Definition and equivalent condition for cellularly stratified

A finite dimensional k-algebra with anti-automorphism involution i
with inflation data $\quad (B_1, V_1, \phi_1; \ldots; B_n, V_n, \phi_n)$

Original definition

(Cellularity) $\quad A$ is iterated inflation of cellular algebras B_l along V_l

(Existence of idem) $\quad \forall l, \exists 0 \neq u_l, v_l \in V_l \quad$ s.t. $\quad e_l := 1_{B_l} \otimes u_l \otimes v_l \quad$ is an idempotent

(Irreversible layer) $\quad l > m \quad \Rightarrow \quad e_l e_m = e_m = e_m e_l$

Remarks

- $e_n = 1; \qquad V_n = k; \qquad \phi_l(u_l, v_l) = 1_{B_l} = \phi_l(v_l, u_l)$
- (ideal within cell chain) $J_l := \bigoplus_{i=1}^{l} B_i \otimes V_i \otimes V_i = A e_l A$

$$\begin{array}{rcl} B_l & \cong & \dfrac{e_l A e_l}{e_l J_{l-1} e_l} \\ 1_{B_l} & \mapsto & e_l \end{array}$$

- B_l is corner split quotient of A/J_{l-1} $\quad$ wrt e_l $\quad$ realized by $S_l := Ae_l \otimes_{e_l A e_l} B_l$

Equivalent condition

(E+I) $\exists$ idempotents $\{e_l : l = 1, \ldots, n\}$ s.t.

- $e_n = 1$
- $l > m \quad \Rightarrow \quad e_l e_m = e_m = e_m e_l$

(J) $\qquad J_l := A e_l A \qquad B_l := \frac{e_l A e_l}{e_l J_{l-1} e_l} \quad$, we have

- $\forall e_l, \quad \exists p_l, q_l \in A$ s.t. $\left\{ \begin{array}{l} e_l = p_l q_l \\ i(e_l) = q_l p_l \text{ mod } J_{l_1} \\ i(p_l) = p_l \end{array} \right.$
- B_l is cellular wrt $\phi_l \circ (i|_{B_l})$
- $J_l = J_{l-1} \oplus X_l \quad$ as vector space with $i(X_l) = X_l$ for $l = 2, \ldots, n$

(Free) $\quad \forall l, \quad S_l := \frac{Ae_l}{J_{l-1} e_l} \quad \in A$ - B_l-bimod, we have

- S_l free of finite rank $m (= \dim_k V_l)$ over B_l
- As right B_l-module, $S_l \cong \left\{ \begin{array}{ll} e_l B_l^{\oplus m} & \text{if } q_l e_l = c e_l \text{ for some } c \in k^\times \\ e_l B_l^{\oplus a} \oplus q_l e_l B_l^{\oplus b} & (a + b = m) \end{array} \right\} = V_l \otimes_k B_l$
- Multiplication induces isomorphism of A-A-bimodule:

$$\frac{Ae_l}{J_{l-1} e_l} \otimes_{B_l} \frac{e_l A}{e_l J_{l-1}} \quad \cong \quad \frac{A e_l A}{A e_{l-1} A} \quad \left(= \frac{J_l}{J_{l-1}} \right)$$