $w_2 = \frac{1}{2}w_1 + i \cdot \mathbb{R}$
- If $\alpha_i \in \mathbb{C}$ arbitrary distinct, can show that $\int_{\gamma_1}, \int_{\gamma_2}$ are still linear independent over $\mathbb{R}$, so they form a lattice Λ (and $\mathbb{C}/\Lambda = E$ by construction)

Corollary 2.3.11
$\deg[m] = m^2$ and $E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ all $m \geq 1$

Proof
$E \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ as abelian group
$E[m] \cong (\frac{1}{m}\mathbb{Z}/\mathbb{Z})^2 \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ $\square$

2.3.5 Homotheties and Isogenies

What are isogenies $E = \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda' = E'$?

- If $\alpha \in \mathbb{C}$ s.t. $\alpha\Lambda \leq \Lambda'$ then

  $$\begin{array}{rcl} \mathbb{C}/\Lambda & \to & \mathbb{C}/\Lambda' \\ z & \mapsto & \alpha z \end{array}$$

  well-defined holomorphic $E \to E'$, $\mathcal{O} \mapsto \mathcal{O}$, given by

  $$\phi_\alpha : (\wp_\Lambda(z), \wp'_\Lambda(z)) \mapsto (\wp_{\Lambda'}(\alpha z), \wp'_{\Lambda'}(\alpha z))$$

  But $z \mapsto \wp_{\Lambda'}(\alpha z)$ is elliptic w.r.t. Λ for $w \in \Lambda$

  $$\wp_{\Lambda'}(\alpha(z+w)) = \wp_{\Lambda'}(\alpha z + \underbrace{\alpha w}_{\in \Lambda'}) = \wp_{\Lambda'}(\alpha z) \quad \text{similar for } \wp'_{\Lambda'}(\alpha z)$$

  $$\Rightarrow \quad \wp_{\Lambda'}(\alpha z), \wp'_{\Lambda'}(\alpha z) \; in \, \mathbb{C}(E) = \mathbb{C}(\wp_\Lambda(z), \wp'_\Lambda(z))$$

  i.e. ϕ_α is a rational map

- Conversely, $\phi : E \to E'$ holomorphic, $\phi(\mathcal{O}) = \mathcal{O}$; e.g. ϕ isogeny.
  $\phi : \mathbb{C} \to \mathbb{C}/\Lambda'$, lifts to the universal cover

  $$\widetilde{\phi} : \mathbb{C} \to \mathbb{C} \quad , \quad \widetilde{\phi}(\Lambda) \subseteq \Lambda'$$

  For $w \in \Lambda$,

  $$z \mapsto \widetilde{\phi}(z+w) - \widetilde{\phi}(z) \qquad \mathbb{C} \to \Lambda' \quad \text{holomorphic}$$

  is constant (dependent on w). So $\widetilde{\phi}'(z)$ is elliptic holomorphic $\Rightarrow \widetilde{\phi}' =$ constant α, i.e.

  $$\widetilde{\phi}(z) = \alpha z + \not{\beta}$$

Corollary 2.3.12

$$\begin{array}{rcl} \{\text{isogenies } \phi : E \to E'\} & = & \{\alpha \in \mathbb{C} \,|\, \alpha\Lambda \subseteq \Lambda'\} \\ \phi & \mapsto & \alpha = \dfrac{\phi^* dz}{dz} = \dfrac{\phi^*(dx/y)}{dx/y} \\ \phi & \leftarrow\!\shortmid & \alpha \quad \text{as above} \end{array}$$

Corollary 2.3.13
$\mathrm{rk}_{\mathbb{Z}}(LHS) \leq 4$ (confirming previous result)

We proved:

Theorem 2.3.14
These categories are equivalent:

- Elliptic curves over $\mathbb{C}$, maps: isogenies
- Elliptic curves over $\mathbb{C}$, maps: analytic maps taking $\mathcal{O}$ to $\mathcal{O}$
- Lattices $\Lambda \subseteq \mathbb{C}$, maps $\{\alpha \in \mathbb{C} \,|\, \alpha\Lambda \subseteq \Lambda'\}$

Corollary 2.3.15
$E \cong E' \Leftrightarrow \Lambda = \alpha\Lambda'$ for some $\alpha \in \mathbb{C}^\times$ (note lattices are homothetic), i.e.

$$\frac{\text{Elliptic curves}}{\cong} = \frac{\text{Lattices}}{\text{homothety}}$$

2.3.6 Curves with Complex Multiplication

Remark. Every $\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$ is homothetic to $\Lambda = \mathbb{Z} + \mathbb{Z}\,\tau$ for some $\tau \in \mathbb{H} = \{z \in \mathbb{C}\,|\mathrm{Im}(z) > 0\}$

Exercise: τ unique up to $SL_2(\mathbb{Z})$-action

Suppose,

$$\begin{aligned} E &= \mathbb{C}\,/\Lambda \text{ has CM, i.e.} \\ R &= \mathrm{End}(E) = \{\alpha \in \mathbb{C}\,|\alpha\Lambda \subseteq \Lambda\} \supsetneqq \mathbb{Z} \end{aligned}$$

We say that E has complex multiplication (CM) by R

$\alpha \in R$, $\alpha \cdot 1 = \alpha \in \Lambda, \alpha \cdot \tau \in \Lambda$
$\Rightarrow \quad \alpha = a + b\tau, \quad \alpha\tau = c + d\tau$ for some $a, b, c, d \in \mathbb{Z}$
$\Rightarrow \quad b\tau^2 + (a - d)\tau - c = 0$ (quadratic equation for τ over $\mathbb{Q}$)
$\Rightarrow \quad \tau \in K = \mathbb{Q}(\sqrt{-D})$, some $D \in \mathbb{Z}_{>0}$; imaginary quadratic field

$R \subseteq \mathbb{Z} + \mathbb{Z}\,\tau$ rank 2 subring $\Rightarrow$ order in K
(Exercise: $R = \mathbb{Z} + f \cdot \mathcal{O}_K$ for some $f \geq 1$, the conductor of R)

Λ an R-module $\subseteq K \Rightarrow$ fractional ideal of R

Conversely, for each order $R \subseteq K$ (any R, any K) e.g. $\Lambda = R$ has CM by R

Generally,

$$\left\{ \begin{array}{c} \text{elliptic curves} \\ \text{with CM by } R \end{array} \right\} \Big/ \text{isom.} = \left\{ \begin{array}{c} \text{fractional ideal} \\ \text{of } R \end{array} \right\} \Big/ \sim \;=\; \text{Class group of } R$$

($I_1 \sim \alpha I_2$ for $\alpha \in K^\times$)
(note the above are finite groups)

Example 2.3.16
$R = \mathbb{Z}[i], K = \mathbb{Q}(i)$
$E : \mathbb{C}\,/\,\mathbb{Z} + \mathbb{Z}\,i \qquad y^2 = x^3 + x$

Example 2.3.17
$R = \mathbb{Z}[\zeta_3], K = \mathbb{Q}(\sqrt{-3})$
$E : \mathbb{C}\,/\,\mathbb{Z} + \mathbb{Z}\,\zeta_3 \qquad y^2 = x^3 + 1$

Example 2.3.18
$R = \mathbb{Z}[\sqrt{-5}], K = \mathbb{Q}(\sqrt{-5})$ (has class number 2)
$E : \mathbb{C}\,/\,\mathbb{Z} + \mathbb{Z}\,\sqrt{-5} \qquad j = 632000 + 282880\sqrt{5}$
$E : \mathbb{C}\,/\,\mathbb{Z} + \mathbb{Z}\,\frac{1+\sqrt{-5}}{2} \qquad j = 632000 - 282880\sqrt{5}$
Beyond Syllabus Fact: j-invariants of elliptic curves with CM by $\mathcal{O}_K$ generate maximal unramified abelian extension, i.e. the Hilbert class field, of K, e.g.:

$$\mathbb{Q}(\sqrt{-5}) \xrightarrow{\text{unramified}} \mathbb{Q}(\sqrt{-5}, \sqrt{5})$$

The study of these is called Theory of CM.

Exercise: If $E \sim E'$ isogenies then E has CM $\Leftrightarrow$ E' has CM; with the same K
Conversely, any 2 elliptic curves with CM by subrings ($\neq \mathbb{Z}$) of $K = \mathbb{Q}(\sqrt{-D})$ with the same D are isogeneous

Exercise: $\mathrm{End}(E) = \mathbb{Z}[\alpha]$, complex conjugation = taking dual isogeny, degree = $|\cdot|^2$

Chapter 3

Arithmetic

3.1 Elliptic Curves over Perfect Field

Ground field K, always perfect

Definition 3.1.1
K is perfect if every finite extension of K is separable $\quad (\Leftrightarrow \overline{K}^{\mathrm{Gal}(\overline{K}/K)} = K)$

Example 3.1.2
Perfect field: char K=0 $K = \overline{K}$
$K = \mathbb{F}_{p^n}$
Non-perfect field: $K = \mathbb{F}_p(X)$

Definition 3.1.3
A curve $C \subseteq \mathbb{P}^n_{\overline{K}}$ is defined over K (written C/K) if it can be give by

$$C : \begin{cases} f_1 = 0 \\ \quad\vdots \\ f_m = 0 \end{cases} \qquad f_i \in K[x_0, \ldots, x_n] \text{ homog. polynomials}$$

The set of K-rational points $C(K)$= $\{(a_0, \ldots, a_n) \in C | a_i \in K\}$

Exercise: $C: \ x^2 + y^2 = -1 \subseteq \mathbb{P}^2_{\mathbb{C}}$ defined over $\mathbb{Q}: C(\mathbb{Q}) = \emptyset$

Definition 3.1.4
K-rational functions: $K(C) = \{\frac{f}{g} \in \overline{K}(C) | f, g \in K(x_0, \ldots, x_n)\}$
K-rational maps: $C_1 \to C_2$ = those defined by K-rational functions

Fact: {non-singular curves over K} $\to$ { f.g. extensions L of K of tr.deg. 1 s.t. $L \cap \overline{K} = K$} (exercise: why $L \cap \overline{K}$) $C \mapsto K(C)$ this is an equivalence of categories

Definition 3.1.5
K-rational divisors

$$\mathrm{Div}_K(C) = (\underbrace{\mathrm{Div}(C)}_{\text{over } \overline{K}})^{\mathrm{Gal}(\overline{K}/K)} \qquad \text{Galois invariants}$$

Clearly $f \in K(C)^\times \ \Rightarrow \ \mathrm{div}(f) \in \mathrm{Div}_K(C)$
(and conversely, the lemma below)

Example 3.1.6
$y^2 = x^3 + 1$ over $\mathbb{Q}$
$\mathrm{div}(x) = (0,1) + (0,-1) - 2(\mathcal{O})$
$\mathrm{div}(y) = (-1,0) + (-\zeta,0) + (-\zeta^2,0) - 3(\mathcal{O})$

Lemma 3.1.7
$D \in \mathrm{Div}_K(C) \Rightarrow \mathcal{L}(D)$ has a basis of functions in $K(C)$

Proof
General fact about vector space with $\mathrm{Gal}(\overline{K}/K)$-action (Silverman III, 5.8.1) □

Definition 3.1.8
An <u>elliptic curve</u> is a pair $(E, \mathcal{O})$, E/K genus 1, $\mathcal{O} \in E(K)$

Example 3.1.9
(Selmer) $C: \ 3x^3 + 4y^3 = 5$ has genus 1, $C(\mathbb{Q}) = \emptyset$, NOT an elliptic curve over $\mathbb{Q}$

- Riemann-Roch + Lemma $\Rightarrow$

$$E \cong y^2 + a_1xy + a_3y = x^3 + \cdots$$

  with $a_i \in K$, unique up to

$$\begin{array}{lll} x & \mapsto & u^2x + r \qquad u, r, s, t \in K \\ y & \mapsto & u^3y + sx + t \qquad u \neq 0 \end{array}$$

- Addition: $E \times E \to E$, inverse: $E \to E$ both defined over K. In particular $(P+Q)^\sigma = P^\sigma + Q^\sigma \ \forall \sigma \in \mathrm{Gal}(\overline{K}/K)$
  Thus $E(K)$ abelian group (main object of study)

Definition 3.1.10

$$\begin{array}{rcl} \mathrm{Hom}_K(E_1, E_2) & = & \underline{K\text{-rational isogenies}} \\ & = & K\text{-morphism s.t. } \mathcal{O} \mapsto \mathcal{O} \\ & = & \mathrm{Hom}(E_1, E_2)^{\mathrm{Gal}(\overline{K}/K)} \\ \mathrm{End}_K(E) & = & \mathrm{Hom}_K(E, E) \overset{\text{subring}}{\subseteq} \mathrm{End}(E) \text{ over } \overline{K} \end{array}$$

Example 3.1.11
$E: y^2 = x^3 + x$ over $\mathbb{Q}$
$\mathrm{End}_{\mathbb{Q}(i)}(E) \cong \mathbb{Z}[i]$
$[i] : (x, y) \mapsto (ix, -y)$
$\mathrm{End}_{\mathbb{Q}}(E) = \mathbb{Z}$
$\dfrac{\phi^* dx/y}{dx/y} \notin \mathbb{Q}$ for $\phi \in \mathbb{Z}[i] \setminus \mathbb{Z}$
$\Rightarrow$ cannot be defined over $\mathbb{Q}$
i.e. E has CM over $\mathbb{Q}(i)$ but not over $\mathbb{Q}$

3.1.1 Torsion and Weil Pairing

E/K, $m \geq 1$, char $K \nmid m$
Recall: <u>m-torsion subgroup</u> $E[m] = \{P \in E(\overline{K}) | mP = \mathcal{O}\} \cong \mathbb{Z}/m\mathbb{Z} + \mathbb{Z}/m\mathbb{Z}$ as abelian group

If $mP = \mathcal{O}$ and $\sigma \in \mathrm{Gal}(\overline{K}/K)$ then
$m(P^\sigma) = (mP)^\sigma = \mathcal{O}^\sigma = \mathcal{O} \Rightarrow P^\sigma \in E[m]$
$\Rightarrow$ $E[m]$ is $\mathrm{Gal}(\overline{K}/K)$-module with linear action, i.e. we have representation:

$$\overline{\rho_m} : \mathrm{Gal}\,\overline{K}/K \to \mathrm{Aut}(E[m]) \cong GL_2(\mathbb{Z}/m\mathbb{Z}) (= GL_2(\mathbb{F}_l) \text{ if prime } m = l)$$

Example 3.1.12
$E/\mathbb{Q}: \quad y^2 = (x-1)(x^2+1) \quad , m = 2$

$$E[2] = \{\mathcal{O}, (1,0), (i,0), (-i,0)\} \quad \cong \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$$

$$\begin{array}{cccc} \overline{\rho_2} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong C_2 \hookrightarrow & S_3 = GL_2(\mathbb{F}_2) \\ \mathrm{id} & \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{complex conjugation} & \longmapsto & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

Example 3.1.13
$E/\mathbb{Q}: \quad y^2 = x^3 - 2$

$$\overline{\rho_2} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}) \cong S_3 = GL_2(\mathbb{F}_2)$$

Remark. Important Theorem (Serre): E/K non-CM, K number field $\Rightarrow$ $\overline{\rho_l}$ surjective $\mathrm{Gal}(\overline{K}/K) \twoheadrightarrow GL_2(\mathbb{F}_l)$ for almost all l

Notation: $\mu_m = m$-th roots of unity in $\overline{K}$ ($\cong \mathbb{Z}/m\mathbb{Z}$ abelian group)

$\bigwedge^2 E[m] \cong \mu_m$ as a Galois module:

Theorem 3.1.14
E/K. There is a bilinear, alternating, non-degenerate, Galois-equivalent pairing

$$e_m : E[m] \times E[m] \to \mu_m \qquad \underline{\text{Weil pairing}}$$

which is adjoint w.r.t. isogenies

$S, T \in E[m]$
bilinear: $e_m(S_1 + S_2, T) = e_m(S_1, T)e_m(S_2, T)$ and $e_m(S, T_1 + T_2) = e_m(S, T_1)e_m(S, T_2)$
alternating: $e_m(T, T) = 1 \quad (\Rightarrow e_m(S, T) = e_m(T, S)^{-1})$
non-degenerate: if $e_m(S, T) = 1 \ \forall S \in E[m]$ then $T = \mathcal{O}$
Galois: $e_m(S^\sigma, T^\sigma) = e_m(S, T)^\sigma \ \forall \sigma \in \mathrm{Gal}(\overline{K}/K)$
adjoint: $\phi : E_1 \to E_2$, $\hat{\phi} : E_2 \to E_1$, $S \in E_1[m], T \in E_2[m]$, then $e_m(S, \hat{\phi}(T)) = e_m(\phi(S), T)$

Over $\mathbb{C}$: $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$

$$e_m(\frac{a}{m}w_1 + \frac{k}{m}w_2, \frac{c}{m}w_1 + \frac{d}{m}w_2) = \exp(2\pi i \frac{ad - bc}{m}) \ \forall a, b, c, d \in \mathbb{Z}/m\mathbb{Z}$$

Proof
Construction:
Say $D_1 = \sum_i a_i(P_i)$, $D_2 = \sum_j b_j(Q_j)$ are disjoint if $P_i \neq Q_j$ (written $D_1 \cap D_2 = \emptyset$)
If $f \in \overline{K}(E)^\times, D = \sum_i a_i(P_i)$ with $\mathrm{div}(f) \cap D = \emptyset$
then define

$$f(D) := \prod_i f(P_i)^{a_i} \quad \in \overline{K}^\times$$

Exercise: (Weil reciprocity) If $\operatorname{div}(f) \cap \operatorname{div} g = \emptyset$, then $f(\operatorname{div}(g)) = g(\operatorname{div}(f))$ (Hint: do $\mathbb{P}^1$ first)

Note:

$$\begin{array}{rcl} E[m] & = & \{D \in \operatorname{Pic}^0(E) | mD \sim 0\} \\ T & \mapsto & (T) - (\mathcal{O}) \\ \sum a_i P_i & \leftarrow & D = \sum a_i(P_i) \end{array}$$

We define e_m on the RHS:
Choose $D_S = \sum a_i(P_i)$, $D_T = \sum b_j(Q_j)$

$$\begin{array}{rcl} mD_S & = & \operatorname{div}(f_S) \\ mD_T & = & \operatorname{div}(f_T) \\ D_S \cap D_T & = & \emptyset \quad \text{(easy using Riemann-Roch)} \end{array}$$

So now we can define:

$$e_m(S,T) := \frac{f_S(D_T)}{f_T(D_S)}$$

Note: $e_m(S,T)^m = \dfrac{f_S(mD_T)}{f_T(mD_S)} = \dfrac{f_S(\operatorname{div}(f_T))}{f_T(\operatorname{div}(f_S))} = 1$
$\Rightarrow \; e_m(S,T) \in \mu_m$

Exercise: e_m is well-defined
Properties: Computation □

3.1.2 Characteristic polynomials of endomorphisms

$E/K, \phi \in \operatorname{End}_K(E), m = \deg \phi$

Lemma 3.1.15
$\exists a_\phi \in \mathbb{Z}$ s.t. the characteristic polynomial

$$f_\phi(T) := T^2 - a_\phi T + m$$

has $f_\phi(\phi) = 0$

Proof
$\deg \phi = \phi\widehat{\phi} = m$
$\deg(1-\phi) = (1-\phi)(1-\widehat{\phi}) = 1 - (\phi + \widehat{\phi}) + m \;\Rightarrow\; \phi + \widehat{\phi} \in \mathbb{Z} \subseteq \operatorname{End}_K(E)$
Let $a_\phi := \phi + \widehat{\phi} \in \mathbb{Z}$
$\Rightarrow \quad f_\phi(T) = T^2 - (\phi + \widehat{\phi})T + \phi\widehat{\phi}$
$\Rightarrow \quad f_\phi(\phi) = 0$ □

Lemma 3.1.16
$f_\phi(T) = (T-\alpha)(T-\overline{\alpha})$ with $\alpha \in \mathbb{C}, |\alpha| = \sqrt{m}$

Proof
Need $\Delta_{f_\phi} = a_\phi^2 - 4m \le 0$
$f(\frac{b}{c}) = \frac{1}{c^2}\deg(c\phi - b) \ge 0 \;\; \forall \frac{b}{c} \in \mathbb{Q}$
$\Rightarrow \quad f(x) \ge 0 \;\; \forall x \in \mathbb{R} \quad \Rightarrow \quad \Delta \le 0$ □

Lemma 3.1.17
$\phi : E \to E$ induces $\phi_l : T_l E \to T_l E$ ($l \neq \text{char } K$)
and

$$\det(\phi_l - TI) = f_\phi(T)$$

i.e. characteristic polynomial of ϕ_l is in $\mathbb{Z}[T]$ (not just $\mathbb{Z}_l[T]$) and is independent of l

Proof
Want: $\det \phi_l = \deg \phi \ \forall \phi \in \text{End}_K(E)$ (so then constant term of $T^2 + a_\phi T + c$ is clear)
Then also

$$\begin{aligned} a_\phi &= 1 - \deg(1-\phi) + \deg \phi \\ \phi_l = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in M_2(\mathbb{Z}_l) \\ \text{tr } \phi_l &= 1 - \det(1-\phi_l) + \det(\phi_l) \end{aligned}$$

Then linear term of the characteristic polynomial are done too.

To prove $\deg \phi_l = \deg \phi$. Write $E[l^n] = \mathbb{Z}/l^n\mathbb{Z} \cdot v_1 + \mathbb{Z}/l^n\mathbb{Z} \cdot v_2$, $\phi_l = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $e = e_{l^n}$ for the Weil pairing

$$\begin{aligned} e(v_1, v_2)^{\deg \phi} &= e(\deg \phi \cdot v_1, v_2) = e(\widehat{\phi}\phi \cdot v_1, v_2) \\ &= e(\phi v_1, \phi v_2) = e(av_1 + cv_2, bv_1 + dv_2) \\ &= e(v_1, v_2)^{ad-bc} = e(v_1, v_2)^{\det \phi_l} \end{aligned}$$

e non-degenerate $\Rightarrow \deg \phi \equiv \det \phi_l \mod l^n$
True for all $n \geq 1 \Rightarrow \deg \phi = \deg \phi_l$ ☐

3.2 Elliptic Curves over Finite Fields

$K = \mathbb{F}_q$ finite, $q = p^d$
$\mathbb{P}^n(K) = \{(a_0 : \ldots : a_n) \in K^{n+1} \setminus \{0\}\}/K^\times$ finite set, size $\frac{q^{n+1}-1}{q-1}$
C/K curve $\Rightarrow C(K)$ finite
E/K elliptic curve $\Rightarrow E(K)$ finite abelian group

Example 3.2.1
$E: \ y^2 = x^3 + 1$ over $K = \mathbb{F}_5$
$|E(\mathbb{F}_5)| = 6, \quad E(\mathbb{F}_5) = \{\mathcal{O}, (0, \pm 1), (2, \pm 3), (4, 0)\} \cong \mathbb{Z}/6\mathbb{Z}$
$|E(\mathbb{F}_{25})| = 36$
$|E(\mathbb{F}_{125})| = 126$, etc.

Definition 3.2.2
Zeta-function of a curve C/K (or a variety)

$$\begin{aligned} Z_{C/\mathbb{F}_q}(T) &:= \exp\left(\sum_{n=1}^{\infty} \frac{\#C(\mathbb{F}_{q^n})}{n} T^n\right) \\ &= 1 + \#C(\mathbb{F}_q)T + \cdots \end{aligned}$$

Example 3.2.3
$C = \mathbb{P}^1$

$\# \mathbb{P}^1(\mathbb{F}_{q^n}) = 1 + q^n$ (since $\{\infty\} \cup K$), so

$$\begin{aligned} Z_{\mathbb{P}^1/\mathbb{F}_{q^n}}(T) &= \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} + \sum_{n=1}^{\infty} \frac{q^n T^n}{n}\right) \\ &= \exp(-\log(1-T) - \log(1-qT)) \\ &= \frac{1}{(1-T)(1-qT)} \end{aligned}$$

Theorem 3.2.4 (Hasse)
For an elliptic $E/\mathbb{F}_q$

$$\begin{aligned} Z_{E/\mathbb{F}_q}(T) &= \frac{(1-\alpha T)(1-\overline{\alpha}T)}{(1-T)(1-qT)} \quad \text{with } |\alpha| = \sqrt{q}, \alpha \in \mathbb{C} \\ &= \frac{1 - aT + qT^2}{(1-T)(1-qT)} \quad \text{with } a = q + 1 - \#E(\mathbb{F}_q) \end{aligned}$$

and $T^2 - aT + q = f_{\mathrm{Frob}_q}(T) =$ characteristic polynomial of Frob_q on $T_l E$ for $l \nmid q$

Corollary 3.2.5
$\#E(\mathbb{F}_q)$ determines $\#E(\mathbb{F}_{q^n})\ \ \forall n \geq 1$

Corollary 3.2.6 (Hasse-Weil Inequality)
$\#E(\mathbb{F}_{q^n}) = 1 - \alpha^n - \overline{\alpha}^n + q^n\ \ \forall n \geq 1$
In particular,

$$|\#E(\mathbb{F}_{q^n}) - q^n - 1| \leq 2\sqrt{q^n}$$

Remark. (Weil:) This is true for all curves, numerator = inverse characteristic polynomial of Frob_q on $T_l(\mathrm{Jac}(C))$ of degree $2g(C)$
"Weil conjectures": Has analogue for all varieties, but this is much harder (Dwork, Deligne, Grothendeck)
$T_l \rightsquigarrow$ étale cohomology

Corollary 3.2.7
$\psi : E \to E'$ isogeny over K, then $\#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$

Proof
ψ induces $T_l E \to T_l E'$, isomorphism of $\mathrm{Gal}(\overline{K}/K)$-modules when $l \nmid \deg \psi$
$\Rightarrow \quad \mathrm{Frob}_q \in \mathrm{Gal}(\overline{K}/K)$ has same characteristic polynomial on both ☐

Remark. Converse also holds (Silverman Chapter V)
Generally for abelian varieties over $\mathbb{F}_q$

$$\mathrm{Hom}(A, A') \otimes \mathbb{Z}_l \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(\overline{K})/K}(T_l A, T_l A')$$

this is the "Tate's Theorem on endomorphisms"
(Faltings:) Also true over number field, but much harder

Can think of $T_l E$ as something that replaces a complex lattice

Proof of Hasse's Theorem 3.2.4
Let $\phi = \mathrm{Frob}_q : E \to E^{(q)} = E$ (Recall $E^{(q)}$ is the E with coefficient in $\mathbb{F}_q = \{a \in \overline{\mathbb{F}_q} | a^q = a\}$)
$\Rightarrow \quad \phi \in \mathrm{End}(E)$
Write $f_\phi(T) = 1 - aT + qT^2 = (1-\alpha T)(1-\overline{\alpha}T)$

$E(\mathbb{F}_q) =$ fixed points of $\phi : E \to E = \ker(1-\phi)$
$1-\phi$ is separable, because $(1-\phi)^* w = w - 0 \neq 0$

$$\begin{aligned} \Rightarrow \quad |\ker(1-\phi)| &= \deg(1-\phi) \\ &= (1-\phi)(1-\widehat{\phi}) \\ &= 1-a+q \\ &= 1-\alpha-\overline{\alpha}+q \end{aligned}$$

Similarly

$$\begin{aligned} |E(\mathbb{F}_{q^n})| &= \deg(1-\phi^n) \\ &= (1-\alpha^n)(1-\overline{\alpha}^n) \quad \alpha, \overline{\alpha} \text{ eigenvalues on } T_l E \\ &= 1-\alpha^n-\overline{\alpha}^n+q^n \end{aligned}$$

Put these in $Z(T)$ and we are done □

Example 3.2.8
$E: \ y^2 = x^3+1$ over $\mathbb{F}_5$
$\phi = \mathrm{Frob}_p$ satisfies $T^2 - aT + q = 0$
$a = 5+1-\#E(\mathbb{F}_5) = 0$
$\Rightarrow \quad f_\phi(T) = 1+5T^2$, $Z_{E/\mathbb{F}_q}(T) = \frac{1+5T^2}{(1-T)(1-5T)}$ $(\alpha, \overline{\alpha} = \sqrt{-5}, -\sqrt{-5})$

$$\begin{aligned} \#E(\mathbb{F}_{5^n}) &= 1-(\sqrt{-5})^n-(-\sqrt{-5})^n+5^n \\ &= 6 \quad \text{if } n=1 \\ &= 36 \quad \text{if } n=2 \\ &= 126 \quad \text{if } n=3 \end{aligned}$$

3.2.1 Reduction mod p

$K = \mathbb{Q}$, p prime, p-adic valuation:

$$\begin{aligned} v = v_p : \mathbb{Q}^\times &\to \mathbb{Z} \\ p^n \frac{a}{b} &\mapsto n \end{aligned}$$

with $(ab, p) = 1$
$\mathcal{O} = \{\frac{a}{b} \in \mathbb{Q} \,|\, p \nmid b\}$
$\mathcal{O}$ mod $p = k = \mathbb{F}_p$ residue field

Generally K field, valuation $v : K^\times \to \mathbb{Z}$
$\mathcal{O} = \{x \in K | v(x) \geq 0\}$ integer ring
$p \rightsquigarrow \pi$ uniformiser, $v(\pi) = 1$
$k = \mathcal{O}/\pi$ residue field

Definition 3.2.9
E/K elliptic curve. A Weierstrass equation

$$E: \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

is integral at p if all $a_i \in \mathcal{O}$ ($\exists$ rescale $a_i \mapsto p^i$ enough times)
Then $\Delta \in \mathcal{O}, v(\Delta) \geq 0$

A minimal model at p is an integral model with $v(\Delta)$ minimal among integer models
THe reduced curve:

$$\widetilde{E}/K: \ y^2 + \overline{a_1}xy + \overline{a_3}y = x^3 + \overline{a_2}x^2 + \overline{a_4}x + \overline{a_6}, \quad \overline{a_i} = a_i \mod p$$

for any minimal model

Easy: minimal model is unique up to $\begin{array}{lll} x & \mapsto & u^2x+r \\ y & \mapsto & u^3y+sx+t \end{array}$; $u,r,s,t \in \mathcal{O}, u \in \mathcal{O}^\times$; induces $\cong$ on reduced curves. When char $k = p \neq 2,3$, may take $y^2 = x^3+ax+b$, $\begin{array}{lll} x & \mapsto & u^2x \\ y & \mapsto & u^3y \end{array}$ as usual.

Example 3.2.10
$y^2 = x^3 - 3\cdot 5^5 x - 3\cdot 5^6$ integral, not minimal at $p=5$, $\Delta = -2^4\cdot 3^3 \cdot 5^{13}$
$x \mapsto 5^2 x$
$y \mapsto 5^3 y$
$a_i \mapsto 5^{-i} a_i$
$\Delta \mapsto 5^{-12}\cdot\Delta$
$\rightsquigarrow$ $y^2 = x^3 - 3x - 3$ integral, $\Delta = -2^4 \cdot 3^3 \cdot 5$ minimal at 5 ($v(\Delta)$ can only change by multiples of 12)
Reduced curve: $\widetilde{E}:\ y^2 = x^3+2x+2 = (x-1)^2(x+2)$ over $\mathbb{F}_5$
Singular ($\Delta \mod p = 0$)

Exercise:
For $p \neq 2,3$ and $j(E) \in \mathcal{O}$ integral model $y^2 = x^3+ax+b$, $a,b \in \mathcal{O}$, is minimal $\Leftrightarrow v(\Delta) < 12$
($p=2,3:$ $\Leftarrow$ still true, but $\Rightarrow$ false, classification is more complicated, need "Tate's algorithm")

Remark. If $K = \mathbb{Q}$ (or number field with class number 1) may choose $a_i \in \mathbb{Z}$ (or $a_i \in \mathcal{O}_K$ resp.) minimal at all primes, global minimal model

3.2.2 Reduction types

Take minimal model ($p \neq 2$), $y^2 = x^3+ax^2+bx+c =: f(x)$, $a,b,c \in \mathcal{O}$
Roots of $\overline{f} = f \mod p$

- Good reduction, $\Delta \not\equiv 0 \mod p$:
  Distinct roots, $\widetilde{E}$ elliptic curve (i.e. non-singular)
- Bad reduction, $\Delta \equiv 0 \mod p$:
  - Multiplicative reduction
    Double root $\widetilde{E}:\ y^2 = x^2(x+\eta)$, this has 2 cases:
    (1) split: $\sqrt{\eta} \in k^\times$
    (2) non-split: $\sqrt{\eta} \notin k^\times$
  - Additive reduction
    Triple root, equivalently, $16a^2 - 28b \not\equiv 0 \mod p$; $\widetilde{E}: y^2 = x^3$

Definition 3.2.11
$\widetilde{E}_{ns}(k) := \widetilde{E}(k) \setminus \{$ singular point if there is one$\}$

In all cases, this is an abelian group with identity $0 = (0:1:0)$
Group law $P+Q+R = 0 \Leftrightarrow P,Q,R$ on a line

Reduction type	$\widetilde{E}_{ns}(k)$ isomorphic to (via $(x,y)\mapsto y/x$)
Additive	$\mathbb{P}^1\setminus\{0\} = \mathbb{G}_a$ additive group
Split multiplicative	$\mathbb{P}^1\setminus\{\pm\sqrt{\eta}\} = \mathbb{G}_m$ multiplicative group
Non-split multiplicative	$k(\sqrt{\eta})^\times/k^\times$ abelian group of order p^n+1

$y^2 = x^3 + \eta x^2$ "looks like" $y^2 - \eta x^2 = 0$ (near (0,0)), so "looks like" $(y-\sqrt{\eta}x)(y+\sqrt{\eta}x) = 0$
$\pm\sqrt{\eta}$ slopes of the two tangent lines (asymtopes)

Proposition 3.2.12
K'/K finite extension, $v' : (K')^\times \twoheadrightarrow \mathbb{Z}$ s.t. $v'|_{K^\times} = ev$ ($e \geq 1$ ramification index)

(1) E good or multiplicative reduction over $K \Rightarrow$ minimal model stays minimal, reduction type stays the same (non-split may become split)

(2) E addictive over $K \Rightarrow \exists K'$ s.t. E/K' either good, $v(j_E) \geq 0$ or multiplicative, $v(j_E) < 0$
We say E/K has potentially good (resp. potentially multiplicative) reduction

Good and multiplicative reduction are called semistable reduction type
Additive also called unstable

Proof
($p \neq 2$)

(1) Clear from the equation

(2) Adjoin roots of $f(x)$ to K, put E in Legendre form (c.f. Example Sheet 2):

$$y^2 = x(x-1)(x-\lambda), \quad \lambda \in \mathcal{O}$$

integral model $j = \dfrac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}$
$\lambda \not\equiv 0, 1 \mod v' \Rightarrow \widetilde{E}$ elliptic; $v(j) \geq 0$
$\lambda \equiv 0, 1 \mod v' \Rightarrow \widetilde{E}$ has double root; $v(j) < 0$

□

3.2.3 Reduction on Points

$\mathbb{P}^n(K) \ni (x_0 : \cdots : x_n) = (\alpha x_0 : \cdots \alpha x_n)$ choose $\alpha \in K^\times$ s.t. $\alpha x_i \in \mathcal{O}$ for some $x_j \in \mathcal{O}^\times$
$\mapsto (\overline{\alpha x_0} : \cdots : \overline{\alpha x_n}) \in \mathbb{P}^n(k)$ (via mod p)

Clearly independent of the choice of α
For E/K elliptic curve, get

$$\begin{aligned} \text{mod } p : E(K) &\rightarrow \widetilde{E}(k) \\ (x,y) &\mapsto \begin{cases} (\overline{x}, \overline{y}) & \text{if } x, y \in \mathcal{O} \\ (0:1:0) = 0 & \text{if } x, y \notin \mathcal{O} \end{cases} \end{aligned}$$

Definition 3.2.13
$E_0(K) = \{P \in E(K) | P \text{ reduces to a point in } \widetilde{E}_{ns}\}$
subgroup of $E(K)$ as $P + Q + R = 0 \Rightarrow P, Q, R$ on a line
$\Rightarrow \overline{P}, \overline{Q}, \overline{R}$ on a line
$\Rightarrow \overline{P} + \overline{Q} + \overline{R} = 0$ in $E_0(K)$
and $E_0(K) \rightarrow \widetilde{E}_{ns}(k)$ is a group homomorphism

Definition 3.2.14
$E_1(K)$ = kernel of above homomorphism
$= \{P \in E(K) | P \text{ reduces to } (0:1:0)\}$
$= \{P = (x,y) \in E(K) | v_p(x) \geq 1, v_p(y) \geq 1\}$ subgroup, so get exact sequence:

$$0 \rightarrow E_1(K) \rightarrow E_0(K) \xrightarrow[\text{}]{\substack{\text{group hom} \\ \text{mod } p}} \widetilde{E}_{ns}(k)$$

Example 3.2.15
$E/\mathbb{Q}: \quad y^2 = x(x+2)(x-3) \quad \Delta = 2^6 3^2 5^2, p = 3$
$\downarrow$
$\widetilde{E}/\mathbb{F}_3: \quad y^2 = x^2(x-1) = x^3 + 2x^2 \quad$ singular ($\sqrt{2} \notin \mathbb{F}_3$ non-split multiplicative reduction)

$\widetilde{E}_{ns}(\mathbb{F}_3) \cong \mathbb{F}_9^\times / \mathbb{F}_3^\times \cong \mathbb{Z}/4\mathbb{Z} = \{\mathcal{O}, (2,1), (1,0), (2,2)\}$
(see picture)
$\mathcal{O} \xrightarrow{\text{mod } 3} \mathcal{O}$
$T_1 = (-2, 0) \mapsto (1, 0)$
$T_2 = (0, 0) \mapsto (0, 0)$ (singular)
$T_3 = (3, 0) \mapsto (0, 0)$ (singular)
$P = (-1, -2) \mapsto (2, 1)$
$2P = (\frac{49}{16}, -\frac{63}{64}) \mapsto (1, 0)$
$2P + T_1 = (-\frac{2}{82}, \frac{280}{729}) \mapsto \mathcal{O}$

$$\begin{array}{rcl} E(\mathbb{Q}) & = & \overbrace{\mathbb{Z}/2}^{T_1} \times \overbrace{\mathbb{Z}/2}^{T_2} \times \overbrace{\mathbb{Z}}^{P} \\ & & \bigcup | \text{ index } 2 \\ E_0(\mathbb{Q}) & = & \underbrace{\mathbb{Z}/2}_{T_1} \times \underbrace{\mathbb{Z}}_{P} \\ & & \bigcup | \text{ index } 4 \quad (\text{in this case, } E_0/E_1 \leftrightarrow \widetilde{E}_{ns}(\mathbb{F}_3)) \\ E_1(\mathbb{Q}) & = & \underbrace{\mathbb{Z}}_{2P+T_1} \end{array}$$

3.3 Elliptic Curves over Local Fields

3.3.1 Completeness and Hensel

$K, v : K^\times \twoheadrightarrow \mathbb{Z}$, $\mathcal{O}, k, \pi$ as above $\rightsquigarrow$ topology on K given by a norm

$$|x| = \left(\frac{1}{\#k}\right)^{v(x)} \qquad x \in K, |0| = 0$$

Properties:

$$\begin{array}{rcl} |xy| & = & |x| \cdot |y| \\ |x+y| & \leq & \max(|x|, |y|) \leq |x| + |y| \quad \text{strong triangle inequality} \\ |x| = 0 & \Leftrightarrow & x = 0 \end{array}$$

$|\cdot|$ is called a non-Archimedean absolute value

Definition 3.3.1
We say $x_n (\in K) \to x (\in K)$ if $|x_n - x| \to 0$
$\Leftrightarrow \quad v(x_n - x) \to \infty$
$\Leftrightarrow \quad x_n \equiv x \mod$ larger and larger powers of π as $n \to \infty$

Definition 3.3.2
The completion $\widehat{K}$ of K (wrt v or $|\cdot|$)
= the completion in topological sense
= {Cauchy sequences x_n, $x_n \in K, |x_n - x_m| \to 0$ as $n, m \to \infty$}/{ sequence $x_n \to 0$}
= field, contains K; $v : \widehat{K} \twoheadrightarrow \mathbb{Z}$ extending one on K with ring of integer $\widehat{\mathcal{O}}$, and same π, k

Definition 3.3.3
K complete $\Leftrightarrow K = \widehat{K} \Leftrightarrow$ every Cauchy sequence converges
(Alternatively: $\widehat{\mathcal{O}} := \varprojlim_{n \geq 1}(\mathcal{O}/\pi^n), \widehat{K} := ff(\widehat{\mathcal{O}})$)

Example 3.3.4
$K = \mathbb{Q}, v = v_p$
$\widehat{\mathcal{O}} = \mathbb{Z}_p = \left\{ \sum_{n=0}^{\infty} a_n p^n | a_n \in \{0, \dots, p-1\} \right\} \supseteq \mathbb{Z}$
$\widehat{K} = \mathbb{Q}_p = \left\{ \sum_{n=n_0}^{\infty} a_n p^n | n_0 \in \mathbb{Z}, a_n \in \{0, \dots, p-1\} \right\}$

Theorem 3.3.5 (Hensel's Lemma)
K complete wrt $v : K^\times \to \mathbb{Z}$, $f(x) \in \mathcal{O}[x], \overline{f} = f \mod \pi \in k[x]$
If $\widetilde{\alpha} \in k$ is s.t. $\overline{f}(\widetilde{\alpha}) = 0, \overline{f}'(\widetilde{\alpha}) \neq 0$
then $\exists! \alpha \in \mathcal{O}$ s.t. $\overline{\alpha} = \widetilde{\alpha}, f(\alpha) = 0$
("simple root lift from k to K")

Proof
Lift $\widetilde{\alpha} \in k$ to any $\alpha_1 \in \mathcal{O}$, let $\alpha_{n+1} = \alpha_n - \dfrac{f(\alpha_n)}{f'(\alpha_n)}$
Check α_n Cauchy so $\alpha_n \to \alpha$ and $f(\alpha) = 0$
(see Newton's method picture) □

3.3.2 Analysis of $E(K)$ for K complete, E/K elliptic curve

Case I: E vs. E_0

Theorem 3.3.6 (Kodaira-Néron)
Write $n = v(\Delta_{\min})$.
Then $E(K)/E_0(K)$ (Néron component group) is finite and

$$\frac{E(K)}{E_0(K)} \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & E \text{ has split multi. reduction} \\ \{1\} & E \text{ has non-split multi. reduction and } n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & E \text{ has non-split multi. reduction and } n \text{ even} \\ \text{Group of order} \leq 4 & E \text{ has additive reduction} \end{cases}$$

The first 3 cases are called reduction type I_n

Remark. Tate's algorithm $\Rightarrow$ more precise description.
Reduction types $II, III, IV, I_n^o, I_n^*, IV^*, III^*, II^*$

Proof
See exercises □

Case II: E_0 vs. E_1

Theorem 3.3.7
K complete, $0 \to E_1(K) \to E_0(K) \to \widetilde{E}_{ns}(k) \to 0$ is exact
i.e. $E_0(K) \twoheadrightarrow \widetilde{E}_{ns}(k)$

Proof
$E: \ g(x,y) = 0$ integral ($g(x,y) = y^2 + a_1xy + a_3y - x^3 - \cdots$)
Take $\widetilde{P} = (\widetilde{x}, \widetilde{y}) \in \widetilde{E}_{ns}(k) \setminus \{0\}$ non-singular
$\Rightarrow \quad \frac{\partial \widetilde{g}}{\partial x}\big|_{\widetilde{P}} \neq 0$ or $\frac{\partial \widetilde{g}}{\partial y}\big|_{\widetilde{P}} \neq 0$

If $\frac{\partial \widetilde{g}}{\partial y}\big|_{\widetilde{P}} \neq 0$, lift $\widetilde{x}$ to any $x \in \mathcal{O}$, solve $g(x,y) = 0$ (as equation of y) by Hensel.
If $\frac{\partial \widetilde{g}}{\partial x}\big|_{\widetilde{P}} \neq 0$, lift $\widetilde{y}$ to $y \in \mathcal{O}$, solve $g(x,y) = 0$ (as equation of x) by Hensel □

Case III: E_1

K complete, $\mathcal{O}$, $\mathfrak{m} = \pi\,\mathcal{O}, \mathcal{O}\,/\,\mathfrak{m} = k$

Proposition 3.3.8
The following map is a bijection

$$\begin{array}{rcll} E_1(K) & \leftrightarrow & \mathfrak{m} & \\ (x,y) & \mapsto & \dfrac{x}{y} & \quad \text{(uniformiser at } \mathcal{O}) \\ \mathcal{O} & \mapsto & 0 & \end{array}$$

Proof
(char $k \neq 2,3$)
$E: \ y^2 = x^3 + ax + b \ \ (\mathbb{A}^2_{Z=1}) \subseteq Y^2Z = X^3 + aXZ^2 + bZ^3 \ \ (\mathbb{P}^2)$
$z = w^3 + awz^2 + bz^3 \ \ (\mathbb{A}^2_{Y\neq 0}) \subseteq Y^2Z = X^3 + aXZ^2 + bZ^3 \ \ (\mathbb{P}^2)$

$$(x,y) \xrightarrow{\text{homogenise}} (x:y:1) = (\frac{x}{y} : 1 : \frac{1}{y}) \mapsto (\underbrace{\frac{x}{y}}_{w}, \underbrace{\frac{1}{y}}_{z})$$

(see pictures)

For each $w \in \mathfrak{m}$ (i.e. $w \equiv 0 \mod \pi$), equation $z = w^3 + awz^2 + bz^3$ has a unique solution, $z(w)$ by Hensel's Lemma ($\frac{\partial}{\partial z}\big|_{(0,0)} = 1 \neq 0$)
$\Rightarrow \quad E_1(K) \ni (w, z(w)) \leftrightarrow w \in \mathfrak{m}$ is a bijection □

Remark. Do Hensel's explicitly $\Rightarrow$ $z(w)$ some explicit power series

$$z(w) = w^3 + aw^7 + bw^9 + 2a^2a^{11} + 5abw^{13} + \cdots \quad \in \mathbb{Z}[a,b][[w]]$$

universal. On $Y = 1$ chart

$$\begin{array}{rcl} y(w) & = & \dfrac{1}{z(w)} = \dfrac{1}{w^3} - aw - bw^3 - a^2w^5 - 3abw^7 + \cdots \\ x(w) & = & \dfrac{w}{z(w)} = \dfrac{1}{w^2} - aw^2 - bw^4 - a^2w^6 + \cdots \end{array}$$

$$\Rightarrow \quad \begin{array}{rcc} E_1(K) & \leftarrow (1:1) \rightarrow & \mathfrak{m} \\ (x,y) & \longmapsto & \frac{x}{y} \\ (x(w), y(w)) & \longleftarrow & w \end{array}$$

3.4 Formal Group

Addition $E_1(K) \times E_1(K) \to E_1(K)$
becomes $\mathcal{F} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$

$$\begin{array}{lll} w_1 & \mapsto & (\overbrace{x(w_1)}^{x_1}, \overbrace{y(w_1)}^{y_1}) \\ w_2 & \mapsto & (\underbrace{x(w_2)}_{x_2}, \underbrace{y(w_2)}_{y_2}) \end{array} \longmapsto \begin{array}{rcl} \kappa & = & \frac{y_2-y_1}{x_2-x_1} \\ x_3 & = & \kappa^2 - x_1 - x_2 \\ y_3 & = & -\kappa(x_3 - x_1) + y_1 (\in K((w_1, w_2))) \end{array} \longmapsto w_3 = \frac{x_3}{y_3}$$

$$\begin{aligned} w_3 &= w_1 + w_2 + 2aw_1w_2(w_1^3 + w_1^2w_2 + w_1w_2^2 + w_2^3) \\ &\quad -3bw_1w_2(w_1^5 + 3w_1^4w_2 + 5w_1^3w_2^2 + 5w_1^2w_2^3 + 3w_1w_2^4 + w_2^5) \\ &\quad + \cdots \\ &=: \mathcal{F}(w_1, w_2) \in K[[w_1, w_2]] \end{aligned}$$

(in fact, $\mathcal{F}(w_1, w_2) \in \mathbb{Z}[a, b][[w_1, w_2]]$ universal for $y^2 = x^3 + ax + b$)

Remark. $\begin{array}{lll} x & \mapsto & x(w) \\ y & \mapsto & y(w) \end{array}$ is the embedding $K(E) \hookrightarrow$ completion of $K(E)$ wrt $v_0 : K(E)^\times \to \mathbb{Z} \cong K[[w]]$

This defines a "kind of addition on $\mathfrak{m}$"

$$w_1, w_2 \in \mathfrak{m} \rightsquigarrow \mathcal{F}(w_1, w_2) \in \mathfrak{m} \quad \text{(converges)}$$

Properties of μ (associative, commutative, etc.)
$\Rightarrow$ $\mathcal{F}$ is a formal group over $\mathcal{O}$

Definition 3.4.1
A (one parameter, commutative) formal group over a ring R is $\mathcal{F} \in R[[X, Y]]$ s.t.

(1) $\mathcal{F}(X, Y) = X + Y +$(terms of deg ≥ 2)
(2) (associative) $\mathcal{F}(X, \mathcal{F}(Y, Z)) = \mathcal{F}(\mathcal{F}(X, Y), Z)$
(3) (commutative) $\mathcal{F}(X, Y) = \mathcal{F}(Y, X)$
(4) (inverse) $\exists! i(T) \in R[[T]]$ s.t. $\mathcal{F}(T, i(T)) = 0 = \mathcal{F}(i(T), T)$
(5) (identity) $\mathcal{F}(X, 0) = X, \mathcal{F}(0, Y) = Y$

"Group law without elements"

Definition 3.4.2
A homomorphism of formal groups $\mathcal{F} \to \mathcal{G}$ is $f \in TR[[T]]$ s.t.

$$f(\mathcal{F}(X, Y)) = \mathcal{G}(f(X), f(Y))$$

$\mathcal{F}$ and $\mathcal{G}$ are isomorphic if $\exists$ hom. $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{F}$ s.t. $f(g(T)) = T$
(Exercise: $\Rightarrow$ $g(f(T)) = T$)

Remark. If $R = \mathcal{O}$ complete, $\mathfrak{m} \subseteq R$ maximal ideal, then

$$\begin{array}{rcl} \mathcal{F} : \mathfrak{m} \times \mathfrak{m} & \to & \mathfrak{m} \\ a, b & \mapsto & a \oplus_{\mathcal{F}} b = \mathcal{F}(a, b) \quad \text{(converges in } \mathfrak{m}) \end{array}$$

makes $(\mathfrak{m}, \oplus_{\mathcal{F}})$ into an abelian group, also denoted $\mathcal{F}(\mathfrak{m})$

Hom. $f : \mathcal{F} \to \mathcal{G}$ induces $\begin{array}{rcl} (\mathfrak{m}, \oplus_{\mathcal{F}}) & \to & (\mathfrak{m}, \oplus_{\mathcal{G}}) \\ a & \mapsto & \end{array}$

Example 3.4.3
Formal addition group: $\widehat{\mathbb{G}_a}(X,Y) = X + Y$
($\rightsquigarrow (\mathfrak{m}, +)$)

Example 3.4.4
Formal multiplicative group: $\widehat{\mathbb{G}_m}(X,Y) = X + Y + XY = (1+X)(1+Y) - 1$
($\rightsquigarrow (1+\mathfrak{m}, \times)$)

Example 3.4.5
Formal group law on $E : y^2 = x^3 + ax + b$, $a, b \in \mathcal{O}$

$$\widehat{E} := \mathcal{F}(X,Y) = X + Y - 2aXY(\cdots) + \cdots$$

$\rightsquigarrow (\mathfrak{m}, \oplus_{\mathcal{F}}) = E_1(K)$

<u>Exercise</u>: Find $i(T)$ in all 3 cases

Example 3.4.6
$\mathcal{F}$ any formal group, denote $\mathcal{F}(X,Y)$ by $X \oplus_{\mathcal{F}} Y$

$$\begin{aligned}
[0](T) &:= 0 \\
[1](T) &:= T \\
[-1](T) &:= i(T) \\
[m](T) &:= \underbrace{T \oplus_{\mathcal{F}} T \cdots \oplus_{\mathcal{F}} T}_{m \text{ times}} \quad \text{(similarly for } m < 0\text{)}
\end{aligned}$$

are homomorphisms $\mathcal{F} \to \mathcal{F}$

<u>E.g.</u>: On $\widehat{E}$

$$[2](T) = 2T - 2aT^5 - 54bT^7 - 140a^2T^9 + O(T'')$$

Example 3.4.7
R field of char. 0

$$\widehat{\mathbb{G}_a} \underset{\log(1+T)}{\overset{\exp(T)-1}{\rightleftarrows}} \widehat{\mathbb{G}_m} \quad \text{isomorphism (check)}$$

Example 3.4.8
$\phi = (\phi_x(x,y), \phi_y(x,y)) : E_1 \to E_2$ isogeny over K
induces $\widehat{E_1} \to \widehat{E_2}$ over K

$$\frac{\phi_y(x(T), y(T))}{\phi_x(x(T), y(T))} \in TK[[T]]$$

3.5 Structure of formal groups

3.5.1 Filtration

$R = \mathcal{O}$ complete, $\mathfrak{m}$ maximal ideal, $\mathcal{F}$ formal group over R, $k = R/\mathfrak{m}$

$$\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \quad \text{sets}$$

$$x, y \in \mathfrak{m}^n \;\Rightarrow\; x \oplus_{\mathcal{F}} y = \underbrace{x + y}_{\in \mathfrak{m}^n} + (\text{something} \in \mathfrak{m}^{n+1}) \in \mathfrak{m}^n$$

$$\mathcal{F}(\mathfrak{m}) \supseteq \mathcal{F}(\mathfrak{m}^2) \supseteq \cdots \quad \text{subgroups}$$

$$\begin{array}{rcl} \mathcal{F}(\mathfrak{m}^n)/\,\mathcal{F}(\mathfrak{m}^{n+1}) & \cong & (\mathfrak{m}^n\,/\,\mathfrak{m}^{n+1}, +) \;\cong\; (k,+) \\ x & \leftrightarrow & x \end{array}$$

So "$\mathcal{F}$ (like $\widehat{\mathbb{G}_a}$) is built up from pieces that look like k"

3.5.2 Invertible Homomorphism

(Work over any R)

Theorem 3.5.1
A homomorphism $f(T) = a_1T + a_2T^2 + \cdots : \mathcal{F} \to \mathcal{F}$ is an isomorphism $\Leftrightarrow a_1 \in R^\times$

Proof
$\underline{\Rightarrow}$:
$f(g(T)) = T, g(T) = b_1T + b_2T^2 + \cdots$
$f(g(T)) = a_1b_1T + \cdots = T$
$\Rightarrow \quad a_1b_1 = 1$
$\Rightarrow \quad a_1 \in R^\times$

$\underline{\Leftarrow}$:
Assume $a_1^{-1} \in R$, let $g_1(T) = a_1^{-1}T$
Want: Construct inductively unique $g_n(T) = g_{n-1}(T) + \lambda_nT^n$
s.t. $f(g_n(T)) \equiv T \mod T^{n+1}$
$\Rightarrow \quad g := \lim g_n \in TR[[T]]$ is unique g s.t. $f(g(T)) = T$

$$\begin{array}{rcl} f(g_n(T)) & = & f(g_{n-1}(T) + \lambda_nT^n) \\ & \equiv & f(g_{n-1}(T)) + a_1\lambda_nT^n \mod T^{n+1} \\ & \equiv & \underbrace{T + bT^n}_{\substack{\text{by induction,} \\ \text{for some } b\in R}} + a_1\lambda_nT^n \mod T^{n+1} \end{array}$$

Now let $b + a_1\lambda_n = 0$ i.e. $\lambda := \frac{-b}{a_1} \Rightarrow$ unique g_n with $f(g_n(T)) \equiv T \mod T^{n+1}$ as required □

Corollary 3.5.2
$R = \mathcal{O}$ complte, $E_1(K)$ has no elts of order m, i.e. no m-torsion, for char $k \nmid m$ (such m are in $\mathcal{O}^\times$)

In general, we have

Corollary 3.5.3
$[m] : \mathcal{F} \to \mathcal{F}$ isom $\Leftrightarrow m \in R^\times$

Proof
$[m](T) = mT + \cdots$ (by induction, true $\forall\, m \in \mathbb{Z}$) □

3.5.3 The Invariant Differential

R ring, $\mathcal{F}\,/R$ formal group

Definition 3.5.4
A <u>differential form</u> ω=expression

$$f(X)dX \quad , \quad f \in R[[X]]$$

for a power series g in X

$$\omega \circ g := f(g(X))g'(X)dX$$

Definition 3.5.5
ω is an <u>invariant differential</u> of $\mathcal{F}/R$

$$\omega \circ \mathcal{F}(X,Y) = \omega$$

as a function of X.
i.e. if

$$f(\mathcal{F}(X,Y)) \cdot \underbrace{\mathcal{F}_1'(X,Y)}_{\text{derivative 1st var.}} dX = f(X)dX$$

ω is <u>normalised</u> if $\omega = (1 + \cdots)dX$ (equivalently, $f(0) = 1$)

Example 3.5.6
$\omega = dX$ on $\widehat{\mathbb{G}_a}$
$\omega = (1+X)^{-1}dX$ on $\widehat{\mathbb{G}_m}$
$\omega_{\widehat{E}} = \frac{x'(w)dw}{y(w)}$; $E: \quad y^2 = x^3 + ax + b$

Proposition 3.5.7
Any $\mathcal{F}/R$ has a unique normalised invariant differential, namely

$$\omega_{\mathcal{F}} := \mathcal{F}_1'(0,Y)^{-1}dY$$

Every invariant differential on $\mathcal{F}$ is of form $a\omega_{\mathcal{F}}$ some $a \in R$

Proof

$$\begin{aligned} & \mathcal{F}(X,\mathcal{F}(Y,Z)) &=& \mathcal{F}(\mathcal{F}(X,Y),Z) \\ \overset{\partial/\partial X}{\Rightarrow} & \mathcal{F}_1'(X,\mathcal{F}(Y,Z)) &=& \mathcal{F}_1'(\mathcal{F}(X,Y),Z) \cdot \mathcal{F}_1'(X,Y) \\ \overset{\text{Put } X=0}{\Rightarrow} & \mathcal{F}_1'(0,\mathcal{F}(Y,Z)) &=& \mathcal{F}_1'(Y,Z) \cdot \mathcal{F}_1'(0,Y) \end{aligned}$$

$\Rightarrow \quad \omega_{\mathcal{F}}$ invariant
$\mathcal{F}_1'(0,Y) = 1 + \cdots \quad \Rightarrow \quad$ normalised

Conversely, $f(X)dX$ invariant
$\Rightarrow \quad$ (by defn) $f(\mathcal{F}(X,Y))\mathcal{F}_1'(X,Y) = f(X)$
$\Rightarrow \quad$ (put $X = 0$) $f(Y) \cdot \mathcal{F}_1'(0,Y) = f(0)$
$\Rightarrow \quad f(Y)dY = f(0) \cdot \omega_{\mathcal{F}}$ $\square$

Corollary 3.5.8
$f : \mathcal{F} \to \mathcal{G}$ homomorphism, $f(T) = a_f T + \cdots$, (i.e. $a_f = f'(0)$) then

$$\omega_{\mathcal{G}} \circ f = a_f \cdot \omega_{\mathcal{F}}$$

Proof

$$\begin{aligned} \omega_{\mathcal{G}} \circ f(\mathcal{F}(X,Y)) &= \omega_{\mathcal{G}}(\mathcal{G}(f(X), f(Y))) \quad (f \text{ hom.}) \\ &= \omega_{\mathcal{G}} \circ f \quad (\omega_{\mathcal{G}} \text{ invariant}) \end{aligned}$$

$\omega_{\mathcal{F}}$ unique $\quad \Rightarrow \quad \omega_{\mathcal{G}} \circ f = \text{constant} \times \omega_{\mathcal{F}}$
constant $= a_f$ $\square$

Corollary 3.5.9
$f, g : \mathcal{F} \to \mathcal{G}$ hom. Then

$$\omega_{\mathcal{G}} \circ (\underbrace{f \oplus g}_{\text{addition form}}) = \omega_{\mathcal{G}} \circ f + \omega_{\mathcal{G}} \circ g$$

(as both equal $(a_f + a_g)\omega_{\mathcal{F}}$) (This was left unproved in Theorem 2.2.25 for isogenies of elliptic curves)

<u>Exercise</u>: p prime, $\mathcal{F}/R$ formal group

$$[p](T) = pf(T) + g(T^p) \quad \text{for some } f, g \in TR[[T]]$$

3.5.4 $\log_{\mathcal{F}}$ and $\exp_{\mathcal{F}}$

•$R = K$ field of characteristic 0, $\mathcal{F}/R$, $\omega_{\mathcal{F}} = (1 + a_1 T + \cdots)dT$

Definition 3.5.10

$$\log_{\mathcal{F}}(T) = \text{“}\int \omega_{\mathcal{F}}\text{”} = T + \frac{a_1}{2}T^2 + \frac{a_2}{3}T^3 + \cdots \in R[[T]]$$

Proposition 3.5.11
$\log_{\mathcal{F}} : \mathcal{F} \to \widehat{\mathbb{G}}_a$ isomorphism of formal groups

Proof
Integrate $\omega_{\mathcal{F}}(\mathcal{F}(X, Y)) = \omega_{\mathcal{F}}(X)$ to X:

$$\log_{\mathcal{F}}(\mathcal{F}(X, Y)) = \log_{\mathcal{F}}(X) + C(Y)$$

where $C(Y) \in R[[Y]]$ const. of integration
$X = 0 \quad \Rightarrow \quad C(Y) = \log_{\mathcal{F}}(Y) \quad \Rightarrow \quad \log_{\mathcal{F}}$ hom. to $\widehat{\mathbb{G}}_a$
Starts with $1 \cdot T + \cdots \quad \Rightarrow \quad$ isom (its inverse called $\exp_{\mathcal{F}}$) □

•K complete wrt $v : K^\times \to \mathbb{Z}$, char K=0, $R = \mathcal{O}$, $\mathfrak{m}$
Now $\log_{\mathcal{F}}, \exp_{\mathcal{F}}$ not necessarily defined over $\mathcal{O}$ (denominators!)
Analyse denominators carefully $\Rightarrow$ still ok on $\mathfrak{m}^n$ for n large enough

Theorem 3.5.12

(1) $\log_{\mathcal{F}} : \mathcal{F}(\mathfrak{m}^r) \xrightarrow{\sim} \widehat{\mathbb{G}}_{\mathrm{a}}(\mathfrak{m}^r)$ for $r > \frac{v(p)}{p-1}$

(2) If $x \in \mathcal{F}(\mathfrak{m})$ has exact order p^n then $p^{n-1}v(x) \leq \frac{v(p)}{p-1}$

Proof
See Silverman, IV 6.4, 61 □

Example 3.5.13
$K = \mathbb{Q}_p$, $\mathcal{F}/\mathbb{Z}_p$
p odd $\quad \Rightarrow \quad \mathcal{F}(p\,\mathbb{Z}_p) \cong (\mathbb{Z}_p, +) \quad (1 > \frac{v(p)}{p-1} = \frac{1}{p-1})$
$p = 2 \quad \Rightarrow \quad \mathcal{F}(4\,\mathbb{Z}_2) \cong (\mathbb{Z}_2, +)$

Example 3.5.14
Set $\mathcal{F} = \widehat{\mathbb{G}}_{\mathrm{m}}$ in the above example, we get:

$$\begin{aligned} (1 + p\,\mathbb{Z}_p, \times) &\cong (\mathbb{Z}_p, +) \quad p \text{ odd} \\ (1 + 4\,\mathbb{Z}_2, \times) &\cong (\mathbb{Z}_2, +) \end{aligned}$$

3.5.5 Consequences for all elliptic curves

WARNING: If E has minimal model $y^2 + a_1xy + a_3y = \cdots$ (may be necessary if char $k = 2$ or 3) then formulae for $x(w), y(w), \mathcal{F}_E(X,Y)$,

$$\omega_{\mathcal{F}} = \frac{x'(w)dw}{2y(w) + a_1x(w) + a_3}$$

more complicated than for $y^2 = x^3 + ax + b$

E/K complete, $K/\mathbb{Q}_p$ finite extension, $v, \mathcal{O}, \mathfrak{m}, k$

Theorem 3.5.15
$E(K)$ contains a subgroup of finite index isomorphic to $(\mathcal{O}, +)$ (even topologically)

Proof

$$E(K) \overset{\text{finite quot.}}{\supseteq} E_0(K) \overset{\text{fin.}\hookrightarrow \widetilde{E}_{ns}(k)}{\supseteq} E_1(K) = \widehat{E}(\mathfrak{m}) \supseteq \widehat{E}(\mathfrak{m}^2) \supseteq \cdots \supseteq \widehat{E}(\mathfrak{m}^r) \cong (\mathcal{O}, +)$$

the containment on the RHS of the qual sign are all finite index, all quotient $\cong (\mathfrak{m}/\mathfrak{m}^2) \cong (k, +)$ $\square$

Corollary 3.5.16
$E(K)/mE(K)$ is finite for any $m > 1$

Proof
r large enough, as before

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{E}(\mathfrak{m}^r) & \longrightarrow & E(K) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [m]} & & \downarrow{\scriptstyle [m]} & & \downarrow{\scriptstyle [m]} & & \\
0 & \longrightarrow & \widehat{E}(\mathfrak{m}^r) & \longrightarrow & E(K) & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

(Note $\widehat{E}(\mathfrak{m}^r) \cong (\mathcal{O}, +)$)
Kernel-cokernel exact sequence:

$$\begin{array}{l}
0 \longrightarrow \mathcal{O}[m] \longrightarrow E(K)[m] \longrightarrow A[m] \\
\qquad \longrightarrow \mathcal{O}/m\,\mathcal{O} \longrightarrow E(K)/mE(K) \longrightarrow A/mA \longrightarrow 0
\end{array}$$

(Top rows are kernels, bottom row are cokernels)
$\mathcal{O}/m\,\mathcal{O}$ finite group of order $(\#k)^{v(m)}$, A/mA finite
$\Rightarrow \quad E(K)/mE(K)$ finite $\square$

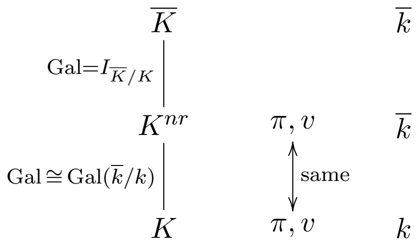
3.6 Néron-Ogg-Shafarevich Criteria

K complete, $p=$ char k, $[K : \mathbb{Q}_p] < \infty$

Definition 3.6.1

$$\begin{aligned}
K^{nr} &= \text{maximal unramified extension of } K \\
&= \bigcup_{(n,p)=1} K(\mu_n) \quad \text{complete, residue field } \overline{k} \\
I_{\overline{K}/K} &= \mathrm{Gal}(\overline{K}/K^{nr}) \quad \text{inertia group} \\
&= \ker\left(\begin{array}{ccc} \mathrm{Gal}(\overline{K}/K) & \to & \mathrm{Gal}(\overline{k}/k) \\ \sigma & \mapsto & \widetilde{\sigma} \end{array} \right)
\end{aligned}$$

(Also, I_K, I_v)



A Gal$(\overline{K}/K)$-module M is <u>unramified</u> if $M^{I_v} = M$
i.e. $\sigma(m) = m \ \ \forall m \in M, \sigma \in I_{\overline{K}/K}$
(i.e. Gal$(\overline{K}/K)$ acts on M through Gal$(\overline{k}/k)$ quotient)

Example 3.6.2
E/K elliptic curve, $M = E[m]$
$F = K(E[m]) = K($ coordinates of all m-torsion points$)$ (this is finite Galois over K)

Then $E[m]$ unramified $\Leftrightarrow$ I_v acts trivially on $E[m]$
$\Leftrightarrow$ I_v acts trivially on $F = K(E[m])$
$\Leftrightarrow$ $F \subseteq K^{nr}$
$\Leftrightarrow$ F/K unramified (in the sense $v_F|_{K^\times} = v_K$)

Example 3.6.3
$E/\mathbb{Q}_p:\ \ y^2 = x^3 - 77$, $M = E[2]$
$F = \mathbb{Q}_p(\text{roots of } x^3 - 77) = \mathbb{Q}_p(\zeta_3, \sqrt[3]{77})$ (this is unramified for $p \neq 3, 7, 11$)
(note: bad primes for $E/\mathbb{Q}$ are $2, 3, 7, 11$)

Theorem 3.6.4
E/K has good reduction, p=char $k \nmid m$. Then

1. mod $p : E(K)[m] \hookrightarrow \widetilde{E}(k)$ is injective
2. $E[m]$ is unramified

Proof

(1) Good reduction $\Rightarrow$ $E = E_0$, $\widetilde{E}_{ns} = \widetilde{E}$
and ker(mod p)=$E_1 = \widehat{E}$ has no torsion (recall, Corollary 3.5.2 $[m] : \widehat{E} \xrightarrow{\sim} \widehat{E}$ for $p \nmid m$)

(2) Let $F = K(E[m])$, $P \in E[m]$, $\sigma \in I_v$
$Q := \sigma(P) \Rightarrow \widetilde{Q} = \widetilde{\sigma}(\widetilde{P})$
$\widetilde{\sigma} = 1$ as $\sigma \in I_v$
$\Rightarrow$ (by (1)) $Q = P$, so $E[m]^{I_v} = E[m]$

$\square$

Remark. In particular, for number field K, $E(K)[m] \hookrightarrow \widetilde{E}(k)$, this help us to determine an upper bound for $E(\mathbb{Q})_{tors}$

Theorem 3.6.5 (Criterion of Néron-Ogg-Shafarevich)
E/K, $l \neq p$
E/K has good reduction $\Leftrightarrow$ $T_l E$ unramified

Remark. This relates two seemingly unrelated things: reduction is a geometric property, and $T_l E$ is purely representation theory

Proof

$\underline{\Rightarrow}$:

By Theorem 3.6.4 (2), all $E[l^n]^{I_v} = E[l^m]$, since $T_l E = \varprojlim E[l^n]$

$\Rightarrow \quad T_l E$ unramified as well

$\underline{\Leftarrow}$:

If $F := K(E[l^n])$ unramified extension over K

so E/K has good reduction $\Leftrightarrow$ E/F does (Exercise)

To find such n, choose n large enough s.t. $l^n > 4, l^n > v(\Delta_E)$

$\Rightarrow \quad l^n > [E(F) : E_0(F)]$ (Kodaira-Néron)

$\Rightarrow \quad E[l^n] \cap E_0[F]$ not cyclic ($E[l^n]$, all defined over F, $\cong \mathbb{Z}/l^n\mathbb{Z} \times \mathbb{Z}/l^n\mathbb{Z}$)

$\Rightarrow \quad \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z} \subseteq E_0(F)$

$\Rightarrow \quad$ (as $l \neq p$, $\widehat{E}$ has no l-torsion point) $\mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z} \subseteq \widetilde{E}_{ns}(k_F)$

But, if E/F has bad reduction,

$$\widetilde{E}_{ns}(k_F) \cong \underbrace{k_F^\times \quad , \quad k_F(\sqrt{\eta})^\times / k_F^\times}_{\text{cyclic}} \quad , \quad \underbrace{k_F^+}_{\text{order = power of } p}$$

□

Corollary 3.6.6

E/K has potentially good reduction (recall, this is equivalent to $v(j) \geq 0$)

$\Leftrightarrow$ E/F has good reduction over some finite F/K

$\Leftrightarrow$ $T_l E^{I_{\overline{F}/F}} = T_l E$ some finite $\overline{F}/K$

$\Leftrightarrow$ $I_{\overline{K}/K}$ acts on $T_l E$ through a finite quotient (i.e. image of $I_{\overline{K}/K} \to \mathrm{Aut}(T_l E)$ is finite)

<u>Exercise</u>:

E/K has potentially good reduction. Then,

(1) if $p \neq 2$, then $E/K(E[4])$ has good reduction

(2) if $p \neq 3$, then $E/K(E[3])$ has good reduction

(3) $I_{\overline{K}/K}$ acts on $T_l E$ through a group of order divides 24 (and 24 may occur when $p = 2$)

3.7 Elliptic curves over number fields

K number field, E/K elliptic curve

Main result:

Theorem 3.7.1 (Mordell-Weil)

E/K elliptic curve over number field

Then $E(K)$ is a finitely generated abelian group

(Asked by Poincaré (1908), proved by Mordell over $\mathbb{Q}$ (1922), then proved by Weil for Jacobians over number fields (1929), Lang-Néron proved for abelian varieties over finite generated fields)

Thus,

$$E(K) \cong \mathbb{Z}^r \oplus T$$

where T is (finite) <u>torsion</u> subgroup

r is the <u>Mordell-Weil rank</u> (or <u>arithmetic rank for E/K</u>)

Proof in 4 steps:

- Torsion is finite

- Existence of a height function on $E(K)$
  (e.g. $\Rightarrow$ $E(K) \not\cong \mathbb{Q}, \mathbb{R}, \ldots$)
- Weak Mordell-Weil Theorem: $E(K)/mE(K)$ is finite
  (e.g. $\Rightarrow$ $E(K) \not\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ sum for infinitely many times)
- The above 3 $\Rightarrow$ $E(K)$ finitely generated

3.7.1 Torsion

<u>Notation</u>:

$$E(K)_{tors} = \bigcup_{m \geq 1} E(K)[m]$$

all points of finite order, subgroup (this is the T in Mordell-Weil)

Theorem 3.7.2
$E(K)_{tors}$ is finite

Proof
$\mathfrak{p} \subseteq \mathcal{O}_K$ any prime, $K \subseteq K_\mathfrak{p}$ completion
For n large, $\widehat{E}(\mathfrak{m}_{K_\mathfrak{p}}^n) \cong (\mathcal{O}_\mathfrak{p}, +)$ torsion-free
$\Rightarrow \quad E(K_\mathfrak{p})_{tors} \hookrightarrow E(K_\mathfrak{p})/(\mathcal{O}_\mathfrak{p}, +)$
But $E(K)_{tors} \hookrightarrow E(K_\mathfrak{p})_{tors}$ and $E(K_\mathfrak{p})/(\mathcal{O}_\mathfrak{p}, +)$ finite (Theorem 3.5.15) □

Theorem 3.7.3 (Cassels)
$E/\mathbb{Q}$ elliptic curve in Weierstrass form with $a_i \in \mathbb{Z}$
If $P = (x, y) \in E(\mathbb{Q})_{tors}$
$\Rightarrow$ either $x, y \in \mathbb{Z}$ or $x \in \frac{1}{4}\mathbb{Z}, y \in \frac{1}{8}\mathbb{Z}$

Proof
May assume E in global minimal model (proves stronger statement)
If p |denominator of x or y
then $P \in E_1(\mathbb{Q}) = \widehat{E}(p\mathbb{Z}_p)$
But $\widehat{E}(p\mathbb{Z}_p) \cong (p\mathbb{Z}_p, +)$ has no torsion for p odd, and
$\widehat{E}(4\mathbb{Z}_2) \cong (4\mathbb{Z}_2, +)$ for $p = 2$ $\quad (\Rightarrow \quad P \in \widehat{E}(2\mathbb{Z}_2) \setminus \widehat{E}(4\mathbb{Z}_2))$

□

Example 3.7.4
Equation $y^2 = (x-5)x(x+5)$ has infinitely many solutions
Proof: $(-\frac{5}{9}, \frac{100}{27}) \in E(\mathbb{Q})$ must have infinite order □

Torsion is generally well-understood:

Theorem 3.7.5 (Nagell-Lutz)
$E/\mathbb{Q}: \quad y^2 = x^3 + ax + b \quad a, b \in \mathbb{Z}$
If $\mathcal{O} \neq P(x, y) \in E(\mathbb{Q})_{tors}$, then

(1) $x, y \in \mathbb{Z}$

(2) either $2P = \mathcal{O}$ or $y | 4a^3 + 27b^2$

Proof
See Silverman VIII 7.2 □

Theorem 3.7.6 (Mazur)

$$E/\mathbb{Q} \text{ has } E(\mathbb{Q})_{tors} \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & n \in \{1, \ldots, 10, 12\} \\ \text{or } \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} & n \leq 4 \end{cases}$$

Proof
Very hard (easy when $j(E) \in \mathbb{Z}$, in example sheet) $\square$

Over number fields $[K:\mathbb{Q}] = d$, $|E(K)_{tors}| \leq C(d)$ (Merel)
$\mathbb{Z}/l\,\mathbb{Z} \subseteq E(K)_{tors} \Rightarrow l \leq (3^{d/2}+1)^2$ (Uesterl)

3.7.2 Heights over $\mathbb{Q}$

Definition 3.7.7
For $\alpha = \frac{p}{q} \in \mathbb{Q}$, define $H_{\mathbb{Q}}(\alpha) = H(\alpha) := \max(|p|, |q|)$, called the height of α
$h_{\mathbb{Q}}(\alpha) = h(\alpha) := \log H(\alpha)$ is logarithmic height

Example 3.7.8
$H(\frac{2}{3})$ small. $H(\frac{20001}{30001})$ large
So the height is not measuring the size of number, but its arithmetic complexity

Properties:

- $h(\alpha) \geq 0$. Equality $\Leftrightarrow \alpha = \pm 1$ or 0
- $\{\alpha | h(\alpha) < c\}$ is finite
- $h(\alpha^d) = dh(\alpha)$, $H(\alpha^d) = H(\alpha)^d$
- Generally, if $f(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} \in \mathbb{Q}(x)$ is of degree d
  (degree of $\mathbb{Q}(x)$ is $\max(m, n)$)
  then $h(f(\alpha)) = dh(\alpha) + O(1)$, i.e.

$$dh(\alpha) - c \leq h(f(\alpha)) \leq dh(\alpha) + c \quad \text{for some } c \text{ independent of } \alpha$$

  Proof
  $\alpha = \frac{p}{q}$. Say $n \geq m$ (otherwise $f \leftrightarrow \frac{1}{f}$), so

$$f(\frac{p}{q}) = \frac{a_n p^n + \cdots a_0 q^n}{(b_m p^m + \cdots + b_0 q^m)q^{n-m}} =: \frac{A(p,q)}{B(p,q)}$$

  has $H(\alpha) \leq (n+1)\max_{i,j}(|a_i|, |b_j|)\max(|p|, |q|)^n \leq cH(\alpha)^{\deg f}$, hence the required upper bound

  For the lower bound, A, B coprime $\Rightarrow$ use Euclidean algorithm:

$$\begin{aligned} A(p,q)r(p,q) + B(p,q)s(p,q) &= p^N d_1 \\ A(p,q)r'(p,q) + B(p,q)s'(p,q) &= q^N d_2 \end{aligned}$$

  with $A, B, r, r', s, s' \in \mathbb{Z}[p,q]$ homogeneous, $d_1, d_2, \in \mathbb{Z}$
  $\Rightarrow$ Cancellation in $A(p,q)/B(p,q)$ is bounded by d_1, d_2
  Triangle inequality $\Rightarrow$ lower bound for $\max(|A|, |B|)$ $\square$

3.7.3 Heights over number fields

If K is a number field. Σ_K set of places (i.e. normalized absolute values) on K

- $|\cdot|_{\mathfrak{p}} := \left|\frac{1}{\#k_{\mathfrak{p}}}\right|^{v_{\mathfrak{p}}(\cdot)}$ for each prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ (finite places)
- $|\cdot|_\sigma := |\sigma(\alpha)|$ (usual real absolute value) for each $\sigma : K \hookrightarrow \mathbb{R}$ (real places)
- $|\cdot|_\sigma = |\sigma(\alpha)|^2$ for each pair $\sigma \neq \overline{\sigma}$ $K \hookrightarrow \mathbb{C}$ (complex places)

Definition 3.7.9
For $\alpha \in K$

$$\begin{aligned} H_K(\alpha) &:= \prod_{v \in \Sigma_K} \max(1, |\alpha|_v) \quad \in \mathbb{R}_{\geq 1} \\ h_K(\alpha) &:= \log H_K(\alpha) \quad \in \mathbb{R}_{\geq 0} \end{aligned}$$

Example 3.7.10
$K = \mathbb{Q}$

$$\begin{aligned} H_K(\frac{2}{3}) &= \max\left\{\left|\frac{2}{3}\right|, 1\right\} \cdot \max\left\{\left|\frac{2}{3}\right|_2, 1\right\} \cdot \max\left\{\left|\frac{2}{3}\right|_3, 1\right\} \cdot 1 \\ &= \max\left\{\frac{|\text{numerator}|}{|\text{denominator}|}, 1\right\} \cdot |\text{denominator}| = \max\{|\text{numer.}|, |\text{denom.}|\} \\ &= \text{same } H \text{ as before} \end{aligned}$$

Example 3.7.11
$K = \mathbb{Q}(\sqrt{5}), \alpha = \frac{1+\sqrt{5}}{2}$

$$\begin{aligned} H_K(\alpha) &= \max\left\{\left|\frac{1+\sqrt{5}}{2}\right|, 1\right\} \cdot \max\left\{\frac{1-\sqrt{5}}{2}, 1\right\} \cdot 1 \\ &= \frac{1+\sqrt{5}}{2} = 1.61\ldots \end{aligned}$$

Remark. For $P = [\alpha : \beta] \in \mathbb{P}^1(K) = K \cup \{\infty\}$, can let

$$H_K(P) = \prod_{v \in \Sigma_K} \max(|\alpha|_v, |\beta|_v)$$

(Analogous for $\mathbb{P}^n(K)$)
Well-defined: $H_K([\alpha : \beta]) = H_K([c\alpha : c\beta])$ as $\prod_v |c|_v = 1$
as this is product formula in number fields:

$$\prod_{v \in \Sigma_K} |c|_v = \prod_{v \in \Sigma_{\mathbb{Q}}} |N_{K/\mathbb{Q}}(c)|_v = 1$$

Remark. H_K, h_K depend on the choice of K, e.g., In $\mathbb{Q}$: $\quad H(\frac{1}{5}) = 5$
In $\mathbb{Q}(i)$: $\quad H_K(\frac{1}{5}) = H_K(\frac{1}{(2+i)(2-i)}) = 5^2 = H_{\mathbb{Q}}(\frac{1}{5})^{[\mathbb{Q}(i):\mathbb{Q}]}$
In $\mathbb{Q}(\sqrt{5})$: $H_K(\frac{1}{5}) = H_K(\frac{1}{(\sqrt{5})^2}) = 5^2 = H_{\mathbb{Q}}(\frac{1}{5})^{[\mathbb{Q}(\sqrt{5}):\mathbb{Q}]}$
Generally, $\alpha \in K \subseteq F$
$H_F(\alpha) = H_K(N_{F/K}(\alpha)) = H_K(\alpha^{[F:K]}) = H_K(\alpha)^{[F:K]}$ so,

Definition 3.7.12
The absolute height

$$H(\alpha) := H_K(\alpha)^{1/[K:\mathbb{Q}]} \quad , \quad h(\alpha) := \frac{1}{[K:\mathbb{Q}]} h_K(\alpha)$$

is independent of $K \ni \alpha$ (i.e. is defined on $\overline{\mathbb{Q}}$)

Properties:

(1) $\{\alpha \in K | h(\alpha) < c\}$ is finite
(2) $h(f(\alpha)) = \deg f \cdot h(\alpha) + O(1)$ for $f \in K(X)$
(3) $h(\alpha) \geq 0$, equality $\Leftrightarrow$ α root of unity or 0

Proof

$\Leftarrow$: All $|\alpha|_v$ are 1 if $\alpha = 0$ or root of unity

$\Rightarrow$: *Proof I*: $\alpha \in \mathcal{O}_K$, $|\sigma(\alpha)| \leq 1\ \forall \sigma : K \hookrightarrow \mathbb{C} \Rightarrow$ a root of unity or 0

Proof II: $h(\alpha) = 0 \Rightarrow \{\alpha^n | n \in \mathbb{Z}\}$ have bounded height
$\Rightarrow$ finite set $\Rightarrow$ two powers are equal $\Rightarrow$ $\alpha = 0$ or root of unity $\square$

3.7.4 Heights of points on elliptic curves

Definition 3.7.13
E/K, $P = (a, b) \in E(\overline{K})$, $h(P) := h(a)$
height relative to $x : E \to \mathbb{P}^1$

Properties:

Lemma 3.7.14

(1) $h(mP) = m^2 h(P) + O(1)$ (the error depends on E/K and m but not P)
"x-coordinate of mP has $\approx m^2$ digits"

(2) $\{P \in E(K) | h(P) < c\}$ finite

(3) Parallelogram law: $h(P+Q) + h(P-Q) = 2h(P) + 2h(Q) + O(1)$ (error depends on E/K not on P, Q)

Proof

(1)

$$\begin{array}{ccc} E & \xrightarrow{[m]} & E \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

$[m] = (\phi(x), \psi(x, y))$ and $\deg \phi = m^2$
$\Rightarrow \quad h(mP) = h(\phi(x(P))) = \deg \phi \cdot h(x(P)) + O(1) = m^2 h(P) + O(1)$

(2) Finite many x-coordinate; ≤ 2 choices for y-coordinates for each

(3) Computation with addition law (see Silvermann III 6.2)

□

3.7.5 Canonical Height

Theorem 3.7.15 (Néron-Tate)
There is a unique function $\widehat{h} : E(\overline{K}) \to \mathbb{R}$ s.t.

(1) $\widehat{h}(P) = h(P) + O(1)$

(2) $\widehat{h}(mP) = m^2 \widehat{h}(P) \ \ \forall P \in E(\overline{K})$

Proof
Uniqueness:
Let $\widehat{h}, \widehat{h}'$ be two such $\Rightarrow |\widehat{h}(P) - \widehat{h}'(P)| \leq 2C \ \ \forall P$
$\Rightarrow \quad |\widehat{h}(2^n P) - \widehat{h}'(2^n P)| \leq 2C \ \ \forall$
$\Rightarrow \quad 4^n |\widehat{h}(P) - \widehat{h}'(P)| \leq 2C \ \ \forall P$
$\Rightarrow \quad$ as $n \to \infty$, $\widehat{h} = \widehat{h}'$

Existence:

$$\widehat{h}(P) := \lim_{n \to \infty} \frac{1}{4^n} h(2^n P) \qquad \text{exists}$$

$a_n := \frac{1}{4^n} h(2^n P)$ check
$|a_n - a_m| \leq \sum_{i=m-n}^{n-1} 4^{-i} C$ as $m \geq n$ both $\to \infty$ get an Cauchy sequence $\Rightarrow$ converge
Finally, $P \mapsto \frac{1}{m^2} \widehat{h}(mP)$ equals $\widehat{h}$ by uniqueness argument □

Lemma 3.7.16 (Properties of Canonical Height)

(1) $\widehat{h} = h + O(1)$

(2) $\widehat{h}(mP) = m^2\widehat{h}(P)$

(3) $\{P \in E(K) | \widehat{h}(P) < C\}$ finite

(4) Parallelogram Law: $\widehat{h}(P+Q) + \widehat{h}(P-Q) = 2\widehat{h}(P) + 2\widehat{h}(Q)$

(5) $\widehat{h}(P) \geq 0$, and $\widehat{h}(P) = 0 \Leftrightarrow P \in E(\overline{K})_{tors}$

Proof

(1) by definition

(2) by definition

(3) True for $h \Rightarrow$ by (1), true for $\widehat{h}$

(4) Replace P, Q by $2^nP, 2^nQ$, divide by 4^n, let $n \to \infty$

(5) $\underline{\geq 0}$: $\widehat{h} := \lim_{n\to\infty} \underbrace{\frac{1}{4^n} h(\cdots)}_{\geq 0}$

$\underline{\Leftarrow}$: $(1+m)P = P$
$\Rightarrow \quad (m+1)\widehat{h}(P) = \widehat{h}(P)$
$\Rightarrow \quad \widehat{h}(P) = 0$

$\underline{\Rightarrow}$: $\{P, 2P, 3P, \ldots\}$ all have height $0 \Rightarrow$ finite set

□

Theorem 3.7.17 (Néron-Tate Pairing)

$$\begin{array}{rcl} E(\overline{K}) \times E(\overline{K}) & \to & \mathbb{R} \\ (P,Q) & \mapsto & \langle P, Q\rangle = \widehat{h}(P+Q) - \widehat{h}(P) - \widehat{h}(Q) \end{array}$$

is bilinear, i.e. $\widehat{h}$ is a quadratic form

Proof

Formal consequences of the parallelgoram law and $\widehat{h}(P) = \widehat{h}(-P)$

Property (4) for $P+R, Q$
$-$ Property (4) for $P-R, Q$
$+$ Property (4) for $P+Q, R$
$-2\times$ Property (4) for $R+Q, R-Q$
$\Rightarrow \quad \langle P+R, Q\rangle = \langle P, Q\rangle + \langle R, Q\rangle$ □

Remark. $\langle\ ,\ \rangle$ can be used to get a lower bound on the Mordell-Weil rank

Example 3.7.18

$E/\mathbb{Q}$, say $P_1 = (2,3), P_2 = (\frac{1}{4}, \frac{1}{8}) \in E(\mathbb{Q})$
say the height pairing matrix is:

$$\begin{pmatrix} \langle P_1, P_1\rangle & \langle P_1, P_2\rangle \\ \langle P_2, P_1\rangle & \langle P_2, P_2\rangle \end{pmatrix} = \begin{pmatrix} 5.3 & 3.1 \\ 3.1 & 4.0 \end{pmatrix}$$

has determinant $\neq 0$
$\Rightarrow \quad P_1, P_2 \in E(\mathbb{Q})$ are linear independent over $\mathbb{Z}$
$\Rightarrow \quad \operatorname{rk}_{\mathbb{Z}} E(\mathbb{Q}) \geq 2$

Remark. Theorem + Property (3) $\Rightarrow$ (see Silverman III 9.5) $\widehat{h}$ positive definite quadratic form on $E(K)\otimes\mathbb{R}$ (a finite dimensional $\mathbb{R}$ vector space as tensor over $\mathbb{Z}$ with $\mathbb{R}$ kills the torsion) once we know $E(K)$ is finitely generated

Definition 3.7.19
The regulator of $E(K) = \mathbb{Z}P_1 \oplus \mathbb{Z}P_2 \oplus \cdots \oplus \mathbb{Z}P_r \oplus$(finite) is

$$\det\left(\langle P_i, P_j\rangle_{1\le i,j\le r}\right) = R \in \mathbb{R}_{>0}$$

Independent of choice of a basis

3.7.6 Descent

Theorem 3.7.20 (Descent Theorem)
K number field E/K elliptic curve
If $E(K)/mE(K)$ is finite for some $m \geq 2$
Then $E(K)$ is finitely generated

Proof
Let $P_1,\ldots,P_n \in E(K)$ be representatives for $E(K)/mE(K)$,

$$M = \max_i \widehat{h}(P_i)$$

Claim: $E(K)$ is generated by $S = \{R \in E(K) \text{ of height } \widehat{h}(R) \leq M\}$

Proof of Claim:
(note S is a finite set)
If not, let $P \in E(K)$ be a point of smallest height not in span(S)
Write $P = mQ + P_j$

$$\begin{aligned} \Rightarrow \quad m^2\widehat{h}(Q) &= \widehat{h}(mQ) = \widehat{h}(P - P_j) \\ &\leq 2\underbrace{\widehat{h}(P)}_{>M} + 2\underbrace{\widehat{h}(P_j)}_{\leq M} \\ &< 4\widehat{h}(P) \\ &\leq m^2\widehat{h}(P) \qquad (\text{as } m \geq 2) \end{aligned}$$

$\Rightarrow \quad \widehat{h}(Q) < \widehat{h}(P)$
$\Rightarrow \quad Q \in \mathrm{Span}(S)$
$\Rightarrow \quad P \in \mathrm{Span}(S) \qquad \# \qquad \blacksquare$

$\square$

Remark. All bounds and constants in $O(1)$'s can be made explicit. So *if* one knows how to find generator for $E(K)/mE(K)$ for some $m \geq 2$, get generators for $E(K)$ (but no such algorithm has been known)

3.8 Group Cohomology

Motivation:

$$0 \to E[m] \to E(\overline{K}) \overset{[m]}{\twoheadrightarrow} E(\overline{K}) \to 0$$

Note for the multiplication by m map $E(\overline{K}) \twoheadrightarrow E(\overline{K})$, every point has m^2 preimages, over algebraically closed field $\Rightarrow \quad E(\overline{K})/mE(\overline{K}) = 0$

Take $\mathrm{Gal}(\overline{K}/K)$-invariants $\Rightarrow$ exact sequence:

$$0 \to E(K)[m] \to E(K) \xrightarrow{[m]} E(K)$$

Failure to be exact on the right is measured by

$$\mathrm{coker}([m] : E(K) \to E(K)) = \frac{E(K)}{mE(K)}$$

In general, say G is a group

Definition 3.8.1
A (left) G-module is an abelian group M with an action of G given by a group homomorphism

$$\begin{array}{rcl} G & \to & \mathrm{Aut}(M) \\ g & \mapsto & (m \mapsto m^g) \end{array}$$

group hom. $\Leftrightarrow \left\{ \begin{array}{l} m^1 = m \ \ \forall m \\ m^{gh} = (m^h)^g \ \forall m \end{array} \right.$ e.g.: $\sigma, \tau \in \mathrm{Gal}(\overline{K}/K), P \in E(\overline{K}) \Rightarrow P^{\tau\sigma} = (P^\sigma)^\tau$

G-invariants

$$M^G := \{m \in M | m^g = m \ \ \forall g \in G\}$$

The functor

$$\begin{array}{rcl} G\text{-modules} & \to & G\text{-module} \\ M & \mapsto & M^G \end{array}$$

is left-exact but not right-exact, i.e. $0 \to A \to B \xrightarrow{\psi} C \to 0$ ses of G-modules $\Rightarrow$
$0 \to A^G \to B^G \xrightarrow{\psi} C^G$ exact (easy to check)

Why $B^G \not\twoheadrightarrow C^G$ in general?
Take $c \in C^G$, $B \twoheadrightarrow C \Rightarrow \exists b$ s.t. $\psi(b) = c$

$$\begin{array}{rcl} \xi : G & \to & B \\ g & \mapsto & b^g - b \end{array}$$

$(\xi = 0 \Leftrightarrow b^g = b \, \forall g)$
The map ξ lands in $A \subseteq B$, since:

$$\psi(b^g - b) = \psi(b)^g - \psi(b) = 0 \Rightarrow b^g - b \in \ker \psi = \mathrm{Im}(A \hookrightarrow B)$$

and satisfies

$$\xi(gh) = b^{gh} - b = (b^h)^g - b^g + b^g - b = \xi(h)^g + \xi(g)$$

ξ is called the crossed homomorphism $G \to A$ or 1-cocylce

Choosing another preimage $b' \in \psi^{-1}(c)$ (so $b' = b + a$ some $a \in A$) changes

$$\xi \to \xi' = \xi + \underbrace{(\text{map } g \mapsto a^g - a)}_{\text{1-coboundary}}$$

Definition 3.8.2
M a G-module

$$\begin{aligned} H^0(G,M) &:= M^G \quad \text{0th cohomology group} \\ H^1(G,M) &:= \frac{\text{1-cocycles}}{\text{1-coboundary}} \quad \text{1st cohomology group} \\ &= \frac{\{\xi : G \to M | \xi(gh) = \xi(g) + \xi(h)^g\}}{\{\text{maps of form } g \mapsto m^g - m \text{ some } m \in M\}} \end{aligned}$$

$\phi : M \to N$ map of G-modules indcues $H^1(G,M) \to H^1(G,N)$ *by composing $G \xrightarrow{\xi} M \xrightarrow{\phi} N$
If G acts trivially on M ($m^g = m \; \forall g, m$), then

$$H^1(G,M) = \frac{\{\xi | \xi(gh) = \xi(g) + \xi(h)\}}{\{0\}} = \mathrm{Hom}(G,M)$$

We constructed a map $\delta : C^G \to H^1(G,A)$ with

$$\ker \delta = \{c \in C^G | \xi : g \mapsto b^g - b = 0, b \in \psi^{-1}(c)\} = \psi(B^G)$$

Generally, we have:

Proposition 3.8.3
ses of G-modules $0 \to A \to B \to C \to 0$ induces a long exact sequence of abelian groups:

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C) \longrightarrow \cdots$$

(the sequence continues to $H^2(G,A)$ etc. note that $H^2(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times)$ Brauer group, important in class field theory and central simple algebras)

Proof
Define maps, checked. Exactness at C^G, checked. Exactness elsewhere, not hard ☐

3.8.1 Galois Cohomology

$G_K = \mathrm{Gal}(\overline{K}/K)$ with K perfect

Definition 3.8.4
A G_K-module M is <u>continuous</u> if $\forall m \in M$, $\mathrm{Stab}_{G_K} m < G_K$ is $\mathrm{Gal}(\overline{K}/L_m)$ for some L_m/K finite extension

(this actuallly means $G_K \times M \to M$ is continuous if M is given discrete topology and G_K profinite topology - $\mathrm{Gal}(\overline{K}/L)$'s fundamental system of open nbhds of id)

Example 3.8.5
$\overline{K}, \overline{K}^\times, E(\overline{K})$

Definition 3.8.6
For continuous G_K-modules, define

- $H^0(G_K, M) := M^G$ as before

- $H^1(G_K, M) := \dfrac{\left\{\begin{array}{c}\text{cts 1-cocycles } \xi : G \to M \text{ s.t.} \\ \forall m \in M \ \ \xi^{-1}(m) = \mathrm{Gal}(\overline{K}/L) \text{ some } L/K \text{ finite}\end{array}\right\}}{\{\text{1-coboundaries}\}}$
(1-coboundaries are continuous automorphism)
- same long exact sequence as before

Theorem 3.8.7
If $\mu_m \subseteq K$ then

$$\begin{array}{rcl} K^\times/(K^\times)^m & \cong & H^1(G_K, \mu_m) \ \ (= \mathrm{Hom}_{\mathrm{cont.}}(G_K, \mu_m)) \\ b & \overset{\delta}{\mapsto} & (\sigma \ \mapsto \ \dfrac{(\sqrt[m]{b})^\sigma}{\sqrt[m]{b}}) \end{array}$$

This is the Kummer map

Proof
$0 \to \mu_m \to \overline{K}^\times \xrightarrow{x \mapsto x^m} \overline{K}^\times \to 0$ induces
$0 \to \mu_m \to K^\times \to K^\times \overset{\delta}{\to} H^1(G_K, \mu_m) \to H^1(G_K, \overline{K}^\times)$, extract:

$$0 \to K^\times/K^{\times m} \overset{\delta}{\hookrightarrow} H^1(G_K, \mu_m) \to H^1(G_K, \overline{K}^\times)$$

for some δ as claimed by definition of connecting homomorphism

To prove δ surjective, either (1) prove $H^1(G_K, \overline{K}^\times) = 0$ “Hilbert ’90 Theorem” (this theorem proves for even when $\mu_m \nsubseteq K$)
or (2) Take $\xi \in H^1(G_K, \mu_m) = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Z}/m\mathbb{Z})$,
$\ker \xi = G_L, L/K$ finite Galois by continuity,

$$\xi : \mathrm{Gal}(L/K) \hookrightarrow \mathbb{Z}/m\mathbb{Z}$$

By Kummer theory, any such L is $K(\sqrt[m]{b})$, some $b \in K^\times$ (as $\mu_m \subseteq K$) □

3.9 Weak Mordell-Weil a lá Mordell

K number field, E/K, our goal is to show $E(K)/2E(K)$ finite (Weak Mordell-Weil). The plan for achieving the goal is as follows:

$E/K: \ \ y^2 = (x - t_1)(x - t_2)(x - t_3), \quad t_i \in K$
Q1: Why may assume $E[2] \subseteq E(K)$

Define the Kummer map

$$\begin{array}{rcll} \kappa : E(K) & \to & (K^\times/K^{\times 2}) \times (K^\times/K^{\times 2}) \times (K^\times/K^{\times 2}) & \\ P & \mapsto & (\kappa_1(P), \kappa_2(P), \kappa_3(P)) & \\ E[2] \not\ni (x, y) & \mapsto & (x - t_1, x - t_2, x - t_3) & \text{(NB: Product of 3 is } 1 \in K^\times/K^{\times 2}) \\ \mathcal{O} & \mapsto & (1, 1, 1) & \\ (t_1, 0) & \mapsto & ((t_1 - t_2)(t_1 - t_3), t_1 - t_2, t_1 - t_3) & \text{(similarly for } (t_2, 0), (t_3, 0)) \end{array}$$

This is a group homomorphism with kerel $2E(K)$, so,
Q2: Why?

$$E(K)/2E(K) \ \hookrightarrow \ (K^\times/K^{\times 2}) \times (K^\times/K^{\times 2})$$

(say $\kappa = (\kappa_1, \kappa_2)$)
The image is trivial at primes $p \nmid 2\Delta_E$, so
$p \nmid 2$ prime of good reduction $\Rightarrow v_p(\kappa_i(P)) \equiv 0 \mod 2$, and so
Q3: Why?

Then proves $E(K)/2E(K)$ finite
Q4: Why?

Example 3.9.1

$$\begin{array}{rcl} \mathbb{Q}^\times/\mathbb{Q}^{\times 2} & \overset{1:1}{\leftrightarrow} & \text{square-free integers} \\ 5 & \mapsto & 5 \\ 5\cdot\left(\frac{7}{8}\right)^2 & \mapsto & 5 \\ -\frac{2}{3} & \mapsto & -6 \end{array}$$

i.e. $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ is a $\mathbb{F}_2$-vector space with basis $-1, 2, 3, 5, 7, \ldots$

3.9.1 Example of 2-descent

$E/\mathbb{Q} \;\; y^2 = x(x+3)(x-6)$
Goal: Determine the structure of $E(\mathbb{Q})$

Step 1: Determine torsion subgroup

- $\Delta = 2^6 3^8$ minimal at all primes as $v_p(\Delta) < 12 \;\; \forall p$
- $\{T_1 = (0,0), T_2 = (-3,0), T_3 = (6,0), \mathcal{O}\} = E[2] \subseteq E(\mathbb{Q}) \Rightarrow \#E(\mathbb{Q})_{tors} \geq 4$

$$\left.\begin{array}{rcl} \#\widetilde{E}(\mathbb{F}_5) & = & 8 \\ \#\widetilde{E}(\mathbb{F}_7) & = & 12 \end{array}\right\} \Rightarrow \#E(\mathbb{Q})_{tors} \leq 4 \text{ (by Theorem 3.6.4)}$$

$\Rightarrow \;\; \#E(\mathbb{Q})_{tors} = 4$

Step 2: Exploit structure using Kummer map
A search for points of small height ($H \leq 2$, i.e. points with x-coordinates $\in \{0, \pm1, \pm2, \pm\frac{1}{2}\}$) yields $P = (-2, 4) \in E(\mathbb{Q})$

Kummer map:	x	$x+3$	$x-6$
$\mathcal{O}$	1	1	1
$T_1 = (0,0)$	-2	3	-6
$T_2 = (-3,0)$	-3	3	~~-9~~, -1
$T_3 = (6,0)$	6	1	6
$P = (-2,4)$	-2	1	-2
$P + T_1 = (9,18)$	1	3	3
$P + T_2 = (24,-108)$	6	3	2
$P + T_3 = (-\frac{3}{4}, -\frac{27}{8})$	-3	1	-3
$2P = (\frac{121}{16}, -\frac{715}{64})$	1	1	$1^{(*)}$

$(*)$: Kernel of the Kummer map is precisely $2E(\mathbb{Q})$, see later

v_p(all entries)=0 for $p \nmid 2\Delta_E = 2^7 3^8$
$\Rightarrow$ all entries $\in \{\pm1, \pm2, \pm3, \pm6\}$ 8 choices
$\Rightarrow$ $E(\mathbb{Q})/2E(\mathbb{Q})$ has order $\leq 8^2 = 64 = 2^6$, hence finite

So Descent Theorem 3.7.20 $\Rightarrow$ $E(\mathbb{Q})$ finitely generated abelian group

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \qquad \text{for some } r \geq 1 \ \ (r \neq 0 \text{ because of } P = (-2,4))$$

This has

$$\frac{E(\mathbb{Q})}{2E(\mathbb{Q})} \cong (\mathbb{Z}/2\mathbb{Z})^{r+2} \qquad \text{order } \leq 2^6$$

$\Rightarrow$ $r \leq 4$. We now bound r further by local analysis.

<u>Over $\mathbb{R}$</u>:
$x + 3 \geq 0 \ \forall (x,y) \in E(\mathbb{Q})$ (can be easily seen by draw a picture)
i.e. the second entry of Kummer map is always ≥ 0

In other words, consider

$$\begin{array}{ccc}
E(\mathbb{Q})/2E(\mathbb{Q}) & \xhookrightarrow{\kappa_{E/\mathbb{Q}}} & \mathbb{Q}^\times/\mathbb{Q}^{\times 2} \times \mathbb{Q}^\times/\mathbb{Q}^{\times 2} \\
\downarrow & & \downarrow \\
E(\mathbb{R})/2E(\mathbb{R}) & \xhookrightarrow{\kappa_{E/\mathbb{R}}} & \mathbb{R}^\times/\mathbb{R}^{\times 2} \times \mathbb{R}^\times/\mathbb{R}^{\times 2} = \{\pm 1\} \times \{\pm 1\}
\end{array}$$

$$E(\mathbb{R})/2E(\mathbb{R}) = \frac{\mathbb{Z}/2\mathbb{Z} \times S^1}{\{1\} \times S^1} = \mathbb{Z}/2\mathbb{Z}$$

$\kappa_{E/\mathbb{Q}}(\mathcal{O}) = (1,1) \qquad \kappa_{E/\mathbb{Q}}(0,0) = (-1,1)$
(This shows that twice of a point always lies on a component of graph)
$\Rightarrow$ $\mathrm{Im}(\kappa_{E/\mathbb{Q}}) \subseteq \{$ anything $\} \times \{$ positive $\}$ because $\mathrm{Im}(\kappa_{E/\mathbb{R}})$ is.

Over $\mathbb{Q}_2$:
Compute $E(\mathbb{Q}_2)/2E(\mathbb{Q}_2) \cong (\mathbb{Z}/2\mathbb{Z})^m$ some $m \geq 1$
$E: \ \ y^2 = x^3 - 3x^2 - 18x$
$\widetilde{E}/\mathbb{F}_2: \ \ (y+x)^2 = x^3$ (additive reduction at 2)
$\widetilde{E}(\mathbb{F}_2) = \{\mathcal{O}, (1,0), (0,0)\}$, (0,0) singular, others non-singular
$\Rightarrow$ $\widetilde{E}_{ns}(\mathbb{F}_2) = \{\mathcal{O}, (1,0)\} \cong (\mathbb{F}_2, +)$ $(\widehat{\mathbb{G}}_a(\mathbb{F}_2))$

The Néron component group: $E(\mathbb{Q}_2)/E_0(\mathbb{Q}_2) \cong \mathbb{Z}/2\mathbb{Z}$ generated by $(0,0) \in E(\mathbb{Q}_2)$ (as $\widetilde{(0,0)}$ singular)

(Tate's algorithm; or directly as in Exercise 52 prove:
if $\widetilde{Q} = (0,0) = \widetilde{Q'}$, then $\widetilde{Q+Q'} \in \widetilde{E}_{ns}(\mathbb{F}_2)$)

<u>3 steps:</u>
Step 1:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_0(\mathbb{Q}_2) & \longrightarrow & E(\mathbb{Q}_2) & \longrightarrow & E(\mathbb{Q}_2)/E_0(\mathbb{Q}_2) & \longrightarrow & 0 \\
 & & \downarrow [2] & & \downarrow [2] & & \downarrow [2] & & \\
0 & \longrightarrow & E_0(\mathbb{Q}_2) & \longrightarrow & E(\mathbb{Q}_2) & \longrightarrow & E(\mathbb{Q}_2)/E_0(\mathbb{Q}_2) & \longrightarrow & 0
\end{array}$$

Kernel-cokernel exact sequence:

$$0 \longrightarrow E_0(\mathbb{Q}_2)[2] \longrightarrow E(\mathbb{Q}_2)[2] \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \frac{E_0(\mathbb{Q}_2)}{2E_0(\mathbb{Q}_2)} \longrightarrow \frac{E(\mathbb{Q}_2)}{2E(\mathbb{Q}_2)} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \underset{\langle(-3,0)\rangle}{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \underset{\langle((0,0),(-3,0))\rangle}{\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}} \twoheadrightarrow \underset{\langle(0,0)\rangle}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\text{zero}} A \hookrightarrow B \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Exactness at $\langle(0,0)\rangle$ and A $\quad\Rightarrow$

$$0 \to \frac{E_0}{2E_0} \to \frac{E}{2E} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Step 2:

$$0 \to E_1(\mathbb{Q}_2) \to E_0(\mathbb{Q}_2) \to \widetilde{E}_{ns}(\mathbb{F}_2) \to 0$$

$\widetilde{E}_{ns}(\mathbb{F}_2) \cong \mathbb{Z}/2\mathbb{Z}$ gen. by $(1,0)$ (the image of $(-3,0)$ under reduction map)
Kernel-cokernel exact sequence for [2] again:

$$0 \longrightarrow E_1(\mathbb{Q}_2)[2] \longrightarrow E_0(\mathbb{Q}_2)[2] \longrightarrow \widetilde{E}_{ns}(\mathbb{F}_2)[2] \longrightarrow \frac{E_1(\mathbb{Q}_2)}{2E_1(\mathbb{Q}_2)} \longrightarrow \frac{E_0(\mathbb{Q}_2)}{2E_0(\mathbb{Q}_2)} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow \frac{\mathbb{Z}/2\mathbb{Z}}{\langle(-3,0)\rangle} \xrightarrow{\cong} \frac{\mathbb{Z}/2\mathbb{Z}}{\langle(\overline{1},\overline{0})\rangle} \xrightarrow{\text{zero}} C \hookrightarrow D \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and get

$$0 \to \frac{E_1}{2E_1} \to \frac{E_0}{2E_0} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Step 3:
$E_1(\mathbb{Q}_2) \cong \widehat{E}(2\mathbb{Z}_2)$ formal group,

$$\begin{array}{ccccccc} 0 \longrightarrow & \widehat{E}(4\mathbb{Z}_2) & \longrightarrow & \widehat{E}(2\mathbb{Z}_2) & \longrightarrow & \frac{\widehat{E}(2\mathbb{Z}_2)}{\widehat{E}(4\mathbb{Z}_2)} & \longrightarrow 0 \\ & \downarrow \cong & & & & \downarrow \cong & \\ & (\mathbb{Z}_2,+) & & & & \frac{2\mathbb{Z}_2}{4\mathbb{Z}_2} \cong \mathbb{Z}/2\mathbb{Z} & \end{array}$$

$\left(\frac{\widehat{E}(2\mathbb{Z}_2)}{\widehat{E}(4\mathbb{Z}_2)} \cong \frac{2\mathbb{Z}_2}{4\mathbb{Z}_2}\right.$ as, from section 3.5.1, $\mathcal{F}(\mathfrak{m}^n)/\mathcal{F}(\mathfrak{m}^{n+1}) \cong (\mathfrak{m}^n/\mathfrak{m}^{n+1},+) \cong (k,+))$
The last $\mathbb{Z}/2\mathbb{Z}$ is generated by $P+T_3$ (any point with v_2(x-coord)=-2)

Kernel-cokernel exact sequence for [2] $\Rightarrow$:

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}_2/2\mathbb{Z}_2 \longrightarrow \frac{\widehat{E}(2\mathbb{Z}_2)}{2\widehat{E}(2\mathbb{Z}_2)} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

So we get

$$E_1/2E_1 \cong \mathbb{Z}/2\mathbb{Z}$$

Combine all 3 steps $\Rightarrow$

$$\frac{E(\mathbb{Q}_2)}{2E(\mathbb{Q}_2)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

generated by $(T_1, T_2, P+T_3) = ((0,0),(-3,0),(-\frac{3}{4},-\frac{27}{8}))$

Because $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} \cong (\mathbb{Z}/2\mathbb{Z})^3$ with representatives $\{\pm1,\pm2,\pm3,\pm6\}$

$$\begin{array}{ccc} E(\mathbb{Q})/2E(\mathbb{Q}) & \hookrightarrow & \{\pm1,\pm2,\pm3,\pm6\} \times \{\pm1,\pm2,\pm3,\pm6\} \\ \downarrow & & \| \\ E(\mathbb{Q}_2)/2E(\mathbb{Q}_2) & \hookrightarrow & (\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}) \times (\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}) \end{array}$$

the image of $\kappa_{E/\mathbb{Q}}$ has size at most $|E(\mathbb{Q}_2)/2E(\mathbb{Q}_2)| = 8 \Rightarrow r \leq 1 \Rightarrow r = 1$

We proved:
$E/\mathbb{Q}:\ y^2 = x(x+3)(x-6)$ has $E(\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/2$

3.9.2 Proof of (Weak) Mordell-Weil Theorem

Theorem 3.9.2 (Weak Mordell-Weil Theorem)
K number field, E/K elliptic curve, $m \geq 2$, then,

$$E(K)/mE(K) \qquad \text{finite}$$

Corollary 3.9.3 (Mordell-Weil Theorem)
$E(K)$ is finitely generated

Proof
Weak Mordell-Weil + Descent Theorem □

Proof of Weak Mordell-Weil Theorem
Each of the step in this proof is to answer each question stated at the start of the section, in the plan for proving Weak Mordell-Weil

Step 1
$F := K(E[m])$. If we show $E(F)/mE(F)$ is finite, then $E(F)$ finitely generated
$\Rightarrow \quad E(K) \hookrightarrow E(F) \quad \Rightarrow \quad E(K)$ also f.g.

Thus, replacing K by F, may assume $E[m] \subseteq E(K)$ $\quad$ ($\Rightarrow \mu_m \subseteq K$ Exercise)

Step 2
Take $G_K = \mathrm{Gal}(\overline{K}/K)$-cohomology of

$$0 \to E[m] \to E(\overline{K}) \xrightarrow{[m]} E(\overline{K}) \to 0$$

Get

$$0 \to E[m] \to E(K) \xrightarrow{[m]} E(K) \xrightarrow{\delta} H^1(G_K, E[m]) \to H^1(G_K, E(\overline{K})) \xrightarrow{[m]} H^1(G_K, E(\overline{K})) \to \cdots$$

Extract

$$0 \to \frac{E(K)}{mE(K)} \xrightarrow{\delta} H^1(G_K, E[m]) \to H^1(G_K, E(\overline{K}))[m] \to 0$$

Kummer sequence for elliptic curve

$\delta(P) = (\sigma \mapsto Q^\sigma - Q)$ for any $Q \in E(\overline{K})$ s.t. $mQ = P$

Now,

$$\overbrace{H^1(\, G_K, \underbrace{E[m]}_{\mathbb{Z}/m \times \mathbb{Z}/m}\,)}^{\mathrm{Hom}_{cont}} = H^1(\, G_K, \underbrace{\mu_m}_{\mathbb{Z}/m}) \times H^1(\, G_K, \underbrace{\mu_m}_{\mathbb{Z}/m}) \cong K^\times/K^{\times m} \times K^\times/K^{\times m} \qquad (3.9.1)$$

The first equality is due to Weil pairing, explicitly:
Let $E[m] = \mathbb{Z}/m\mathbb{Z}T_1 \oplus \mathbb{Z}/m\mathbb{Z}T_2$, have two Weil pairings:

$$\begin{array}{rcl} E[m] & \to & \mu_m \\ \alpha_1 : T & \mapsto & e_m(T, T_1) \\ \alpha_2 : T & \mapsto & e_m(T, T_2) \end{array}$$

and

$$(\alpha_1, \alpha_2) : E[m] \xrightarrow{\sim} \mu_m \times \mu_m$$

becuase e_m bilinear, non-degenerate.

The isomorphism in the ses (3.9.1) is due to Kummer map in Theorem 3.8.7, and we now construct:

$$\kappa = (\kappa_1, \kappa_2) : E(K)/mE(K) \hookrightarrow K^\times/K^{\times m} \times K^\times/K^{\times m} \tag{3.9.2}$$

which is a group homomorphism, given by $\kappa_i = H^1(\alpha_i) \circ \delta$

(Exercise: For $m = 2$, use definition of e_m (relies on function f s.t. $\mathrm{div}(f) = 2(T) - 2(\mathcal{O})$ e.g. $f = x - x_T$) to show $\kappa = (x - x(T_1), x - x(T_2))$ for $m = 2$)

Step 3
Let $p \nmid m\Delta_E$ be a prime of good reduction, K_p completion, valuation v_p, residue field k (finite) K_p^{nr} maximal unramified extension, same valuation, residue field $\overline{k}$

Claim: $E(K_p^{nr})/mE(K_p^{nr}) = 0$

Proof of Claim:
(Note: $E = E_0, \widetilde{E} = \widetilde{E}_{ns}$, good reduction at P)

$$0 \to E_1(K_p^{nr}) \to E(K_p^{nr}) \to \widetilde{E}(\overline{k}) \to 0$$

Kernel-cokernel exact sequence for $[m]$ $\Rightarrow$

$$\cdots \to \underbrace{\frac{E_1(K_p^{nr})}{mE_1(K_p^{nr})}}_{=0} \to \frac{E(K_p^{nr})}{mE(K_p^{nr})} \to \underbrace{\frac{\widetilde{E}(\overline{k})}{m\widetilde{E}(\overline{k})}}_{=0} \to 0$$

exact sequence.
First cancelling due to: $p \nmid m \Rightarrow [m]$ isom of formal groups
Second cancelling due to: $\widetilde{E}$ elliptic curve over algebraically closed field $\Rightarrow [m]$ surjective
$\Rightarrow$ the middle group is zero; proves the claim ■

Now consider the following commute diagram

$$\begin{array}{ccc} E(K)/mE(K) & \overset{\kappa}{\hookrightarrow} & K^\times/K^{\times m} \times K^\times/K^{\times m} \\ \downarrow & & \downarrow \\ \cancel{E(K_p^{nr})/mE(K_p^{nr})}=0 & \hookrightarrow & (K_p^{nr})^\times/(K_p^{nr})^{\times m} \times (K_p^{nr})^\times/(K_p^{nr})^{\times m} \end{array}$$

$\Rightarrow$ $\mathrm{Im}(\kappa_i)$ are elements of $K^\times$ which are in $(K_p^{nr})^{\times m}$
In particular, they have v_p, which is same on K and K_p^{nr}, multiple of m

We proved $v_p(\kappa_i(P)) \equiv 0 \mod m \quad \forall p \nmid m\Delta_E$

Step 4
Let $\mathfrak{p}, \dots, \mathfrak{p}_n$ be prime divisors of $m\Delta_E$

Claim: $H_{\mathfrak{p}_1,\dots,\mathfrak{p}_n} = \{\alpha \in K^\times/K^{\times m} | v_\mathfrak{p}(\alpha) \equiv 0 \mod m \,\forall \mathfrak{p} \neq \mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is finite

Proof of Claim:
Enough to show

$$H_{\mathfrak{p}_1,\dots,\mathfrak{p}_{n-1}} = \ker(H_{\mathfrak{p}_1,\dots,\mathfrak{p}_n} \xrightarrow{v_{\mathfrak{p}_n}} \mathbb{Z}/m\mathbb{Z})$$

is finite. Inductively, need that

$$H_\emptyset = \{\alpha \in K^\times/K^{\times m} | v_\mathfrak{p} \equiv 0 \mod m \forall \mathfrak{p}\}$$

For $\alpha \in H_\emptyset$,

$$\underbrace{(\alpha)}_{\text{ideal}\subseteq\mathcal{O}_K} = \prod_{\mathfrak{p}} \mathfrak{p}^{mn_\mathfrak{p}} = \left(\prod_{\mathfrak{p}} \mathfrak{p}^{n_\mathfrak{p}}\right)^m =: I_\alpha^m$$

So enough to show

$$U = \ker\begin{pmatrix} H_\emptyset & \to & \text{class gp of } \mathcal{O}_K \\ \alpha & \mapsto & I_\alpha \end{pmatrix}$$

is finite. (note the class group is finite)

For $\alpha \in U, I_\alpha = (x_\alpha)$ principal (as it is in the principal ideal class)
$\Rightarrow \quad (\alpha) = (x_\alpha^m) \quad \Rightarrow \quad \frac{\alpha}{x_\alpha^m} \in \mathcal{O}_K^\times$

If $\frac{\alpha}{x_\alpha^m} = u^m \in (\mathcal{O}_K^\times)^m$
$\Rightarrow \quad \alpha = (ux_\alpha)^m \in K^{\times m}$ (i.e. trivial element in U). So

$$\begin{array}{rcl} U & \hookrightarrow & \mathcal{O}_K^\times / \mathcal{O}_K^{\times\, m} \\ \alpha & \mapsto & \dfrac{\alpha}{x_\alpha^m} \end{array}$$

Note $\mathcal{O}_K^\times / \mathcal{O}_K^{\times\, m}$ is finite, since $\mathcal{O}_K^\times$ is finite generated (by Dirichlet Unit Theorem, c.f. Algebraic Number Theory course) ■

Given claim $\quad\Rightarrow\quad E(K)/mE(K) \hookrightarrow H_{\mathfrak{p}_1,\ldots,\mathfrak{p}_n} \times H_{\mathfrak{p}_1,\ldots,\mathfrak{p}_n}$
$\Rightarrow$ DONE □

Remark. Same strategy works for many finitely generated field (e.g. $\mathbb{Q}(t_1, t_2), \mathbb{F}_q(t), \ldots$)

Remark. To actually find $E(K)/mE(K)$ is hard:
There may be classes in $H^1(G_K, E[m])$ that are in the image of $E(K_v)/mE(K_v)$ for all places v, but not the image of $E(K)/mE(K)$
The "Local-Global Principle" (Hasse Principle) may fail for elliptic curve.

Example 3.9.4
$y^2 = x(x+3)(x-6)$ over $\mathbb{Q}$, $m = 2$
Here the local-global principle works

If the local-global principle does work, can find $E(K)/mE(K)$ and therefore can find $E(K)$

Remark. In practice, $m = 2$ (may be $m = 3$, just)
A general $E/\mathbb{Q}$ would have, e.g. for $m = 3$, $\mathrm{Gal}(\mathbb{Q}(E[3])/\mathbb{Q}) \cong GL_2(\mathbb{F}_3)$
And $\mathbb{Q}(E[3])$ is too large to compute its class group, unit group in practice.

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