Derived Equivalences of Block Algebras
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For information of the course, see `http://www.maths.abdn.ac.uk/~spark/2010derived.html`

Broué's Abelian Defect Group Conjecture:
G finite group
k algebraically closed field of characteristic $p\big||G|$
b block of kG with defect group P
c the Brauer correspondent of b (a block of $kN_G(P)$ with defect group P)
If P is abelian, then $\mathbf{D}(kGb) \cong \mathbf{D}(kN_G(P)c)$ as triangulated categories.

This is verified for P cyclic, for $G = \Sigma_n$, and some other cases.

1 Homological Algebra

1.1 Adjoint functors

Definition 1.1.1
$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$
F is left adjoint to G (or G is right adjoint to F) if $\exists$ bijection $\theta_{x,y} : \mathrm{Hom}_{\mathcal{D}}(Fx, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, Gy)$ $\forall x \in \mathrm{Ob}(\mathcal{C}), y \in \mathrm{Ob}(\mathcal{D})$, which is natural in x and y

Example 1.1.2
A, B rings, M a B-A-bimodule, $M \otimes_A - : \mathbf{Mod}(A) \rightleftarrows \mathbf{Mod}(B) : \mathrm{Hom}_B(M, -)$

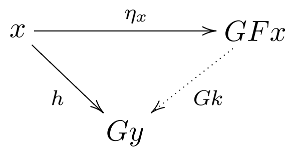
Proposition 1.1.3
$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ and F left adjoint to G. Note that:
$\forall x \in \mathrm{Ob}(\mathcal{C}) \quad \theta_{x,Fx}(1_{Fx}) = \eta_x : \ x \ \to GFx$ in $\mathcal{C}$
$\forall y \in \mathrm{Ob}(\mathcal{D}) \quad \theta_{Gy,y}(1_{Gy}) = \epsilon_y : FGy \to \ y$ in $\mathcal{D}$
Then, the assignment $x \mapsto \eta_x$ and $y \mapsto \epsilon_y$ induces natural transformations:
$\eta : 1_{\mathcal{C}} \to GF$ (unit of the adjuction)
$\epsilon : FG \to 1_{\mathcal{D}}$ (counit of the adjuction), and

$$\begin{array}{rcl} \theta_{x,y} : \mathrm{Hom}_{\mathcal{D}}(Fx, y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(x, Gy) \\ (h : Fx \to y) & \mapsto & (x \xrightarrow{\eta_x} GFx \xrightarrow{Gh} Gy) \\ (Fx \xrightarrow{Fk} FGy \xrightarrow{\epsilon_y} y) & \leftarrow\!\shortmid & (x \xrightarrow{k} Gy) \end{array}$$

Proposition 1.1.4
$F : C \rightleftarrows D : G$, TFAE:

(1) F is left adjoint to G

(2) There is a natural transformation, $\eta : 1_{\mathcal{C}} \to GF$ s.t. for each $x \in \mathrm{Ob}(\mathcal{C})$, $\eta_x : x \to GFx$ is universal in the sense that $\forall y \in \mathrm{Ob}(\mathcal{D})$ $\forall h \in \mathrm{Hom}_{\mathcal{C}}(x, Gy)$ $\exists!\, k \in \mathrm{Hom}_{\mathcal{D}}(Fx, y)$ s.t. the following diagram commutes



(3) Dual condition: $\exists \epsilon : FG \to 1_{\mathcal{D}}$ s.t. for each $y \in \mathrm{Ob}(\mathcal{D})$, ϵ_y is universal

(4) $\exists \eta : 1_{\mathcal{C}} \to GF, \epsilon : FG \to 1_{\mathcal{D}}$ (natural transformations) s.t.

$$\begin{aligned} F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F &= 1_F \\ G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G &= 1_G \end{aligned}$$

If these equivalent conditions are satisfied then the adjuction map is given as in Proposition 1.1.3.

1.2 Equivalences

Definition 1.2.1
$F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if $\exists G : \mathcal{D} \to \mathcal{C}$ s.t. $GF \cong 1_{\mathcal{C}}, FG \cong 1_{\mathcal{D}}$

Definition 1.2.2
$F : \mathcal{C} \to \mathcal{D}$

(1) F is faithful (resp. full, resp. fully faithful) if $F_{x,x'} : \mathrm{Hom}_{\mathcal{C}}(x, x') \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fx')$ is injective (resp. surjective, resp. bijective) $\forall x, x'$

(2) F is essentially surjective if $\forall y \in \mathrm{Ob}(\mathcal{D})$, $\exists x \in \mathrm{Ob}(\mathcal{C})$ s.t. $Fx \cong y$ in $\mathcal{D}$

Proposition 1.2.3
$F : \mathcal{C} \to \mathcal{D}$ TFAE:

(1) F is an equivalence

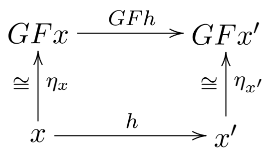
(2) F is fully faithful and essentially surjective

(3) F is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ with unit $\eta : 1_{\mathcal{C}} \xrightarrow{\sim} GF$ and counit $\epsilon : FG \xrightarrow{\sim} 1_{\mathcal{D}}$ which are natural isomorphisms

Proof

(1)⇒(2): Suppose $G : \mathcal{D} \to \mathcal{C}$ s.t. $\eta : 1_{\mathcal{C}} \cong GF, \epsilon : FG \cong 1_{\mathcal{D}}$
(F essentially surjective) $\forall y \in \mathrm{Ob}(\mathcal{D})$, $F(Gy) \cong y$ (via ϵ_y)

(F faithful) Suppose $h, k : x \to x'$ in $\mathcal{C}$



$Fh = Fk \quad \Rightarrow \quad GFh = GFk \quad \Rightarrow \quad$ (by above diagram) $h = k$

(G faithful) ✓

(F full) Suppose $k : Fx \to Fx'$ in $\mathcal{D}$. Define $h = \eta_{x'}^{-1} \circ Gk \circ \eta_x$

$$\begin{array}{ccc} GFx & \xrightarrow{Gk} & GFx' \\ \cong \uparrow \eta_x & & \cong \uparrow \eta_{x'} \\ x & \overset{h}{\dashrightarrow} & x' \end{array}$$

Compare this with the previous diagram
$\Rightarrow$ $GFh = Gk$
$\Rightarrow$ (as G faithful) $Fh = k$

(2)$\Rightarrow$(1): Want $G : \mathcal{D} \to \mathcal{C}$ s.t. $GF \cong 1_{\mathcal{C}}, FG \cong 1_{\mathcal{D}}$
For $y \in \mathrm{Ob}(\mathcal{D})$, F essentially surjective $\Rightarrow \exists Gy \in \mathrm{Ob}(\mathcal{C})$ $\exists \epsilon_y : FGy \xrightarrow{\sim} y$ in $\mathcal{D}$
For $k : y \to y'$ in $\mathcal{D}$. Since F fully faithful, $\exists!\, h (= Gk) : Gy \to Gy'$ in $\mathcal{C}$ s.t.

$$\begin{array}{ccc} FGy & \underset{\epsilon_y}{\overset{\cong}{\longrightarrow}} & y \\ Fh \downarrow & & \downarrow k \\ FGy' & \underset{\epsilon_{y'}}{\overset{\cong}{\longrightarrow}} & y' \end{array}$$

$\Rightarrow$ this define G as functor and ϵ as natural isomorphism $\epsilon : FG \to 1_{\mathcal{D}}$
$\Rightarrow$ $\epsilon F : FGF \xrightarrow{\sim} F$
$\Rightarrow$ (as F faithful) $\exists \eta : 1_{\mathcal{D}} \xrightarrow{\sim} GF$ s.t. $\epsilon F \circ F\eta = 1_F$

□

Corollary 1.2.4
Any equivalence of categories has an inverse which is both left and right adjoint.

Proposition 1.2.5
$F : C \rightleftarrows D : G$, F left adjoint to G with unit $\eta : 1_{\mathcal{C}} \to GF$, counit $\epsilon : FG \to 1_{\mathcal{D}}$
F is fully faithful $\iff$ η natural isomorphism
In this case, let $F(\mathcal{C})$ = essential image of $F = \{y \in \mathcal{D} \,|\, Fx \cong y \, \exists x \in \mathcal{C}\}$
then the induced functors $F : \mathcal{C} \rightleftarrows F(\mathcal{C}) : G$ are equivalences, inverse to each other.

Proof

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x, x') & \xrightarrow{F_{x,x'}} & \mathrm{Hom}_{\mathcal{D}}(Fx, Fx') \\ & {\scriptstyle (\eta_{x'})_*} \searrow \quad \swarrow {\scriptstyle \cong} & \\ & \mathrm{Hom}_{\mathcal{C}}(x, GFx') & \end{array}$$

(The isomorphism on the right is the adjunction map) □

1.3 Limits and Colimits

Definition 1.3.1
$F : \mathcal{I} \to \mathcal{C}$, $\mathcal{I}$ small (i.e. $\mathrm{Ob}(\mathcal{I})$ is a set). For $x \in \mathrm{Ob}(\mathcal{C})$, let

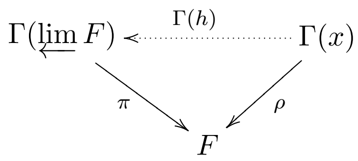
$$\Gamma(x) : \mathcal{I} \to \mathcal{C}$$

be the constant functor which sends every object to x and every morphism to 1_x

(1) The limit of F is an object $\varprojlim F$ of $\mathcal{C}$ together with a natural transformation

$$\pi : \Gamma(\varprojlim F) \to F$$

which is universal in the sense that:
if $x \in \mathrm{Ob}(\mathcal{C}), \rho : \Gamma(x) \to F$ natural transformation
then $\exists!\, h : x \to \varprojlim F$ in $\mathcal{C}$ s.t. it induces natural transformation $\Gamma(h)$ making below diagram commutes:



(2) Define colimit $\varinjlim F$ dually

If $\mathcal{I}$ is discrete (i.e. the only morphisms are the identity morphisms), then

$$\begin{aligned} \varprojlim_{\mathcal{I}} F &= \prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i) \qquad \text{product} \\ \varinjlim_{\mathcal{I}} F &= \coprod_{i \in \mathrm{Ob}(\mathcal{I})} F(i) \qquad \text{coproduct} \end{aligned}$$

Example 1.3.2
A a ring, every $F : \mathcal{I} \to \mathbf{Mod}(A)$ with $\mathcal{I}$ small has limit and colimit.

$\mathcal{I}$ discrete $\varprojlim_{\mathcal{I}} F$ is direct product of modules, and $\varinjlim_{\mathcal{I}} F$ is direct sum of modules. In general,

$$\begin{aligned} \varprojlim_{\mathcal{I}} F &= \left\{ (x_i) \in \prod_i F(i) \middle| f(x_i) = x_j \ \forall f : i \to j \text{ in } \mathcal{D} \right\} \\ \varinjlim_{\mathcal{I}} F &= \bigoplus_{i \in \mathrm{Ob}(\mathcal{D})} F(i) \Big/ \left\langle f(x_i) - x_i \,\middle|\, f : i \to j \text{ in } \mathcal{D}, x_i \in F(i) \right\rangle \end{aligned}$$

Definition 1.3.3
$F : \mathcal{C} \to \mathcal{D}$, F preserves limits if whenever $H : \mathcal{I} \to \mathcal{C}$ is the functor with $\mathcal{I}$ small which has limit

$$\Gamma(\varprojlim H) \xrightarrow{\pi} H$$

then

$$\Gamma(F(\varprojlim H)) \xrightarrow{F\pi} FH$$

is a "universal" natural transformation

Proposition 1.3.4
Right (resp. left) adjoints preserve limits (resp. colimits)

Remark. In particular, left adjoints preserves coproducts.

1.4 Additive categories and abelian categories

Definition 1.4.1
Category $\mathcal{C}$ is additive if

(1) $\mathcal{C}$ is preadditive: $\mathrm{Hom}_{\mathcal{C}}(x, x)$ is an abelian group, and the composition map:
$\mathrm{Hom}_{\mathcal{C}}(x', x'') \times \mathrm{Hom}_{\mathcal{C}}(x, x') \to \mathrm{Hom}_{\mathcal{C}}(x, x'')$ is bilinear $\forall x, x', x'' \in \mathrm{Ob}(\mathcal{C})$

(2) $\mathcal{C}$ has a zero object 0: i.e. $\forall x \in \mathrm{Ob}(\mathcal{C})$, $|\operatorname{Hom}_{\mathcal{C}}(0,x)| = |\operatorname{Hom}_{\mathcal{C}}(x,0)| = 1$

(3) $\mathcal{C}$ has finite coproducts: i.e. for every finite family $\{x_i\}$ of objects of $\mathcal{C}$, $\coprod_i x_i$ exists in $\mathcal{C}$

Remark. (a) $(x \to 0 \to y) = 0 \in \operatorname{Hom}_{\mathcal{C}}(x,y)$
(b) If (1),(2) above holds, then (3) $\iff$ $\mathcal{C}$ has finite product and they are the same as coproduct

Example 1.4.2
Let A be ring
Mod(A)=category of (left) A-modules
Proj(A)=category of projective A-modules
Inj(A)=category of injective A-modules
mod(A)=category of f.g. A-modules
proj(A) = category of f.g. proj. A-modules
inj(A) = category of f.g. inj. A-modules

All these are additive categories, the first three categories has arbitrary (infinite) direct sums
Mod(A) is an abelian category (i.e. additive; kernel, cokernel exists; and first isom theorem hold)
mod(A) is an abelian category *if* A is Noetherian

Definition 1.4.3
$\mathcal{C}, \mathcal{D}$ additive categories, $F : \mathcal{C} \to \mathcal{D}$ additive if

(1) $F_{x,x'} : \operatorname{Hom}_{\mathcal{C}}(x,x) \to \operatorname{Hom}_{\mathcal{D}}(Fx, Fx')$ is a group hom $\forall x, x'$

(2) $F(0) = 0$

(3) F preserves finite coproducts.

$\mathcal{C}, \mathcal{D}$ are equivalent as additive categories if $\exists$ additive equivalence(functor) $\mathcal{C} \to \mathcal{D}$

Definition 1.4.4
$\mathcal{C}, \mathcal{D}$ abelian categories, $F : \mathcal{C} \to \mathcal{D}$ additive functor.
F is exact if ($x' \xrightarrow{h} x \xrightarrow{k} x''$ exact in $\mathcal{C}$) $\Rightarrow$ ($Fx' \xrightarrow{Fh} Fx \xrightarrow{Fk} Fx''$ in $\mathcal{D}$)
F is left exact if ($0 \to x' \xrightarrow{h} x \xrightarrow{k} x''$ exact in $\mathcal{C}$) $\Rightarrow$ $(0 \to Fx' \xrightarrow{Fh} Fx \xrightarrow{Fk} Fx''$ in $\mathcal{D}$)
Right exact is defined similarly.

Proposition 1.4.5
In abelian categories, right (resp. left) adjoint is left (resp. right) exact

Corollary 1.4.6
An equivalence of abelian categories is exact

1.5 Morita theory for module categories

Definition 1.5.1
Rings A, B are Morita equivalent if **Mod**(A), **Mod**(B) are equivalent as abelian categories

Definition 1.5.2
A progenerator for **Mod**(B) is a f.g. projective B-module M s.t. every B-module is a quotient of a direct sum of copies of M

Remark. (a) B itself is a progenerator for **Mod**(B)
(b) M progenerator for **Mod**(B) $\Leftrightarrow$ $M | B^m$, $B | M^n$ for some m, n (Think! Use (a))

Theorem 1.5.3 (Morita)
A, B rings, TFAE:

(1) A, B Morita equivalent

(2) $\mathbf{Mod}(A) \cong \mathbf{Mod}(B)$ as additive categories

(3) $\mathbf{mod}(A) \cong \mathbf{mod}(B)$ as additive categories

(4) $\mathbf{Proj}(A) \cong \mathbf{Proj}(B)$ as additive categories

(5) $\mathbf{proj}(A) \cong \mathbf{proj}(B)$ as additive categories

(6) $\exists M$ progenerator for $\mathbf{Mod}(B)$ s.t. $\mathrm{End}_B(M)^{op} \cong A$ as rings

(7) $\exists M, B$ - A-bimodule, and N, A - B-bimodule, s.t.

$$\begin{array}{rcll} M \otimes_A N & \cong & B & \text{as } B\text{-}B\text{-bimodule} \\ N \otimes_B M & \cong & A & \text{as } A\text{-}A\text{-bimodule} \end{array}$$

(8) The same as (7) with an additional condition:
M f.g. projective as left B-module and as right A-module
N f.g. projective as left A-module and as right B-module

When these conditions are satisfied, the following maps are equivalence inverse to each other:

$$\mathbf{Mod}(A) \underset{N \otimes_B -}{\overset{M \otimes_A -}{\rightleftarrows}} \mathbf{Mod}(B)$$

Definition 1.5.4
Let $\mathcal{C}$ be an additive category with arbitrary coproducts (denote by $\oplus$).
An object x of $\mathcal{C}$ is called <u>compact</u> if $\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathbf{Ab}$ preserves arbitrary coproducts, i.e. for any family of objects $\{y_i\}_{i \in I}$ of $\mathcal{C}$, the map

$$\begin{array}{rcl} \theta : \bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{C}}(x, y_i) & \to & \mathrm{Hom}_{\mathcal{C}}\left(x \, , \bigoplus_{i \in I} y_i \right) \\ (f_i : x \to y_i)_{i \in I} & \mapsto & f \quad \text{s.t. } \pi_i f = f_i \end{array}$$

is an isomorphism, where $\pi_i : \bigoplus_{i \in I} y_i \to y_i$ the canonical projection.

Remark. θ is always injective, since, $f = 0 \;\Rightarrow\; f_i = \pi_i f = 0 \;\; \forall i \in I$. Therefore:
θ is an isomorphism
$\Leftrightarrow$ θ surjective
$\Leftrightarrow$ any map $f : x \to \bigoplus_{i \in I} y_i$ factors through $\bigoplus_{i \in I_0} y_i \hookrightarrow \bigoplus_{i \in I} y_i$ for some finite $I_0 \subseteq I$

Proposition 1.5.5
$\mathbf{proj}(A) = \mathbf{Proj}(A)^c$ (subcategory of $\mathbf{Proj}(A)$ with objects being compact)

Proof

$\subseteq$: Obvious

$\supseteq$: Suppose $M \in \mathbf{Proj}(A)^c$, $\exists A$-hom. $f : \bigoplus_{i \in I} A \twoheadrightarrow M$
M projecitve $\Rightarrow$ $\exists g : M \to \bigoplus_{i \in I} A$ s.t. $fg = 1_M$
M compact $\Rightarrow$ $g(M) \subseteq \bigoplus_{i \in I_0} A$ for some $I_0 \subseteq I$ $\Rightarrow$ $f : \bigoplus_{i \in I_0} A \twoheadrightarrow M$ $\Rightarrow$ M f.g.

□

Proof of Theorem 1.5.3

(1)$\Leftrightarrow$(2): by Corollary 1.4.6

(2)$\Rightarrow$(4): $\mathbf{Proj}(A)$ is characterized purely categorically,so preserved by an equivalence.

(4)$\Rightarrow$(5): by Proposition 1.5.5

(5)$\Rightarrow$(6): Let F :proj$(A) \leftrightarrows$proj$(B) : G$
Set $M = F(A)$ f.g. projective B-module
A progenerator of $\mathbf{Mod}(A)$ $\Rightarrow$ $\bigoplus A \twoheadrightarrow G(B)$ $\Rightarrow$ $\bigoplus M \twoheadrightarrow FG(B) \xrightarrow{\sim} B$
$\Rightarrow$ M is a progenerator for $\mathbf{Mod}(B)$

$$\begin{array}{rcl} \mathrm{End}_B(M)^{op} \xrightarrow[F]{\sim} \mathrm{End}_A(A)^{op} & \cong & A \\ f & \mapsto & f(1) \\ (x \mapsto xa) & \leftarrow\!\shortmid & a \end{array}$$

(6)$\Rightarrow$(7): Set $N = \mathrm{Hom}_B(M, B)$
Write $\lambda : A \xrightarrow{\sim} \mathrm{End}_B(M)^{op}$ (ring isom)
M is B-A-bimodule: $ma = \lambda(a)(m)$
N is A-B-bimodule: $(an)(m) = n(ma)$, $(nb)(m) = n(m)b$
($\lambda : A \to \mathrm{Hom}_B(M_A, M_A)$ is in fact an isom. of A-A-bimodule)

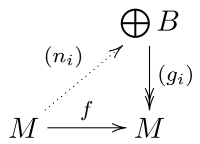
$$\begin{array}{rcl} \phi : M \otimes_A N & \to & B \\ m \otimes n & \mapsto & n(m) \end{array}$$

is a well-defined hom of A-A-bimodule

$$\begin{array}{rcl} \psi : N \otimes_B M & \to & \mathrm{Hom}_B(M, M) \xrightarrow[\lambda^{-1}]{\sim} A \\ n \otimes m & \mapsto & (m' \mapsto n(m')m) \end{array}$$

is a well-defined hom of B-B-bimodule
ϕ surjective: M progenerator $\Rightarrow$ $\exists(n_i) : \bigoplus_{\text{finite}} M \twoheadrightarrow B$ $\Rightarrow$ $1_B = \sum_i n_i(m_i) = \phi\left(\sum_i m_i \otimes n_i\right)$
ψ surjective: $\exists(g_i) : \bigoplus_{\text{finite}} B \twoheadrightarrow M$

$\Rightarrow$



$\Rightarrow f = \sum_i g_i n_i = \psi(\sum_i n_i \otimes g_i(1_B))$
$\forall x, z \in M, \quad y, w \in N$

$$\begin{array}{rcl} \phi(x \otimes y)z & = & x\psi(y \otimes z) \\ \psi(y \otimes z)w & = & y\phi(z \otimes w) \end{array}$$

Claim: ϕ, ψ are isoms
Proof of Claim:
ϕ is isom:
Suppose $\phi(\sum_i x_i \otimes y_i) = 0$
ϕ surjective $\Rightarrow \phi(\sum_j z_j \otimes w_j) = 1_B$

$$
\begin{aligned}
\sum_i x_i \otimes y_i &= (\sum x_i \otimes y_i)\phi(\sum_j z_j \otimes w_j) \\
&= \sum_{i,j} x_i \otimes y_i \phi(z_j \otimes w_j) \\
&= \sum_{i,j} x_i \otimes \psi(y_i \otimes z_j) w_j \\
&= \sum_{i,j} x_i \psi(y_i \otimes z_j) \otimes w_j \\
&= \sum_{i,j} \phi(x_i \otimes y_i) z_j \otimes w_j \\
&= \phi(\sum_i x_i \otimes y_i)(\sum_j z_j \otimes w_j) = 0 \qquad \blacksquare
\end{aligned}
$$

(7) ⇒(2): $M \otimes_A -$ and $N \otimes_B -$ are equivalence inverse to each other.

(6) ⇒(8): Use the same construction as in (7). Then we are left to show M, N f.g. projective as left/right modules.

By assumption, M is f.g. projective as left B-module. This implies N is f.g. projective as right B-module, because of the following claim (Note this is obvious for $M = B$)

Claim: $\mathrm{Hom}_{\mathcal{C}}(-, B)$ preserves finite direct sums "by additivity"

Proof of Claim:
$M \otimes M' \cong \bigoplus_{\text{finite}} B$ as left B-modules
$\Rightarrow \quad \mathrm{Hom}_B(M, B) \oplus \mathrm{Hom}_B(M', B) \cong \bigoplus_{\text{fintie}} \mathrm{Hom}_B(B, B) \cong \bigoplus_{\text{finite}} B \quad$ (as right B-modules) $\blacksquare$

$M \cong \mathrm{Hom}_A(N_B, A_A)$ as B - A-bimoudles:

$$
\begin{aligned}
\mathrm{Hom}_A(N, A) &\cong \mathrm{Hom}_A(N, \mathrm{Hom}_B(M, M)) \\
&\cong \mathrm{Hom}_B(M \otimes_A N, M) \\
&\cong \mathrm{Hom}_B(B, M) \;\cong\; M \text{ (as } B\text{ - }A\text{-bimodule)}
\end{aligned}
$$

N is f.g. projective as left A-module:
$N = \mathrm{Hom}_B(M, B), A \cong \mathrm{Hom}_B(M, M)$ progenerator for $\mathbf{Mod}(B)$
$\Rightarrow \quad B \otimes X \cong \bigoplus_{\text{finite}} M$ for some X
$\Rightarrow \quad \mathrm{Hom}_B(M, B) \oplus \mathrm{Hom}_B(M, X) \cong \bigoplus_{\text{finite}} \mathrm{Hom}_B(M, M) \cong \bigoplus_{\text{fiinte}} A$

M is f.g. projective as right A-module: as before

(8) ⇒(1): ✓

(8) ⇒(3): because M, N are f.g.

(3) ⇒(5): the same as (2)⇒(4)

□

Proposition 1.5.6

(1) M is f.g. projective as left B-module
$\Rightarrow \quad \mathrm{Hom}_B(M, B \otimes_B -) \;\cong\; \mathrm{Hom}_B(M, -)$

(2) M is f.g. projective as right A-module
$\Rightarrow \quad - \otimes_A \mathrm{Hom}_{A^{op}}(M, A) \;\cong\; \mathrm{Hom}_{A^{op}}(M, -)$

Proof
We only show (1) here: For left B-module X

$$\begin{aligned}\phi : \mathrm{Hom}_B(M,B) \otimes_B X &\to \mathrm{Hom}_B(M,X) \\ f \otimes x &\mapsto (m \mapsto f(m)x)\end{aligned}$$

is a well-defined group homomorphism, natural in X
ϕ is an isom when $M = B$
In general, ϕ is an isomorphism "by additiviy" (check) □

Proposition 1.5.7
$A \sim_{\text{Morita}} B \;\Rightarrow\; A^{op} \sim_{\text{Morita}} B^{op}$

Proposition 1.5.8
$A \sim_{\text{Morita}} B \;\Rightarrow\; Z(A) \cong Z(B)$ as rings

Remark. We will get these result using derived categories, so proofs are not given here

1.6 Triangulated Categories

Definition 1.6.1
$\mathcal{C}$ additive category, $\Sigma : \mathcal{C} \to \mathcal{C}$ equivalence

(1) A triangle in $\mathcal{C}$ is a sequence in $\mathcal{C}$ of the form

$$X \to Y \to Z \to \Sigma X$$

(2) A morphism of triangles from $X \to Y \to Z \to \Sigma X$ to $X' \to Y' \to Z' \to \Sigma X'$ is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

It is an isomorphism if f, g, h are isomorphisms in $\mathcal{C}$

Definition 1.6.2
$\mathcal{C}$ additive category, $\Sigma : \mathcal{C} \to \mathcal{C}$ equivalence
$(\mathcal{C}, \Sigma)$ is a triangulated category if there is a class of triangles (called exact triangles) satisfying:

(T1):
- A triangle isomorphic to an exact triangle is exact
- $\forall f : X \to Y$ in $\mathcal{C}$, $\exists$ exact triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$
- $\forall X \in \mathrm{Ob}(\mathcal{C})$, $X \xrightarrow{1_X} X \to 0 \to \Sigma X$ is exact triangle

(T2): If we have the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

with $X \xrightarrow{f} Y \to Z \to \Sigma X$ and $X' \xrightarrow{g} Y' \to Z' \to \Sigma X'$ are exact triangles
then $\exists$ (not necessarily unique) morphism $h : Z \to Z'$ s.t. the following diagram commutes, and

have exact rows (i.e. giving a morphism of exact triangles)

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \exists h} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{g} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

(T3) (Turning triangles) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is exact, then $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is exact.

(T4) (Octahedral axiom) If we have commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

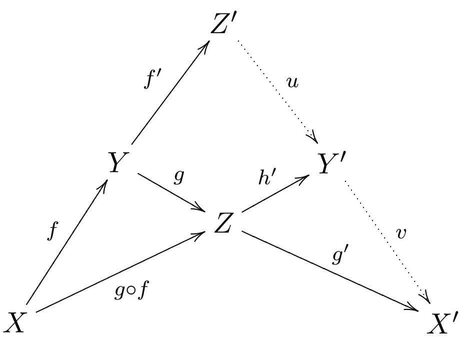
and with following exact triangles in $\mathcal{C}$

$$X \xrightarrow{f} Y \xrightarrow{f'} Z' \xrightarrow{f''} \Sigma X$$
$$Y \xrightarrow{g} Z \xrightarrow{g'} X' \xrightarrow{g''} \Sigma Y$$
$$X \xrightarrow{h} Z \xrightarrow{h'} Y' \xrightarrow{h''} \Sigma X$$

then $\exists$ commutative diagram of exact rows: then $\exists$ commutative diagram of exact rows:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \\ \| & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} & & \| \\ X & \longrightarrow & Z & \longrightarrow & Y' & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \Sigma f} \\ Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & \Sigma X \\ {\scriptstyle f'}\downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle \Sigma f'} \\ Z' & \xrightarrow{u} & Y' & \xrightarrow{v} & X' & \xrightarrow{w} & \Sigma X \end{array}$$

This axiom is usually memorised using this diagram:



Remark. (T4) can be viewed as "third isomorphism theorem"

$$\left.\begin{array}{rcl} Z' & \approx & Y/X \\ Y' & \approx & Z/X \\ X' & \approx & Z/Y \end{array}\right\} \Rightarrow (Z/X)/(Y/X) \approx Z/Y$$

Fix $(\mathcal{C}, \Sigma)$ triangulated category

Proposition 1.6.3
If $X \xrightarrow{u} Y \xrightarrow{v} Z \to \Sigma X$ is exact, then $v \circ u = 0$

Proof
(T1) + (T2) $\Rightarrow \quad 0 \to Z = Z \to 0$ exact
(T3) + (T2) $\Rightarrow$

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

morphism of exact triangles. So $v \circ u = 0$ □

Definition 1.6.4
Let $\mathcal{A}$ be an additive category. An additive functor $F : \mathcal{C} \to \mathcal{A}$ is cohomological if:
$X \to Y \to Z \to \Sigma X$ exact in $\mathcal{C} \quad \Rightarrow \quad FX \to FY \to FZ$ exact in $\mathcal{A}$

Remark. By (T3), in fact we get a long exact sequence in $\mathcal{A}$: $\cdots \to FX \to FY \to FZ \to F\Sigma X \to \cdots$

Proposition 1.6.5
For any $U \in \mathrm{Ob}(\mathcal{C})$, Hom functors $\mathrm{Hom}_{\mathcal{C}}(U, -), \mathrm{Hom}_{\mathcal{C}}(-, U)$ are cohomological

Proof
Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be exact in $\mathcal{C}$
Consider $\mathcal{C}(U, X) \xrightarrow{f_*} \mathcal{C}(U, Y) \xrightarrow{g_*} \mathcal{C}(U, Z)$
$g_* \circ f_* = (g \circ f)_* = 0$ by Proposition 1.6.3

Let $u \in \ker(g_*)$ i.e. $u : U \to Y$ in $\mathcal{C}$ s.t. $g \circ u = 0$
(T2) $\Rightarrow$

$$\begin{array}{ccccccc} U & = & U & \longrightarrow & 0 & \longrightarrow & \Sigma U \\ {\scriptstyle v}\vdots & & {\scriptstyle u}\downarrow & & \downarrow & & \vdots{\scriptstyle \Sigma v} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

morphism of exact triangles $\Rightarrow \quad u = f_*(v)$ □

Definition 1.6.6
$(\mathcal{C}, \Sigma)$ a triangulated category

(1) A full triangulated subcategory of $\mathcal{C}$ is a full additive subcategory $\mathcal{D}$ of $\mathcal{C}$
(i.e. $0 \in \mathcal{D}$, $\mathcal{D}$ closed under taking finite coproducts) which is closed under Σ and taking triangles:
if $X \to Y \to Z \to \Sigma X$ exact in $\mathcal{C}$ and $X, Y \in \mathcal{D}$, then $Z \in \mathcal{D}$
(by turning triangle axiom, can use: $X, Z \in \mathcal{D}$, resp. $Y, Z \in \mathcal{D}$, then $Y \in \mathcal{D}$, resp. $X \in \mathcal{D}$)

(2) A thick subcategory of a full triangulated subcategory $\mathcal{D}$ of $\mathcal{C}$ which is closed under taking direct summands:
if $X, Y \in \mathcal{C}$ and $X \oplus Y \in \mathcal{D}$, then $X, Y \in \mathcal{D}$

(3) Suppose $\mathcal{C}$ has arbitrary coproducts. Then a localizing subcategory of $\mathcal{C}$ is a full triangulated subcategory of $\mathcal{C}$ closed under taking arbitrary coproduct

Remark. Localizing $\Rightarrow$ thick

Proposition 1.6.7
Let the following diagram be a morphism of exact triangles in triangulated categories $(\mathcal{C}, \Sigma)$

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle \cong}\downarrow{\scriptstyle f} & & {\scriptstyle \cong}\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

If f, g are isomorphisms, then h is an isomorphism.

Proof
Let U be an arbitrary object in $\mathcal{C}$. Proposition 1.6.5 $\Rightarrow$ $\mathrm{Hom}_{\mathcal{C}}(U, -)$ is cohomological.

$$\begin{array}{ccccccccccc}
\mathcal{C}(U,X) & \longrightarrow & \mathcal{C}(U,Y) & \longrightarrow & \mathcal{C}(U,Z) & \longrightarrow & \mathcal{C}(U,\Sigma X) & \longrightarrow & \mathcal{C}(U,\Sigma Y) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle (\Sigma f)_*} & & \downarrow{\scriptstyle (\Sigma g)_*} & & \\
\mathcal{C}(U,X') & \longrightarrow & \mathcal{C}(U,Y') & \longrightarrow & \mathcal{C}(U,Z') & \longrightarrow & \mathcal{C}(U,\Sigma X') & \longrightarrow & \mathcal{C}(U,\Sigma Y) & \longrightarrow & \cdots
\end{array}$$

is a commutative diagram in **Ab** with exact rows.
f, g isom $\Rightarrow$ f_*, g_* isom $\Rightarrow$ (By Five lemma) h_* is isom $\Rightarrow$ h is isom. □

Proposition 1.6.8
In a triangulated category $(\mathcal{C}, \Sigma)$, products and coproducts of exact triangles are exact.

Proof
Let $X_i \to Y_i \to Z_i \to \Sigma X_i$ be a family of exact triangles in $\mathcal{C}$ indexed by $i \in I$. We need to show that

$$\bigoplus_i X_i \to \bigoplus_i Y_i \to \bigoplus_i Z_i \to \bigoplus_i \Sigma X_i \cong \Sigma \bigoplus_i X_i$$

is exact triangle.
(Note that $\bigoplus_i \Sigma X_i \cong \Sigma \bigoplus_i X_i$ since Σ is an equivalence.)
(T2) $\Rightarrow$ for each i, there exists a commutative diagram with first row exact:

$$\begin{array}{lccccccc}
\text{exact row:} & \bigoplus_i X_i & \longrightarrow & \bigoplus_i Y_i & \longrightarrow & W & \longrightarrow & \Sigma \bigoplus_i X_i \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& X_i & \longrightarrow & Y_i & \longrightarrow & Z_i & \longrightarrow & \Sigma X_i
\end{array}$$

fNote that the bottom row is not necessarily exact. Now let $U \in \mathrm{Ob}(\mathcal{C})$. Taking coproducts in the bottom row of the above diagram and applying $\mathrm{Hom}_{\mathcal{C}}(-, U)$ gives us a long exact sequence

$$\begin{array}{ccccccccccc}
\prod \mathcal{C}(X_i,U) & \longleftarrow & \prod \mathcal{C}(Y_i,U) & \longleftarrow & \mathcal{C}(W,U) & \longleftarrow & \prod \mathcal{C}(\Sigma X_i,U) & \longleftarrow & \prod \mathcal{C}(\Sigma Y_i,U) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
\prod \mathcal{C}(X_i,U) & \longleftarrow & \prod \mathcal{C}(Y_i,U) & \longleftarrow & \prod \mathcal{C}(Z_i,U) & \longleftarrow & \prod \mathcal{C}(\Sigma X_i,U) & \longleftarrow & \prod \mathcal{C}(\Sigma Y_i,U) & \longleftarrow & \cdots
\end{array}$$

This is a commutative diagram of exact rows in **Ab**.
Five Lemma $\Rightarrow$ $\mathcal{C}(W, U) \xrightarrow{\sim} \prod_i \mathcal{C}(Z_i, U)$ $\Rightarrow$ $\bigoplus_i Z_i \xrightarrow{\sim} W$ □

Definition 1.6.9
Let $(\mathcal{C}, \Sigma)$ and $(\mathcal{D}, \Sigma')$ be triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is exact if it satisfies the following two conditions.

(1) There exists a natural isomorphism $\Sigma' F \cong F\Sigma$.

(2) If $X \to Y \to Z \to \Sigma X$ is an exact triangle in $\mathcal{C}$, then

$$FX \to FY \to FZ \to \Sigma' FX$$

is an exact triangle in $\mathcal{D}$.

Proposition 1.6.10
Left (resp. right) adjoints of an exact functor of triangulated categories are exact.

Proof
We prove for left adjoint, right adjoint is similar.
Let $(\mathcal{C}, \Sigma)$ and $(\mathcal{D}, \Sigma')$ be triangulated categories
Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor that is left adjoint to $G : \mathcal{D} \to \mathcal{C}$
Let $\epsilon : FG \to 1_{\mathcal{D}}$ be the counit of the adjunction.

First, one can check that:

$$\begin{array}{rll} F\Sigma & \text{is left adjoint to} & \Sigma^{-1}G \\ \Sigma' F & \text{is left adjoint to} & G\Sigma'^{-1} \end{array}$$

Since $F\Sigma \cong \Sigma' F \quad \Rightarrow \quad \Sigma^{-1} G \cong G\Sigma'^{-1}$.

Next, we need to show that if $X \to Y \to Z \to \Sigma' X$ is an exact triangle in $\mathcal{D}$, then

$$GX \to GY \to GZ \to \Sigma GX$$

is exact in $\mathcal{C}$:
By (T2), there exists an exact triangle $GX \to GY \to W \to \Sigma GX$
Since F is exact, there exists a commutative diagram (in $\mathcal{D}$)

$$\begin{array}{ccccccc} FGX & \longrightarrow & FGY & \longrightarrow & FW & \longrightarrow & \Sigma' FGX \\ \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} & & \vdots{\scriptstyle \theta} & & \downarrow{\scriptstyle \Sigma\epsilon_X} \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma' X \end{array}.$$

For every object U in $\mathcal{C}$, we have a commutative diagram

$$\begin{array}{ccccccccccc} \mathcal{C}(U, GX) & \longrightarrow & \mathcal{C}(U, GY) & \longrightarrow & \mathcal{C}(U, W) & \longrightarrow & \mathcal{C}(U, \Sigma GX) & \longrightarrow & \mathcal{C}(U, \Sigma GY) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \epsilon_X \circ F(-)} & & \downarrow{\scriptstyle \epsilon_Y \circ F(-)} & & \downarrow{\scriptstyle \theta \circ F(-)} & & \downarrow{\scriptstyle \epsilon_{\Sigma' X} \circ F(-)} & & \downarrow{\scriptstyle \epsilon_{\Sigma' Y} \circ F(-)} & & \\ \mathcal{D}(FU, X) & \longrightarrow & \mathcal{D}(FU, Y) & \longrightarrow & \mathcal{D}(FU, Z) & \longrightarrow & \mathcal{D}(FU, \Sigma' X) & \longrightarrow & \mathcal{D}(FU, \Sigma' Y) & \longrightarrow & \cdots \end{array}$$

with exact rows (as $\mathrm{Hom}_{\mathcal{C}}(U, -)$ is cohomological), where we know that every vertical arrow except possibly the middle one are adjunction isomorphisms. Hence by the five lemma, $\theta \circ F(-)$ is also an isomorphism. $\square$

1.7 Differential graded algebras and modules

Throughout this section, k will denote a fixed commutative ring.

Definition 1.7.1
A graded k-algebra is a k-algebra A with a k-module decomposition

$$A = \bigoplus_{i \in \mathbb{Z}} A^i$$

such that $A^i A^j \subseteq A^{i+j}$ for all $i, j \in \mathbb{Z}$.

If A is a graded k-algebra, then A^0 is an ordinary k-algebra with $1_{A^0} = 1_A$. Also, if A is any k-algebra, then A may be viewed as a graded k-algebra concentrated in degree zero, i.e., $A = A^0$.

Definition 1.7.2
Let $A = \bigoplus_{i \in \mathbb{Z}} A^i$ be a graded k-algebra.

(1) A graded module is an A-module M with decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M^i$$

such that $A^i M^j \subseteq M^{i+j}$ for all $i, j \in \mathbb{Z}$.

(2) If M and N are graded A-modules then an A-module morphism $f : M \to N$ is a morphism of graded A-modules of degree n if $f(M^i) \subseteq N^{i+n}$ for all $i \in \mathbb{Z}$.

Definition 1.7.3
A differential graded k-algebra is a graded k-algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$ together with a morphism of graded k-modules $d : A \to A$ of degree one, called the differential, such that $d \circ d = 0$ and

$$d(ab) = d(a)b + (-1)^i a d(b) \quad \forall a \in A^i, b \in A^j$$

Definition 1.7.4

(1) Let A be a differential graded k-algebra. A differential graded A-module is a graded A-module M together with a morphism of graded k-modules $d : M \to M$ (note the abuse of notation!) of degree one such that $d \circ d = 0$ and

$$d(am) = d(a)m + (-1)^i a d(m) \quad \forall a \in A^i, m \in M^j$$

(2) Let M and N be differential graded A-modules. A morphism of differential graded A-modules is an A-module morphism $f : M \to N$ of degree zero such that $d_N \circ f = f \circ d_M$.

Remark. If A is an ordinary k-algebra, then a differential graded A-module is just a complex of A-modules, and a morphism of differential graded A-algebras is a chain map.

Definition 1.7.5
Let A be a differential graded k-algebra. We define $\mathbf{C}(A)$ to be the category whose objects are differential graded A-modules, and whose morphisms are morphisms of differential graded A-modules, i.e. homogeneous morphisms of graded A-modules of degree 0 commuting with differentials.

Definition 1.7.6
If A is a differential graded k-algebra, M is a differential graded A-module and $n \in \mathbb{Z}$, we define the differential graded A-module $M[n]$ to be differential graded A-module with grading given by $M[n]^i = M^{i+n}$, and differential given by $d_{M[n]} = (-1)^n d_M$.

Definition 1.7.7
Let A and B be differential graded (dg) k-algebras

(1) M, N dg A-module, we define the dg k-module (i.e. complexes of k-modules) $\mathcal{H}\mathrm{om}_A(M, N)$ by:

$$\begin{aligned} \mathcal{H}\mathrm{om}_A(M,N)^n &= \{f : M \to N \mid f \text{ is a morphism of dg } A\text{-modules of degree } n\} \\ d^n : \mathcal{H}\mathrm{om}_A(M,N)^n &\to \mathcal{H}\mathrm{om}_A(M,N)^{n+1} \\ f &\mapsto d_N \circ f - (-1)^n f \circ d_M \end{aligned}$$

Moreover if M is a dg A-B-bimodule, then $\mathcal{H}\mathrm{om}_A(M, N)$ is a dg B-module.

(2) M, N dg k-modules, $M \otimes_k N$ a dg k-module:

$$\begin{aligned} (M \otimes_k N)^n &= \bigoplus_{i+j=n} M^i \otimes N^j \\ d : (M \otimes_k N)^n &\to (M \otimes_k N)^{n+1} \\ x \otimes y &\mapsto d(x) \otimes y + (-1)^{|x|} x \otimes dy \end{aligned}$$

(3) M dg right A-module, N dg left A-module $\Rightarrow$ $M \otimes_A N$ dg k-module

$$(M \otimes_A N)^n = M \otimes_k N / \langle xa \otimes y - x \otimes ay \mid x \in M, y \in N, a \in A \rangle$$

Moreover, if M is a dg B - A-bimodule, then $M \otimes_A N$ is a dg B-module

Definition 1.7.8
A dg k-algebra, we define $\mathbf{Diff}(A)$ to be the category whose objects are dg A-modules and whose morphisms are $\mathrm{Hom}_{\mathbf{Diff}(A)}(M, N) := \mathcal{H}\mathrm{om}_A(M, N)$.

Lemma 1.7.9
A dg k-algebra, M, N dg A-modules, the i-th cocycle:

$$Z^i(\mathcal{H}\mathrm{om}_A(M, N)) = \mathrm{Hom}_{\mathbf{C}(A)}(M, N[i])$$

Proof
Let $f \in \mathcal{H}\mathrm{om}_A(M, N)^i$, then
$f \in Z^i(\mathcal{H}\mathrm{om}_A(M, N))$
$\Leftrightarrow$ $d(f) = d_N \circ f - (-1)^i f \circ d_M = 0$
$\Leftrightarrow$ $d_{N[i]} \circ f = f \circ d_M$
$\Leftrightarrow$ $f \in \mathrm{Hom}_{\mathbf{C}(A)}(M, N[i])$ □

Proposition 1.7.10
A dg k-algebra
$\mathbf{C}(A)$ is an abelian category where kernels and cokernels are given degreewise
$\mathbf{C}(A)$ has arbitrary colimits given degreewise

Proposition 1.7.11
A, B dg k-algebra, M a dg B - A-bimodule

$$\mathbf{Diff}(A) \underset{\mathcal{H}\mathrm{om}_A(M,-)}{\overset{M\otimes_A -}{\rightleftarrows}} \mathbf{Diff}(B) \quad , \quad \mathcal{H}\mathrm{om}_B(M \otimes_A X, Y) \cong \mathcal{H}\mathrm{om}_A(X, \mathcal{H}\mathrm{om}_B(M, Y))$$

$$\mathbf{C}(A) \underset{\mathcal{H}\mathrm{om}_A(M,-)}{\overset{M\otimes_A -}{\rightleftarrows}} \mathbf{C}(B) \quad , \quad \mathrm{Hom}_{\mathbf{C}(B)}(M \otimes_A X, Y) \cong \mathrm{Hom}_{\mathbf{C}(A)}(X, \mathcal{H}\mathrm{om}_B(M, Y))$$

1.8 Homotopy and derived categories

Let A be dg k-algebra

Definition 1.8.1 (Homotopy category)
Homotopy category $\mathbf{K}(A)$ is defined as:
objects: dg A-modules
morphisms: homotopy classes $[f]$ of morphisms of dg A-module $f : X \to Y$
For $f, g : X \to Y$,

- $[f] = [g] \;\Leftrightarrow\;$ $f - g = d_Y \circ h + h \circ d_X$ for some $h : X \to Y$ morphism of graded A-module of degree -1
- $[f] + [g] = [f + g]$

Proposition 1.8.2
$\mathbf{K}(A)$ is a triangulated category w.r.t. $[1] : \mathbf{K}(A) \to \mathbf{K}(A)$

Exact triangles are those triangles isomorphic to those of the form

$$X \xrightarrow{[f]} Y \xrightarrow{[i(f)]} \operatorname{cone}(f) \xrightarrow{[p(f)]} X[1]$$

$$\begin{array}{ccccccc}
X^n & \xrightarrow{f} & Y^n & \xrightarrow{\binom{0}{1}} & X^{n+1} \oplus Y^n & \xrightarrow{(1\ 0)} & X^{n+1} \\
\downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_Y} & & \downarrow{\scriptstyle \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}} & & \downarrow{\scriptstyle -d_X} \\
X^{n+1} & \xrightarrow[f]{} & Y^{n+1} & \xrightarrow[\binom{0}{1}]{} & X^{n+2} \oplus Y^{n+1} & \xrightarrow[(1\ 0)]{} & X^{n+2}
\end{array}$$

$\mathbf{K}(A)$ has arbitrary coproducts induced from those in $\mathbf{C}(A)$

Proposition 1.8.3
Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a degreewise split ses in $\mathbf{C}(A)$

$$\left(\text{i.e. } \exists \begin{array}{c} u : Y \to X \\ v : Z \to Y \end{array} \text{ morphisms of graded } A\text{-module of degree 0, s.t. } \begin{cases} uf = 1_X \\ gv = 1_Z \\ fu + vg = 1_Y \end{cases} \right)$$

Then $\exists r : Z \to X[1]$ morphism in $\mathbf{K}(A)$ s.t. $X \xrightarrow{[f]} Y \xrightarrow{[g]} Z \xrightarrow{[r]} X[1]$ is exact in $\mathbf{K}(A)$

Proof
$r = u \circ d_Y \circ v$ □

Remark. OR: $\mathbf{K}(A)$ is the stable category of the Frobenius category $(\mathbf{C}(A), \mathcal{S})$, where $\mathcal{S} = \{$degreewise split ses in $\mathbf{C}(A)\}$

Proposition 1.8.4
$H^n : \mathbf{K}(A) \to \mathbf{Mod}(k)$ is a cohomological functor

Definition 1.8.5 (Derived category)
Derived category $\mathbf{D}(A)$ is a category together with $\pi : \mathbf{K}(A) \to \mathbf{D}(A)$ which sends quisms (quasi-isomorphisms) in $\mathbf{K}(A)$ to isomorphisms in $\mathbf{D}(A)$ and universal w.r.t. this property:
if $\phi : \mathbf{K}(A) \to \mathcal{D}$ is a functor which sends quisms to isomorphisms
then $\exists !\, \mathbf{D}(A) \to \mathcal{D}$ s.t. the following diagram commutes:

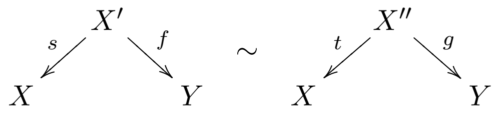
$$\begin{array}{ccc}
\mathbf{K}(A) & \xrightarrow{\pi} & \mathbf{D}(A) \\
& {\scriptstyle \phi} \searrow \quad \swarrow {\scriptstyle \exists ! \psi} & \\
& \mathcal{D} &
\end{array}$$

Construction of $\mathbf{D}(A)$:
objects: the same as $\mathbf{K}(A)$
morphisms: the equivalence classes of "roofs":

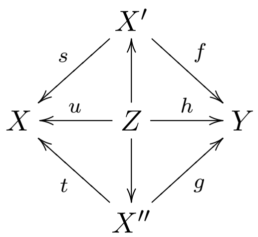
$$\begin{array}{ccc}
& X' & \\
{\scriptstyle s} \swarrow & & \searrow {\scriptstyle f} \\
X & & Y
\end{array}$$

where f morphism in $\mathbf{K}(A)$ and s quism in $\mathbf{K}(A)$.

Equivalence:

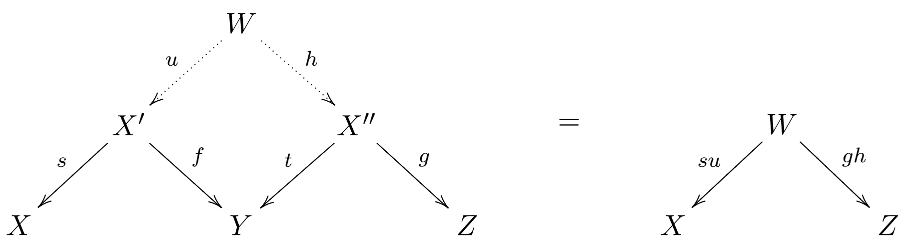


$\iff$ ∃commutative diagram



with u a quism

Composition of roofs is defined as follows:



Note: su is quism; u, h always exists in $\mathbf{K}(A)$ (see Gelfand-Manin III.2.8)

Proposition 1.8.6
$\mathbf{D}(A)$ is a triangulated category w.r.t. $[1] : \mathbf{D}(A) \to \mathbf{D}(A)$

- exact triangles are triangles isomorphic to images of exact triangles of $\mathbf{K}(A)$
- $\mathbf{D}(A)$ has arbitrary coproducts induced from $\mathbf{K}(A)$
- $\pi : \mathbf{K}(A) \to \mathbf{D}(A)$ exact, preserves coproducts

1.9 Projective resolutions of bounded right complexes

Let A be k-algebra

Definition 1.9.1

(1) $\mathbf{K}^-(A) = \{X \in \mathbf{K}(A) \mid X^i = 0 \ \forall i \gg 0\}$
$\mathbf{K}^+(A) = \{X \in \mathbf{K}(A) \mid X^i = 0 \ \forall i \ll 0\}$
$\mathbf{K}^{-,b}(A) = \{X \in \mathbf{K}^-(A) \mid H^i(X) = 0 \ \forall |i| \gg 0\}$
$\mathbf{K}^{+,b}(A) = \{X \in \mathbf{K}^+(A) \mid H^i(X) = 0 \ \forall |i| \gg 0\}$

(2) $\mathbf{D}^-(A) = \{X \in \mathbf{D}(A) \mid H^i(X) = 0 \ \forall i \gg 0\}$
$\mathbf{D}^+(A) = \{X \in \mathbf{D}(A) \mid H^i(X) = 0 \ \forall i \ll 0\}$
$\mathbf{D}^b(A) = \{X \in \mathbf{D}(A) \mid H^i(X) = 0 \ \forall |i| \gg 0\}$

All these are triangulated categories
$\mathbf{D}^-(A)$ is the derived categories of $\mathbf{K}^-(A)$
$\mathbf{D}^+(A)$ is the derived categories of $\mathbf{K}^+(A)$
$\mathbf{D}^b(A)$ is the derived categories of $\mathbf{K}^{-,b}(A)$
$\mathbf{D}^b(A)$ is the derived categories of $\mathbf{K}^{+,b}(A)$

If $X \in \mathbf{D}(A)$ s.t. $H^i(X) = 0 \ \ \forall i > n$, then $\tau_{\leq n}X \hookrightarrow X$ is a quism:

$$\begin{array}{ccccccccccc}
\tau_{\leq n}X : & & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & Z^n(X) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \| & & \downarrow & & \downarrow & & \\
X : & & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots
\end{array}$$

Definition 1.9.2
X right bounded complex of A-modules. A projective resolution of X is a right bounded complex P of projective A-modules with a quism $\epsilon : P \to X$

This definition is not the same as the "usual" definition of projective resolution, which is of form like the following:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where P_i and M are A-modules; but this "usual" projective resolution can be viewed as a quism of complexes (our projective resolution) by:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Proposition 1.9.3
Every bounded right complex has projective resolution which is unique up to homotopy equivalence. Taking projective resolution is functorial in $\mathbf{K}^-(A)$ (which we denote using p from now on):
if $\epsilon_X : pX \to X$ is a projective resolution then $\forall f : X \to Y$ chain map
$\exists !$(up to homotopy) $pf : pX \to pY$ s.t. the diagram commutes in $\mathbf{K}^-(A)$:

$$\begin{array}{ccc}
pX & \xrightarrow{\epsilon_X} & X \\
{\scriptstyle pf}\,\vdots & & \downarrow{\scriptstyle f} \\
pY & \xrightarrow{\epsilon_Y} & Y
\end{array}$$

Lemma 1.9.4
P, Q bounded right complexes of projective A-modules

(1) $\mathrm{Hom}_{\mathbf{K}(A)}(P, N) = 0 \ \ \forall N$ acyclic (i.e. $H^i(N) = 0 \ \forall i > 0$)

(2) If $f : X \to Y$ is a quism, then $f_* : \mathrm{Hom}_{\mathbf{K}(A)}(P, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(A)}(P, Y)$

(3) If $f : X \to P$ is a quism, then $\exists g : P \to X$ s.t. $f \circ g \simeq 1_P$

(4) If $f : P \to Q$ is a quism, then f a homotopy equivalence

Now, if $f : X \to Y$ quism $\Rightarrow$ get commutative diagram
$$\begin{array}{ccc}
pX & \xrightarrow{\epsilon_X} & X \\
{\scriptstyle pf}\downarrow & & \downarrow{\scriptstyle f} \\
pY & \xrightarrow{\epsilon_Y} & Y
\end{array}$$

$\Rightarrow$ pf quism
$\Rightarrow$ (by (4) above) pf homotopy equivalence

$\Rightarrow$ diagram commutes:
$$\begin{array}{ccc}
\mathbf{K}^-(A) & \xrightarrow{\quad p \quad} & \mathbf{K}^-(A) \\
 & {\scriptstyle \pi}\searrow \quad \nearrow{\scriptstyle p'} & \\
 & \mathbf{D}^-(A) &
\end{array}$$

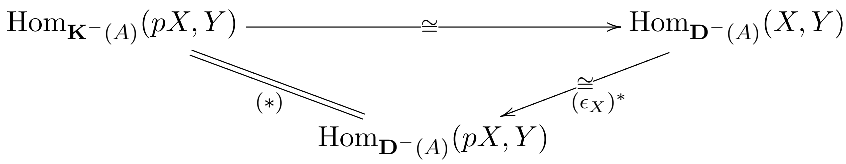
Proposition 1.9.5

(1) $p' : \mathbf{D}^-(A) \to \mathbf{K}^-(A)$ is left adjoint to $\pi : \mathbf{K}^-(A) \to \mathbf{D}^-(A)$

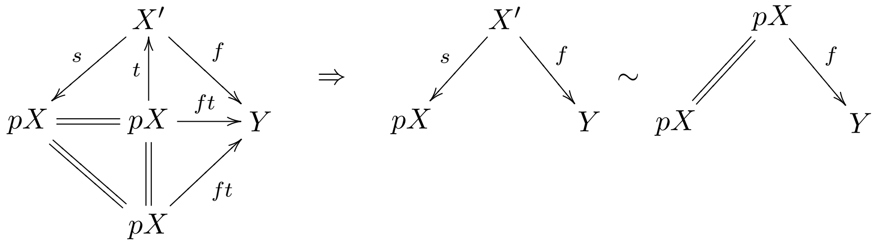
(2) $p' : \mathbf{D}^-(A) \xrightarrow{\sim} \mathbf{K}^-(\mathbf{Proj}\, A)$ is an equivalence of categories

Proof

(1) WANT:

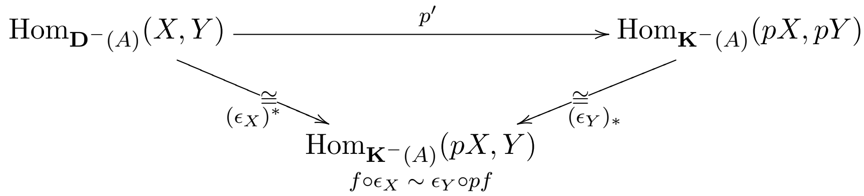


A morphism in $\mathrm{Hom}_{\mathbf{D}^-(A)}(pX, Y)$ is represented by the roof $pX \xleftarrow{[s]} X' \xrightarrow{[f]} Y$ (with s quism)
By Lemma 1.9.4(3)
$\Rightarrow\ \exists t : pX \to X'$ s.t. $s \circ t \sim 1_{pX}$:

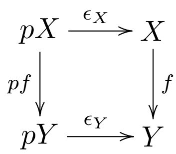


$\Rightarrow$ equality at (∗)

(2) <u>p' fully faithful</u>:
WANT:



(1) $\Rightarrow$ isom of $(\epsilon_X)^*$ above
Lemma 1.9.4 (2) $\Rightarrow$ isom of $(\epsilon_Y)_*$ above
And the following diagram commutes:



$\Rightarrow$ p' bijective

<u>p' essentially surjective</u>:
Proposition 1.9.3 $\Rightarrow$ $\forall P \in \mathbf{K}^-(\mathbf{Proj}\, A)$, $\exists \epsilon_P : pP \to P$ qusim
$\Rightarrow$ (by Lemma 1.9.4 (4)) ϵ_P homotopy equivalence.

□

Proposition 1.9.6
Dually: The injective resolution functor

$$i : \mathbf{K}^+(A) \to \mathbf{K}^+(A)$$

induces a functor

$$i' : \mathbf{D}^+(A) \to \mathbf{K}^+(A)$$

which is right adjoint to $\pi : \mathbf{K}^+(A) \to \mathbf{D}^+(A)$, and

$$i' : \mathbf{D}^+(A) \to \mathbf{K}^+(\mathbf{Inj}\, A)$$

is an equivalence of categories

1.10 Homotopically projective resolutions of unbounded complexes

Let A be dg k-algebra

Definition 1.10.1
A dg A-module X is homotopically projective if $\mathrm{Hom}_{\mathbf{K}(A)}(X, N) = 0\ \forall N$ acyclic

Definition 1.10.2
A dg A-module X satisfies the property (P)

$$\Leftrightarrow \qquad \underset{\text{in } \mathbf{K}(A)}{X \cong} \underbrace{\varinjlim}_{\text{in } \mathbf{C}(A)} (P_0 \xrightarrow{i_0} P_1 \xrightarrow{i_1} P_2 \xrightarrow{i_2} \cdots)$$

where

- each i_k $(k \geq 0)$, $0 \to P_k \overset{i_k}{\hookrightarrow} P_{k+1} \to P_{k+1}/P_k \to 0$ is a degreewise split ses
- each P_k/P_{k-1} $(k \geq 0 : P_{-1} = 0)$ is "relatively projective"
  i.e. a direct summand of a direct sum of copies of shifts of A
  (e.g. If $A = A^0$: complexes of projective A-modules with 0 differentials)

Denote $\mathbf{K}_p(A) = \{X \in \mathbf{K}(A) | X \text{ homotopically projecitve}\}$

Proposition 1.10.3
$\mathbf{K}_p(A)$ is a localizing subcategory of $\mathbf{K}(A)$

Proof
Closed under [1]: ✓

Closed under taking triangle:
If $X \to Y \to Z \to X[1]$ is exact in $\mathbf{K}(A)$ and if X, Z are homotopically projective, then for any acyclic N, take $\mathrm{Hom}_{\mathbf{K}(A)}(-, N)$:

$$\underbrace{\mathrm{Hom}_{\mathbf{K}(A)}(Z, N)}_{=0} \to \mathrm{Hom}_{\mathbf{K}(A)}(Y, N) \to \underbrace{\mathrm{Hom}_{\mathbf{K}(A)}(X, N)}_{=0} \qquad \text{exact}$$

$\Rightarrow \quad \mathrm{Hom}_{\mathbf{K}(A)}(Y, N) = 0 \quad \Rightarrow \quad Y$ homotopically projective

Closed under $\oplus$:
If X_i are homotopically projective

$$\mathrm{Hom}_{\mathbf{K}(A)}(\bigoplus_i X_i, N) \cong \prod_i \mathrm{Hom}_{\mathbf{K}(A)}(X_i, N) = 0 \ \ \forall N \text{ acyclic}$$

$\Rightarrow \quad \bigoplus_i X_i$ homotopically projective □

Proposition 1.10.4
If a dg A-module X satisfies (P), then X is homotopically projective

Proof
Each P_k/P_{k-1} is homotopically projective:
Lemma 1.7.9 $\Rightarrow$ $\mathrm{Hom}_{\mathbf{C}(A)}(A,N) = Z^0(\mathcal{H}\mathrm{om}_A(A,N)) = Z^0(N)$
$\mathrm{Hom}_{\mathbf{K}(A)}(A,N) = H^0(\mathcal{H}\mathrm{om}_A(A,N)) = H^0(N) = 0$

Each P_k is homotopically projective:

$$0 \to P_{k-1} \xrightarrow{i_{k-1}} P_k \to P_k/P_{k-1} \to 0$$

is a degreewise split ses in $\mathbf{C}(A)$
$\Rightarrow$ (by Proposition 1.8.3) $P_{k-1} \to P_k \to P_k/P_{k-1} \to P_{k-1}[1]$ exact in $\mathbf{K}(A)$

$\varinjlim(P_0 \to P_1 \to \cdots)$ is homotopically projective:
$\exists$ ses in $\mathbf{C}(A)$:

$$\begin{array}{ccccccc} 0 \to & \bigoplus P_k & \xrightarrow{1-i} & \bigoplus P_k & \to & \varinjlim P_k & \to 0 \\ & (x_0, x_1, \ldots) & \mapsto & (x_0, x_1 - i_0(x_0), x_2 - i_1(x_1), \ldots) & & & \end{array}$$

This is degreewise split
$\Rightarrow$ (by Proposition 1.8.3) $\bigoplus P_k \to \bigoplus P_k \to \varinjlim P_k \to \bigoplus P_k[1]$ exact in $\mathbf{K}(A)$

□

Theorem 1.10.5
$\forall X \in \mathbf{K}(A)$, $\exists$exact triangle in $\mathbf{K}(A)$

$$pX \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} aX \xrightarrow{\delta_X} X[1]$$

where pX satisfies (P) and aX acyclic.

Some consequences:

- ϵ_X is a quism:
  H^n is cohomological, so
  $$\underbrace{(H^n(aX)[-1])}_{=0} \to H^n(pX) \xrightarrow{\sim} H^n(X) \to \underbrace{H^n(aX)}_{=0} \quad \text{exact}$$
- $X \in \mathbf{K}(A)$. X satisfies (P) $\Leftrightarrow$ X homotopically projecitve:
  ($\Rightarrow$): Proposition 1.10.4
  ($\Leftarrow$): X homotopically projective.
  By Theorem 1.10.5, $pX \to X \to aX \to pX[1]$ exact
  Apply $\mathrm{Hom}_{\mathbf{K}(A)}(-, aX)$
  $\Rightarrow$ $\underbrace{\mathrm{Hom}_{\mathbf{K}(A)}(pX[1], aX)}_{=0} \to \mathrm{Hom}_{\mathbf{K}(A)}(aX, aX) \to \underbrace{\mathrm{Hom}_{\mathbf{K}(A)}(X, aX)}_{=0}$
  as pX satisfies (P) implies pX homotopically projective $\Rightarrow$ $\mathrm{Hom}_{\mathbf{K}(A)}(aX, aX) = 0$
  $\Rightarrow$ $aX \cong 0$ in $\mathbf{K}(A)$
  $\Rightarrow$ $pX \xrightarrow{\sim} X$ in $\mathbf{K}(A)$ □
- For eact $X \in \mathbf{K}(A)$, there is a functor
  $$\begin{array}{rcl} T : \mathbf{K}(A) & \to & T(\mathbf{K}(A)) = \text{category of exact triangles in } \mathbf{K}(A) \\ X & \mapsto & T_X = (pX \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} aX \xrightarrow{\delta_X} pX[1]) \text{ as in Theorem 1.10.5} \end{array}$$

Proof Suppose $f : X \to Y$ is chain map. Then

$$\begin{array}{ccccccc} pX & \xrightarrow{\epsilon_X} & X & \xrightarrow{\eta_X} & aX & \xrightarrow{\delta_X} & pX[1] \\ {\scriptstyle pf}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle af} & & \downarrow{\scriptstyle pf[1]} \\ pY & \xrightarrow{\epsilon_Y} & Y & \xrightarrow{\eta_Y} & aY & \xrightarrow{\delta_Y} & pY[1] \end{array}$$

aY acyclic $\Rightarrow$ $\mathrm{Hom}_{\mathbf{K}(A)}(pX, aY) = 0$ $\Rightarrow$ $\eta_Y \circ f \circ \epsilon_X = 0$

Proposition 1.6.5: $\mathrm{Hom}_{\mathbf{K}(A)}(X, -)$ is cohomological $\Rightarrow$ $\exists pf : pX \to pY$ s.t. $f \circ \epsilon_X \sim \epsilon_Y \circ pf$

Then by (T2), $\exists af : aX \to aY$ making the above diagram commutative in $\mathbf{K}(A)$

Now need to show pf is unique up to homotopy (equivalently, $f \sim 0 \Rightarrow pf \sim 0$):
If $f \sim 0$ $\Rightarrow$ $\epsilon_Y \circ pf \sim f \circ \epsilon_Y = 0$
$\Rightarrow$ pf factors through $aY[-1]$ as $\mathrm{Hom}(pX, -)$ is cohomological and the bottom row is exact triangle $\Rightarrow$ $pf \sim 0$ as $aY[-1]$ acyclic

Therefore, T (and p, a) are functors. (And ϵ, η are transformations) □

Proposition 1.10.6
$p : \mathbf{K}(A) \to \mathbf{K}_p(A)$ is right adjoint to the inclusion functor $i : \mathbf{K}_p(A) \to \mathbf{K}(A)$

Proof
Need: $\mathrm{Hom}_{\mathbf{K}_p(A)}(X, pY) \xrightarrow[(\epsilon_Y)_*]{\cong} \mathrm{Hom}_{\mathbf{K}(A)}(X, Y)$

$pY \xrightarrow{\epsilon_Y} Y \to aY \to pY[1]$ exact
Apply $\mathrm{Hom}_{\mathbf{K}(A)}(X, -)$:

$$\underbrace{\mathrm{Hom}_{\mathbf{K}(A)}(X, aY[-1])}_{=0} \to \mathrm{Hom}_{\mathbf{K}(A)}(X, pY) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(A)}(X, Y) \to \underbrace{\mathrm{Hom}_{\mathbf{K}(A)}(X, aY)}_{=0}$$

□

$p : \mathbf{K}(A) \to \mathbf{K}(A)$ sends quism to homotopic equivalence:

$$\begin{array}{ccc} pX & \xrightarrow{\epsilon_X} & X \\ {\scriptstyle pf}\downarrow & & \downarrow{\scriptstyle f} \\ pY & \xrightarrow{\epsilon_Y} & Y \end{array}$$

(ϵ_X, ϵ_Y quisms) f quism $\Rightarrow$ pf quism
$\Rightarrow$ (as pX, pY homotopically projective) pf homotopic equivalence. Thus,

$$\begin{array}{ccc} \mathbf{K}(A) & \xrightarrow{p} & \mathbf{K}(A) \\ & {\scriptstyle \pi}\searrow \quad \nearrow{\scriptstyle p'} & \\ & \mathbf{D}(A) & \end{array}$$

Proposition 1.10.7
$p' : \mathbf{D}(A) \to \mathbf{K}(A)$ is left adjoint to $\pi : \mathbf{K}(A) \to \mathbf{D}(A)$ (c.f. Proposition 1.9.5)

Proof

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{K}(A)}(pX, Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{D}(A)}(X, Y) \\ & \diagdown\!\!\diagdown \quad \swarrow (\epsilon_X)^* & \\ & \mathrm{Hom}_{\mathbf{D}(A)}(pX, Y) & \end{array}$$

□

Proposition 1.10.8
$p' : \mathbf{D}(A) \to \mathbf{K}_p(A)$ is an equivalence of categories
In fact p' is exact and preserves direct sums

Proof
p' exact because p' is left adjoint to π which is exact
p' preserves direct sum □

Definition 1.10.9
$X \in \mathbf{K}(A)$ is homotopically injective if $\mathrm{Hom}_{\mathbf{K}(A)}(N, X) = 0 \ \forall N$ acyclic
Similarly denote $\mathbf{K}_i(A) = \{X \in \mathbf{K}(A) | X \text{ homotopically injective}\}$

Theorem 1.10.10
$\forall X \in \mathbf{K}(A)$, $\exists$ exact triangle

$$a'X \to X \to iX \to a'X[1]$$

iX homotopically injective, $a'X$ acyclic

Proposition 1.10.11

(1) $i : \mathbf{D}(A) \to \mathbf{K}(A)$ is right adjoint to $\pi : \mathbf{K}(A) \to \mathbf{D}(A)$

(2) $i : \mathbf{D}(A) \to \mathbf{K}_i(A)$ is an equivalence

(3) i is exact, preserves direct sum

Sketch Proof of Theorem 1.10.5
Suppose $A = A^0$
Given a complex X of A-modules
$\exists$"full projective resolution" of X. i.e.

$$\mathbf{A} = (\cdots \to P^{-2} \to P^{-1} \to P^0 \to X \to 0)$$

a sequence of complexes s.t.

$$\begin{array}{cccccccccc} \cdots & \to & P^{-2,j} & \to & P^{-1,j} & \to & P^{0,j} & \to & X^j & \to & 0 \\ \cdots & \to & Z^j(P^{-2}) & \to & Z^j(P^{-1}) & \to & Z^j(P^0) & \to & Z^j(X) & \to & 0 \\ \cdots & \to & B^j(P^{-2}) & \to & B^j(P^{-1}) & \to & B^j(P^0) & \to & B^j(X) & \to & 0 \\ \cdots & \to & H^j(P^{-2}) & \to & H^j(P^{-1}) & \to & H^j(P^0) & \to & H^j(X) & \to & 0 \end{array}$$

all the above sequences are projective resolutions $\forall j \in \mathbb{Z}$

This is because:
$0 \to Z^j(X) \to X^j \xrightarrow{d} B^{j+1}(X) \to 0$ and
$0 \to B^j(X) \to Z^j(X) \to H^j(X) \to 0$
are two ses's $\forall j \in \mathbb{Z}$
Take projective resolutions of $B^j(X), H^j(X)$ and use Horseshoe lemma twice

Now, we have a sequences of double complexes:

$$\begin{array}{cccccccccc}
\mathbf{P} & : & \cdots \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^{0} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{X} & : & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \| & & \\
\mathbf{A} & : & \cdots \longrightarrow & P^{-1} & \longrightarrow & P^{0} & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Take total complexes: $P^n = \bigoplus_{i+j=n} P^{i,j}$, (co)boundary map: $\epsilon = d + (-1)^j \delta$ where

$$\begin{array}{ccc}
P^{i,j+1} & \longrightarrow & P^{i+1,j+1} \\
\delta \uparrow & & \uparrow \\
P^{i,j} & \xrightarrow{d} & P^{i+1,j}
\end{array}$$

and get a sequence $P \to X \to A$, where $A = \mathrm{cone}(P \to X)$

$$\Rightarrow \qquad P \to X \to A \to P[1] \text{ exact triangle}$$

(Exercise)Check: P satisfies (P), and A acyclic ☐

Definition 1.10.12 (Derived functors)

$$\begin{array}{ccc}
\mathbf{K}(A) & \xrightarrow{F} & \mathbf{K}(B) \\
p \uparrow \ \downarrow \pi \ \uparrow i & & p \uparrow \ \downarrow \pi \ \uparrow i \\
\mathbf{D}(A) & \overset{LF=\pi\circ F\circ p}{\underset{RF=\pi\circ F\circ i}{\rightrightarrows}} & \mathbf{D}(B)
\end{array}$$

LF total left derived functor of F
RF total right derived functor of F
(Note: the p here, and from now on, is the p' previously)

Example: A, B dg k-algebras, M dg B-A-bimodule. Then,

$$\mathbf{K}(A) \underset{\mathcal{H}\mathrm{om}_B(M,-)}{\overset{M\otimes A-}{\rightleftarrows}} \mathbf{K}(B)$$

$$\mathbf{D}(A) \underset{R\,\mathcal{H}\mathrm{om}_B(M,-)}{\overset{L(M\otimes A-)=M\otimes^L_A-}{\rightleftarrows}} \mathbf{D}(B)$$

Proposition 1.10.13
If $F : \mathbf{K}(A) \to \mathbf{K}(B)$ is left adjoint to $G : \mathbf{K}(B) \to \mathbf{K}(A)$
then $LF : \mathbf{D}(A) \to \mathbf{D}(B)$ is left adjoint to $RG : \mathbf{D}(B) \to \mathbf{D}(A)$

Proof
$\mathrm{Hom}_{\mathbf{D}(B)}(LFX, Y) = \mathrm{Hom}_{\mathbf{D}(B)}(\pi FpX, Y)$
$\cong \mathrm{Hom}_{\mathbf{K}(B)}(FpX, iY)$
$\cong \mathrm{Hom}_{\mathbf{K}(A)}(pX, GiY)$
$\cong \mathrm{Hom}_{\mathbf{D}(A)}(X, \pi GiY)$
$= \mathrm{Hom}_{\mathbf{D}(A)}(X, RGY)$ ☐

Consequences: $\mathrm{Ext}^n_A(X,Y) \cong \mathrm{Hom}_{\mathbf{D}(A)}(X, Y[n])$
(Because LHS= $H^n(R\,\mathcal{H}\mathrm{om}_A(X,-)(Y)) = H^n(\mathcal{H}\mathrm{om}_A(X, iY)) = \mathrm{Hom}_{\mathbf{K}(A)}(X, Y[n])$ =RHS)

1.11 Morita theorem for derived module categories

Definition 1.11.1
Let A be algebra over commutative ring k. A complex of A-module X is perfect if it is a bounded complex of f.g. projective A-modules.
Let the subcategory $\mathrm{Perf}(A) = \{X \in \mathbf{D}(A) | \exists X' \text{ perfect, } X \text{ is quasi-isomorphic to } X'\}$

If T is a complex, denote $\mathrm{Thick}(T)$ as the smallest thick subcategory of $\mathbf{D}(A)$ that contains T

Proposition 1.11.2
Let A be algebra over commutative ring k

$$\mathrm{Perf}(A) = \mathrm{Thick}(A) = \mathbf{D}(A)^c = \{X \in \mathbf{D}(A) | \operatorname{Hom}_{\mathbf{D}(A)}(X, -) \text{ preserves arbitrary direct sums}\}$$

Proof
First equality:

(1) $\mathrm{Perf}(A) \subseteq \mathrm{Thick}(A)$
Let $P = (0 \to P^0 \to P^1 \to \cdots \to P^n \to 0), P^i$ f.g. projective
Induction on n to show $P \in \mathrm{Thick}(A)$:
$n = 0$: Proposition 1.5.5 $\mathbf{proj}(A) = \mathbf{Proj}(A)^c$
$n > 0$: Let $P' = (0 \to P^0 \to P^1 \to \cdots \to P^{n-1} \to 0)$
$\Rightarrow$ $0 \to P' \to P \to P^n[-n] \to 0$ is a degreewise split ses
$\Rightarrow$ $P' \to P \to P^n[-n] \to P'[1]$ exact in $\mathbf{K}(A)$
$\Rightarrow$ $P' \to P \to P^n[-n] \to P'[1]$ exact in $\mathbf{D}(A)$
Induction hypothesis: $P', P^n[-n] \in \mathrm{Thick}(A)$
$\Rightarrow$ $P \in \mathrm{Thick}(A)$

(2) Show $\mathrm{Perf}(A)$ is a thick subcategory of $\mathbf{D}(A)$ (Exercise)

Second equality:

(1) $\mathrm{Thick}(A) \subseteq \mathbf{D}(A)^c$:
Suffice to show that $\mathbf{D}(A)^c$ thick subcategory of $\mathbf{D}(A)$ and $A \in \mathbf{D}(A)^c$

Claim: $\mathbf{D}(A)^c$ thick subcategory of $\mathbf{D}(A)$
Proof of Claim:
Closed under shift:
X compact $\Rightarrow$ $\operatorname{Hom}_{\mathbf{D}(A)}(X[1], -) = \operatorname{Hom}_{\mathbf{D}(A)}(X, -[-1])$ preserves $\bigoplus$ $\Rightarrow$ $X[1]$ compact
Closed under taking triangles:
Suppose $X \to Y \to Z \to X[1]$ exact in $\mathbf{D}(A)$ and $X, Z \in \mathbf{D}(A)^c$
For arbitrary $\bigoplus M_\lambda$ in $\mathbf{D}(A)$

$$\begin{array}{ccccc} \operatorname{Hom}_{\mathbf{D}(A)}(Z, \bigoplus M_\lambda) & \longrightarrow & \operatorname{Hom}_{\mathbf{D}(A)}(Y, \bigoplus M_\lambda) & \longrightarrow & \operatorname{Hom}_{\mathbf{D}(A)}(X, \bigoplus M_\lambda) \\ \downarrow \cong & & \uparrow & & \downarrow \cong \\ \bigoplus \operatorname{Hom}_{\mathbf{D}(A)}(Z, M_\lambda) & \longrightarrow & \bigoplus \operatorname{Hom}_{\mathbf{D}(A)}(Y, M_\lambda) & \longrightarrow & \bigoplus \operatorname{Hom}_{\mathbf{D}(A)}(X, M_\lambda) \end{array}$$

is a commutative diagram of exact sequences (in $\mathbf{Ab}$)
$\Rightarrow$ the middle map is also an isomorphism by Five Lemma
Closed under finite direct sums:
$\operatorname{Hom}_{\mathbf{D}(A)}(\bigoplus_{\text{finite}} X_i, -) \cong \bigoplus_{\text{finite}} \operatorname{Hom}_{\mathbf{D}(A)}(X_i, -)$ ■

(2) $\underline{\text{Thick}(A) \supseteq \mathbf{D}(A)^c}$:
Let $X \in \mathbf{D}(A)^c$
Proposition 1.10.4: $X \simeq \varinjlim(P_0 \to P_1 \to \cdots)$
$\Rightarrow$ $\bigoplus P_k \to \bigoplus P_k \to X \to \bigoplus P_k[1]$ exact triangle in $\mathbf{K}(A)$, so exact triangle in $\mathbf{D}(A)$
$\Rightarrow$ $\bigoplus \mathrm{Hom}_{\mathbf{D}(A)}(X, P_k) \hookrightarrow \bigoplus \mathrm{Hom}_{\mathbf{D}(A)}(X, P_k) \to \mathrm{Hom}_{\mathbf{D}(A)}(X, X) \to * \hookrightarrow *$
$\Rightarrow$ $0 \to \bigoplus \mathrm{Hom}(X, P_k) \to \bigoplus \mathrm{Hom}(X, P_k) \to \mathrm{Hom}(X, X) \to 0$ ses in $\mathbf{Ab}$
(The two injection is due to the degreewise split property of X)
$\Rightarrow$ $\mathrm{Hom}(X, X) \cong \varinjlim(X, P_k)$
(This $\cong$ is in $\mathbf{Ab}$, by definition of $\varinjlim$ and maps induced by the injections before)
$\Rightarrow$ $1_X = (X \to P_k \to X)$ for some k
$\Rightarrow$ X summand of P_k and so done according to Keller
(question, there was no restriction on P_k, why is this done?)

□

Remark. We can get around our question at the end of the proof using some other technique, but we will not present them here.

Proposition 1.11.3 (Infinite dévissage)
$A = k$-algebra, then $\mathbf{D}(A)$=Loc(A) := smallest localising subcategory of $\mathbf{D}(A)$ containing A
(dévissage means unscrewing, see proof for "intuition")

Proof
Let $X \in \mathbf{D}(A)$, $X \simeq \varinjlim(P_k)$
$\Rightarrow$ $\bigoplus P_k \to \bigoplus P_k \to X \to \bigoplus P_k[1]$ exact in $\mathbf{D}(A)$
By construction each $P_k \in$Loc(A) $\Rightarrow$ $X \in$Loc(A) □

Theorem 1.11.4 (Rickard)
A, B are k-algebras, B flat as k-module. TFAE:

(1) $\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories (so are the $\mathbf{D}^+, \mathbf{D}^-, \mathbf{D}^b$)

(2) $\mathrm{Perf}(A) \cong \mathrm{Perf}(B)$ as triangulated categories

(3) $\exists T$ bounded complex of fg projective B-module (i.e. perfect B-module) s.t.

- $\mathrm{End}_{\mathbf{D}(B)}(T)^{op} \cong A$ as k-algebra
  $\mathrm{Hom}_{\mathbf{D}(B)}(T, T[n]) = 0 \ \forall n \neq 0$
- Perf(B)=Thick(T)

(4) $\exists X$ bounded complex of B - A-bimodules s.t. $\mathbf{D}(A) \xrightarrow{X \otimes^L_A -} \mathbf{D}(B)$ is an equivalence

We first consider the following remarks before proving this theorem.

Remark. A, B are k-algebras, X complex of B - A-bimodules

- $X \otimes_A - : \mathbf{C}(A) \to \mathbf{C}(B)$ right exact functor
- If the components of X are flat as right A-modules, then $X \otimes_A - : \mathbf{C}(A) \to \mathbf{C}(B)$ is exact
- $X \otimes_A - : \mathbf{K}(A) \to \mathbf{K}(B)$ is exact
  (Because, if $f : M \to N$ is a chain map, then cone$(X \otimes_A M \to X \otimes_A N) \cong X \otimes_A$ cone$(M \to N)$)
- $X \otimes^L_A - : \mathbf{D}(A) \to \mathbf{D}(B)$ is exact
  (Becuase, $X \otimes^L_A - = \pi \circ (X \otimes_A -) \circ p$ and these three maps are exact)

- If the components of X are flat as right A-modules
  then $X \otimes_A - : \mathbf{D}(A) \to \mathbf{D}(B)$ (now well-defined on complexes)
  and $X \otimes_A - \;\cong\; X \otimes_A^L -$
  (Because, $X \otimes_A -$ exact $\;\Rightarrow\;$ preserves quisms)

Remark. X, Y complexes of A-modules

$$\begin{array}{ccc} H^n(R\,\mathcal{H}\mathrm{om}_A(X,Y)) & = & \mathrm{Hom}_{\mathbf{D}(A)}(X, Y[n]) \\ \| & & \downarrow \cong \\ H^n(\mathcal{H}\mathrm{om}_A(X, iY)) & = & \mathrm{Hom}_{\mathbf{K}(A)}(X, iY[n]) \end{array}$$

Proof of Theorem 1.11.4

(1)⇒(2): Implication of Proposition 1.11.2

(2)⇒(3): Let $F : \mathrm{Perf}(A) \xrightarrow{\sim} \mathrm{Perf}(B)$, set $T = F(A)$, then

$$\begin{array}{rcll} \mathrm{End}_{\mathbf{D}(B)}(T)^{op} & \cong & \mathrm{End}_{\mathbf{D}(A)}(A)^{op} & \text{via } F \\ & \cong & \mathrm{End}_{\mathbf{K}(A)}(A)^{op} & \text{as } A \text{ homotopically projective } (*) \\ & \cong & \mathrm{End}_A(A)^{op} & \\ & \cong & A & \end{array}$$

(*): In general, P homotopically projective $\;\Rightarrow\; \mathrm{Hom}_{\mathbf{K}(A)}(P, X) = \mathrm{Hom}_{\mathbf{D}(A)}(P, X) \;\; \forall X$, this is a extension of Lemma 1.9.4 (3)
This is because any root $P \overset{s}{\leftarrow} Y \to X$ with s a quism, we get an exact triangle $Y \overset{s}{\to} P \to N \to Y[1]$
P homotopically projective $\;\Rightarrow\; N$ acyclic $\;\Rightarrow\; P \to N$ is zero map $\;\Rightarrow\; \exists t$ s.t. $st \simeq 1_P$
$\Rightarrow\;$ a morphism from P in $\mathbf{D}(A)$ and in $\mathbf{K}(A)$ is the same thing

(3)⇒(4): We want to construct X bounded complex of B - A-bimodules and a quism $\phi : T \to X$ of complexes of B-modules s.t. the following diagram commutes in $\mathbf{K}(B)$

$$\begin{array}{ccc} T & \xrightarrow{\phi} & X \\ {\scriptstyle \sigma(a)} \downarrow & & \downarrow {\scriptstyle \cdot a} \\ T & \xrightarrow{\phi} & X \end{array}$$

where $\sigma : A \xrightarrow{\sim} \mathrm{End}_{\mathbf{D}(B)}(T)^{op}$

Let $C := \mathcal{H}\mathrm{om}_B(T, T)$, a dg k-algebra, i.e.

$$\begin{array}{rcl} C^n & = & \displaystyle\prod_{i \in \mathbb{Z}} \mathrm{Hom}_B(T^i, T^{i+n}) \\ d(f) & = & d \circ f - (-1)^{|f|} f \circ d \end{array}$$

Then T is a dg B - C-bimodule. Note that

$$\begin{array}{rcl} H^n(C) & = & H^n(\mathcal{H}\mathrm{om}_B(T, T)) \\ & = & \mathrm{Hom}_{\mathbf{K}(B)}(T, T[n]) \\ & = & \mathrm{Hom}_{\mathbf{D}(B)}(T, T[n]) \\ & = & \begin{cases} A & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} \end{array}$$

Let $C^- = \tau_{\leq 0}(C) = (\cdots \to C^{-2} \to C^{-1} \to Z^0(C) \to 0 \to \cdots)$
This is a dg subalgebra of C

$$\begin{array}{rcccccccccc}
C & : & \cdots \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow \cdots \\
{\scriptstyle \text{quism}} \uparrow & & & \| & & \| & & \uparrow & & \uparrow & \\
C^- & : & \cdots \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & Z^0(C) & \longrightarrow & 0 & \longrightarrow \cdots \\
{\scriptstyle \text{quism}} \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
A & : & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

(Note $\sigma : A \xrightarrow{\sim} H^0(C)$, so $Z^0(C) \twoheadrightarrow A$ via map induced by σ^{-1}, call this $\widetilde{\sigma^{-1}}$)

Define $X := T \otimes C^-$, dg B - A-bimodule (i.e. complexes of B - A-bimodules)

$$\begin{array}{ccccccc}
\overset{t}{T} & \xrightarrow{\sim} & \overset{t \otimes \mathrm{id}_T}{T \otimes_{C^-} C^-} & \xrightarrow{\phi} & \overset{t \otimes 1_A}{T \otimes_{C^-} A} & = & X \\
 & & \downarrow {\scriptstyle \sigma(a)} & & \downarrow {\scriptstyle \cdot a} & & \\
T & \xrightarrow{\sim} & \underset{\sigma(a)(t) \otimes \mathrm{id}_T}{T \otimes_{C^-} C^-} & \longrightarrow & \underset{\sigma(a)t \otimes 1 = t \otimes a}{T \otimes_{C^-} A} & = & X
\end{array}$$

$\sigma(a)t \otimes 1_A = t \otimes a$ because the map $C^- \to A$ is induced by $\widetilde{\sigma^{-1}} : Z^0(C) \to A$

Claim: ϕ is a quism

Proof of Claim:
Let $\mathcal{U} = \{U \in \mathbf{D}(B \otimes_k (C^-)^{op}) | U \otimes_{C^-} C^- \xrightarrow{\sim} U \otimes_{C^-} A \text{ quism}\}$
Our aim is to show $T \in \mathcal{U}$
We use infinite dévissage 1.11.3 to show $\mathcal{U} = \mathbf{D}(B \otimes_k (C^-)^{op})$

$\mathcal{U}$ is a localizing subcategory of $\mathbf{D}(B \otimes_k (C^-)^{op})$:
$\Delta : U \to V \to W \to U[1]$ exact triangle in $\mathbf{D}(B \otimes_k (C^-)^{op})$,

$$\Rightarrow \quad \begin{array}{ccccccc}
U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & U[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U \otimes_{C^-} A & \longrightarrow & V \otimes_{C^-} A & \longrightarrow & W \otimes_{C^-} A & \longrightarrow & U \otimes_{C^-} A[1]
\end{array}$$

diagram commutes
If $U, V \in \mathcal{U}$ then consider taking the H^n so we get exact sequences of abelian groups, then invoke by Five Lemma and eventually we get $W \in \mathcal{U}$

$B \otimes_k (C^-)^{op} \in \mathcal{U}$:

$$\begin{array}{ccc}
(B \otimes_k (C^-)^{op}) \otimes_{C^-} C^- & \longrightarrow & (B \otimes_k (C^-)^{op}) \otimes_{C^-} A \\
\downarrow \cong & & \downarrow \cong \\
B \otimes_k C^- & \xrightarrow{\text{quism}} & B \otimes_k A
\end{array}$$

The quism at the bottom is due to the fact that B is flat as k-module ■

Claim: $X \otimes^L_A -$ is an equivalence

Proof of Claim:
Essentially Surjective:
$T \overset{\text{quism}}{\cong} X \overset{\text{homotopy equiv}}{\cong} X \otimes^L_A A \in$ essential image
$\mathrm{Perf}(B) = \mathrm{Thick}(T) \subseteq$ essential image

$\underline{X \otimes^L_A - \text{ is fully faithful:}}$

$$\mathbf{D}(A) \underset{RG = R\mathcal{H}\text{om}_B(X,-)}{\overset{LF = X\otimes^L_A -}{\rightleftarrows}} \mathbf{D}(B)$$

Need: The unit of the adjunction $\text{id}_{\mathbf{D}(A)} \to RGLF$ is a natural isomorphism
Strategy: Use infinite dévissage

Let $\mathcal{U} = \{U \in \mathbf{D}(A) | U \xrightarrow{\sim} RGLFU \text{ in } \mathbf{D}(A)\}$
$\mathcal{U}$ is a localising subcategory:
$\mathcal{U}$ is a triangulated subcategory of $\mathbf{D}(A)$ because LF and RG are exact functors between triangulated categories

Preserve direct sums:
LF preserves direct sum because it is a left adjoint
For RG: Need to check

$$\bigoplus_\lambda RGU_\lambda \xrightarrow{\sim} RG(\bigoplus_\lambda U_\lambda) \quad \text{in } \mathbf{D}(A)$$

indeed $\forall n$,

$$\begin{array}{ccc} H^n(\bigoplus R\mathcal{H}\text{om}_B(X, U_\lambda)) & \xrightarrow{\sim} & H^n(R\mathcal{H}\text{om}_B(X, \bigoplus U_\lambda)) \\ \downarrow \cong & & \| \\ \bigoplus H^n(R\mathcal{H}\text{om}_B(X, U_\lambda)) & & \| \\ \| & & \| \\ \bigoplus \text{Hom}_{\mathbf{D}(B)}(X, U_\lambda[n]) & \xrightarrow{\sim} & \text{Hom}_{\mathbf{D}(B)}(X, \bigoplus U_\lambda[n]) \end{array}$$

Note we have isomorphism on the bottom row because $X \overset{\text{quism}}{\cong} T$ perfect $\Rightarrow$ X compact in $\mathbf{D}(B)$

$\underline{A \in \mathcal{U}}$:
Need: $A \xrightarrow{\sim} RGLFA = R\mathcal{H}\text{om}_B(X, X) \quad \text{in } \mathbf{D}(A)$
i.e. $H^n(A) \to \text{Hom}_{\mathbf{D}(B)}(T, T[n])$
This is true as $\mathcal{U}$ is localizing subcategory ■

□

Lemma 1.11.5
A is k-algebra

$$\mathbf{D}^-(A) = \{X \in \mathbf{D}(A) | \forall P \text{ perfect}, \ \text{Hom}_{\mathbf{D}(A)}(P, X[n]) = 0 \ \forall n \gg 0\}$$

Corollary 1.11.6
A, B are k-algebras, f.g. projective as k-modules
k commutative Noetherian ring. Then TFAE:

(1) $\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories

(2) $\exists X$ bounded complexes of B - A-bimodules, f.g. projective as left B-modules and right A-modules
$\exists Y$ bounded complexes of A - B-bimodules, f.g. projective as left A-modules and right B-modules
s.t. $Y \otimes_B X \cong A$ in $\mathbf{D}(A \otimes_k A^{op})$
and $X \otimes_A Y \cong B$ in $\mathbf{D}(B \otimes_k B^{op})$

In this case $\mathbf{D}(A) \underset{Y\otimes_B -}{\overset{X\otimes_A -}{\rightleftarrows}} \mathbf{D}(B)$ are equivalences inverse to each other

Remark. This is like the Morita theorem 1.5.3 for derived categories.

Lemma 1.11.7
A, B are k-algebras, B flat k-module
Let X be complex of B - A-bimodules s.t. $X \otimes_A^L - : \mathbf{D}(A) \to \mathbf{D}(B)$ is an equivalence
Let S be a perfect complex of A-modules s.t. $B \xrightarrow[\text{quism}]{\sim} X \otimes_A S$ in $\mathbf{D}(B)$ (such S always exists)
Then $X \xrightarrow[\text{quism}]{\sim} \mathcal{H}\text{om}_A(S, A)$ in $\mathbf{D}(A^{op})$

Proof

$$\begin{array}{rcl} X \otimes_A^L - : \mathbf{D}(A) & \to & \mathbf{D}(B) \\ A & \mapsto & {}_B X \\ S & \mapsto & B \end{array}$$

is an equivalence, so $\text{Hom}_{\mathbf{D}(A)}(S, A[n]) \cong \text{Hom}_{\mathbf{D}(B)}(B, X[n]) \ \forall n$
$\Rightarrow$ the composition $\mathcal{H}\text{om}_A(S, A) \xrightarrow{X \otimes_A -} \mathcal{H}\text{om}_B(X \otimes_A S, X) \xrightarrow{\sim} \mathcal{H}\text{om}_B(B, X) \cong X$ is a quism □

Consequence: $\mathcal{H}\text{om}_A(S, A)$ is a perfect complex of right A-modules

Lemma 1.11.8
Let X, Y be complexes as in (2) in Corollary 1.11.6 and give equivalences on $\mathbf{D}(A), \mathbf{D}(B)$

(1) $\begin{array}{rcl} A & \to & \mathcal{H}\text{om}_B(X, X)^{op} \\ a & \mapsto & (x \mapsto xa) \end{array}$ is a quism of dg k-algebras

(2) $Y \cong \mathcal{H}\text{om}_B(X, B)$ in $\mathbf{D}(A \otimes_k B^{op})$
$X \cong \mathcal{H}\text{om}_A(Y, A)$ in $\mathbf{D}(B \otimes_k A^{op})$

Proof

(1) By considering taking H^*, this is the same as saying $\begin{array}{rcl} A & \xrightarrow{\sim} & \text{End}_{\mathbf{D}(B)}(X)^{op} \\ a & \mapsto & (x \mapsto xa) \end{array}$ as k algebras, and $0 \cong \text{Hom}_{\mathbf{D}(B)}(X, X[n]) \ \forall n \neq 0$
This is true because $X \otimes_A -$ is an equivalence by assumption

(2) $X \otimes_A - : \mathbf{D}(A) \leftrightarrows \mathbf{D}(B) : \mathcal{H}\text{om}_B(X, -)$ adjoint pairs
and $X \otimes_A -$ is an equivalence with inverse $Y \otimes_B -$
$\Rightarrow$ $\mathcal{H}\text{om}_B(X, -) \cong Y \otimes_B -$ (natural isom)
$\Rightarrow$ $\mathcal{H}\text{om}_B(X, B) \cong Y$ in $\mathbf{D}(A \otimes_k B^{op})$ This is an isom. in $\mathbf{D}(A \otimes_k B^{op})$ by the naturality

□

Corollary 1.11.9
A, B are k-algebras, f.g. projective as k-modules. k commutative Noetherian ring.
If $\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories $\Rightarrow$ $\mathbf{D}(A^{op}) \cong \mathbf{D}(B^{op})$

Proof
$X \otimes_A - : \mathbf{D}(A) \leftrightarrows \mathbf{D}(B) : Y \otimes_B - \quad \Rightarrow \quad - \otimes_B X : \mathbf{D}(B^{op}) \leftrightarrows \mathbf{D}(A^{op}) : - \otimes_A Y$ □

Corollary 1.11.10
A, B are k-algebras, f.g. projective as k-modules. k commutative Noetherian ring.
$\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories $\Rightarrow$ $Z(A) \cong Z(B)$ as k-agebras

Proof
Corollary 1.11.6 $\Rightarrow$ $\exists X, Y$ giving equivalences of derived categories.
Define $\phi : Z(A) \to Z(B)$ as follows
First, $B \xrightarrow{\sim} \mathrm{End}_{\mathbf{D}(A)}(Y)^{op}$ (via $b \mapsto (y \mapsto yb)$)

The "left multiplication by a" $(a\cdot)$ on Y is contained in

$$\begin{array}{rcl} Z(\mathrm{End}_{\mathbf{D}(A)}(Y)) & \cong & Z(B) \\ a\cdot & \leftrightarrow & \phi(a) \end{array}$$

i.e. $[a\cdot] = [\cdot\phi(a)]$ in $\mathrm{End}_{\mathbf{D}(A)}(Y)$
Check ϕ is a k-algebra homomorphism
Similarly $\psi : Z(B) \to Z(A)$ is given by $A \xrightarrow{\sim} \mathrm{End}_{\mathbf{D}(B)}(X)^{op}$ and we identify $[b\cdot] = [\cdot\psi(b)]$ in $\mathrm{End}_{\mathbf{D}(B)}(X)$

Aim: show $\psi \circ \phi = 1_{Z(A)}$
Let $a \in Z(A)$ $\Rightarrow$ $[\cdot\psi\phi(a)] = [\phi(a)\cdot]$ in $\mathrm{End}_{\mathbf{D}(B)}(X)$
Lemma 1.11.8 (2) $\Rightarrow$ $X \cong \mathcal{H}\mathrm{om}_A(Y, A)$ in $\mathbf{D}(B \otimes_k A^{op})$

> **Claim:** $[\phi(a)\cdot] = [\cdot a]$ in $\mathrm{End}_{\mathbf{D}(B)}(X)$
> **Proof of Claim:**
> If $f \in \mathcal{H}\mathrm{om}_A(Y, A)$
> $\Rightarrow$ $\phi(a)f = f(\cdot\phi(a)) \cong f(a\cdot) = af = fa$
> (Note that left B action of $\mathcal{H}\mathrm{om}_A(Y, A)$ comes from right B-action of Y) ■

Since $A \xrightarrow{\sim} \mathrm{End}_{\mathbf{D}(B)}(X)^{op}$ is an isom, $\psi\phi(a) = a$ □

Corollary 1.11.11
Setup as before, $\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories
$\Rightarrow$ the Grothendieck groups $K_0(A) \cong K_0(B)$

Corollary 1.11.12
Setup as before, $\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories
$\Rightarrow$ the Hochschild cohomology $HH^*(A) \cong HH^*(B)$ as graded k-algebra

Corollary 1.11.13
Setup as before. Suppose A, B are self-injective (i.e. $\Leftrightarrow {}_AA$ is injective)
Then $\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories $\Rightarrow$ $\underline{\mathbf{Mod}}(A) \cong \underline{\mathbf{Mod}}(B)$
(i.e. $\mathrm{Ob}(\underline{\mathbf{Mod}}(A)) = \mathrm{Ob}(\mathbf{Mod}(A))$, $\underline{\mathrm{Hom}}_A(M, N) = \mathrm{Hom}_A(M, N)/\{f | f$ factors through projectives$\}$)

Proof of Corollary 1.11.6
Suppose $\mathbf{D}(A) \cong \mathbf{D}(B)$ as triangulated categories
By Rickard's Theorem 1.11.4, $\exists X$ complex of B-A-bimodules s.t. $X \otimes^L_A - : \mathbf{D}(A) \xrightarrow{\sim} \mathbf{D}(B)$
Note: ${}_BX \cong T$ (quism) in $\mathbf{D}(B)$ and T perfect

Lemma 1.11.7: $X_A \cong \mathcal{H}\mathrm{om}_A(S, A)$ (quism) and S perfect
$\Rightarrow$ $H^n(X) \cong H^n(T)$ f.g. k-modules, nonzero only for finitely many n
$\exists$"projection resolution" of X as complex of B-A-bimodules
i.e. P bounded right complex of f.g. projective B-A-bimodules $P \xrightarrow{\text{quism}} X$

Take n s.t. $\forall i \leq n$, $T^i = 0 = \mathcal{H}\mathrm{om}_A(S, A)^n$. Consider

$$\begin{array}{ccccccccc} P = & (\cdots \longrightarrow & P^{n-1} & \longrightarrow & P^n & \longrightarrow & P^{n+1} & \longrightarrow \cdots) \\ & & \uparrow & & \uparrow & & \| & \\ P' = & (\cdots \longrightarrow & 0 & \longrightarrow & Z^n(P) & \longrightarrow & P^{n+1} & \longrightarrow \cdots) \end{array}$$

This is a quism and P' is a bounded complex of f.g. bimodules, all projective (as bimodules) except for $Z^n(P)$

Claim: $Z^n(P)$ is projective as left B-modules and right A-module

Proof of Claim:
We have a quism $P' \to T$
Then $\text{cone}(P' \to T) = (0 \to Z^n(P) \to P^{n+1} \oplus T^n \to P^{n+2} \oplus T^{n+1} \to \cdots \to 0)$ is acyclic
By splitting off projective left B-modules terms from the right
$\Rightarrow \quad Z^n(P)$ is projective as left B-modules
Similarly for A-module ■

So $P' \cong X$ in $\mathbf{D}(B \otimes_k A^{op})$
P' bounded complex of f.g. B - A-bimodules, all projective except for the left most nonzero term which is projective as left/right module
$\Rightarrow \quad P' \otimes^L_A - \cong P' \otimes_A -$ is an equivalence $\mathbf{D}(A) \to \mathbf{D}(B)$ □

Proof of Corollary 1.11.13
Suppose $\mathbf{D}(A) \cong \mathbf{D}(B)$
By the proof of Corollary 1.11.6
$\exists X$ bounded complex of f.g. B - A-bimodules all projective as bimodules, except for the leftmost nonzero term (X^0) which projective as left B-module and as right A-modules
s.t. $X \otimes_A - : \mathbf{D}(A) \leftrightarrows \mathbf{D}(B) : \mathcal{H}\text{om}_B(X, -)$ (equivalence)
Note: $\mathcal{H}\text{om}_B(X, -) \cong \mathcal{H}\text{om}_B(X, B) \otimes_B -$ by additivity argument and X projective as left B-module

Let $Y = \mathcal{H}\text{om}_B(X, B)$. Then Y is a bounded complex of f.g. A - B-bimodules, all projective except for the rightmost nonzero term, $\text{Hom}_B(X^0, B)$ which is projective as left and right modules

$X = (0 \to X^0 \to X^1 \to \cdots \to X^n \to 0)$
$Y = (0 \to \text{Hom}_B(X^n, B) \to \cdots \text{Hom}_B(X^1, B) \to \text{Hom}_B(X^0, B) \to 0)$

$Y \otimes_B X \cong A$ in $\mathbf{D}(A \otimes_k A^{op})$
$X \otimes_A Y \cong B$ in $\mathbf{D}(B \otimes_k B^{op})$
All terms of $Y \otimes_B X$, except for the degree zero, term are projective as A - A-bimodules
Note: M projective B - A-bimodule, and N is A - B-bimodule projective as left and right module
$\Rightarrow \quad M \otimes_A N$ projecitve as B - B-bimodule, and $N \otimes_B M$ projective as A - A-bimodule

$$\begin{array}{ccccccccccccc}
Y \otimes_A X & : & (0 & \longrightarrow & Z^{-n} & \longrightarrow \cdots \longrightarrow & Z^0 & \longrightarrow \cdots \longrightarrow & Z^n & \longrightarrow & 0) \\
\downarrow \cong & & & & \downarrow & & \downarrow & & \downarrow & & \\
A & : & (0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & A & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0)
\end{array}$$

Since A, B are self-injective, $A \otimes_k B$ is self-injective
We can split off projective (=injective) terms from the right and from the left to get $Z^0 \cong A$ in $\underline{\mathbf{Mod}}(A \otimes A^{op})$

But $Z^0 = X^0 \otimes_A \text{Hom}_B(X^0, B) \oplus$ projectives
$\Rightarrow \quad X^0 \otimes_A \text{Hom}_B(X^0, B) \cong A$ in $\underline{\mathbf{Mod}}(A \otimes A^{op})$
$\text{Hom}_B(X^0, B) \otimes_B X^0 \cong B$ in $\underline{\mathbf{Mod}}(B \otimes B^{op})$ □

2 Modular Representation Theory

2.1 Idempotents and blocks

Let A be f.d. k-algebra where k a field. Recall:

- $$\begin{aligned} J(A) &= \bigcap \text{ max left ideals of } A \\ &= \bigcap \{\mathrm{Ann}_A(S) | S \text{ simple } A\text{-modules}\} \\ &= \bigcap \text{ max right ideals of } A \\ &= \bigcap \{\mathrm{Ann}_A(S) | S \text{ simple right } A\text{-modules}\} \\ &= \{a \in A | 1 - xay \in A^\times \ \forall x, y \in A\} \quad (1 + J(A) \subseteq A^\times) \\ &= \text{ the largest nilpotent ideal of } A \text{ if } A \text{ is Noethernian} \end{aligned}$$

- M an A-module, then
$$\begin{aligned} M \text{ semisimple} &\Leftrightarrow M = \bigoplus \text{ simple } A\text{-modules} \\ &\Leftrightarrow M = \sum \text{ simple } A\text{-modules} \\ &\Leftrightarrow \text{ Every submodule } N \text{ of } M \text{ is direct summand} \\ &\qquad \text{i.e. } \exists \text{ submodule } N' \text{ of } M \text{ s.t. } N \oplus N' = M \end{aligned}$$

- All submodules and quotient modules of a semisimple module are semisimple

- **Artin-Wedderburn's Theorem**
$$\begin{aligned} A \text{ semisimple} &\Leftrightarrow {}_AA \text{ semisimple} \\ &\Leftrightarrow \text{ Every } A\text{-module is semisimple} \\ &\Leftrightarrow \text{ Every } A\text{-module is projective} \\ &\Leftrightarrow \text{ Every } A\text{-module is injective} \\ &\Leftrightarrow \text{ Every ses of } A\text{-module is split} \\ &\Leftrightarrow J(A) = 0 \\ &\Leftrightarrow A \cong \prod_{i=1}^{r} M_{n_i}(D_i) \quad (D_i \text{ division algebras}) \end{aligned}$$

- $J(A/J(A)) = 0 \Rightarrow A/J(A) \cong \prod_{i=1}^r M_{n_i}(D_i)$
In this case, $\exists$ exactly r isoclasses (isomorphism classes) of simple A-modules correspond to columns of $M_{n_i}(D_i)$, $1 \leq i \leq r$

- M is A-module
$$\begin{aligned} \mathrm{Rad}(M) &= \bigcap \text{ max submodules of } M \\ &= \text{ the smallest submodule of } M \text{ s.t. } M/\mathrm{Rad}(M) \text{ semisimple} \\ &= J(A)M \quad (\text{as } M \text{ f.g. } A\text{-module}) \\ \mathrm{Soc}(M) &= \sum \text{ simple submodules of } M \\ &= \text{ the largest semisimple submodule of } M \\ &= \{m \in M | J(A)m = 0\} \end{aligned}$$

Definition 2.1.1

(1) An element e of A is an idempotent if $e^2 = e \neq 0$

(2) Two idempotents e, f of A are orthogonal if $ef = 0 = fe$

(3) An idempotent e of A has an orthogonal decomposition if $\exists$ an orthogonal pair of idempotents i, j s.t. $e = i + j$

(4) An idempotent e of A is primitive if e has no orthogonal decomposition

(5) Two idempotents e, f of A are conjugate in A if $\exists x \in A^\times$ s.t. $f = xex^{-1}$

(6) A central idempotent of A is an idempotent of $Z(A)$

(7) A block is a primitive idempotent of $Z(A)$

Lemma 2.1.2

(1) If $e \neq 1$ is an idempotent of A, then $1 - e$ is an idempotent which is orthogonal to e. In this case, $A = Ae \oplus A(1-e)$ as left A-modules

(2) Let e, i be idempotents of A
i appears in an orthogonal decomposition of $e \Leftrightarrow ei = e = ie \Leftrightarrow i = eie$

(3) Let e be central idempotent and i be a primitive idempotent of A
$ei \neq 0 \Leftrightarrow ei = i$

(4) b, b' are blocks of A. Then $bb' \neq 0 \Leftrightarrow b = b'$

Proposition 2.1.3

(1) 1_A has a primitive orthogonal decomposition
i.e. a finite set I of pairwise orthogonal primitive idempotents of A s.t. $\sum_{i \in I} i = 1_A$

(2) (**Krull-Schmidt Theorem**) If I, J are two primitive orthogonal idempotents of 1_A, then $\exists$ bijection $f : I \to J, \exists x \in A^\times$ s.t. $f(i) = xix^{-1} \ \forall i \in I$

(3) The following maps are bijections:

$$\left\{ \begin{array}{c} \text{ccl of primitive} \\ \text{idem. of } A \end{array} \right\} \to \left\{ \begin{array}{c} \text{iso-classes of} \\ \text{proj. ind. } A\text{-modules} \end{array} \right\} \to \left\{ \begin{array}{c} \text{iso-classes of} \\ \text{simple } A\text{-modules} \end{array} \right\}$$
$$e \mapsto Ae$$
$$P \mapsto P/\operatorname{Rad}(P) = P/J(A)P$$

(4) If $M = M_1 \oplus \cdots \oplus M_r$ is a decomposition of A-modules into indecomposable summands, let $\pi_i : M \to M$ be the projection on M_i
$\Rightarrow \quad \{\pi_1, \ldots, \pi_r\}$ is a primitive orthogonal decomposition of id_M in $\operatorname{End}_A(M)$

Proposition 2.1.4
The set $\mathcal{B}$ of the blocks of A is a primitive orthogonal decomposition of 1_A in $Z(A)$, i.e.

- $\mathcal{B}$ is a finite set
- $bb' = 0 \ \forall b, b' \in \mathcal{B}, b \neq b'$
- $\sum_{b \in \mathcal{B}} b = 1_A$

Proof
By dimension argument, there is a primitive orthogonal decomposition of 1_A in $Z(A)$: $1_A = b_1 + \cdots + b_r$

If $b \in \mathcal{B}$, then $b = bb_1 + \cdots + bb_r$
$\Rightarrow \quad b = bb_i$ (for some i) $= b_i$ by Lemma 2.1.2 (4) □

Definition 2.1.5
A is local if $A/J(A)$ is a division algebra

Proposition 2.1.6
TFAE:

(1) A is local

(2) $A \setminus A^\times = J(A)$

(3) $A \setminus A^\times$ is an ideal of A

(4) 1_A is a primitive idempotent of A

(5) 1_A is the only idempotent of A

(6) (**Fitting's Lemma**) Every element of A is either invertible or nilpotent

Proof
(1)$\Rightarrow$(2): $x \in A \setminus J(A) \Rightarrow x + J(A) \neq 0$ in $A/J(A)$
But $A/J(A)$ division algebra by Artin-Wedderburn $\Rightarrow \exists y \in A \setminus J(A)$ s.t. $xy - 1, yx - 1 \in J(A)$
$\Rightarrow \quad yx, xy \in 1 + J(A) \subseteq A^\times$
$\Rightarrow \quad x \in A^\times$

(2)$\Rightarrow$(1), (2)$\Rightarrow$(3), (4)$\Leftrightarrow$(5): Trivial

(3)$\Rightarrow$(2): If $A \setminus A^\times$ is an ideal of A, then it is a unique max. left ideal of $A \Rightarrow J(A) = A \setminus A^\times$

(2)$\Rightarrow$(6): $J(A)$ is nilpotent

(5)$\Rightarrow$(6)$\Rightarrow$(3): See any book with proof of Fitting's Lemma □

Consequences:
- M an A-module, then $\quad$ M indecomposable $\Leftrightarrow \mathrm{End}_A(M)$ local
- i idempotent of A, then $\quad$ i primitive $\Leftrightarrow iAi$ local
- e central idempotent of A, then $\quad$ e is block $\Leftrightarrow eZ(A)e$ local

Corollary 2.1.7
All subalgebras and quotient algebras of local algebra are local

Proof
Subalgebras: by Proposition 2.1.6 (5)
Quotients: Suppose $A \twoheadrightarrow B \Rightarrow A/J(A) \twoheadrightarrow B/J(B)$ □

Lemma 2.1.8 (Rosenberg's Lemma)
Let i be primitive idempotent of A. Let $\{I_\lambda | \lambda \in \Lambda\}$ be set of ideals of A
If $i \in \sum_{\lambda \in \Lambda} I_\lambda$, then $i \in I_\lambda$ for some $\lambda \in \Lambda$

Proof
Suppose $i \in \sum_{\lambda \in \Lambda} I_\lambda$
$\Rightarrow \quad i \in \sum_{\lambda \in \Lambda} iI_\lambda i$ and $iI_\lambda i$ is an ideal of iAi

Since i primitive $\Rightarrow$ iAi local
$\Rightarrow$ either $iI_\lambda i = iAi$ or $iI_\lambda i \subseteq J(iAi)$
If $iI\lambda i \subseteq J(iAi)$ $\forall \lambda \in \Lambda$
$\Rightarrow$ $i \in J(iAi)$ nilpotent
$\Rightarrow$ $i = i^2 = i^3 = \cdots = i^n = 0$ which is absurd

$\therefore$ $\exists \lambda \in \Lambda$ s.t. $I_\lambda \supseteq iI_\lambda i = iAi \ni i$ $\square$

Lemma 2.1.9 (Idempotent Lifting)
A, B f.d. k-aglebras, where ka field. I, J ideals of A, B respectively
Let $f : A \to B$ be k-algebra homomorphism s.t. $f(I) = J$

(1) If i is primitive idempotent of A contained in I, then $f(i) = 0$ or $f(i)$ is a primitive idempotent of B (contained in J)

(2) If j is a primitive idempotent of B contained in J, then $\exists i$ primitive idempotent of A contained in I s.t. $f(i) = j$

(3) If i, i' are primitive idempotents of A contained in I s.t. $f(i) \neq 0 \neq f(i')$, then $i \sim_A i'$ $\Leftrightarrow f(i) \sim_B f(i')$

($\sim_R$ means conjugate in R)

Sketch of Proof
First consider $A = I, B = J$
Case 1: (Usual version of Idempotent Lifting)
$B = A/J(A)$, $f : A \twoheadrightarrow B$ canonical surjection

(1) If i is a primitive idempotent of A s.t. $f(i) \neq 0$
$\Rightarrow$ iAi is local (by Fitting's Lemma)
$\Rightarrow$ $f(iAi) = f(i)Bf(i)$ is local
$\Rightarrow$ $f(i)$ is a primitive idempotent of B

(2) May assume $J(A)^2 = 0$ (This is because, by nilpotency of taking Jacobson radical, then consider successive surjection $A/J(A)^i \twoheadrightarrow A/J(A)^{i+1}$ and the identity $J(A/J(A)^i) = J(A)/J(A)^i$)

Let j be a primitive idempotent of B
Choose $x \in A$ s.t. $f(x) = j$
$\Rightarrow$ $f(x^2) = j^2 = j = f(x)$ $\Rightarrow$ $x^2 - x \in \ker f = J(A)$ $J(A)^2 = 0$ $\Rightarrow$ $(x^2 - x)^2 = 0$
Set $i = 3x^2 - 2x^3$ $\Rightarrow$ $f(i) = j$ and $i^2 - i = 0$

If i is not primitive then j is not primitive, contradiction. Therefore i is primitive.

(3) Omitted/Exercise

Case 2: $f : A \to B$ surjective algebra homomorphism. Consider

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/J(A) & \overset{\overline{f}}{\twoheadrightarrow} & B/J(B) \end{array}$$

Artin-Wedderburn: $A/J(A)$ and $B/J(B)$ are finite products of matrix algebras over division algebras
$\Rightarrow$ $\ker \overline{f}$ is a finite product of some matrix algebra factors of $A/J(A)$
$\Rightarrow$ $\overline{f}$ satisfies (1)-(3)
By Case 1, $A \twoheadrightarrow A/J(A)$ and $B \twoheadrightarrow B/J(B)$ satisfy (1)-(3). So we are done my commutative of the diagram.

Case 3: (General case)

(1) Let i be a primitive idempotent of A contained in I
Apply Case 2 to $f : A \to f(A)$
$\Rightarrow$ $f(i)$ primitive idempotent of $f(A)$ contained in J
Suppose $f(i) = j_1 + j_2$ is orthogonal idem. decomposition in B
Then $j_1 f(i) = j_1$ (by orthogonality), which is in $J \subseteq f(A)$
Similarly, $j_2 f(i) = j_2 \in J \subseteq f(A)$, contradicting primitivity of $f(i)$ in $f(A)$
$\Rightarrow$ $f(i)$ is primitive in B

$\square$

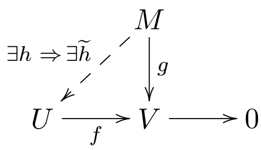
2.2 Vertices and sources

G finite group, k a field of char p

Proposition 2.2.1 (Maschke's Theorem)
kG semisimple $\Leftrightarrow$ $p \nmid |G|$

Modular representation theory = study of kG-modules when $p \big| |G|$
In this case, not all kG-modules are projective

Definition 2.2.2
Let $H \leq G$, M a kG-module.
Say M is <u>relatively H-projective</u> if f surjective kG-hom, g a kG-hom and if $\exists h$ a kH-hom s.t. $fh = g$, then $\exists \widetilde{h} : M \to U$ a kG-hom s.t. $f\widetilde{h} = g$



Note that relatively 1-projective is the usual notion of projective module

For $H \leq G$ and N a kH-module, M a kG-module

$$\begin{aligned} \operatorname{Ind}_H^G(N) &= kG \otimes_{kH} N \\ \operatorname{Res}_H^G(M) &= M \quad \text{(viewed as } kH\text{-module)} \end{aligned}$$

define the functors $\operatorname{Ind}_H^G : \mathbf{Mod}(kH) \leftrightarrows \mathbf{Mod}(kG) : \operatorname{Res}_H^G$

For $L \leq H \leq G$, we have

$$\begin{aligned} \operatorname{Ind}_H^G \operatorname{Ind}_L^H &= \operatorname{Ind}_L^G \\ \operatorname{Res}_L^H \operatorname{Res}_H^G &= \operatorname{Res}_L^G \\ \operatorname{Ind}_H^G(N) = kG \otimes_{kH} N &= \bigoplus_{x \in [G/H]} x \otimes_{kH} kG \text{ (as } k\text{-vector space)} \end{aligned}$$

In general, let ${}^xN$ be the $k({}^xN)$-module whose underlying k-vector space is N and $k({}^xH)$-module structure is given by

$$xhx^{-1} \cdot n = hn \qquad (h \in H, n \in N)$$

Then $x \otimes_{kH} N \cong {}^xN$ (since $xhx^{-1}(x \otimes n) = xh \otimes n = x \otimes hn$)

For $H \le G$, and M a kG-module. Define the trace map,

$$\begin{aligned} \mathrm{Tr}_H^G : \mathrm{End}_{kH}(M) &\to \mathrm{End}_{kG}(M) \\ \phi &\mapsto \left(m \mapsto \sum_{x \in [G/H]} x\phi(x^{-1}m) \right) \end{aligned}$$

Lemma 2.2.3 (Mackey Decomposition Formula)
$H, L \le G$, and M a kL-module

$$\mathrm{Res}_H^G \mathrm{Ind}_L^G(M) \cong \bigoplus_{x \in [H\backslash G/L]} \mathrm{Ind}_{H \cap {}^xL}^H \mathrm{Res}_{H \cap {}^xL}^{{}^xL}({}^xM)$$

(Proof can be found in most representation theory books, e.g. Benson's Rep and Cohom vol. 1)

Proposition 2.2.4
$H \le G$, M a kG-module. TFAE:

(1) $M$is relatively H-projective

(2) $M | \mathrm{Ind}_H^G \mathrm{Res}_H^G(M)$

(3) $M | \mathrm{Ind}_H^G(N)$ for some kH-module N

(4) (**Higman's Criterion**) $\mathrm{id}_M = \mathrm{Tr}_H^G(\phi)$ for some $\phi \in \mathrm{End}_{kH}(M)$

Definition 2.2.5
M an indecomposable kG-module

(1) A vertex of M is a minimal subgroup Q of G s.t. M is relatively Q-projective

(2) If Q is a vertex of M, then kQ-source of M is an indecomposable kQ-module V s.t. $M | \mathrm{Ind}_Q^G(V)$
(Such V always exists by Proposition 2.2.4 and Krull-Schmidt Theorem)

Lemma 2.2.6
M an indecomposable kG-module with vertex Q
Then $\exists$ kQ-source S s.t. $S | \mathrm{Res}_Q^G(M)$
Moreover, $\forall$ kQ-source V of M, $\exists x \in N_G(Q)$ s.t. $V \cong {}^xS$

Proof
By Proposition 2.2.4 (2), $M | \mathrm{Ind}_Q^G \mathrm{Res}_Q^G(M)$
By Krull-Schmidt Theorem, such S exists
Let V be a kQ-source of M. We have:

$$M | \mathrm{Ind}_Q^G(S) \quad , \quad S | \mathrm{Res}_Q^G(M) \qquad M | \mathrm{Ind}_Q^G(V)$$

$$\mathrm{Res}_Q^G(M) \mid \mathrm{Res}_Q^G \mathrm{Ind}_Q^G(V) \overset{\text{Mackey}}{=} \bigoplus_{x \in [Q\backslash G/Q]} \mathrm{Ind}_{Q \cap {}^xQ}^Q \mathrm{Res}_{Q \cap {}^xQ}^{{}^xQ}({}^xV)$$

S indecomposable $\Rightarrow$ $S \mid \mathrm{Res}_Q^G \mathrm{Ind}_Q^G(V) = \bigoplus_{x \in [Q\backslash G/Q]} \mathrm{Ind}_{Q \cap {}^xQ}^Q \mathrm{Res}_{Q \cap {}^xQ}^{{}^xQ}({}^xV)$ for some $x \in G$
$\Rightarrow$ $M | \mathrm{Ind}_Q^G(S) | \cdots$
$\Rightarrow$ M is relative $Q \cap {}^xQ$-projective

Q vertex $\Rightarrow$ $Q = Q \cap {}^xQ$
$\Rightarrow$ $Q = {}^xQ$
$\Rightarrow$ $x \in N_G(Q)$ and $S | {}^xV$
$S, {}^xV$ are indecomposable $\Rightarrow$ $S \cong {}^xV$ □

Proposition 2.2.7
M indecomposable kG-module. Let $(Q,V),(R,W)$ be vertex-source pairs for M
Then $\exists x \in G$ s.t. $(R,W) = {}^x(Q,V)$
i.e. $R = {}^xQ, W \cong {}^xV$

Proof
By Lemma 2.2.6:
$\exists S$ indecomposable kQ-modules s.t. $S | \operatorname{Res}^G_Q(M), M | \operatorname{Ind}^G_Q(S)$
$\exists T$ indecomposable kR-modules s.t. $T | \operatorname{Res}^G_R(M), M | \operatorname{Ind}^G_R(T)$
$S | \operatorname{Res}^G_Q(M) | \operatorname{Res}^G_Q \operatorname{Ind}^G_R(T)$
$\Rightarrow$ (by Mackey formula) S is relatively $Q \cap {}^xR$-projective for some $x \in G$
But S has vertex $Q \quad \Rightarrow \quad Q = Q \cap {}^xR \subseteq {}^xR$

Similarly $\exists y \in G,\ R \subseteq {}^yQ \qquad \Rightarrow \qquad Q = {}^xR$
(Note this x is not necessarily the same x as stated in the proposition)
The rest follows from Lemma 2.2.6 □

Notations:
$A =_G B$ means A equal to a G-conjugate of B
$A \leq_G B$ means A is G-conjugate to a subgroup of B

Lemma 2.2.8
$H \leq G$, M indecomposable kG-module with vertex $Q \leq H$
$\Rightarrow \quad \exists V$ indecomposable kH-module satisfying any two of the following:

$$(1)\ \ M | \operatorname{Ind}^G_H(V) \qquad (2)\ \ V | \operatorname{Res}^G_H(M) \qquad (3)\ \ V \text{ has vertex } Q$$

Proof
(1), (2):
M relatively Q-projective $\ \Rightarrow\ $ M relatively H-projecitve
$\Rightarrow \quad M | \operatorname{Ind}^G_H \operatorname{Res}^G_H(M) \quad \Rightarrow$ done by Krull-Schmidt

(1), (3):
Lemma 2.2.6 $\ \Rightarrow\ \exists S$ indecomposable kQ-module s.t. $S | \operatorname{Res}^G_Q(M),\ M | \operatorname{Ind}^G_Q(S) = \operatorname{Ind}^G_H \operatorname{Ind}^H_Q(S)$
Krull-Schmidt $\ \Rightarrow\ \exists V$ indecomposable kH-module s.t. $V | \operatorname{Ind}^H_Q(S),\ M | \operatorname{Ind}^G_H(V)$
$\Rightarrow\ M$ has vertex $Q \ \Rightarrow\ V$ has vertex Q

(2), (3):
Lemma 2.2.6 + Krull-Schmidt $\ \Rightarrow\ \exists V$ indecomposable kH-module s.t. $S | \operatorname{Res}^H_Q(V),\ V | \operatorname{Res}^G_H(M)$

[WANT: V has vertex Q]
$V | \operatorname{Res}^G_H(M) | \operatorname{Res}^G_H \operatorname{Ind}^G_Q(S)$
$\Rightarrow$ (by Mackey) V is relatively $H \cap {}^xQ$-projective for some $x \in G$
$\Rightarrow$ V has vertex $R \leq H \cap {}^xQ$ (so $R \subseteq {}^xQ$) and source T

$S | \operatorname{Res}^H_Q(V) | \operatorname{Res}^H_Q \operatorname{Ind}^H_R(T)$
$\Rightarrow\ S$ is relatively $Q \cap {}^yR$-projective for some $y \in H$

S has vertex Q
$\Rightarrow\ Q \leq_H R$
By comparing orders, we get $Q =_H R$ (i.e. Q is equal to a H-conjugate of R) □

2.3 Green's Correspondence

Fix G finite group, k a field of characteristic p

Definition 2.3.1
χ a set of subgroups of G
A kG-module M is relatively $\mathcal{X}$-projective or projective relative to $\mathcal{X}$ if M is a direct sum of modules which are projective relative to some $Q \in \mathcal{X}$

Lemma 2.3.2
$Q \leq H \leq G$. Let V be a kH-module, relatively Q-projective. Then:

$$\operatorname{Res}^G_H \operatorname{Ind}^G_H(V) \cong V \oplus Y$$

where Y is projective relatively to $\mathcal{Y} = \{H \cap {}^xQ | x \in G - H\}$

Proof
Mackey:

$$\begin{aligned}\operatorname{Res}^G_H \operatorname{Ind}^G_H(V) &\cong \bigoplus_{x \in [H\backslash G/H]} \operatorname{Ind}^{{}^xH}_{H\cap {}^xH}({}^xV) \\ &\cong V \oplus \underbrace{\bigoplus_{\substack{x \in [H\backslash G/H] \\ x \notin H}} \operatorname{Ind}^{{}^xH}_{H\cap {}^xH}({}^xV)}_{Y}\end{aligned}$$

V relatively Q-projective $\Rightarrow$ $V | \operatorname{Ind}^G_Q(S)$ for some kQ-module S, i.e. $V \oplus W \cong \operatorname{Ind}^H_Q(S)$

As for V, we have $\operatorname{Res}^G_H \operatorname{Ind}^G_H(W) \cong W \oplus Z$
Take $\operatorname{Res}^G_H \operatorname{Ind}^G_H$ to $V \oplus W \cong \operatorname{Ind}^H_Q(S)$:

$$\begin{aligned}V \oplus Y \oplus W \oplus Z &\cong \operatorname{Res}^G_H \operatorname{Ind}^G_Q(S) \\ &\cong \bigoplus_{x \in [H\backslash G/H]} \operatorname{Ind}^G_{H\cap {}^xQ} \operatorname{Res}^{{}^xQ}_{H\cap {}^xQ}({}^xS) \\ &\cong \underbrace{\operatorname{Ind}^G_Q(S)}_{V\oplus W} \oplus \underbrace{\bigoplus_{x\notin H} \cdots}_{U}\end{aligned}$$

Krull-Schmidt implies:

$$Y \,\Bigg|\, \bigoplus_{\substack{x \in [H\backslash G/H] \\ x \notin H}} \operatorname{Ind}^G_{H\cap {}^xQ} \operatorname{Res}^{{}^xQ}_{H\cap {}^xQ}({}^xS)$$

So Y is relatively $\mathcal{Y}$-projecitve $\square$

Theorem 2.3.3 (Green's Correspondence)
Let Q be a p-subgroup of G. $N_G(Q) \leq H \leq G$. Then

$$\left\{ \begin{array}{c} \text{indecomposable } kG\text{-module} \\ \text{with vertex } Q \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{indecomposable } kH\text{-module} \\ \text{with vertex } Q \end{array} \right\}$$
$$U \quad \leftrightarrow \quad V$$

such that

(1) $\mathrm{Res}^G_H(U) \cong V \oplus Y$ with Y projective relative to $\mathcal{Y} = \{H \cap {}^xQ | x \in G - H\}$

(2) $\mathrm{Ind}^G_H(V) \cong U \oplus X$ with X projective relative to $\mathcal{X} = \{Q \cap {}^xQ | x \in G - H\}$

Remark. The above characterises U and V as follows:

- U is a unique indecomposable summand of $\mathrm{Ind}^G_H(V)$ with vertex Q
- V is a unique indecomposable summand of $\mathrm{Res}^G_H(U)$ with vertex Q
- No indecomposable summand of Y (resp. of X) has vertex Q
  Proof: (We prove for Y, for X is similar)
  Suppose Y_1 is an indecomposable summand of Y with vertex Q
  Proposition 2.2.7 $\Rightarrow$ ${}^yQ \leq H \cap {}^xQ$ for some $x \in G - H$ and some $y \in H$
  ${}^yQ = {}^xQ \Rightarrow y^{-1}x \in N_G(Q) \leq H \Rightarrow x \in H$ (contradiction) □

Proof
Let V be an indecomposable kH-module with vertex Q
Lemma 2.3.2 $\Rightarrow$ $\mathrm{Res}^G_H \mathrm{Ind}^G_H(V) \cong V \oplus Y'$ with Y' relatively $\mathcal{Y}$-projective
$\Rightarrow$ $\exists!\, U$ (up to isom. by Krul-Schmidt) indecomposable kG-module s.t. $U | \mathrm{Ind}^G_H(V)$, $V | \mathrm{Res}^G_H(U)$

Write $\quad \mathrm{Ind}^G_H(V) \cong U \oplus X \quad , \quad \mathrm{Res}^G_H(U) \cong V \oplus Y$
$\Rightarrow \quad \underbrace{\mathrm{Res}^G_H(U)}_{V \oplus Y} \oplus \mathrm{Res}^G_H(X) \cong V \oplus Y'$

$$\Rightarrow \left\{ \begin{array}{l} Y | Y' \\ \mathrm{Res}^G_H(X) | Y' \end{array} \right. \text{ with } Y \text{ relatively } \mathcal{Y}\text{-projective}$$

Claim: X is relative $\mathcal{X}$-projective

Proof of Claim:
Let X_1 be an indecomposable summand of X
$\Rightarrow$ $\mathrm{Res}^G_H(X_1) | \mathrm{Res}^G_H(X) | Y'$
$\Rightarrow$ $X_1 | X | \mathrm{Ind}^G_H(V)$
V has vertex Q
$\Rightarrow$ X_1 is relatively Q-projective $\Rightarrow$ X_1 has vertex $R \leq Q$

$$\Rightarrow \text{ (by Lemma 2.2.8) } \left\{ \begin{array}{l} \exists \text{ indecomposable } kH\text{-module } W | \mathrm{Res}^G_H(X_1) \\ W \text{ has vertex } R \end{array} \right.$$

$\Rightarrow$ $R \leq_H H \cap {}^xQ$ for some $x \in G - H$
$R \leq Q \Rightarrow R \leq Q \cap {}^xQ \Rightarrow X$ is relatively $\mathcal{X}$-projective ■

Similarly for the other direction □

2.4 Trace, Brauer homomorphism, defect groups

G finite group, k field of characteristic p
$\mathcal{B}$=set of blocks of kG

$\Rightarrow \quad 1_{kG} = \sum_{b \in \mathcal{B}} b$
$\Rightarrow \quad kG = \prod_{b \in \mathcal{B}} kGb, \qquad kGb$ indecomposable k-algebra (called the block algebra)

Let M be a (f.g.) kG-module
$\Rightarrow \quad M = \bigoplus_{b \in \mathcal{B}} bM, \qquad bM$ a kG-module
In particular, if M is indecomposable, then $M = bM$ for some $b \in \mathcal{B}$
then $b'M = 0 \ \forall b' \in \mathcal{B}$ s.t. $b' \neq b$
We say that a kG-module M belongs to $b \in \mathcal{B}$ if $M = bM$

If M belongs to $b \in \mathcal{B}$ and N belongs to $b' \in \mathcal{B}$, and $b \neq b'$
then $\mathrm{Hom}_{kG}(M, N) = 0$
$\rightsquigarrow \quad \mathbf{Mod}(kG) = \mathbf{Mod}(kGb_1) \times \cdots \times \mathbf{Mod}(kGb_r)$

Definition 2.4.1
The unique block $b \in \mathcal{B}$ to which the trivial kG-module k belongs is called the principal block of kG

Let $\epsilon : kG \to k$ be the augmentation map $(\sum_{x \in G} \lambda_x x \mapsto \sum_{x \in G} \lambda_x)$
and $I(kG) = \ker(\epsilon) =$ augmentation ideal
Then $b \in \mathcal{B}$ is principal $\Leftrightarrow \ \epsilon(b) = 1 \ \Leftrightarrow \ \epsilon(b) \neq 0 \ \Leftrightarrow \ b \notin I(kG)$
(Because $bk = k \Rightarrow \epsilon(b) = 1$; $b' \neq b, b'k = 0 \Rightarrow \epsilon(b') = 0$)

Definition 2.4.2
$K \leq H \leq G$

$$\begin{aligned} (kG)^H &:= \{a \in kG \mid {}^x a = xax^{-1} = a \ \ \forall x \in H\} \\ \mathrm{Tr}_K^H : (kG)^H &\to (kG)^H \\ a &\mapsto \sum_{x \in [H/K]} {}^x a \\ \mathrm{Res}_K^H : (kG)^H &\hookrightarrow (kG)^K \end{aligned}$$

Proposition 2.4.3
$K \leq H \leq G, L \leq G$

(1) $\mathrm{Tr}_H^G \mathrm{Tr}_K^H = \mathrm{Tr}_K^G$

(2) If $a \in (kG)^K, b \in (kG)^H$, then $b \,\mathrm{Tr}_K^H(a) = \mathrm{Tr}_K^H(ba)$ and $\mathrm{Tr}_K^H(a) b = \mathrm{Tr}_K^H(ab)$
In particular, $(kG)_K^H := \mathrm{Tr}_K^H((kG)^K)$ is an ideal in $(kG)^H$

(3) (**Mackey decomposition**)

$$\mathrm{Res}_H^G \mathrm{Tr}_L^G(a) = \sum_{x \in [H \backslash G / L]} \mathrm{Tr}_{H \cap {}^x L}^H \mathrm{Res}_{H \cap {}^x L}^{{}^x L}({}^x a)$$

i.e. $\mathrm{Tr}_L^G(a) = \sum_{x \in [H \backslash G / L]} \mathrm{Tr}_{H \cap {}^x L}^H({}^x a)$

Definition 2.4.4
A defect group of a block b of kG is a minimal subgroup P of G s.t. $b \in (kG)_P^G$
(Note: $(kG)^G = Z(kG)$)

Proposition 2.4.5
Let b be a block of kG

(1) Defect groups of b are p-subgroup of G

(2) Defect groups of b are G-conjugate to each other

Proof

(1) Let $S \in \mathrm{Syl}_p(G)$. Then $p \nmid |G:S| \in k^\times$
Then $b = \mathrm{Tr}_S^G\left(\frac{1}{|G:S|}b\right)$ (Think carefully)

(2) See later, using Brauer homomorphism

□

Definition 2.4.6 (Brauer homomorphism)
Let Q be a p-subgroup of G. Consider the k-linear map

$$\begin{array}{rcl} kG & \to & kC_G(Q) \\ \sum_{x \in G} \lambda_x x & \mapsto & \sum_{x \in C_G(Q)} \lambda_x x \end{array}$$

The restriction of this map to $(kG)^Q$ is called the Brauer homomorphism

$$\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$$

Lemma 2.4.7
$Q \leq G$. $(kG)^Q$ has as a k-basis $\{\mathrm{Tr}_{C_Q(x_i)}^Q(x_i)\}$
where $\{x_i\}$ is a set of representatives of Q-ccls of G

Proof

$$\begin{aligned} (kG)^Q &= \{a = \sum_{x\in G} \lambda_x x \Big| yay^{-1} = a \ \ \forall y \in Q\} \\ &= \{a = \sum_{x\in G} \lambda_x x \Big| \sum_{x\in G} \lambda_x yxy^{-1} = \sum_{x\in G} \lambda_x x \ \ \forall y \in Q\} \\ &= \{\sum_{x\in G} \lambda_x x \Big| \lambda_x = \lambda_{y^{-1}xy} \ \ \forall y \in Q\} \end{aligned}$$

□

Then, we have

$$\begin{aligned} (kG)^Q &= k\text{-span}\{\mathrm{Tr}_{C_Q(x_i)}^Q(x_i) \Big| Q = C_Q(x_i) \ (\text{i.e.} x_i \in C_G(Q))\} \oplus \underbrace{k\text{-span}\{\mathrm{Tr}_{C_Q(x_i)}^Q(x_i) \Big| Q > C_Q(x_i)\}}_{=:I} \\ &= kC_G(Q) \oplus I \qquad \text{as } k\text{-vector space} \end{aligned}$$

Proposition 2.4.8

$$I = \sum_{R<Q} (kG)_R^Q$$

Hence Br_Q is a surjective algebra homomorphism, with $\ker(\mathrm{Br}_Q) = \sum_{R<Q} (kG)_R^Q$
(Note: We use char $k = p$, and Q a p-subgroup)

Proof
($\supseteq$:) By definition of I

($\supseteq$:) Let $R \subset Q$. Then $(kG)^Q_R = \mathrm{Tr}^Q_R((kG)^R)$
Lemma 2.4.7 $\Rightarrow$ $(kG)^Q = k\text{-span}\{\mathrm{Tr}^R_{C_R(y_j)}(y_j)\}$
$\Rightarrow$ $(kG)^Q_R = k\text{-span}\{\mathrm{Tr}^Q_{C_R(y_j)}(y_j)\}$

$$\Rightarrow \quad \mathrm{Tr}^Q_{C_R(y_j)}(y_j) = \mathrm{Tr}^Q_{C_Q(y_j)} \mathrm{Tr}^{C_Q(y_j)}_{C_R(y_j)}(y_j) = |C_Q(y_j) : C_R(y_j)| \,\mathrm{Tr}^Q_{C_Q(y_j)}(y_j) = \left\{ \begin{array}{l} 0 \\ \mathrm{Tr}^Q_{C_R(y_j)}(y_j) \end{array} \right. \in I \qquad \square$$

Lemma 2.4.9
b block of kG. Q, R p-subgroups of G
Suppose $b \in (kG)^G_Q, \mathrm{Br}_R(b) \neq 0$
Then $R \leq_G Q$

Proof
Have $b = \mathrm{Tr}^G_Q(b)$ and $c \in (kG)^Q$
$\Rightarrow$ (by Mackey) $\sum_{\lambda \in [R\backslash G/Q]} \mathrm{Tr}^R_{R\cap {}^xQ}({}^xc)$
If $R \cap {}^xQ < R \ \ \forall x \in G$, then $b \in \ker(\mathrm{Br}_R)$ contradiction.
$\therefore$ $R \cap {}^xQ = R \subseteq {}^xQ$ for some $x \in G$ $\square$

Proposition 2.4.10
b block of kG. P a p-subgroup of G. TFAE:

(1) P is a defect group of b

(2) $b \in (kG)^G_P$, $\mathrm{Br}_P(b) \neq 0$

(3) P is a maximal p-subgroup of G s.t. $\mathrm{Br}_P(b) \neq 0$

Proof
<u>(2)⇒(1)</u>:
Use Lemma 2.4.9. Suppose P is not minimal, then $\exists Q < P$ s.t. $b \in (kG)^G_Q$
Lemma 2.4.9 $\Rightarrow$ $P \leq_G Q$ contradiction

<u>(1)⇒(2)</u>:
Have $b = \mathrm{Tr}^G_P(c), c \in (kG)^P$ (WANT: $\mathrm{Br}_P(b) \neq 0$)
Suppose $\mathrm{Br}_P(b) = 0$. $b \in \ker(\mathrm{Br}_P) = \sum_{Q<P}(kG)^P_Q$
Then $b = b^2 = b\,\mathrm{Tr}^G_P(c) = \mathrm{Tr}^G_P(bc)$
Now $b \in \sum_{Q<P}(kG)^P_Q$
$\Rightarrow$ $bc \in \sum_{Q<P}(kG)^P_Q$ (as $c \in (kG)^P$)
$\Rightarrow$ $b = \mathrm{Tr}^G_P(bc) \in \sum_{Q<P}(kG)^G_Q$ Have $\left\{ \begin{array}{l} (kGb)^G_Q \text{ are ideal of } (kG)^G = Z(kG) \\ b \text{ is a primitive idempotent of } Z(kG) \end{array} \right.$
Rosenberg Lemma $\Rightarrow$ $b \in (kG)^G_Q$ contradiction

<u>(2)⇒(3)</u>: Obvious by the same trick

<u>(3)⇒(2)</u>:
Let Q be a defect group, $b \in (kG)^G_Q$, $P \leq_G Q$ by Lemma 2.4.9
Similarly for other way. $\square$

Corollary 2.4.11
All defect groups are G-conjugate

Proposition 2.4.12
Let b be a block of kG with defect group P
Let V be an indecomposable kG-module with vertex Q
Suppose V belongs to b. Then $Q \leq_G P$

Proof
Have: $b = \mathrm{Tr}_P^G(c), \quad c \in (kG)^P$ and $V = bV$
Higman's criterion: $\mathrm{id}_V = \mathrm{Tr}_P^G(\phi)$ some $\phi \in \mathrm{End}_{kP}(V)$
Define $\phi : V \to V$ by $\phi(v) = cv$
Then $\phi \in \mathrm{End}_{kP}(V)$, $\mathrm{Tr}_P^G(\phi)(v) = \sum_{x \in [G/P]} x\phi(x^{-1}v) = \sum_{x\in[G/P]} xcx^{-1}v = \mathrm{Tr}_P^G(c)v = bv = v$ □

Corollary 2.4.13
If b is a block of kG with trivial defect group then kGb is simple
(Note: We will show that the converse holds using Brauer's second main theorem)

Corollary 2.4.14
Let b be the principal block of kG
Then defect groups of b are the Sylow p-subgroups of G

Proof
Show: the trivial kG-module k has the Sylow p-subgroup of G as vertices
Use Higman's criterion □

2.5 Brauer's First Main Theorem

Theorem 2.5.1 (Brauer's first main theorem)
Let P be a p-subgroup of G. Then there exists one-to-one correspondence give by Br_P:

$$\left\{ \begin{array}{l} \text{blocks of } kG \\ \text{with defect group } P \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{blocks of } kN_G(P) \\ \text{with defect group } P \end{array} \right\}$$

Recall: Brauer homomorphism Br_P where P is a p-subgroup of G. Then

$$(kG)^P = \underbrace{kC_G(P)}_{k-\text{subalg. of } (kG)^P} \oplus \underbrace{\sum_{Q<P} (kG)_Q^P}_{\text{ideal of } (kG)^P}$$

and both terms on right hand side are invariant under $N_G(P)$-conjugation

$\Rightarrow \quad \mathrm{Br}_P : (kG)^P \to kC_G(P)$ is a algebra hom. preserving $N_G(P)$-conjugation ("$N_G(P)$-algebra homomorphism")

Remark. G-algebras can be used to unify defect group theory and vertex theory:
If M is an indecomposable kG-module, then $A = \mathrm{End}_k(M)$ is a G-algebra s.t. $A^G = \mathrm{End}_{kG}(M)$ is local
Similarly if b is a block of kG then $A = kGb$ is a G-algebra and $A^G = Z(kGb)$ is local
Define Trace as in kG and Brauer hom. $\mathrm{Br}_P^A : A^P \to A(P) = A^P / \sum_{Q<P} A_Q^P$
Then the same formalism applies to A

Lemma 2.5.2
Let P be a p-subgroup of G. Then

$$\mathrm{Br}_P \, \mathrm{Tr}_P^G(a) = \mathrm{Tr}_P^{N_G(P)} \, \mathrm{Br}_P(a) \qquad (a \in (kG)^P)$$

So in particular:

$$\mathrm{Br}_P\,|_{(kG)^G_P} : (kG)^G_P \twoheadrightarrow (kC_G(P))^{N_G(P)}_P$$

Proof

Note: $\mathrm{Br}_P \mathrm{Tr}^G_P$ should be written as $\mathrm{Br}_P \mathrm{Res}^G_P \mathrm{Tr}^G_P(a)$ formally, but there is no loss of generality as the restriction map is just embedding.

$$\begin{aligned}
\mathrm{Br}_P \mathrm{Res}^G_P \mathrm{Tr}^G_P(a) &= \mathrm{Br}_P\left(\sum_{x\in[P\backslash G/P]} \mathrm{Tr}^P_{P\cap {}^xP}({}^xa)\right) \quad \text{by Mackey} \\
&= \mathrm{Br}_P\left(\sum_{x\in[N_G(P)/P]} {}^xG\right) \\
&\quad \text{(because: } P\cap {}^xP = P \Leftrightarrow {}^xP = P \Leftrightarrow x\in N_G(P)) \\
&= \sum_{x\in[N_G(P):P]} {}^x\,\mathrm{Br}_P(a) \quad \text{as } \mathrm{Br}_P \text{ is an } N_G(P)\text{-algebra hom.} \\
&= \mathrm{Tr}^{N_G(P)}_P \mathrm{Br}_P(a)
\end{aligned}$$

□

Lemma 2.5.3 (Cliffod's Theorem)

Let $N \trianglelefteq G$

If S is a simple kG-module, then $\mathrm{Res}^G_N(S)$ is semisimple

Proof

Let $S_1 = \mathrm{Soc}(\mathrm{Res}^R_N(S))$ = the sum of all simple kN-submodules of $\mathrm{Res}^G_N(S)$

Consider $\mathcal{X} = \{$simple kN-submodule of $\mathrm{Res}^G_N(S)\}$, it is G-invariant

i.e. if $T\in\mathcal{X}, g\in G$ then $gT\in\mathcal{X}$

$\Rightarrow$ S_1 is in fact kG-submodule of S. But S simple, so $S_1 = S$ □

Lemma 2.5.4

$N \trianglelefteq G$. $\pi : kG \to kG/N$ via $x \mapsto xN$

Then $\ker\pi = I(kN)kG$ (recall $I(kN)$ is augmentation ideal of kN)

(Easy/Exercise)

Lemma 2.5.5

P finite p-group. Then $I(kP) = J(kP)$

In particular, $kP/J(kP) \cong k$, so

- kP is local
- kP has unique simple module k
- kP has unique projective indecomposable module kP

Proof

Induction on $|P|$

$\underline{|P| = p}$: $\quad P = \langle g | g^p = 1\rangle$

$I(kP) = (g-1)kP$ is nilpotent because $(g-1)^p = g^p - 1 = 0$

$\underline{|P| > p}$: $\quad$ Take $Z < Z(P)$, $|Z| = p$

Then $I(kZ) = J(kZ)$ by induction hypothesis

$\pi : kP \twoheadrightarrow kP/Z$ has kernel $\ker\pi = I(kZ)kP$, which is nilpotent

$\Rightarrow \quad \pi^{-1}(I(kP/Z)) = I(kP)$

But by induction: $J(kP/Z) = I(kP/Z)$
$\Rightarrow \quad J(kP/Z)$ nilpotent
$\Rightarrow \quad I(kP)$ nilpotent □

Lemma 2.5.6
P normal p-subgroup of G

(1) Every central idempotent of kG is contained in $kC_G(P)$

(2) Every defect group of every block of kG contains P

Proof

Claim: $\sum_{Q<P}(kG)_Q^P \subseteq J(kG)$
Proof of Claim:
Let S be a simple kG-module.
Then S is semisimple as kP-module (by Clifford), hence P acts trivially on S (as P is p-group)
Then (for $s \in S$), $\mathrm{Tr}_Q^P(a)s = \sum_{x\in P/Q} xax^{-1}s = \sum_{x\in P/Q} xas = |P:Q|as = 0$ ■

$\Rightarrow \quad (kG)^P = kC_G(P) \oplus \sum_{Q<P}(kG)_Q^P$ (corresponding to idempotent decomposition $e = e_1 + e_2$)
Let $e = e^2 \in Z(kG)$
$\Rightarrow \quad e = e^{p^n} = e_1^{p^n} + e_2^{p^n} = e_1^{p^n}$
(as $e_2^{p^n} = 0$ for sufficiently large n, and e_1, e_2 commutes by centrality of e)
$\Rightarrow \quad \mathrm{Br}_P(e) = e \neq 0$ □

Proof of Brauer First Main Theorem 2.5.1
{ blocks of kG with defect group P} = { primitive idempotents of $Z(kG)$ contained in $(kG)_P^G$ not contained in $\ker(\mathrm{Br}_P)$}

$\mathrm{Br}_P : (kG)^P \twoheadrightarrow kC_G(P)$ restricts to
$Z(kG) \to Z(kC_G(P))$ and by Lemma 2.5.2 we get a surjective map: $(kG)_P^G \twoheadrightarrow (kC_G(P))_P^{N_G(P)}$

So by idempotent lifting lemma, we get { primitive idempotents of $Z(kG)$ contained in $(kG)_P^G$ not contained in $\ker(\mathrm{Br}_P)$} $\leftrightarrow$ { primitive idempotents of $(kC_G(P))^{N_G(P)}$ contained in $(kC_G(P))_P^{N_G(P)}$} = { primitive idempotents of $Z(kN_G(P))$ contained in $(kN_G(P))_P^{N_G(P)}$} = { blocks of $kN_G(P)$ with defect group P}

First equality is by Lemma 2.5.6: Every central idempotent of $kN_G(P)$ is contained in $kC_G(P)$
Second equality is again by Lemma 2.5.6: Every defect group of a block of $kN_G(P)$ contains P □

2.6 Brauer's Second Main Theorem

Lemma 2.6.1
Q is a p-subgroup of G (Recall: $(kG)^Q = kC_G(Q) \oplus \sum_{R<Q}(kG)_R^Q$) We have:

$$(kG)^{N_G(Q)} = (kC_G(Q))^{N_G(Q)} \oplus \sum_{Q \nleq H \leq N_G(Q)} (kG)_H^{N_G(Q)}$$

Proof
$(kG)^{N_G(Q)}$ has as k-basis the set of $N_G(Q)$-conjugacy class sums, which are of the form

$$\mathrm{Tr}_{C_{N_G(Q)}(x)}^{N_G(Q)}(x)\ , \quad x \in G$$

Have $Q \leq C_{N_G(Q)}(x) \;\Leftrightarrow\; x \in C_G(Q)$. So:

$$(kG)^{N_G(Q)} = (kC_G(Q))^{N_G(Q)} + \sum_{Q \not\leq H \leq N_G(Q)} (kG)_H^{N_G(Q)}$$

Claim: $\sum_{Q \not\leq H \leq N_G(Q)} (kG)_H^{N_G(Q)} \subseteq \sum_{R<Q} (kG)_R^Q$

Proof of Claim:

Mackey decomposition: for $Q \not\leq H \leq N_G(Q)$,

$$\mathrm{Tr}_H^{N_G(Q)}(a) = \sum_{x \in [Q \backslash N_G(Q)/H]} \mathrm{Tr}^Q_{Q \cap {}^x H}({}^x a) \in \sum_{R<Q} (kG)_R^Q$$

$(Q \cap {}^x H = Q \;\Leftrightarrow\; Q \subseteq {}^x H \;\Leftrightarrow\; Q \subseteq H$ since $x \in N_G(Q))$ ■

So we have a direct sum and we are done. □

Theorem 2.6.2 (Brauer's Second Main Theorem: Nagao version)
Let e be central idempotent of kG
M be a kG-module s.t. $M = eM$
Q be a p-subgroup of G. Then:

$$\mathrm{Res}^G_{N_G(Q)}(M) = \mathrm{Br}_Q(e)M \oplus M'$$

where M' is projective relative to $\mathcal{X} = \{H | Q \not\leq H \leq N_G(Q)\}$

Proof
$M = \mathrm{Br}_Q(e)M \oplus \underbrace{(1 - \mathrm{Br}_Q(e))M}_{M'}$ as $kN_G(Q)$-module
Since $M = eM$, we have

$$\begin{aligned}(e - \mathrm{Br}_Q(e))M' &= (e - \mathrm{Br}_Q(e))(1 - \mathrm{Br}_Q(e))M \\ &= (e - \mathrm{Br}_Q(e) - e\,\mathrm{Br}_Q(e) + \mathrm{Br}_Q(e)^2)M \\ &= (1 - \mathrm{Br}_Q(e))eM \;=\; M'\end{aligned}$$

By Lemma 2.6.1 and by $e \in Z(kG) \;\Rightarrow\; e - \mathrm{Br}_Q(e) \in \sum_{H \in \mathcal{X}} (kG)_H^{N_G(Q)}$

Using Higman's Criterion and Rosenberg Lemma:
$\Rightarrow$ M' is projective relative to $\mathcal{X}$ □

This can let us match Brauer's and Green's correspondence

Corollary 2.6.3
Let U be an indecomposable kG-module with vertex Q
Let V be an indecomposable $kN_G(Q)$-module with vertex Q, corresponding to U under Green correspondence
e be a central idempotent of kG
Then $U = eU \;\Leftrightarrow\; V = \mathrm{Br}_Q(e)V$

Proof
$\underline{\Rightarrow}$:
Suppose $U = eU$. Then by Brauer's second main theorem 2.6.2:

$$\mathrm{Res}^G_{N_G(Q)}(U) = \mathrm{Br}_Q(e)U \oplus U'$$

where U' is projective relative to $\mathcal{X} = \{H | Q \nleq H \leq N_G(Q)\}$
$\Rightarrow$ no indecomposable summand of U' has vertex Q
$\Rightarrow$ $V | \operatorname{Br}_Q(e)U$
$\Rightarrow$ $V = \operatorname{Br}_Q(e)V$

$\underline{\Leftarrow}$:
Suppose $V = \operatorname{Br}_Q(e)V$ and $U = e'U$ s.t. $e'e = 0$
$\Rightarrow$ $V = \operatorname{Br}_Q(e')V = \operatorname{Br}_Q(e')\operatorname{Br}_Q(e)V = \operatorname{Br}_Q(e'e)V = 0$ contradiction □

Corollary 2.6.4
b be block of kG with defect group P
Then $\exists$ indecomposable kGb-module with vertex P

Proof
By Corollary 2.6.3, if V is an indecomposable $kN_G(P)\operatorname{Br}_P(b)$-module with vertex P, then the Green correspondence U of V is an indecomposable kGb-module with vertex P.

So suffice to prove for G with $P \lhd G (= N_G(P))$
Let V be a projective indecomposable of $k(G/P)$-module, s.t. $\bar{b}V = V$ (where $\bar{b}$ = image of b under canonical projection $kG \to kG/P$)
Viewed as a kG-module, V is an indecomposable kG-module s.t. $V | \operatorname{Ind}_P^G(k)$ and $V = bV$

Claim: V has vertex P

Proof of Claim:
Suppose V has vertex $Q < P$
Lemma 2.2.8: $\exists$ indecomposable kP-module W s.t. $W | \operatorname{Res}_P^G(V)$ and W has vertex Q
But $W | \operatorname{Res}_P^G \operatorname{Ind}_P^G(k) = \bigoplus_{x \in [P\backslash G/P]} \operatorname{Ind}_{P \cap {}^xP}^P(k) = \bigoplus_{x \in [G/P]} k$
$\Rightarrow$ $W \cong k$ as kP-module

But k_P has vertex P which is a contradiction ■

□

Definition 2.6.5
An algebra A has <u>finite representation type</u> if $\exists$ only finitely many isomorphism classes of indecomposable A-module

Definition 2.6.6
M is kG-module. Say M is <u>uniserial</u> if M has a unique composition series
i.e. $\operatorname{Rad}^i(M)/\operatorname{Rad}^{i+1}(M)$ simple $\forall i$
i.e. $\operatorname{Soc}^{i+1}(M)/\operatorname{Soc}^i(M)$ simple $\forall i$

Proposition 2.6.7
P finite cyclic p-group, then kP has finite representation type and all the indecomposable kP-modules are uniserial.

Proof
$P = \langle g | g^{p^n} = 1 \rangle$
$\Rightarrow$ $I(kP) = (g-1)kP = J(kP)$
$\Rightarrow$ $kP \cong k[X]/(X^{p^n})$ (via $g - 1 \leftarrow\!\shortmid X + (X^{p^n})$)
$\Rightarrow$ the indecomposable kP-modules are: $V_i := kP/J(kP)^i \quad (1 \leq i \leq p^n)$
$\Rightarrow$ $\operatorname{Rad}^j(V_i)/\operatorname{Rad}^{j+1}(V_i) \cong k$
$\Rightarrow$ V_i are uniserial. □

Proposition 2.6.8
Let b be block of kG with defect P
If P is cyclic, then kGb has finite representation type

Proof
Let M be an indecomposable kGb-module
Then M has vertex $Q \leq P$
P cyclic $\Rightarrow$ Q cyclic
$\Rightarrow$ $\exists$ only finitely many indecomposable kQ-modules, and $M | \operatorname{Ind}_Q^G(S)$, S indecomposable kQ-module
$\square$

Remark. In fact, P cyclic $\Leftrightarrow kGb$ finite representation type

2.7 Extended Brauer's First Main Theorem

Let b be block of kG with defect group P
Then $\operatorname{Br}_P(b)$ a block of $kN_G(P)$ with defect group P
In fact, $\operatorname{Br}_P(b) \in (kC_G(P))_P^{N_G(P)} \subseteq Z(kC_G(P))$
i.e. $\operatorname{Br}_P(b)$ is a central idempotent of $kC_G(P)$
$\Rightarrow$ $\exists e$ a block of $kC_G(P)$ s.t. $\operatorname{Br}_P(b)e = e$

Now $C_G(P) \lhd N_G(P)$ $\Rightarrow$ $N_G(P)$-conjugation induces an algebra automorphism of $kC_G(P)$, hence permutes the blocks of $kC_G(P)$:
If $x \in N_G(P)$, then $\operatorname{Br}_P(b)^x e = {}^x e$
Set $N_G(P, e) := \{x \in N_G(P) | {}^x e = e\}$
If $x \in N_G(P) - N_G(P, e)$, then ${}^x e$ is a block of $kC_G(P)$ different from e
$\Rightarrow$ $({}^x e)e = 0$
$\Rightarrow$ $\operatorname{Tr}_{N_G(P,e)}^{N_G(P)}(e)$ is a sum of distinct blocks of $kC_G(P)$
$\Rightarrow$ e is a central idempotent of $kN_G(P)$
But $\operatorname{Br}_P(b) = \operatorname{Br}_P(b) \operatorname{Tr}_{N_G(P,e)}^{N_G(P)}(e) = \operatorname{Tr}_{N_G(P,e)}^{N_G(P)}(e)$

Proposition 2.7.1

(1) e has defect group $Z(P)$

(2) for k algebraically closed, $p \nmid |N_G(P, e) : PC_G(P)|$

We first need the following lemma to prove the proposition:

Lemma 2.7.2
Let A be G-algebra (i.e. A is an algebra with $G \to \operatorname{Aut}(A)$), $N \lhd G$, $N \leq H \leq G \Rightarrow$ $A_N^G \subseteq A_N^H$
Using Mackey, we get

$$\operatorname{Tr}_N^G(a) = \sum_{x \in H \backslash G / N} \operatorname{Tr}_{H \cap {}^x N}^H({}^x a) \quad \in \ A_N^H$$

Proof of Proposition 2.7.1

(1) $\operatorname{Br}_P(b) \in (kC_G(P))_P^{N_G(P)} \subseteq (kC_G(P))_P^{PC_G(P)}$
$e = \operatorname{Br}_P(b)e \in (kC_G(P)e)_P^{PC_G(P)} = (kC_G(P)e)_{Z(P)}^{C_G(P)}$
$\Rightarrow$ e has defect group contained in $Z(P)$, but $Z(P)$ is a normal p-subgroup of $C_G(P)$, so $Z(P)$ is contained in any defect group of e
$\Rightarrow$ e has defect group $Z(P)$

(2) $\mathrm{Br}_P(b) \in (kC_G(P))_P^{N_G(P)} \subseteq (kC_G(P))_P^{N_G(P,e)}$ using Lemma 2.7.2
$e = \mathrm{Br}_P(b)e \in (kC_G(P)e)_P^{N_G(P,e)}$
Write $e = \mathrm{Tr}_P^{N_G(P,e)}(a) = \mathrm{Tr}_{PC_G(P)}^{N_G(P,e)} \mathrm{Tr}_P^{PC_G(P)}(a)$ for some $a \in kC_G(P)e$
$\mathrm{Tr}_P^{PC_G(P)}(a) \in (kC_G(P)e)^{PC_G(P)} = Z(kC_G(P)e)$
Since e is a block of $kC_G(P)$
$\Rightarrow$ $Z(kC_G(P)e)$ is a local algebra
k algebraically closed $\Rightarrow$ $Z(kC_G(P)e)/J(Z(kC_G(P)e)) \cong k$
$\Rightarrow$ $\mathrm{Tr}_P^{PC_G(P)}(a) = \lambda e + r$, some $r \in J(Z(kC_G(P)e))$ and $\lambda \in k$

$$\Rightarrow \quad e = \mathrm{Tr}_{PC_G(P)}^{N_G(P,e)}(\lambda e + r) = \lambda |N_G(P,e) : PC_G(P)| e + \underbrace{\mathrm{Tr}_{PC_G(P)}^{N_G(P,e)}(r)}_{\in JZ(kC_G(P)e)}$$

If $r \big| |N_G(P,e) : PC_G(P)|$, then $e = r$, contradicting nilpotentcy of $r \Rightarrow$ $p \nmid |N_G(P,e) : PC_G(P)|$

□

Pictorial summary: G b let $\mathrm{Br}_P(b)e = e, \mathrm{Br}_P(e) = \mathrm{Tr}_{N_G(P,e)}^{N_G(P)}(e)$
$N_G(P)$ $\mathrm{Br}_P(b)$
$N_G(P,e)$
$PC_G(P)$ (index p' subgroup of $N_G(P,e)$)
$C_G(P)$ e

Definition 2.7.3
$|N_G(P,e) : PC_G(P)|$ = the inertial index of b

Definition 2.7.4
A Brauer pair for kG is a pair (Q,e) where Q is a p-subgroup of G and e is a block of $kC_G(Q)$

Lemma 2.7.5
Let (Q,e) be a Brauer pair for kG
Then $\exists!$ block b of kG s.t. $\mathrm{Br}_Q(b)e = e$

Proof
$\mathrm{Br}_Q : (kG)^Q \twoheadrightarrow kC_G(Q)$ is a surjective algebra homomorphism
For any block b of kG, $\mathrm{Br}_Q(G)$ is either zero or a central idempotent of $kC_G(Q)$
Let $\mathcal{B} = \{$ blocks of $kG\}$
then $1_{kG} = \sum_{b \in \mathcal{B}} b$
$\Rightarrow$ $1_{kC_G(Q)} = \sum_{b \in \mathcal{B}_0} \mathrm{Br}_Q(b)$ where $\mathcal{B}_0 = \{b \in \mathcal{B} | \mathrm{Br}_Q(b) \neq 0\}$
$\Rightarrow$ $e = \sum_{b \in \mathcal{B}} \mathrm{Br}_Q(b)e$
$\Rightarrow$ $\mathrm{Br}_Q(b)e \neq 0$ form some $b \in \mathcal{B}$
$$\Rightarrow \begin{cases} \mathrm{Br}_Q(b)e = e \\ \mathrm{Br}_Q(b')e = 0 \ \ \forall b' \neq b \end{cases}$$
□

In the above case, we say
either (Q,e) is a b-Brauer pair of kG
or b is the induced block of e

If (Q,e) is a b-Brauer pair (i.e. $\mathrm{Br}_Q(b)e = e$)
then $\mathrm{Br}_Q(b) \neq 0$, so $Q \leq_G P$ where P is defect group of b

Definition 2.7.6
Let $(Q, e), (R, f)$ be Brauer pairs for kG

Define $(R, f) \lhd (Q, e) \quad \Leftrightarrow \quad \begin{cases} R \lhd Q \\ f \text{ is } Q\text{-invariant and } \mathrm{Br}_Q^{kC_G(R)}(f)e = e \end{cases}$

(Note: $e \in kC_G(Q) \hookrightarrow kC_G(R) \ni f$)

$(R, f) \leq (Q, e) \quad \Leftrightarrow \quad \exists (R, f) = (R_0, f_0) \lhd (R_1, f_1) \lhd \cdots \lhd (R_n, f_n) = (Q, e)$

Theorem 2.7.7 (Alperin-Broué)
Let b be block of kG with defect group P

(1) Let Q, R be p-subgroup of G s.t. (Q, e) a b-Brauer pair and $R \leq Q$
Then $\exists!$ block f of $kC_G(R)$ s.t. $(R, f) \leq (Q, e)$

(2) Let (Q, e) be a b-Brauer pair
(Q, e) is maximal $\quad \Leftrightarrow \quad Q$ is a defect group of b

(3) Let $(Q, e), (R, f)$ maximal b-Brauer pair
Then $\exists x \in G$ s.t. $(R, f) = {}^x(Q, e)$

(Proof omitted)
This theorem leads to:
Let b be block of kG with defect group P
Let e be a block of $kC_G(P)$ s.t. $\mathrm{Br}_Q(b)e = e$
Then (P, e) is a maximal b-Brauer pair
$\forall Q \leq P, \exists! e_Q$ s.t. $(Q, e_Q) \leq (P, e)$

(This is like giving a Sylow structure for blocks)

Definition 2.7.8
$\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$ be a category s.t.
objects are the subgroups of P
morphisms are $\mathrm{Hom}_{\mathcal{F}}(Q, R) = \{c_x : Q \hookrightarrow R \mid x \in G \text{ s.t. } {}^x(Q, e_Q) \leq (R, e_R) \text{ and } c_x(y) = {}^x y \ \ \forall y \in Q\}$
This category is the fusion system of the block b

Note that $\mathrm{Aut}_{\mathcal{F}}(P) = N_G(P, e)/C_G(P), \mathrm{Out}_{\mathcal{F}}(P) = N_G(P, e)/PC_G(P)$
$\Rightarrow \quad |\mathrm{Out}_{\mathcal{F}}(P)|$ =the inertial index of b

2.8 Cyclic blocks

Setup for this section:
G a finite group, k is algebraically closed field of characteristic p
b block of kG with cyclic defect group P of order p^n $(n \geq 1)$
$\mathrm{Br}_P(b)$ is (the Brauer correspondent of b) block of $kN_G(P)$
e block of $kC_G(P)$ s.t. $\mathrm{Br}_P(b)e = e$

$\Rightarrow \quad \begin{cases} e \text{ has defect group } P = Z(P) \\ \mathrm{Br}_P(b) = \mathrm{Tr}_{N_G(P,e)}^{N_G(P)}(e) \\ |N_G(P, e) : C_G(P)| \not\equiv 0 \mod p \end{cases}$ (Brauer Extended First Main Theorem)

P_1 the unique subgroup of P of order P
$\Rightarrow \quad P_1$ char P
$\Rightarrow \quad N_G(P) \leq N_G(P_1)$ and $C_G(P) \leq C_G(P_1)$
e_1 (unique) block of $kC_G(P_1)$ s.t. $\mathrm{Br}_P(e_1)e = e$ (note: $C_G(P) = C_{C_G(P_1)}(P)$, c.f. Lemma 2.7.5)
b_1 (unique) block of $kN_G(P_1)$ with defect group P s.t. $\mathrm{Br}_P(b_1) = \mathrm{Br}_P(b)$

Lemma 2.8.1
$Q \lhd R$ p-subgroup of G
$a \in (kG)^R \;\Rightarrow\; \mathrm{Br}_R(\mathrm{Br}_Q(a)) = \mathrm{Br}_R(a)$

Proof
$Q \leq R \;\Rightarrow C_G(R) \leq C_G(Q)$
$a \in (kG)^R \;\Rightarrow\; \mathrm{Br}_Q(a) \in (kC_G(Q))^R$ Since Brauer homomorphism is the truncation map, so by the above two reasons, we get the result. □

Lemma 2.8.2
$\mathrm{Br}_{P_1}(b_1)e_1 = e_1$

Proof
By Lemma 2.7.5, $\exists!$ b_1' block of $kN_G(P_1)$ s.t. $\mathrm{Br}_{P_1}(b_1')e_1 = e_1$
$P_1 \lhd P$, so invoke Lemma 2.8.1 for Br_P
$\Rightarrow \quad \mathrm{Br}_P(b_1')\,\mathrm{Br}_P(e) = \mathrm{Br}_P(e_1)$
$\Rightarrow \quad \mathrm{Br}_P(b_1')e = e$ (multiplying by e)
$\Rightarrow \quad \mathrm{Br}_P(b_1')\,\mathrm{Br}_P(b_1)e = e$ (using $\mathrm{Br}_P(b_1)e = e$)
$\Rightarrow \quad b_1'b_1 \neq 0$
$\Rightarrow \quad b_1' = b_1$ □

Lemma 2.8.3
e_1 has defect group P

Proof
$\mathrm{Br}_P(e_1)e = e \quad \Rightarrow \quad \mathrm{Br}_P(e_1) \neq 0 \quad \Rightarrow \quad e_1$ has defect group $Q \leq P$ since:
Suppose $Q > P$, take Br_Q to Lemma 2.8.2, $0 = \mathrm{Br}_Q(b_1)\,\mathrm{Br}_Q(e_1) = \mathrm{Br}_Q(e_1)$ contradiction □

Note that
(P, e) is maximal b-Brauer pair
$(P_1, e_1) \lhd (P, e)$ (because $\mathrm{Br}_P(e_1)e = e$)
$\Rightarrow \; N_G(P, e) \leq N_G(P_1, e_1)$
(becuase, if $x \in N_G(P, e) \;\Rightarrow\; \mathrm{Br}_P({}^x e_1)e = e$
$\Rightarrow \; {}^x e_1 = e_1$ by uniqueness of e_1)

Proposition 2.8.4
$N_G(P, e)/C_G(P) \cong N_G(P_1, e_1)/C_G(P_1)$

Proof
Let $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$
$\Rightarrow \quad N_G(P, e)/C_G(P) \cong \mathrm{Aut}_{\mathcal{F}}(P)$ and $N_G(P_1, e_1)/C_G(P_1) \cong \mathrm{Aut}_{\mathcal{F}}(P_1)$
Consider the restriction map $\begin{array}{rcl} \phi : \mathrm{Aut}_{\mathcal{F}}(P) & \rightarrow & \mathrm{Aut}_{\mathcal{F}}(P_1) \\ \alpha & \mapsto & \alpha|_{P_1} \end{array}$

We use the following result (Burnside's theorem) from fusion system without proof: P is abelian $\Rightarrow \; \mathcal{F} = N_{\mathcal{F}}(P)$ and ϕ surjective

Check that $\ker \phi$ is a p-group (as P cyclic)
$\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{F}}(P)$ is a p'-group (as P abelian)
But $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\,\mathrm{Inn}(P)$
$\Rightarrow \quad \ker \phi = 1$
$\Rightarrow \quad \phi$ isom □

Theorem 2.8.5
There is a one-to-one correspondence:

$$\left\{\begin{array}{c}\text{non-projective indecomposable}\\ kGb\text{-modules}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{non-projective indecomposable}\\ kN_G(P_1)b_1\text{-modules}\end{array}\right\}$$
$$U \longleftrightarrow V$$

s.t.
$U\downarrow_{N_G(P_1)} \cong V \oplus Y, Y =$projective$\oplus$module not in b_1
$V\uparrow^G \cong U \oplus X, X$ projective

If U_1, U_2 are kGb_1-modules
corresponding to V_1, V_2 $kN_G(P_1)b_1$-module
then $\underline{\mathrm{Hom}}_{kG}(U_1, U_2) \cong \underline{\mathrm{Hom}}_{kN_G(P_1)}(V_1, V_2)$
Moreover, $\underline{\mathbf{mod}}(kGb) \cong \underline{\mathbf{mod}}(kN_G(P_1)b_1)$

Proof
Use Green's correspondence:

$$\left\{\begin{array}{c}\text{non-projective indecomposable}\\ kGb\text{-modules}\end{array}\right\} = \coprod_{P_1\leq Q\leq P}\left\{\begin{array}{c}\text{indecomposable } kN_G(P_1)b_1\text{-modules}\\ \text{with vertex } Q\end{array}\right\}$$

(becuase b has defect group P and P_1 is the unique subgroup of P of order p)
Same for $kN_G(P_1)b_1$, so enough to show:

For each $P_1 \leq Q \leq P$

$$\left\{\begin{array}{c}\text{indecomposable } kGb\text{-modules}\\ \text{with vertex } Q\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{indecomposable } kN_G(P_1)b_1\text{-modules}\\ \text{with vertex } Q\end{array}\right\}$$

$P_1 \leq Q \leq P \Rightarrow N_G(Q) \leq N_G(P_1)$
So Green correspondence says:

$$\left\{\begin{array}{c}\text{indecomposable } kG\text{-modules}\\ \text{with vertex } Q\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{indecomposable } kN_G(P_1)\text{-modules}\\ \text{with vertex } Q\end{array}\right\}$$

s.t. $U\downarrow_{N_G(P_1)} \cong V \oplus Y$, Y projective relative to $\mathcal{Y} = \{N_G(P_1) \cap {}^xQ | x \in G - N_G(P_1)\}$
$V\uparrow^G \cong U \oplus X$, X projective relative to $\mathcal{X} = \{Q \cap {}^xQ | x \in G - N_G(P_1)\}$

$\underline{\mathcal{X} = \{1\}}$:
Suppose $Q \cap {}^xQ \neq 1$ for some $x \in G - N_G(P_1)$
Then $\left.\begin{array}{c}P_1 \leq Q \cap {}^xQ\\ {}^xP_1 \leq Q \cap {}^xQ\end{array}\right\} \Rightarrow P_1 = {}^xP_1$ (by uniqueness)
$\Rightarrow$ $x \in N_G(P_1)$ which is a contraction

$\underline{Y = \text{projective} \oplus \text{modules not in } b_1}$
This is to show: $N_G(P_1) \cap {}^xQ \neq 1$ for some $x \in G - N_G(P_1)$ then no non-projective $kN_G(P_1)b_1$-module is projective relative to $N_G(P_1) \cap {}^xQ$
So suppose this is not ture
b_1 has defect group P
$\Rightarrow$ such module has vertex $P_1 \leq R \leq P$
$\Rightarrow$ $P_1 \leq R \leq_{N_G(P_1)} {}^xQ$

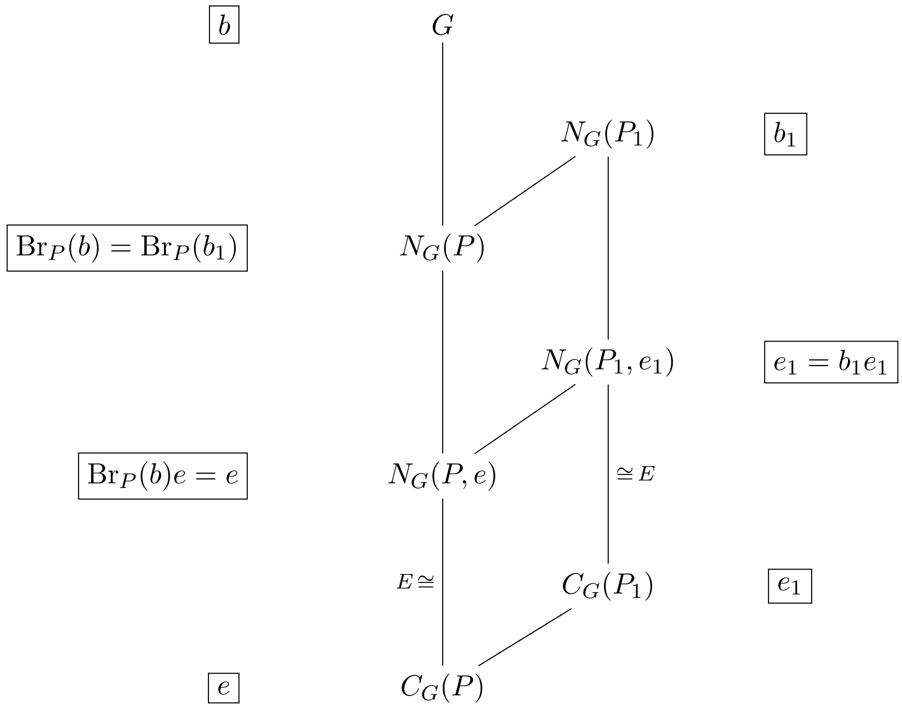
$\Rightarrow \quad P_1 \leq {}^xQ$
${}^xQ \supseteq {}^xP_1 \;\; \Rightarrow \;\; {}^xP_1 = P_1 \;\; \Rightarrow \;\; x \in N_G(P_1)$ contradiction

And the Green's correspondence respect blocks (by Nagao's theorem, see Benson)
Now let U_1, U_2 be kGb_1-modules corresponding to V_1, V_2 the $kN_G(P_1)b_1$-modules
$U_i \downarrow_{N_G(P_1)} \cong V_i \oplus Y_i$
$V_i \uparrow^G \cong U_i \oplus X_i$, then we get:

$$\begin{aligned}\underline{\mathrm{Hom}}_{kG}(U_1, U_2) &\cong \underline{\mathrm{Hom}}_{kG}(\mathrm{Ind}^G_{N_G(P_1)}(V_1), U_2)\\ &\cong \underline{\mathrm{Hom}}_{kN_G(P_1)}(V_1, \mathrm{Res}^G_{N_G(P_1)}(U_2))\\ &\cong \underline{\mathrm{Hom}}_{kN_G(P_1)}(V_1, V_2)\end{aligned}$$

□

<u>Synopsis</u>:



Also, we have

$$\begin{aligned}\mathrm{Br}_P(b)e &= e\\ \mathrm{Br}_{P_1}(e_1)e &= e\\ \mathrm{Br}_{P_1}(b_1)e_1 = b_1e_1 &= e_1\end{aligned}$$

Note:
e is a block of $kN_G(P, e)$
e_1 is a block of $kN_G(P_1, e_1)$
becuase $P \lhd N_G(P, e)$: If $e = e_1 + e_2$ is a decomposition in $Z(kN_G(P, e))$, then it is in fact a decomposition in $Z(kC_G(P))$

(1) $\underline{\mathbf{mod}}(kGb) \cong \underline{\mathbf{mod}}(kN_G(P_1)b_1)$

(2) $\mathbf{Mod}(kN_G(P_1)b_1) \cong \mathbf{Mod}(k(P \rtimes E))$ (where $E := N_G(P, e)/C_G(P) \cong N_G(P_1, e_1)/C_G(P_1)$)

(3) $k(P \rtimes E)$ is a Brauer tree algebra of star-shaped Brauer tree with an exceptional vertex in the centre.

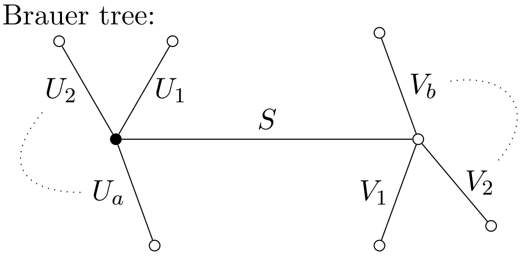
(4) An algebra which is stably equivalent to an algebra in (3) is a <u>Brauer tree algebra</u>. kGb is a Brauer tree algebra.

(5) (Rickard) $D^b(\mathbf{Mod}(kGb)) \cong D^b(\mathbf{Mod}(kN_G(P)\operatorname{Br}_P(b)))$

2.9 Brauer tree algebra

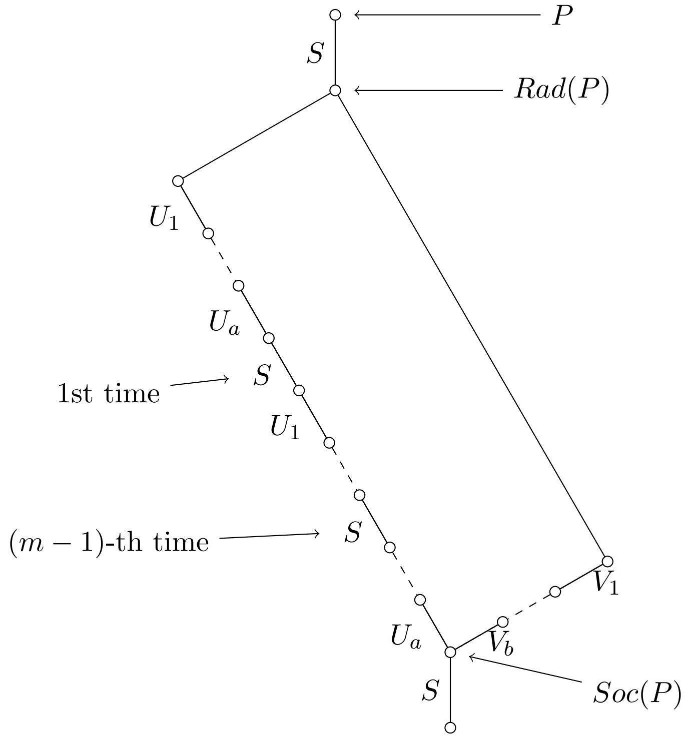
- tree = finite connected graph without loop
- plannar tree = tree with plannar embedding = tree with circular counter-clockwise ordering of edges eminating from each vertex
- Brauer tree = plannar tree with an exceptional vertex with exceptional multiplicity ≥ 1
  Denote a Brauer tree as triple (Γ, v_0, m) = (plannar tree, exceptional vertex, exceptional multiplicity)
- Brauer tree algebra A of a Brauer tree (Γ, v_0, m) is given by specifying the radical quotients of projective indecomposable A-module:
  - simples $\leftrightarrow$ edges of Γ
  - simples $\leftrightarrow$ projecitve indecomposable P_i s.t.
    $P_i/\operatorname{Rad}(P_i) \cong \operatorname{Soc}(P_i) \cong S_i$ (A is a symmetric algebra)
    $\operatorname{Rad}(P_i)/\operatorname{Soc}(P_i) \cong U \oplus V$, where U, V are uniserial, with composition factor specified by edges eminating from the two ends of the vertex, s.t. from top to bottom, the composition factor correspond to the edges eminating from a vertex in a counter-clockwise direction.
    Suppose U correspond to a vertex of multiplicity m, then S_i appears in U for $m-1$ times

Brauer tree:



with black vertex being the multiplicity m exceptional vertex.

Then the projective indecomposable corresponding to simple S has filtration diagram:



Proposition 2.9.1
This recipe determines the Brauer tree algebra up to Morita equivalence.

Remark. It is NOT true in general that Lowey structure determines algebra structure.

Theorem 2.9.2
Let b be a block of kG with cyclic defect group P of order p^n $(n \geq 1)$ and with inertial index ϵ
Then kGb is a Brauer tree algebra with a Brauer tree of ϵ edges and the exceptional multiplicity $(p^n-1)/\epsilon$

Remark. $\epsilon = |E| \Big| |Aut(P_1)| = p-1$

Proposition 2.9.3
Let b be a block of kG with cyclic defect group P of order p^n $(n \geq 1)$ with inertial index 1.
Then kGb is a Brauer tree algebra of a Brauer tree with 1 edge and an exceptional vertex with multiplicity $p^n - 1$
i.e. $\exists!$ simple module and projective cover P of S is a uniserial module of Lowey length p^n
(i.e. $\mathrm{Rad}^i(P)/\mathrm{Rad}^{i+1}(P) \cong S \ \ i = 0, \ldots, p^n - 1$)

Remark. This proposition applies to e_1:
The inertial index of e_1 is $|N_{C_G(P_1)}(P,e) : C_G(P)| = |N_G(P,e) \cap C_G(P_1) : C_G(P)| = 1$ because $N_G(P,e)/C_G(P) \cong N_G(P_1,e_1)/C_G(P_1)$, and maximal e_1-Brauer pair is the same as that of b, and $PC_G(P) = C_G(P)$ as P abelian

Lemma 2.9.4
Suppose that A is finite dimensional algebra s.t. every injective indecomposable and projective indecomposable A-module are uniserial.
Then every indecomposable A-module is uniserial

Proof
Let M be an indecomposable A-module
Let U/V be a maximal uniserial subquotient of M (this always exists)
Let S be the top of U/V, P be the projective cover of S

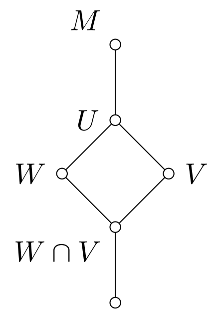
Then we get

$$\begin{array}{ccc} P & \dashrightarrow & U/V \\ \downarrow & & \downarrow \\ P/\operatorname{Rad}(P) & \xrightarrow{\sim} & (U/V)/\operatorname{Rad}(U/V) \end{array}$$

(The surjection on top row is by Nakayama's Lemma), then

$$\begin{array}{ccc} & & U \\ & \nearrow & \downarrow \\ P & \longrightarrow & U/V \end{array}$$

Let W be the image of $P \to U$
Then W is uniserial module, $W + V = U$
Then $U/V \cong W/(V \cap W)$



$\Rightarrow \quad W \cap V = 0$ by maximality
$\Rightarrow \quad U = V \oplus W$

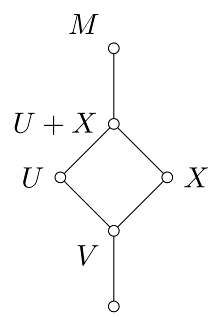
Dually, let $T = \operatorname{Soc}(U/V)$, $I =$ injective hull of T
Then

$$\begin{array}{ccc} U/V & \dashrightarrow & I \\ \uparrow & & \uparrow \\ \operatorname{Soc}(U/V) & \xrightarrow{\sim} & \operatorname{Soc}(I) \end{array}$$

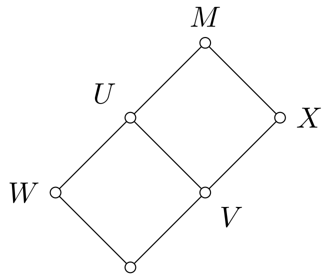
Then

$$\begin{array}{ccc} U/V & \hookrightarrow & I \\ \downarrow & \nearrow & \\ M/V & & \end{array}$$

Let $X/V = \ker(M/V \to I)$
Then $M/X \hookrightarrow I$, so M/X is uniserial.
If $X/V \cap U/V \neq 0$, then X/V contains $\operatorname{Soc}(U/V)$, contradiction
Therefore, $X/V \cap U/V = 0$, so $X \cap U = V$



$\Rightarrow \quad U/V \cong (U+X)/X \hookrightarrow M/X$
$\Rightarrow \quad M = U + X$ by maximality
$\Rightarrow \quad M = W \oplus X$



$\Rightarrow \quad M = W$ as $W \neq 0$ and M indecomposable □

2.10 Covering blocks

Let $N \lhd G$

Definition 2.10.1
b a block of kG
e a blcok of kN
Say b covers $e \quad \Leftrightarrow be \neq 0$

Lemma 2.10.2
b block of kG, e block of kN
b covers $e \quad \Leftrightarrow \quad b\,\mathrm{Tr}_H^G(e) = b$, where $H = \mathrm{Stab}_G(e) = \{x \in G | {}^x e = e\}$

Proof
Since $N \lhd G$, G-conjugate induces an algebra automorphism of kN, hence a permutation of the blocks of kN
$\Rightarrow \quad \mathrm{Tr}_H^G(e) =$ the G-orbit sum of blocks of kN containing e
$\Rightarrow \quad \mathrm{Tr}_H^G(e)$ is a central idempotent of kG

$be \neq 0 \quad \Leftrightarrow \quad b\,\mathrm{Tr}_H^G(e) \neq 0 \quad \Leftrightarrow \quad b\,\mathrm{Tr}_H^G(e) = b$ □

Proposition 2.10.3
b block of kG, e block of kN, $H = \mathrm{Stab}_G(e) \Rightarrow \quad c = be$ block of kH s.t. $kGb \cong M_n(kHc)$ as k-algebras $(n = |G:H|)$
In particular, kGb is Morita equivalent to kHc

Proof
Previous Lemma: $b = b\,\mathrm{Tr}_H^G(e) \quad \Rightarrow \quad kGb | kG\,\mathrm{Tr}_H^G(e)$ as k-algebras
Let $[G/H] = \{x_i | 1 \leq i \leq n\}$ set of coset representatives

$$\begin{aligned} kG\,\mathrm{Tr}_H^G(e) &= \bigoplus_{1 \leq j \leq n} kGx_j e x_j^{-1} \\ &= \bigoplus_{1 \leq j \leq n} kGe x_j^{-1} \\ &= \bigoplus_{1 \leq i,j \leq n} x_i (kHe) x_j^{-1} \\ &\cong M_n(kHe) \qquad (k\text{-algebra isom via } x_i a x_j^{-1} \mapsto a e_{ij}) \end{aligned}$$

The image of b is a block of $M_n(kHe)$
$Z(M_n(kHe)) = Z(kHe)I_n \cong Z(kHe)$

$$\begin{aligned} b &= \sum_i x_i c x_i^{-1} \quad \text{for some block } c \text{ of } kHe\text{, which is of form } \sum ce_{ii} \text{ in } Z(M_n(kHe)) \\ &= \mathrm{Tr}_H^G(e) \;\; \text{(use the inverse map of the } k\text{-algebra isom)} \end{aligned}$$

$\Rightarrow \quad kGb \cong M_n(kHc)$

Note: If $x \in G - H$, then $e \neq {}^x e$, both blocks of kN
$\Rightarrow \quad e^x e = exe^{-1} = 0$
$\Rightarrow \quad exe = 0 \;\; \forall x \in G - H$

$$\begin{aligned} be &= \mathrm{Tr}_H^G(c)e \;=\; \sum_i x_i c x_i^{-1} e \\ &= \sum_i x_i c e x_i^{-1} e \\ &= ce \;=\; c \end{aligned}$$

□

Theorem 2.10.4
Let Q be a central p-subgroup of G i.e. p-subgroup contained in $Z(G)$
Then the canonical map $\pi : kG \to kG/Q$ induces a bijection between $\mathcal{B}(kG)$ and $\mathcal{B}(kG/Q)$
where if $b \in \mathcal{B}(kG)$ with defect group P, then $\pi(b) \in \mathcal{B}(kG/Q)$ with defect group P/Q

(Proof omitted, also note that this is generally NOT true if Q is just normal in G)

Proposition 2.10.5 (Proposition 2.9.3)
Let b be a block of kG with cyclic defect group P of order p^n $(n \geq 1)$ and with inertial index 1
Then kGb has only one simple module S, and its projective cover P_S is uniserial with length p^n, i.e. Brauer tree with 1 edge and an exceptional vertex with multiplicity $m = p^n - 1$

Lemma 2.10.6
Let A be a finite representation type algebra
Suppose A has only one simple module S
Then the projective cover P_S of S is uniserial

Proof
Have $P_S/\operatorname{Rad}(P_S) \cong S$
Suppose $\operatorname{Rad}(P_S) \neq 0$

Claim: $\operatorname{Rad}(P_S)/\operatorname{Rad}^2(P_S) \cong S$

Proof of Claim:
Suppose not, $\operatorname{Rad}(P_S)/\operatorname{Rad}^2(P_S) \cong \underbrace{S \oplus \cdots \oplus S}_{r \geq 2 \text{ times}}$

$\Rightarrow \quad P_S/\operatorname{Rad}^2(P_S) = \begin{matrix} S \\ S \oplus \cdots \oplus S \end{matrix}$

$\Rightarrow \quad \operatorname{End}_A(P_S) \twoheadrightarrow \operatorname{End}_A(P_S/\operatorname{Rad}^2(P_S))$ because :

$$\begin{array}{ccc} P_S & \twoheadrightarrow & P_S/\operatorname{Rad}^2(P_S) \\ \downarrow & & \downarrow \\ P_S & \twoheadrightarrow & P_S/\operatorname{Rad}^2(P_S) \end{array}$$

$\operatorname{End}_A(P_S/\operatorname{Rad}^2(P_S)) \cong k[X_1, \ldots, X_r]/(X_iX_j : 1 \leq i \leq j \leq r)$ (where X_i sends the top, S, to the i-th copy of S in the 2nd layer), this algebra is well-known to have infinite representation type.
$\Rightarrow$ $\operatorname{End}_A(P_S/\operatorname{Rad}^2(P_S))$ has infinite representation type
$\Rightarrow$ A has infinite representation type as A is Morita equivalent to $\operatorname{End}_A(P_S)$, contradiction ■

This Morita equivalent comes from: $A = P_S \oplus \cdots \oplus P_S$, so $A \cong \operatorname{End}_A(A)^{op} \cong \operatorname{End}_A(P_S \oplus \cdots \oplus P_S)^{op} \cong M_n(\operatorname{End}_A(P_S))^{op}$ which is Morita equivalent to $\operatorname{End}_A(P_S)^{op}$

Note in general, A f.d. algebra, $P_1, \ldots, P_s$ projective indecomposables, then A is Morita equivalent to $\operatorname{End}_A(P_1 \oplus \cdots \oplus P_s)$

With the claim, we get

$$\begin{array}{ccc} P_S & \longrightarrow & \operatorname{Rad}(P_S) \\ \downarrow & & \downarrow \\ P_S/\operatorname{Rad}(P_S) & \xrightarrow{\sim} & \operatorname{Rad}(P_S)/\operatorname{Rad}^2(P_S) \end{array}$$

$\Rightarrow$ $S \cong \operatorname{Rad}(P_S)/\operatorname{Rad}^2(P_S) \twoheadrightarrow \operatorname{Rad}^2(P_S)/\operatorname{Rad}^3(P_S)$
$\Rightarrow$ $\operatorname{Rad}^2(P_S)/\operatorname{Rad}^3(P_S) \cong S$ or 0
Repeat this procedure until we get 0, so done □

We need another lemma before proving Proposition 2.10.5:

Lemma 2.10.7
If M, N are non-projective indecomposable modules of self-injective algebra A, then $M \hookrightarrow N$ and $M \twoheadrightarrow N$ have non-zero image in the stable module category $\underline{\textbf{mod}}(A)$

Proof
Suppose $f : M \twoheadrightarrow N$ factors through some projective P
Then f factors through the projective cover P_N of N by projectiveness of P, because:

$$\begin{array}{ccc} M & \stackrel{f}{\twoheadrightarrow} & N \\ \downarrow & \nearrow & \uparrow \\ P & \dashrightarrow & P_N \end{array}$$

$$\Rightarrow \quad \begin{array}{ccc} M & \stackrel{f}{\twoheadrightarrow} & N \\ & {}_g\searrow \quad \nearrow_\pi & \\ & P_N & \end{array}$$

f is surjective and $\operatorname{Im} g + \operatorname{Rad}(N) = P_N$
$\Rightarrow$ $\operatorname{Im} g = P_N$ (by Nakayama's Lemma) $\Rightarrow$ g is surjective

P_N is projecitve $\Rightarrow$ $P_N | M$, contradicting indecomposability □

Proof of Proposition 2.10.5
Induction on $|G|$
Base case: $G \cong \mathbb{Z}/p\mathbb{Z}$

<u>Case 1</u>: $P_1 \leq Z(G)$ (In particular, the base case)
By Theorem 2.10.4, $\pi : kG \to kG/P_1$ sends b to a block $\pi(b) = \overline{b}$ of kG/P_1 with defect group P/P_1

By induction (or by structure theory of blocks of trivial defect group)
$kG/P_1\overline{b}$ only has one simple module $\overline{S}$, with projective cover $P_{\overline{S}}$ is uniserial of length p^{n-1}

Since $P_1 \lhd G$, by Clifford's Theorem 2.5.3, simples of kGb corresponds to simples of $kG/P_1\bar{b}$
$\Rightarrow \quad S(=\overline{S}$ viewed as kGb-module) is the only simple module of kGb

Apply Lemma 2.10.6: P_S is uniserial
Need to show: length of $P_S = p\cdot$ (length of $P_{\overline{S}}$)
$\pi : kG \to kG/P_1$, and let $P_1 = \langle g | g^p = 1\rangle$
$\ker\pi = J(kP_1)kG = (g-1)kG$
$\Rightarrow \quad kG/(g-1)kG \cong kG/P_1$ (as algebra, also as kG-module)

Claim: $(g-1)^j kG/(g-1)^{j+1}kG \cong kG/P_1$ as kG-modules

Proof of Claim:

$$\begin{array}{rrcl} & kG & \overset{(g-1)^j}{\twoheadrightarrow} & (g-1)^j kG \\ & (g-1)kG & \twoheadrightarrow & (g-1)^{j+1}kG \\ \Rightarrow & kG/P_1 \cong kG/(g-1)kG & \twoheadrightarrow & (g-1)^j kG/(g-1)^{j+1}kG \end{array}$$

By comparing dimension

$$kG/P_1 \cong (g-1)^j kG/(g-1)^{j+1}kG \qquad \blacksquare$$

This restricts to

$$kG/P_1\bar{b} \cong (g-1)^j kGb/(g-1)^{j+1}kGb$$

and to projective indecomposables:

$$P_{\overline{S}} \cong (g-1)^j P_S/(g-1)^{j+1}P_S$$

using dimension argument, get length of $P_S = (p)$(length of $P_{\overline{S}}) = p^n$

<u>Case 2</u>: $P_1 \nleq Z(G), P_1 \lhd G$
$\Rightarrow \quad N_G(P_1) = G, N_G(P_1, e_1) = C_G(P_1) \lhd G$ $(C_G(P_1) \neq G)$(inertial index 1)
$\Rightarrow \quad b = b_1$ covers e_1 and $kGb$is Morita equivalent to$kC_G(P_1)e_1$
e_1 is a block of $kC_G(P_1)$ with defect group P, inertial index 1 and $C_G(P_1) \lneqq G$

By induction (apply Case 1): $kC_G(P_1)e_1$ has the desired property
$\Rightarrow \quad kGb$ has the desired property

<u>Case 3</u>: $P_1 \ntrianglelefteq G$
$\Rightarrow \quad N_G(P_1) \lneqq G$
By induction: $kN_G(P_1)b_1$ has only one simple module T and its projective cover P_T is uniserial of length p^n

We know that $\underline{\mathbf{mod}}(kGb) \cong \underline{\mathbf{mod}}(kN_G(P_1)b_1)$ (via Green correspondence)
Want to show: kGb has only one simple module
Suppose S_1, S_2 are kG-simple. They are not projective because defect group is not trivial
$\Rightarrow \quad kGb$ not semisimple
Note that, an indecomposable algebra A has a injective projective simple module $\Leftrightarrow$ A is semisimple (see Remark for proof). In particular if A is injective, any projective simple is also injective.

Let V_1, V_2 be the $kN_G(P_1)b_1$-module which are the Green correspondence of S_1, S_2 respectively
Recall Lemma 2.10.6: if every projective indecomposable and injective indecomposable is uniserial, then every indecomposable is uniserial

In particular, they have simple top $\cong S$
$\Rightarrow$ they are quotients of P_S, which is uniserial with all composition factors $\cong S$

$$\begin{array}{ccc} P_S & \dashrightarrow & U \\ \downarrow & & \downarrow \\ S & \xrightarrow{\sim} & U/\operatorname{Rad}(U) \end{array}$$

Thus $V_1 \twoheadrightarrow V_2$

By Lemma 2.10.7: $0 \neq \underline{\operatorname{Hom}}_{kN_G(P_1)}(V_1, V_2) \cong \underline{\operatorname{Hom}}_{kG}(S_1, S_2)$
$\Rightarrow$ $\operatorname{Hom}_{kG}(S_1, S_2) \neq 0$
$\Rightarrow$ $S_1 \cong S_2$

Then P_S is uniserial because kGb has finite representation type
$\Rightarrow$ length of P_S = number of kGb-indecomposable
= 1 + number of non-projective kGb-indecomposables
= 1 + number of non-projective $kN_G(P_1)b_1$-indecomposable
= length of P_T = p^n □

Remark. To prove: A indecomposable algebra, A is semisimple $\Leftrightarrow$ there is a projecitve injective simple
Proof: $\Rightarrow$: clear
$\Leftarrow$: $A \cong nS \oplus P$ where P direct sum of projective indecomposables
$A^{op} \cong \operatorname{End}_A(A) \cong \operatorname{End}_A(S) \oplus \operatorname{Hom}_A(S, P) \oplus \operatorname{End}_A(P)$
But $\operatorname{Hom}_A(S, P) = 0$ because otherwise, for any projective indecomposable $P'|P$, $0 \neq f : S \hookrightarrow P'$, by injectiveness of S, f splits so $S|P'$, contradicting indecomposability of P'. Hence $A \cong \operatorname{End}_A(S)^{op} \oplus \operatorname{End}_A(P)^{op}$ contradicting indecomposability of A

Proposition 2.10.8
b block with cyclic defect group P of order p^n $(n \geq 1)$ with inertial index ϵ. Suppose $P_1 \lhd G$
Then kGb is the Brauer tree algebra of the star-shaped Brauer tree with exceptional vertex in the centre with ϵ edges and exceptional multiplicity $(p^n - 1)/\epsilon$

Remark. This applies to b_1. Rephrase this proposition: kGb has ϵ simple modules $S_1, \ldots, S_\epsilon$, with uniserial projective cover P_j of S_j, all of length p^n and $\operatorname{Rad}^i(P_j)/\operatorname{Rad}^{i+1}(P_j) \cong S_{i+j}$
i.e. kGb is Morita equivalent to $k(P \rtimes E)$ where $E \cong N_G(P, e)/C_G(P)$

Proof
Since $P_1 \lhd G$, $b = b_1$ covers e_1 and kGb is Morita equivalent to $kN_G(P_1, e_1)e_1$
By induction, may assume: e_1 is G-stable

Known: $kC_G(P_1)e_1$ ("nilpotent block") has a unique simple module S and the projective cover P_S of S is uniserial of length p^n

To show: S "extends" to a kG-module in exactly ϵ distinct ways and they are the simple kGb-modules.
Want to introduce G-action on S which is compatible with $C_G(P_1)$-action.
Show $G/C_G(P_1) \leq \operatorname{Aut}(P_1) \cong C_{p-1}$ and $|G/C_G(P_1)| = \epsilon$
$\Rightarrow$ $G/C_G(P_1) \cong C_\epsilon$
Write $G/C_G(P_1) = \langle gC_G(P_1) \rangle, g \in G, g^\epsilon \in C_G(P_1)$
Consider ${}^gS$ (i.e. $(kC_G(P_1) =)k^gC_G(P_1)$-module s.t. $gcg^{-1}.s = cs$ for $c \in C_G(P_1)$)
S simple $kC_G(P_1)e_1$-module
$\Rightarrow$ ${}^gS$ simple $kC_G(P_1)e_1$-module (as e is G-invariant)
$\Rightarrow$ $S \cong {}^gS$
i.e. $\exists \theta : S \to S$ a k-linear isomorphism s.t. $\theta(xs) = gxg^{-1}\theta(s)$ $\forall x \in C_G(P_1), \forall s \in S$
$\Rightarrow$ $\theta^2(xs) = \theta((gxg^{-1}\theta(s))) = g^2xg^{-2}\theta^2(s)$

etc.
$\Rightarrow \quad \theta^\epsilon(xs) = g^\epsilon x g^{-\epsilon}\theta^\epsilon(s)$
$\Rightarrow \quad g^{-\epsilon}\theta^\epsilon \in \mathrm{End}_{C_G(P_1)}(S) \cong k$ by simplicity (and assume k algebraically closed)
$\Rightarrow \quad \exists \mu \in k^\times$ s.t. $g^{-\epsilon}\theta^\epsilon(s) = \mu s \;\; \forall s \in S$
$\Rightarrow \quad g^\epsilon s = \mu^{-1}\theta^\epsilon(s)$
Take $\lambda \in k^\times$ s.t. $\lambda^\epsilon = \mu^{-1}$, then $g^\epsilon s = (\lambda\theta)^\epsilon(s)$
$\Rightarrow \quad g = \lambda\theta$ gives a compatible G-action on S
(defines $G = \coprod g^i C_G(P_1) \to GL_k(S)$)
There are ϵ choices of λ because k is algebraically closed
$\exists \epsilon$ extensions of S to G, write $S_1, \ldots, S_\epsilon$

Now $\mathrm{Hom}_{kG}(S_i, \mathrm{Ind}^G_{C_G(P_1)}(S)) \cong \mathrm{Hom}_{kC_G(P_1)}(\underbrace{\mathrm{Res}^G_{C_G(P_1)}(S_i)}_{S}, S) \cong k \;\; \forall i$

$\Rightarrow \quad \mathrm{Ind}^G_{C_G(P_1)}(S) \cong S_1 \oplus \cdots \oplus S_\epsilon$

We need to show these are the only simples.
Now let T be a simple kGb-module $\mathrm{Hom}_{kG}(T, \mathrm{Ind}^G_{C_G(P_1)}(S)) \cong \mathrm{Hom}_{kG}(\mathrm{Res}^G_{C_G(P_1)}(T), S)$
Clifford Theorem says $\mathrm{Res}^G_{C_G(P_1)}(T)$ is semisimple, hence is isomorphic to S^n for some n
$\Rightarrow \quad T \cong S_i$ some i

Now need to show the Lowey structure. It is enough to show:
$\mathrm{Rad}\, P_i / \mathrm{Rad}^2\, P_i \cong S_{\pi(i)}$ for some (cyclic) permutation π of $\{1, \ldots, \epsilon\}$
Compute $\mathrm{Ext}^1_{kG}(S_j, \underbrace{S_1 \oplus \cdots \oplus S_\epsilon}_{\mathrm{Ind}^G_{C_G(P_1)}(S)}) \cong \mathrm{Ext}^1_{kC_G(P_1)}(\underbrace{\mathrm{Res}^G_{C_G(P_1)}(S_j)}_{S}, S) \cong k$
$\Rightarrow \quad \exists i$ s.t. $\mathrm{Ext}^1_{kG}(S_j, S_i) \cong k$ and $\mathrm{Ext}^1_{kG}(S_j, S_l) = 0 \; \forall l \neq i$
And since kGb is a block, the assignment $j \mapsto i$ is a transitive cyclic permutation (think carefully, or see Landrock/Alperin)

Now remain to show Lowey length of P_i = Lowey length of $P_S = p^n$ (P_S=projective cover of S)
$\mathrm{Ind}^G_{C_G(P_1)}(P_S)$ projective, surjects onto $\mathrm{Ind}^G_{C_G(P_1)}(S) \cong S_1 \oplus \cdots \oplus S_\epsilon$
Let $P_1 \oplus \cdots \oplus P_\epsilon$ be projective cover of $S_1 \oplus \cdots \oplus S_\epsilon$ (P_i projective cover of S_i)
$\Rightarrow \quad P_1 \oplus \cdots \oplus P_\epsilon | \mathrm{Ind}^G_{C_G(P_1)}(P_S)$
$\mathrm{Res}^G_{C_G(P_1)}(P_i)$ projective, surjects onto $\mathrm{Res}^G_{C_G(P_1)}(S_i) \cong S$
$\Rightarrow \quad \mathrm{Res}^G_{C_G(P_1)}(P_i) \cong a_i P_S$ some $a_i \geq 1$
By compare dimension: $\Rightarrow \quad \mathrm{Res}^G_{C_G(P_1)}(P_i) \cong P_S \;\; \forall i$ and $\mathrm{Res}^G_{C_G(S_1)}(S_i) \cong P_S \;\; \forall i$
$\Rightarrow \quad$ Lowey length of P_i = Lowey length of P_S □

Theorem 2.10.9 (Theorem 2.9.2, Dade et al)
Cyclic block kGb is a Brauer tree algebra with $\epsilon = |N_G(P, e) : C_G(P)| = |N_G(P_1, e_1) : C_G(P_1)|$ and exceptional multiplicity $(p^n - 1)/\epsilon$

2.11 Derived equivalence of group algebra

Theorem 2.11.1 (Rickard)
Brauer tree algebras are derived equivalent if and only if they have the same number of edges and same exceptional multiplicity.

Proof
$\Rightarrow$:
Recall the corollaries of Rickard's Theorem 1.11.4: If two algebras A and B are derived equivalent, then $K_0(A) \cong K_0(B)$ (in particular, they have the same number of simple modules) and $\underline{\mathbf{mod}}(A) \cong \underline{\mathbf{mod}}(B)$

as triangulated categories.

Let $K_0^{pr}(A) = \langle [P] \in K_0(A) | P$ projective indecomposable A-modules$\rangle$
$\Rightarrow \quad K_0(A)/K_0^{pr}(A) \cong K_0(B)/K_0^{pr}(B)$
Let $S_1, \dots, S_r$ be the simple A-modules and $P_1, \dots P_r$ be the corresponding projective indecomposables.
If $c_{ij} = \dim_k \operatorname{Hom}_A(P_i, P_j) =$ number of composition factor of P_j isomorphic to S_i (i.e. the Cartan matrix), then

$$[P_i] = \sum_j c_{ji}[S_j]$$

$\Rightarrow \quad \det(c_{ij}) = |K_0(A)/K_0^{pr}(A)|$ is an invariant of derived equivalence.

If B is ϵ edges, multiplicity m, star-shaped Brauer tree algebra with exceptional vertex in the centre then $\operatorname{Rad}^j(S_i)/\operatorname{Rad}^{j+1}(S_i) \cong S_{i+j}$
so the Cartan matrix:

$$(c_{ij}) = \begin{pmatrix} m+1 & m & \cdots & m \\ m & m+1 & & \vdots \\ m & & \ddots & \vdots \\ \vdots & & & \vdots \\ m & \cdots & m & m+1 \end{pmatrix}$$

and $\det(c_{ij}) = \epsilon m + 1$
This result holds in general (see last part of Alperin/Benson), i.e. any Brauer tree algebra derived equivalent to B has ϵ edges and exceptional multiplicity m

$\Leftarrow$:
Let Γ be a Brauer tree with ϵ edges and exceptional multiplicity m and A the Brauer tree algebra of Γ
$S_1, \dots, S_\epsilon$ the simple A-modules
$P_1, \dots, P_\epsilon$ the projective indecomposable A-modules
Recall Rickard's Theorem 1.11.4: A and B are derived equivalent, if and only if, $\exists T$ bounded complex of f.g. projective A-modules s.t.

- $\operatorname{End}_A(T)^{op} \cong B$
- $\operatorname{Hom}_A(T, T[n]) = 0 \ \forall n \neq 0$
- $\operatorname{Thick}_{\mathbf{D}(A)}(T) = \operatorname{Perf}(A) \ni A$

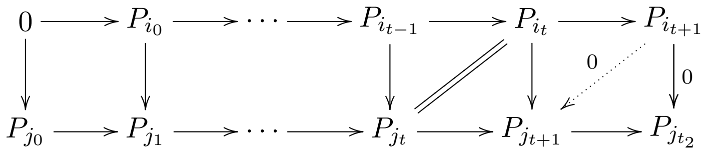
We will construct T ((one-sided) tilting complex of A) with endomorphism isomorphic to B where B is a Brauer star.

For each edge i of Γ, there is a unique path from the exceptional vertex to i, label edge of this path as $i_1, i_2, \dots, i_r = i$
Consider the Lowey structure of P_{i_j} and $P_{i_{j+1}}$ ($j = 1, \dots, r-1$), Brauer tree algebra implies that $\dim_k(P_{i_j}, P_{i_{j+1}}) = 1$, so there is a uniserial module with top S_{i_j} and socle $S_{i_{j+1}}$ which is isomorphic to a quotient of P_{i_j} and to a submodule of $P_{i_{j+1}}$, i.e. there is a non-zero map between P_{i_j} and $P_{i_{j+1}}$

Let $Q_i = (0 \to P_{i_0} \to P_{i_1} \to \cdots \to P_{i_r} \to 0)$
where the maps between P_{i_j}'s are non-zero (by previous paragraph) and P_{i_0} is in degree 0.
Let $Q = \oplus_i Q_i$

Claim: Q is a tilting complex with $B = \operatorname{End}_{\mathbf{D}(A)}(Q)^{op}$ is a Brauer tree algebra of star

Proof of Claim:
Check that $A \in \text{Thick}(Q)$ and $\text{Hom}_{\mathbf{D}(A)}(Q, Q[n]) = 0 \ \ \forall n \neq 0$
Easy when $|n| \geq 2$, not so difficult when $n = \pm 1$
For example, $\underline{n = 1}$: Suppose $\text{Hom}(Q, Q[1]) \neq 0$
Suppose $P_{i_t} \to P_{j_{t+1}}$ is non-zero, considering how this is possible on the Brauer tree, we get $i_t = j_t$, and so



To show $B = \text{End}(Q_i)^{op}$
Let $\pi_i \in B$ the projection onto Q_i
π_i is a projective indecomposable of B (as $\pi_i B \pi_i \cong \text{End}_{\mathbf{D}(A)}(Q_i) \cong \text{End}_A(P_{i_0})$ which is local, and $\pi_1 + \cdots + \pi_\epsilon = 1_B$)

$\square$

<u>**Broué's Abelian Defect Group Conjecture**</u>:
G finite group
k algebraically closed field of characteristic $p \big| |G|$
b block of kG with defect group P
$c = \text{Br}_P(b)$
If P is abelian, then $\mathbf{D}(kGb) \cong \mathbf{D}(kN_G(P)c)$ as triangulated categories (derived equivalence)

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