Algebraic Topology

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1 Homotopy Equivalence

X, Y topological spaces
$\text{Map}(X, Y) = \{f : X \to Y \mid f \text{ continuous}\}$
Convention: all spaces are topological, all maps are continuous

Definition 1.1
$f_0, f_1 : X \to Y$ are homotopic ($f_0 \sim f_1$) if there is a continuous map $F : X \times [0,1] \to Y$ s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$
i.e. $f_t(x) = F(x,t)$ is a path from f_0 to f_1 in $\text{Map}(X,Y)$

Examples See notes

Lemma 1.2
Homotopy is a equivalence relation

Definition 1.3
X, Y spaces $[X,Y] = \text{Map}(X,Y)/\sim = \{$ homotopy classes $X \to Y\} = \{$ path component of $\text{Map}(X,Y)\}$

Lemma 1.4
If $f_0, f_1 : X \to Y$, $g_0, g_1 : Y \to Z$ and $f_0 \sim f_1, g_0 \sim g_1$
then $g_0 f_0 \sim g_1 f_1$

Corollary 1.5
For any space X, $[X, \mathbb{R}^n]$ has a unique element

Proof
Let $0_X : X \to \mathbb{R}^n \qquad 0_X(x) = 0$
If $f : X \to \mathbb{R}^n$ then $f = 1_{\mathbb{R}^n} f \sim 0_{\mathbb{R}^n} f = 0_X$ □

Definition 1.6
X is contractible if $1_X \sim c$ where $c : X \to X$ is a constant map

Corollary 1.7
X is contractible $\Leftrightarrow$ $[Y, X]$ has a unique element

Proof
$\Rightarrow$: as before

$\Leftarrow$: $[X, X]$ has 1 element $1_X \sim c$ □

Question: If X is contractible, what is $[X,Y]$?

Definition 1.8
Spaces X, Y are homotopy equivalence $(X \sim Y)$ if there are maps
$f : X \to Y, g : Y \to X$ s.t. $fg \sim 1_Y, gf \sim 1_X$

Example X contractible $\Leftrightarrow X \sim P = \{p\}$
X is contractible $1_X \sim c, c(x) \equiv c$
Take $f : X \to P \qquad f(x) \equiv p$
$g : P \to X \qquad g(p) = c$
then $fg = 1_P, gf = c \sim 1_X$

Fundamental Question of Algebraic Topology:
Given spaces X and Y,
(i) Can I tell if $X \sim Y$ (ii) What is $[X,Y]$?

2 Homotopy Groups

Definition 2.1 (Map of Pairs)
$f : (X,A) \to (Y,B)$ means
(1) $A \subseteq X, B \subseteq Y$
(2) $f : X \to Y$
(3) $f(A) \subseteq B$ $f_0, f_1 : (X,A) \to (Y,B)$
$f_0 \sim f_1$ means $\exists F : (X \times [0,1], A \times [0,1]) \to (Y,B)$ with
$F(x,0) = f_0, F(x,1) = f_1$, i.e. $f_t = F(x,t), f_t : (X,A) \to (Y,B)$

Definition 2.2
X is a space, $p \in X$. $\pi_n(X,p) = [(D^n, S^{n-1}), (X,p)] = [(S^n, N), (X,p)]$

Facts about π_n

1. $\pi_0(X,p) = \{\text{path components of } X\}$ (Exercise)
   $\pi_1(X,p)$ is a group
   $\pi_n(X,p) \quad (n > 1)$ is an abelian group

2. π_n is a functor from

$$\begin{aligned} \{\text{pointed space, pointed maps}\} &\longrightarrow \{\text{groups and homomorphisms}\} \\ \text{spaces } (X,p) &\longrightarrow \text{group } \pi_n(X,p) \\ f : (X,p) \to (Y,q) &\longrightarrow \text{hom. } f_* : \pi_n(X,p) \to \pi_n(Y,q) \end{aligned}$$

   $\gamma : (S^n, N) \to (X,p)$
   $f\gamma = f_*(\gamma) : (S^n, N) \to (Y,q)$ satisfies
   (a) $(1_{(X,p)})_* = 1_{\pi_n(X,p)}$
   (b) $(fg)_* = f_* g_*$

3. $f, g : (X,p) \to (Y,q)$
   If $f \sim g$ then $f_* = g_*$ Proof: $f_*(\gamma) = f\gamma \sim g\gamma = g_*(\gamma)$ □

   Corollary 2.3
   Suppose $(X,p) \sim (Y,q)$, then $\pi_n(X,p) \cong \pi_n(Y,q)$

   Proof
   $f : (X,p) \to (Y,q) \qquad g : (Y,q) \to (X,p)$

$fg \sim 1_(Y,q) \qquad gf \sim 1_(X,p)$
$\Rightarrow (fg)_* = f_* g_* = 1_{\pi_n(Y,q)}$
$(gf)_* = g_* f_* = 1_{\pi_n(X,p)}$ □

4. For nice spaces X, X is contractible $\Leftrightarrow \pi_n(X,p) = 0 \forall n$

5. (columns are n, rows are m)

$\pi_n(S^m)$	1	2	3	4	$\leftarrow n$
1	$\mathbb{Z}$	0	0	0	$\dots$
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/12\,\mathbb{Z}/2, \dots$
3	0	0	$\mathbb{Z}$	$\dots$	

Facts:
$\pi_n(S^m)$ is a finitely generated (f.g.) abelian group

$$\text{rk } \pi_n(S^m) = \begin{cases} 1 & m = n \\ 1 & m = 2k, n = 2m-1 \\ 0 & \text{otherwise} \end{cases}$$

3 Category and Functors

Category is composed of objects and morphisms
object = "set" with some structure
morphism = function from one object to another that respect this structure
Example: {Vector space, linear maps}, {Groups, homomorphism}, {Topological space, continuous map}

Functor = morphism from one category to another

$$\begin{array}{rcl} F : \mathscr{C}_1 & \to & \mathscr{C}_2 \\ A \text{ object of } \mathscr{C}_1 & \mapsto & F(A) \text{ object of } \mathscr{C}_2 \\ (f : A_1 \to A_2) \text{ morphism of } \mathscr{C}_1 & \mapsto & (f_* : F(A_1) \to F(A_2))\text{morphism of } \mathscr{C}_2 \\ & \text{s.t.} & \begin{cases} (1_A)_* = 1_{F(A)} \\ (fg)_* = f_* g_* \end{cases} \end{array}$$

4 Ordinary Homology

We will construct a functor

$$\begin{array}{rcl} H_* : \{\text{space,maps}\} & \to & \{\mathbb{Z}-\text{modules}, \mathbb{Z}-\text{linear map}\} \\ \text{Sapce } X & \mapsto & \text{abelian group } H_*(X) = \bigoplus_{i \geq 0} H_i(X) \\ (f : X \to Y) & \mapsto & \text{homomorphism } f_* : H_*(X) \to H_*(Y) \end{array}$$

4.1 Important properties of H_*

(1) Homotopy Invariance:

$$f, g : X \to Y \qquad f \sim g \Rightarrow f_* = g_*$$

(2) Dimension Axiom:

$$H_*(X) = 0 \quad \forall * > \dim X$$

Corollary 4.1
$X \sim Y \quad \Rightarrow \quad H_*(X) \cong H_*(Y)$

Proof
$f : X \to Y, g : Y \to X$ s.t. $fg \sim 1_Y, gf \sim 1_X$
$1_{H_*(X)} = (1_X)_* = (gf)_* = g_* f_*$
Similarly, $1_{H_*(Y)} = f_* g_*$
$\Rightarrow f_*, g_*$ are inverse homomorphism □

5 Chain Complexes

R is a commutative ring with 1 (think $R = \mathbb{Z}$ or R=field)

Definition 5.1
A chain complexes (C_*, d) over R is

1. An R-module $C_* = \bigoplus_{n \in \mathbb{Z}} C_n$
2. An R-linear map $d : C_* \to C_*$
   $d = \bigoplus d_n \qquad d_n : C_n \to C_{n-1}$ s.t. $d^2 = 0$

i.e.

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots, \qquad d_{n-1} d_n = 0 \; \forall n$$

5.1 Terminology

n is the grading, often $C_n \equiv 0 \quad \forall n < 0$ and often d is the differential as boundry map
$x \in \ker d \;\Rightarrow\; x$ is closed or a cycle
$x \in \operatorname{Im} d \;\Rightarrow\; x$ is a boundry

Lemma 5.2
$d^2 = 0 \Leftrightarrow \operatorname{Im} d_{n+1} \subseteq \ker d_n \quad \forall n$

Definition 5.3
The homology of C is the module $\frac{\ker d_n}{\operatorname{Im} d_{n+1}}$, denote $H_n(C)$
$x \in \ker d_n, [x]$ is its image in $H_n(C)$

Examples: see notes

6 Chian Complex of a Simplex

Definition 6.1
The <u>n-dimensional simplex</u> $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i \geq 0} x_i = 1\}$

<u>Examples</u>: see notes

Δ^n has $n+1$ <u>vertices</u> $v_0, \ldots, v_n$= intersections with coordinate axes

k-dimensional faces of $\Delta^n \longleftrightarrow$ collections of $k+1$ vertices

Definition 6.2
$S_*(\Delta^n)$=<u>simplicial complex of</u> Δ^n
$S_k(\Delta^n)$ =free $\mathbb{Z}$-module generated by the k-dimensional faces
$=< e_I \mid I$ is a $(k+1)$-element subset of $\{0, \ldots, n\} >$ $\quad (I = \{i_0, \ldots, i_k \mid i_0 < i_1 < \cdots < i_k\})$

$$d(e_I) = \sum_{j=0}^{k} (-1)^j e_{I - i_j}$$

<u>$n = 1$</u>: $d(e_{01}) = e_1 - e_0$
<u>$n = 2$</u>: $d(e_{012}) = e_{12} - e_{02} + e_{01}$
$d^2(e_{012}) = (e_2 - e_1) - (e_2 - e_0) + (e_1 - e_0) = 0$

Lemma 6.3
$d^2 = 0$

Proof
STP $d^2(e_I) = 0$

$$\begin{aligned} d^2(e_I) &= d\left(\sum_{j=0}^{k} (-1)^j e_{I-i_j}\right) \\ &= \sum_{j=0}^{k} (-1)^j \left(\sum_{l<j} (-1)^l e_{I-i_j-i_l} + \sum_{l>j} (-1)^{l-1} e_{I-i_j-i_l}\right) \\ &= \cdots = 0 \end{aligned}$$

□

<u>What is $H_*(X)$?</u>

Definition 6.4
$C_*(X)$ = singular chain complex of X
$C_k(X) =< e_\sigma \mid \sigma : \Delta^k \to X$ is any continuous map$>$
$d(e_\sigma) = \sum (-1)^j e_{\sigma \circ F_j}$
where $F_j : \Delta^{k-1} \to \Delta^k$
$(x_0, \ldots, x_{k-1}) \mapsto (x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{k-1})$

7 Singular Chain

Face map:

$$\begin{array}{rcl} F_j^n : \Delta^{n-1} & \to & \Delta^n \\ (x_0, \ldots, x_{n-1}) & \mapsto & (x_0, \ldots, x_{j-1}, 0, x_j, \ldots, x_{n-1}) \end{array}$$

Example:
$n = 1 \quad \Delta^0 \to \Delta^1$ (see notes for picture)

$n = 2 \quad \Delta^1 \to \Delta^2$ (see notes for picture)

What is d?

$$d(e_\sigma) = \sum_{j=0}^{n} (-1)^j e_{\sigma \circ F_j^n} \qquad \sigma \circ F_j : \Delta^{n-1} \xrightarrow{F_j} \Delta^n \xrightarrow{\sigma} X$$

Proposition 7.1
$d^2 = 0$

Proof
There is a homomorphism $\alpha_\sigma : S_*(\Delta^n) \to C_*(X)$ with face f gives $e_f \mapsto e_{\sigma F_f}$

$$\begin{array}{rcl} d_{C_*} \circ \alpha_\sigma & = & \alpha_\sigma \circ d_{S_*} \\ e_\sigma & = & \alpha_\sigma(e_{\Delta^n}) \\ \Rightarrow d^2_{C_*}(e_\sigma) & = & d^2_{\alpha_\sigma}(e_{\Delta^n}) \\ & = & \alpha_\sigma(d^2_{S_*}(e_{\Delta^n})) = \alpha_\sigma(0) = 0 \end{array}$$

□

Example:

$$\begin{array}{rcl} C_0(X) & = & < e_\sigma | \sigma : \Delta^0 \to X > \\ & = & < e_p | p \in X > \ \ p \text{ a point in } X \text{ as } \Delta^0 \text{ a point} \\ C_1(X) & = & < e_\sigma | \sigma : \Delta^1 \to X > \\ & = & < e_\gamma | \gamma : [0,1] \to X \text{ is a path} > \\ d(e_\gamma) & = & e_{\gamma(1)} - e_{\gamma(0)} \end{array}$$

(see notes for pictures of Singular chain in $\mathbb{R}^2$/cycle in $C_1(\mathbb{R}^2)$)

2 cycles in $C_1(S^1)$: (picture)

They represent the same element of $H_1(S^1)$, namely $[e_1 - e_2] = [f_1]$. $f_1 - e_1 + e_2 = d(e_\sigma)$ $\quad \sigma : \Delta^2 \to S^1$

Lemma 7.2
If X path connected, $H_0(X) \cong \mathbb{Z}$

Proof

$$\begin{aligned} H_0(X) &= \frac{\ker d_0}{\operatorname{Im} d_1} \\ &= \frac{< e_p | p \in X >}{\operatorname{span}\{e_{\gamma(1)} - e_{\gamma(0)} | \gamma : [0,1] \to X\}} \\ &= \frac{< e_p | p \in X >}{\operatorname{span}\{e_p - e_q | p, q \in X\}} = \mathbb{Z} \end{aligned}$$

(via $\sum a_i e_{p_i} = \sum a_i \in \mathbb{Z}$) □

Lemma 7.3
$H_*(X) \cong \bigoplus_\alpha H_*(X_\alpha)$, X_α are path compoenents of X

Proof
$\sigma : \Delta^n \to X$, Δ^n is path connected
$\Rightarrow \operatorname{Im} \sigma \subseteq X_\alpha$ some α
$\Rightarrow C_*(X) = \oplus_\alpha C_*(X_\alpha)$ as a group

$\operatorname{Im} \sigma \subseteq X_\alpha \Rightarrow$ all faces of e_σ are $\subseteq X_\alpha$
$\Rightarrow d(e_\sigma) \subseteq C_*(X_\alpha)$
$(C_*(X), d) \cong \bigoplus (C_*(X_\alpha), d)$
$\Rightarrow H_*(X) \cong \bigoplus H_*(X_\alpha)$ □

Corollary 7.4
$H_0(X) = \mathbb{Z}^{\#}$ of path component of X

Lemma 7.5
$P = \{p\} \Rightarrow H_*(P) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$

Proof
Thre is a unique map $\sigma_n : \Delta^n \to P$
$d(e_{\sigma_n}) = \sum_{j=0}^n (-1)^j e_{\sigma_n F_j^n} = \sum_{j=0}^n (-1)^j e_{\sigma_{n-1}} = \begin{cases} e_{\sigma_{n-1}} & 0 < n \text{ is even} \\ 0 & n \text{ odd} \end{cases}$
$C_*(P)$:
$\ldots \to C_2 \to C_1 \to C_0 \to 0$

etc....
$\Rightarrow H_*(P)$ is generated by e_{σ_0} $\square$

8 Induced Maps

(R is any ring, but usually think as $\mathbb{Z}$)

Definition 8.1
Suppose (C_*, d_C) and (D_*, d_D) are chain complexes
A chain map $C_* \to D_*$ is a R-lenear map $f : C_* \to D_*$ $\quad f = \oplus f_i \quad f_i : C_i \to D_i$ s.t. $fd_C = d_D f$
(see notes for commutative diagram)

Lemma 8.2
1_{C_*} is a chain map. If $f : C_* \to D_*, g : D_* \to E_*$ are chian maps, then so is gf (i.e. {chain complexes, chain maps} is a category)

Definition 8.3
Suppose $f : C_* \to D_*$ is a chain map. Define

$$\begin{aligned} f_* : H_*(C) &\to H_*(D) \\ [x] &\mapsto [f(x)] \end{aligned}$$

Have to check this works:

1. $f(x)$ is closed

$$d(f(x)) = f(d(x)) = f(0) = 0 \text{ (since } x \text{ is closed)}$$

2. $[x] = [y]$ in $H_*(C) \Rightarrow [f(x)] = [f(y)]$ in $H_*(D)$

$$\begin{gathered} [x] = [y] \Rightarrow x - y = dz \text{ some } z \\ \Rightarrow f(x) - f(y) = f(x-y) = f(dz) = df(z) \\ \Rightarrow [f(x)] = [f(y)] \end{gathered}$$

Notice:
If $f : C_* \to D_*, g : D_* \to E_*$
$(gf)_*([x]) = [g(f(x))] = g_*([f(x)]) = g_*(f_*([x])) \Rightarrow (gf)_* = g_* f_*$
i.e. H_n is a functor:

$$\begin{aligned} \{\text{chian complexes, chain maps}\} &\xrightarrow{H_*} \{R\text{-modules, } R\text{-linear maps}\} \\ C_* &\mapsto H_*(C) \\ (f : C_* \to D_*) &\mapsto (f_* : H_*(C) \to H_*(D)) \end{aligned}$$

To define $f_* : H_*(X) \to H_*(Y)$, we first define chain map $f_\# : C_*(X) \to C_*(Y)$ (see picture)

$f_\#(e_\sigma) = e_{f\sigma}$ and extend linearly
Check $f_\#$ is a chain map ($df_\# = f_\# d$)

$$d(f_\#(e_\sigma)) = d(e_{f\sigma}) = \sum_{j=0}^{n}(-1)^j e_{f\sigma F_j}$$
$$f_\#(d(e_\sigma)) = f\left(\sum_{j=0}^{n}(-1)^j e_{\sigma F_j}\right) = \sum_{j=0}^{n}(-1)^j e_{f\sigma F_j}$$

Notice that $(fg)_\# = f_\# g_\#, \qquad (1_X)_\# = 1_{C_*(X)}$, so we have a functor

$$\begin{array}{rclcl} \{\text{Spaces, Maps}\} & \to & \{\text{Chain complexes, Chain maps}\} & \to & \{\mathbb{Z}-\text{modules}, \mathbb{Z}-\text{linear maps}\} \\ X & \mapsto & C_*(X) & \mapsto & H_*(X) \\ (f : X \to Y) & \mapsto & (f_\# : C_*(X) \to C_*(Y)) & \mapsto & (f_* : H_*(X) \to H_*(Y)) \end{array}$$

Composition of functors is a functor, so H_* is a functor

9 Homotopy Invariance

Definition 9.1
Suppose $f, g : C_* \to D_*$ are chain maps, we say $f \sim g$ (f is chain homotopic to g) if $\exists R$-lnear map $\begin{array}{lccc} h: & C_* & \to & D_* \\ h: & C_n & \to & D_{n+1} \end{array}$ with $d_D h + h d_C = f - g$

Lemma 9.2
Suppose $f, g : C_* \to D_*$ chain maps, $f \sim g$, then $f_* = g_*$

Proof
Suppose $[x] \in H_n(C)$ so $dx = 0$
Then

$$f_*([x]) - g_*([x]) = [f(x) - g(x)] = [d(h(x)) + h(d(x))] = [dh(x)] = [0]$$

□

Given $f, g : X \to Y$ with $f \sim g$, let's show $f_\# \sim g_\#$. We have $h : X \times [0,1] \to Y$
$h(x,0) = f(x) \quad h(x,1) = g(x)$

Claim: $\operatorname{Im}\alpha$ is the chain homotopy we want

If I work mod 2: $dh(\Delta^n) + hd(\Delta^n)$ =top+bottom=$f+g$

Formally, define affine linear maps P_i:

$$\begin{aligned} P_i : \Delta^{n+1} &\to \Delta^n \times [0,1] (\text{ with vertices } x_i\text{'s}, y_i\text{'s. see picture}) \\ v_0, \dots, v_i &\mapsto x_0, \dots, x_i \\ v_{i+1}, \dots, v_{n+1} &\mapsto y_i, \dots, y_n \end{aligned}$$

Example: $\Delta^2 \to \Delta \times [0,1]$
$P_0 : [v_0, v_1, v_2] \mapsto [x_0, y_0, y_1]$
$P_1 : [v_0, v_1, v_2] \mapsto [x_0, x_1, y_1]$

Now we define $h_\#(e_\sigma) = \sum_{i=0}^n (-1)^j e_{h\sigma P_i}$
Have to check $dh_\# + h_\# d(e_\sigma) = e_{f\sigma} - e_{g\sigma}$
Obvious when h is the identity map.

$$\begin{aligned} dh_\#(e_\sigma) &= d\left(\sum_i (-1)^i [x_0 \cdots x_i y_i \cdots y_n]\right) \\ &= \sum_i (-1)^i \left(\sum_{j \le i} (-1)^j [x_0 \cdots \hat{x_j} \cdots x_i y_i \cdots y_n] + \sum_{j \ge i} (-1)^{j+1} [x_0 \cdots x_i y_i \cdots \hat{y_j} \cdots y_n]\right) \\ h_\# d(e_\sigma) &= h_\# \left(\sum (-1)^j [v_0 \cdots \hat{v_j} \cdots v_{n+1}]\right) \\ &= \sum (-1)^j \left(\sum_{j > i} (-1)^i [x_0 \cdots x_i y_i \cdots \hat{y_j} \cdots y_n] + \sum (-1)^{i-1} [x_0 \cdots \hat{x_j} \cdots x_i y_i \cdots y_n]\right) \\ \Rightarrow dh_\# + h_\# d(e_\sigma) &= \sum_i (-1)^i \sum_{j=i} (-1)^j [x_0 \cdots x_{i-1} y_i \cdots y_n] + \sum (-1)^i (-1)^{j+1} [x_0 \cdots x_i y_{i+1} \cdots y_n] \\ &= [x_0 \cdots x_n] - [y_0 \cdots y_n] \end{aligned}$$

Corollary 9.3
If X is contractible, then

$$H_*(X) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases}$$

Proof
$X \sim P = \{\text{point}\}$ □

Now we move on to computing $H_*(X)$ for $X \nsim \{\text{point}\}$

10 Exact Sequence

Definition 10.1
The sequence

$$\ldots \to A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{i-1} A_{i-2} \to \ldots$$

where $A_i = R$-modules, $f_i = R$-linear map, is exact at A_i if

$$\ker f_i = \operatorname{Im} f_{i+1} \qquad \text{(i.e. } f_{i+1} f_i = 0)$$

The whole sequence is exact if it is exact $\forall A_i$ ($\Leftrightarrow (A_*, f)$ is a chain complex with zero homology)

Exercise

$$\begin{aligned} 0 \to A \xrightarrow{f} B \text{ is exact} &\quad \Leftrightarrow \quad f \text{ injective} \\ B \xrightarrow{g} C \to 0 \text{ is exact} &\quad \Leftrightarrow \quad g \text{ surjective} \\ 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \text{ is exact} &\quad \Leftrightarrow \quad f \text{ injective, } g \text{ surjective} \\ &\quad \Leftrightarrow \quad A \subseteq B,\ A = \ker g = \operatorname{Im} f \\ &\quad \Leftrightarrow \quad C \simeq B/A \end{aligned}$$

We say

$$0 \to A_* \xrightarrow{\iota} B_* \xrightarrow{\pi} C_* \to 0$$

is a short exact sequence (s.e.s) of chain complexes if

1. ι, π are chain maps
2. $0 \to A_n \xrightarrow{\iota} B_n \xrightarrow{\pi} C_n \to 0$ is exact $\forall n$

Lemma 10.2 (Snake Lemma)
Suppose $0 \to A_* \xrightarrow{\iota} B_* \xrightarrow{\pi} C_* \to 0$ is a s.e.s. of chain complexes. Then there is a long exact sequence on homology with ∂ (the boundry map) (see diagram)

Proof
Let's construct ∂: (see diagram)

Given $[x] \in H_n(C) \quad (dx = 0)$
π surjective $\Rightarrow$ pick y with $\pi(y) = x$, then $\pi(dy) = d(\pi(y)) = dx = 0$
$\Rightarrow$ I can find $z \in A_{n-1}$ with $\iota(z) = dy$, and $\iota(dz) = d(\iota z) = d(dy) = 0 \Rightarrow dz = 0$ as ι is injective

Define $\partial([x]) = [z] \in H_{n-1}(A)$
Need to check

1. ∂ does not depends on my choice of y and x (Exercise)
2. The sequence is exact at each term (Exercise)
3. Exactness at $H_n(C)$:

Suppose $[x] \in \ker \partial$ i.e. $[z] = 0 \Leftrightarrow z = dw$ some $w \in A_n$
Look at $y - \iota(w)$:

$$\begin{aligned} d(y - \iota(w)) &= dy - d(\iota(w)) = dy - \iota dw \\ &= dy - \iota(z) = dy - dy = 0 \end{aligned}$$

i.e. $y - \iota(w)$ is closed in B_n

$$\pi(y - \iota(w)) = \pi(y) - 0 = \pi(y) = x$$

i.e. $\pi_*([y - \iota(w)]) = [x] \Rightarrow x \in \operatorname{Im} \pi_*$

Conversely, if $[x] \in \operatorname{Im} \pi_*$, can choose $[y] \in H_n(B)$ s.t. $\pi_*([y]) = [x] \Rightarrow \pi(y) = x$
$dy = 0 \Rightarrow z = 0 \Rightarrow \partial([x]) = 0$ □

11 Topology

Suppose $\{U_i\}$ open cover of X (i.e. $X = \bigcup U_i$)

Definition 11.1

$$C_n^{\{U_i\}} = \langle e_\sigma \mid \sigma : \Delta^n \to X, \ \operatorname{Im} \sigma \subseteq U_i \text{ for some } i \rangle$$

$\operatorname{Im} \sigma \subseteq U_i \Rightarrow \operatorname{Im}(\sigma F_j) \subseteq U_i$
$e_\sigma \in C_n^{\{U_i\}} \Rightarrow de_\sigma \in C_{n-1}^{\{U_i\}}$
i.e. $\iota : C_*^{\{U_i\}}(X) \hookrightarrow C_*(X)$ is a subcomplex (ι a chain map)

Lemma 11.2 (Key Lemma on Subdivision)
$\iota : C_*^{\{U_i\}}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence, i.e.

$$\begin{aligned} \pi : C_*(X) &\to C_*^{\{U_i\}}(X) \\ \iota \circ \pi &\sim 1_{C_*(X)} \\ \pi \circ \iota &\sim 1_{C_*^{\{U_i\}}} \end{aligned}$$

(To be proved later)

12 Mayer-Vietoris sequence

: Suppose $\{A, B\}$ is an open cover of X. (See notes for diagram of inclusion maps).

Then we have a s.e.s

$$0 \longrightarrow C_*(A\cap B) \xrightarrow{f_{A\#}\oplus f_{B\#}} C_*(A)\oplus C_*(B) \xrightarrow{g_{A\#}-g_{B\#}} C_*^{\{A,B\}}(X) \longrightarrow 0$$

(Note, $C_*^{\{A,B\}}(X)$ has elements e_σ with $\operatorname{Im}\sigma \subseteq A$ or $\operatorname{Im}\sigma \subseteq B$)

Corollary 12.1
There is a long exact sequence (see notes):

Example: $X = S^1$ (see notes for pictures)
$A \sim$point, $B \sim$point, $A\cap B \sim 2$ points

Firstly, know that $H_n(A\cap B), H_n(A)\oplus H_n(B) = 0,\ n = 1,2 \Rightarrow H_2(S^1) = 0$ (by exactness)
Then, know that $H_0(A\cap B) = \mathbb{Z}\oplus\mathbb{Z}, H_0(A)\oplus H_0(B) = \mathbb{Z}\oplus\mathbb{Z}, H_0(S^1) = \mathbb{Z}$ i.e.

$$\begin{array}{cccc} H_0(A\cap B) & \xrightarrow{f_{A_*}\oplus f_{B_*}} & H_0(A)\oplus H_0(B) & \rightarrow & H_0(S^1) \\ \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z} \\ e_x \quad e_y & & 1_A \quad 1_B & & \mathbb{Z} \end{array}$$

$$\begin{array}{c} f_{A_*}(e_x) = 1_A \quad , \quad f_{A_*}(e_y) = 1_A \\ f_{B_*}(e_x) = 1_B \quad , \quad f_{B_*}(e_y) = 1_B \\ \ker f_{A_*}\oplus f_{B_*} = \langle e_x - e_y\rangle \simeq \mathbb{Z} \\ \Rightarrow H_1(S^1) \simeq \mathbb{Z} \end{array}$$

13 Examples

13.1 $H_*(S^1)$

Know that $H_*(S^1) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

Exercise: Check $\partial(\alpha)$ generates $\ker(f_A \oplus f_B)$
(Picture of cycle in $C_1(S^1)$)

13.2 $H_*(S^n)$

Theorem 13.1
$H_*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{otherwise} \end{cases}$

Proof
Induction on n

Mayer-Vietoris sequence with

$$\begin{aligned} A &= \{\overrightarrow{x} \in S^n \mid x_{n+1} > -\epsilon\} \cong \text{Int}(D^n) \sim \text{point} \\ B &= \{\overrightarrow{x} \in S^n \mid x_{n+1} < \epsilon\} \sim \text{point} \\ A \cap B &= (-\epsilon, \epsilon) \times S^{n-1} \sim S^{n-1} \end{aligned}$$

$$\cdots \longrightarrow H_k(A \cap B) \longrightarrow H_k(A) \oplus H_k(B) \longrightarrow H_k(S^n) \longrightarrow \cdots$$

$k > 1$:
$H_k(A \cap B) = H_k(S^{n-1}) \to 0 \to H_k(S^n) \to H_{k-1}(S^{n-1}) \to 0$
$\Rightarrow$ $H_k(S^n) \cong H_{k-1}(S^{n-1})$
$\Rightarrow$ by induction, $H_n(S^n) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ for $n \leq 2$
and $H_{k-1}(S^{n-1}) \cong H_k(S^n) = 0$ for $n > k > 1$

Bottom of the sequence:
$$0 \to H_1(S^n) \to H_0(S^{n-1}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$
Know: $H_0(S^{n-1}) = \mathbb{Z}$
Exercise: $H_1(S^n) = 0$ □

Corollary 13.2
$S^n \not\approx S^m$ for $n \neq m$

Corollary 13.3
$\nexists$ continuous $f : D^n \to S^{n-1}$ s.t $f|_{S^{n-1}} = 1_{S^{n-1}}$

Proof
Let $\iota : S^{n-1} \to D^n$ be the inclusion

$$f\iota = 1_{S^{n-1}}$$
$$1_{H_{n-1}(S^{n-1})} = (f\iota)_* = f_*\iota_*$$
$$H_{n-1}(S^{n-1}) \to H_n(D^n) \to H_{n-1}(S^{n-1})$$
$$\Rightarrow \qquad f_*\iota_* = 0$$
$$\text{But} \qquad 1_{\mathbb{Z}} \neq 0_{\mathbb{Z}} \quad \#$$

□

Corollary 13.4 (Brouwer's Fixed Point Theorem)
Any continuous map $g : D^n \to D^n$ has a fixed point (i.e. $\exists x \in D^n$ s.t. $g(x) = x$)

Proof
Supose g has no fixed point, we will construct $f : D^n \to S^{n-1}$
$$f|_{S^{n-1}} = 1_{S^{n-1}}$$
(see picture)

$f(x) =$ Intersection of ray from $g(x)$ to x with S^{n-1}
f continuous # □

13.3 Induced maps

(see notes)

13.4 Wedge products

$(X, p), (Y, q)$ are pointed space (i.e. $p \in X, q \in Y$)

$$X \vee Y = X \sqcup Y / p \sim q$$

Now try to compute $H_*(S^1 \vee S^1)$ (see notes for picture of cover A and B)
<u>Exercise</u>: What is the homotopy euqivalence

$A \sim S^1$
$B \sim S^1$
$A \cap B \cong (0,1) \sim$ point (see notes for chain)

We get $H_1(S^1 \vee S^1) = \mathbb{Z} \oplus \mathbb{Z}$
$H_k(S^1 \vee S^1) = 0 \qquad k > 1$

$$\begin{aligned} H_*(S^n \vee S^m) &= \begin{cases} \mathbb{Z} & * = 0, n, m; n \neq m \\ 0 & \text{otherwise} \end{cases} \\ \text{or} &= \begin{cases} \mathbb{Z}^2 & * = n = m \\ \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In general, for $* > 0$

$$H_* \left(\bigvee_{i=1}^{k} S^{n_i} \right) = \bigoplus_{i=1}^{k} H_* \left(S^{n_i} \right)$$

13.5 Torus

$X = T^2 = S^1 \times S^1$ (see notes)

We get

$$H_*(T^2) = \begin{cases} 0 & * > 2 \\ \mathbb{Z} & * = 2 \\ \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 0 \end{cases}$$

<u>Exercise</u>: If you know covering spaces, $H_*(T^2) \cong H_*(S^2 \vee S^1 \vee S^1)$, then is $T^2 \sim S^2 \vee S^1 \vee S^1$?

Every map $S^2 \to T^2$ is null homotopic (via $S^2 \to \mathbb{R}^2 \to T^2$) How can $H_2(T^2) \neq 0$?
Not every $x \in H_n(X)$ comes from $f : S^n \to X$ ($H^n(S^n) = \mathbb{Z}$)
(Remark: If M is an oriented manifold. $H_n(M) = \mathbb{Z}$)
There is a homomorphism

$$\begin{array}{rcl} \pi_n(X,p) & \longrightarrow & H_n(X) \\ f : S^n \to & \mapsto & f_*(1) \end{array}$$

In general, this map is neither injective or surjective.

14 Homology of a Pair

Suppose $A \subseteq X$, $\iota : A \to X$, $\iota_\# : C_*(A) \to C_*(X)$ is injective

Definition 14.1

$$C_*(X,A) = C_*(X)/C_*(A)$$

Short exact sequence:

$$0 \to C_*(A) \xrightarrow{\iota_\#} C_*(X) \to C_*(X,A) \to 0$$

Gives long exact sequence (see notes):

Remark:
$f : (X,A) \to (Y,B)$, $f(A) \subseteq B$
Then Im $f_\#(C_*(A)) \subseteq C_*(B)$
So there is a well-defined map

$$\begin{array}{rcl} f_\# : C_*(X)/C_*(A) = C_*(X,A) & \to & C_*(Y)/C_*(B) = C_*(Y,B) \\ f_* : H_*(X,A) & \to & H_*(Y,B) \end{array}$$

Theorem 14.2 (Excision)
Suppose $\bar{B} \subseteq \text{Int}A$. Then $\iota_* : H_*(X-B, A-B) \to H_*(X,A)$ is an isomorphism
(This is equivalent to Subdivision Lemma 11.2)

15 Collapsing a Subset

Let $A \subseteq X$
Define $X/A = X/\sim \qquad a \sim b$ if $a, b \in A$
$\pi : X \to X/A$ the projection
$f : X/A \to Y$ continuous $\quad \Leftrightarrow \quad f\pi$ continuous

Example: $S^{n-1} \subseteq D^n \qquad D^n/S^{n-1} \simeq S^n$
$n = 2$ (see notes for picture)

Definition 15.1
The pair (X, A) is good if there is an open set $U \supseteq \bar{A}$ s.t. A is a strong deformation retract of U i.e. $\exists \pi : (U, A) \to (A, A)$ s.t. $\pi \sim 1_{(U,A)}$ as map of pairs (note: homotopy restricts to 1_A at all times)
(In particular, π is a homotopy equivalence and $H_*(A) \cong H_*(U)$)

Examples:

1. $U = D^n \times A \to \{0\} \times A$

2. (Smooth closed manifold, Smooth closed submanifold) is an example of good pair

Theorem 15.2
Suppose (X, A) is a good pair. Then

$$\begin{aligned} \pi : (X, A) &\to (X/A, A/A = \text{point}) \\ \pi_* : H_*(X, A) &\to H_*(X/A, \text{point}) \end{aligned}$$

is an isomorphism.

Exercise:
If X path-connected

$$\text{Reduced Homology } \widetilde{H}_*(X) = H_*(X, \text{point}) = \begin{cases} H_*(X) & * > 0 \\ 0 & * = 0 \end{cases}$$

15.1 Example

(1) $\widetilde{H}_*(S^n)$

$$\widetilde{\mathrm{H}}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise} \end{cases}$$

Proof
Induction on n.
Use exact sequence of (D^n, S^{n-1})

For $* > 1$, $H_*(D^n), H_{*-1}(D^n) = 0$
$H_{*-1}(S^{n-1}) = \widetilde{\mathrm{H}}_{*-1}(S^{n-1})$. So,

$$0 \to \widetilde{\mathrm{H}}_*(S^n) \to \widetilde{\mathrm{H}}_{*-1}(S^{n-1}) \to 0$$

$\Rightarrow \quad \widetilde{\mathrm{H}}_*(S^n) \cong \widetilde{\mathrm{H}}_{*-1}(S^{n-1})(\cong \mathbb{Z}$ by induction) □

(2) M^n topological n-manifold
i.e. every $x \in m$ has $U \ni x, U \cong \mathrm{Int}(D^n)$
Then $H_*(M, M - X) \cong H_*(D^n_{1/2}, D^n_{1/2} - \mathrm{pt.})$ (by Excision)
Let $A = M - X$ and $B = M - \mathrm{Int}D^n_{1/2}$, then $H_*(D^n_{1/2}, D^n_{1/2} - \mathrm{pt.}) \cong \widetilde{\mathrm{H}}_*(S^n)$

Corollary 15.3
You can see the dimension of M near every point of M
$\Rightarrow$ $M \simeq N$, then $\dim M = \dim N$

(3) Complex Projective n-space

$$\begin{aligned} \mathbb{C}P^n &= \{(z_0 : \ldots : z_n) | z_i \in \mathbb{C}, \text{ not all zero}\}/\sim \\ & \qquad a \sim b \text{ if } a = \lambda b, \text{ some } \lambda \in \mathbb{C}^\times \\ &= \{\overrightarrow{z} \in \mathbb{C}^{n+1} \Big| \|z\| = 1\}/\sim' \\ & \qquad a \sim' b \text{ if } a = \lambda b, \text{ some } \lambda \in S^0 \\ &= S^{2n+1}/\sim \end{aligned}$$

Example: $\mathbb{C}P^1 \simeq S^2 \qquad [z, w] \mapsto z/w$

Claim:

$$H_*(\mathbb{C}P^n) \simeq \begin{cases} \mathbb{Z} & * = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Proof

$$\begin{array}{ccc} \mathbb{C}P^{n-1} & \subseteq & \mathbb{C}P^n \\ (z_0 : \ldots : z_{n-1}) & \mapsto & (z_0 : \ldots : z_{n-1} : 0) \end{array}$$

Consider the long exact sequence of pair $(\mathbb{C}P^n, \mathbb{C}P^{n-1})$:

We claim that $\mathbb{C}\,P^n/\,\mathbb{C}\,P^{n-1} \simeq S^{2n}$, a sketch proof is to consider this:

$$\begin{array}{ccc} & [z_0,\dots,z_n] \mapsto (\frac{z_0}{z_n},\dots,\frac{z_{n-1}}{z_n}) & \\ \mathbb{C}\,P^n & \longrightarrow & S^{2n} \subseteq \mathbb{R}^{2n} \\ {\scriptstyle z_n = 0}\ \Big\uparrow & & \Big\uparrow \\ \mathbb{C}\,P^{n-1} & \longrightarrow & \text{point} \end{array}$$

By induction, we have

□

16 Proofs

Theorem 16.1 (Collapsing a Pair)
If (X, A) is a good pair with $\pi : (X, A) \to (X/A, A/A)$
then $\pi_* : H_*(X, A) \to H_*(X/A, A/A)$ is an isomorphism.

We will now prove that subdivision lemma (11.2) $\Rightarrow$ Excision Theorem (14.2) $\Rightarrow$ Theorem of Collapsing Pair (16.1). And then prove subdivision lemma (11.2)

16.1 Subdivision Lemma ⇒ Excision Theorem

Definition 16.2
If $\{U_i\}$ is an open cover of X

$$C_*^{\{U_i\}}(X,A) = \frac{\mathrm{C}_*^{\{\mathrm{U_i}\}}(X)}{C_*^{\{U_i\cap A\}}(A)}$$

Subdivision $\quad\Rightarrow\quad H_*^{\{U_i\}}(X,A) \cong H_*(X,A)$ (not obvious)

If $\bar{B} \subseteq \mathrm{Int}A, \{U_i\} = \{\mathrm{Int}A, X - \bar{B}\}$

$$\begin{aligned}
{\mathrm{C}_*^{\{\mathrm{U_i}\}}}_n(X-B) &= \langle e_\sigma | \sigma : \Delta^n \to X - B \quad \mathrm{Im}\,\sigma \subseteq U_i \text{ some } i\rangle \\
&\cong \frac{\mathrm{C}_*^{\{\mathrm{U_i}\}}(X)}{\langle e_\sigma | \,\mathrm{Im}\,\sigma \cap B \neq \emptyset, \mathrm{Im}\,\sigma \subseteq U_i \text{ some } i\rangle} \\
&= \frac{\mathrm{C}_*^{\{\mathrm{U_i}\}}(X)}{\langle e_\sigma | \,\mathrm{Im}\,\sigma \cap B \neq \emptyset, \mathrm{Im}\,\sigma \subseteq \mathrm{Int}A\rangle} \\
\frac{\mathrm{C}_*^{\{\mathrm{U_i}\}}(X-B)}{\mathrm{C}_*^{\{\mathrm{U_i}\}}(A-B)} &= \frac{\mathrm{C}_*^{\{\mathrm{U_i}\}}(X)}{\langle e_\sigma | \,\mathrm{Im}\,\sigma \cap B \neq \emptyset, \mathrm{Im}\,\sigma \subseteq \mathrm{Int}A\rangle + \langle e_\sigma | \,\mathrm{Im}_\sigma \subseteq A - B\rangle} \\
&= \frac{\mathrm{C}_*^{\{\mathrm{U_i}\}}(X)}{\langle e_\sigma | \,\mathrm{Im}\,\sigma \subseteq A\rangle} \quad = \quad \mathrm{C}_*^{\{\mathrm{U_i}\}}(X,A)
\end{aligned}$$

$$\Rightarrow \quad \begin{array}{ccc}
H_*^{\{U_i\}}(X-B, A-B) & \cong & H_*^{\{U_i\}}(X,A) \\
\| & & \| \\
H_*(X_i, A-B) & = & H_*(X,A)
\end{array} \qquad \square$$

16.2 Excision Theorem ⇒ Collapsing Pair

Given a god pair (X,A), pick U with $\bar{A} \subseteq U$, $H_*(A) \cong H_*(U)$. Consider this digram:

$$\begin{array}{ccccc}
H_*(X,A) & \xrightarrow{i_{1*}} & H_*(X,U) & \xleftarrow{j_{1*}} & H_*(X-A, U-A) \\
\downarrow \pi_{1*} & & \pi_{2*}\downarrow & & \pi_{3*}\downarrow \text{Homeo} \\
H_*(X/A, A/A) & \xrightarrow{i_{2*}} & H_*(X/A, U/A) & \xleftarrow{j_{2*}} & H_*((X-A)/A, (U-A/A))
\end{array}$$

It commutes $\quad\Rightarrow\quad \pi_{3*}$ is isomorphism since it comes from a homeomorphism, and j_{1*}, j_{2*} are isomorphism by Excision Theorem

Our goal is to show that i_{1*}, i_{2*} are both isomorphism.

Consider the long exact sequence of a triple $A \subseteq U \subseteq X$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{C_*(A)}{C_*(B)} & \hookrightarrow & \dfrac{C_*(X)}{C_*(B)} & \twoheadrightarrow & \dfrac{C_*(X)}{C_*(A)} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & C_*(A,B) & \to & C_*(X,B) & \to & C_*(X,A) & \longrightarrow & 0
\end{array}$$

So now we have:

$$H_*(U,A) \to H_*(X,A) \xrightarrow{i_{1*}} H_*(X,U) \to H_{*-1}(U,A)$$

where $H_*(U, A) = H_{*-1}(U, A) = 0$ (since $H_*(U) \cong H_*(A)$ by Excision, then look at the long exact sequence of the pair (U, A))
$\Rightarrow \qquad i_{1*}$ is an isomorphism
Similarly for i_{2*} □

16.3 Proof of Subdivision Lemma (11.2)

From now on, we work over $\mathbb{Z}/2\mathbb{Z}$ i.e. $-1 = 1$

Outline of proof:

1. Define a map $B : C_*(X) \to C_*(X)$ (the barycentric subdivision)
2. B is a chain map
3. Show $B \sim 1_{C_*(X)}$ (chain homotopic)
4. If $e_\sigma \in C_*(X)$, then $B^n(e_\sigma) \subseteq \mathrm{C}_*^{\{\mathrm{U_i}\}}(X)$ for some n
5. Use the above to prove ι_* is bijective

16.3.1 Define Barycentric Subdivision

Let $e_n \in C_n(\Delta^n)$ be the simplex represented by the identity map
Define $B(e_n)$ inductively and set $B(e_\sigma) = \sigma_\#(B(e_n))$ where $\sigma : \Delta^n \to X$
Notice $(\Delta^n, \partial\Delta^n) \cong (B^n, S^{n-1})$
Given a simplex $\sigma \in C_k(\partial\Delta^n)$, We have cone on σ: $c(\sigma) \in C_{k+1}(\Delta^n)$, then take the new vertex to origin in B^n and extend linearly. (see picture)

Now define $d(c\sigma) = \sigma + c(d\sigma)$ and $B(e_0) = e_0$
So inductively, $B(e_n) = c(B(de_n))$
$n = 1$:

$n = 2$:

16.3.2 B is a chain map

By induction on n

$$\begin{aligned} dB(e_n) &= d(c(B(de_n))) \\ &= Bd(e_n) + c(d(Bde_n)) \\ &= Bd(e_n) + c(Bd^2e_n) \qquad \text{(by induction)} \\ &= Bd(e_n) \end{aligned}$$

16.3.3 $B \sim 1_{C_*(X)}$

Want $H : C_*(X) \to C_{*+1}(X)$ s.t. $dH + Hd = B + \text{id}$
Again, it is enough to define $H(e_n)$, then use $H(e_\sigma) = \sigma_\#(H(e_n))$

Let $P_n : \Delta^{n+1} \to \Delta^n$, a map that sends the last vertex to the centre of Δ^n, and also $dP_n = e_n + c(de_n)$
$H(e_n) = P_n + c(H(de_n))$
Exercise: Check that $dH + Hd = B + \text{id}$ and proof step 4 of proof.

16.3.4 ι_* is bijective

Step 1-4 of the proof implies
ι_* surjective

Let $[x] \in H_*(X)$
By Step 3, $[x] = [B^n x] \subseteq H_*^{\{U_i\}}(X)$ for n large (by Step 4)

ι_* injective

$x - y = dz \quad \Rightarrow \quad B^n x - B^n y = B^n dz = dB^n z \in \mathrm{C}_*^{\{\mathrm{U_i}\}}(X)$, and $[x] = B^n x$ and $[y] = B^n y$

17 Cell Complexes (CW complexes)

Definition 17.1
$A \subseteq X, f : A \to Y$

$$\begin{aligned} Y \cup_f X &= (X \sqcup Y)/ \quad a \; f(a) \quad \forall a \in A \\ &= \text{attaching } X \text{ to } Y \text{ along } A \text{ using } f \end{aligned}$$

If $(X, A) = (D^n, S^{n-1})$, we say $Y \cup_f D^n$ is the result of adding an n-cell to Y.

Example: $(X, A) = (D^2, S^1), \qquad Y = \mathbb{R}^2, \qquad f = \iota : S^1 \hookrightarrow D^2$

$f : S^1 \to D^2$
$\quad z \mapsto 0$

Definition 17.2
A 0-dimensional cell complex is a disjoint union of points.
A n-dimensional cell complex is the result of adding some n-cells to an $(n-1)$-dimensional cell complex

Example:

Notation: X finite if the number of cells is finite

Definition 17.3
$A \subseteq X$ is a subcomplex if it is a union of cells in X s.t. it is closed under attaching maps
The n-skeleton of X, $X_{(n)}$, is the union of all cells of dimesnion $\leq n$

Fact: If X is finite complex, $A \subseteq X$ subcomplex, then (X, A) is a good pair.

Example:
0-cell $\cup$ n-cell $=$ point$\cup D^n = D^n/S^{n-1} = S^n$
0-cell $\cup$ two n-cells $= S^n \vee S^n$

Notice: A given space will have many different structure as cell complexes. Example:

1. $S^1 =$ 0-cell $\cup$ 1-cell
   $S^1 =$ two 1-cell $\cup$ two 0-cell
2. $S^n = S^{n-1} \cup D^n_{\text{North}} \cup D^n_{\text{South}} = S^{n-1} \cup$ two n-cells
3. Product of 2 cell complexes with $\{e_i\}, \{f_j\}$ is a cell complex with cells $e_i \times f_j$

$$\begin{aligned} T^2 = S^1 \times S^1 &= (0\,\text{cell } \cup 1\,\text{cell}) \times (0 \text{ cell } \cup 1\,\text{cell}) \\ &= (0\,\text{cell})^2 \cup (1\,\text{cell} \times 0\,\text{cell}) \cup (1\,\text{cell} \times 0\,\text{cell}) \cup 2\,\text{cell} \\ &= 0\,\text{cell } \cup \text{ two } 1\,\text{cell } \cup 2\,\text{cell} \end{aligned}$$

4. Real projective space, $\mathbb{R}\,P^4 = \{x \in \mathbb{R}^{n+1}\,|x \neq 0\}/ \qquad x\ \lambda x \quad \lambda \in \mathbb{R}^\times$
   $= \{x \in S^n\}/\sim' \qquad x \sim' -x$

Claim: $\mathbb{R}P^n$ has a cell decomposition with one 0 cell, one 1 cell, ..., one n cell

Proof
$S^n = S^{n-1} \cup D^n_N \cup D^n_S = \{x_{n+1} = 0\} \cup \{x_{n+1} \geq 0\} \cup \{x_{n+1} \leq 0\}$

$x \mapsto -x$ preserves S^{n-1} and switches D^n_N and D^n_S. So divide by $\sim'$ we get:
$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_f D^n$
$f : S^{n-1} \to \mathbb{R}P^{n-1}$ projection
Use induction on n and notice $\mathbb{R}P^0$ =point, $\mathbb{R}P^1 = S^1$ □

5. Similarly, $\mathbb{C}P^n = S^{2n+1}/\sim = \{z \in \mathbb{C}^{n+1} \,|\, \|z\| = 1\}/\sim \qquad z \sim \lambda z, \quad \lambda \in S^1$
has decomposition with one 0 cell, one 2 cell, ... =, one $2n$ cell

Proof
$S^{2n+1} = S^{2n-1} \cup X = \{z_{n+1} = 0\} \cup \{z_{n+1} \neq 0\}$
Divide out by $\sim$:

$$\begin{aligned} \mathbb{C}P^n &= \mathbb{C}P^{n-1} \cup (X/\sim) \\ X/\sim &= \{(z_1, \dots, z_{n+1}) | z_{n+1} \neq 0, \|z\| = 1\}/\sim \\ &= \left\{ (\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_n + 1}) \right\} \simeq \mathbb{R}^{2n} = \text{Int}(D^{2n}) \end{aligned}$$

So we get $\mathbb{C}P^{n+1} = \mathbb{C}P^n \cup_f D^{2n+2}$
$f : S^{2n} \to \mathbb{C}P^n$ projection map

$$\begin{array}{lccc} & \mathbb{C}^{n-1} \cup_f D^2 n & \to & \mathbb{C}P^n \\ \mathbb{C}^{n-1} = \{z \in S^{2n-1}/\sim\} & z & \mapsto (\vec{z}, 0) \in S^{2n+1}/\sim & \\ D^{2n} = \{w \in \mathbb{C}^n \,|\, \|w\| \leq 1\} & w & \mapsto & (\vec{w}, \sqrt{1 - \|w\|^2}) \in S^{2n+1}/\sim \end{array}$$

Note that when $\|w\| = 1$, this agrees with ιf
Easy to see that this is bijective and continuous, so it is a homeomorphism

Now use induction on n.
Notice $\mathbb{C}P^1$ =0 cell$\cup$ 2 cell=S^2
$\mathbb{C}P^2 = \mathbb{C}P^1 \cup_f D^4$
(here $f : S^3 \to S^2$, and it generates $\pi_3(S^2)$
it is called the Hopf fibration
$f(z, w) = z/w \in$Riemann Sphere=$\mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$
e.g. $f^{-1}(\text{point}) \simeq S^2$ but $S^3 \neq S^2 \times S^1$) □

Theorem 17.4
Suppose X is finite cell complex
Then there is a finitely generated chain complexes $C^{\text{cell}}_*(X)$ with $H^{\text{cell}}_*(X) \simeq H_*(X)$
And C^{cell}_n has 1 generator for each n cell in X

Example:
S^n=0 cell *cup* n cell

$$C^{\text{cell}}_*(S^n) = \begin{array}{ccccccccc} \mathbb{Z} & \to & 0 & \to & \cdots & \to & 0 & \to & \mathbb{Z} \\ C_n & & C_{n-1} & & & & & & C_0 \end{array}$$

$$\Rightarrow H^{\text{cell}}_*(X) = \begin{cases} \mathbb{Z} & * = 0, n > 1 \\ 0 & \text{otherwise} \end{cases}$$

Example:
$\mathbb{C}P^n$ has 1 cell of dimension 0,2,4,..., $2n$

$$C_*^{\text{cell}}(\mathbb{C}P^n) = \begin{array}{ccccccccc} \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \to & \cdots & \to & \mathbb{Z} \\ C_{2n} & & C_{2n-1} & & C_{2n-2} & & & & C_0 \end{array} \qquad \text{so } d \equiv 0$$

$$\Rightarrow H_*^{\text{cell}}(X) = \begin{cases} \mathbb{Z} & * = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

What about $\mathbb{R}P^n$?

$$C_*^{\text{cell}}(\mathbb{R}P^n) = \begin{array}{ccccccc} \mathbb{Z} & \to & \mathbb{Z} & \to & \cdots & \to & \mathbb{Z} \\ C_n & & C_{n-1} & & & & C_0 \end{array}$$

18 Cellular Chain Complex

If X is a finite cell complex, there is a chain complex $C_*^{\text{cell}}(X)$ with $H_*^{\text{cell}}(X) \cong H_*(X)$
$C_n^{\text{cell}}(X)$ has 1 generator for each n-cell in X.

Definition 18.1

$$\begin{aligned} C_n^{\text{cell}}(X) &= H_n(X_{(n)}, X_{(n-1)}) \\ &= H_n(X_{(n)}, X_{(n-1)}) \\ &\cong H_n(\bigvee_\tau S_\tau^n) \qquad \tau \text{ runs over } n\text{-cells of } X \\ &\cong \langle e_\tau | \tau \text{ an } n\text{-cell} \rangle \end{aligned}$$

Attaching map $f_\tau : S^{n-1} \to X_{(n-1)}$

$d(e_\tau) = \alpha_*(1) \in H_{n-1}(X_{(n-1)}/X_{(n-2)}) \cong C_{n-1}^{\text{cell}}(X)$ $\qquad$ $1 \in H_{n-1}(S^{n-1})$
Matrix entries: τ is an n-cell, τ' is an $(n-1)$-cell
The coefficient of $e_{\tau'}$ in $d(e_\tau)$ is $\beta_*(1) \in H_{n-1}(S^{n-1}) \cong \mathbb{Z}$

19 Degrees of Maps $S^n \to S^n$

Definition 19.1
Given $f : S^n \to S^n$, degree of f=$f_*(1) \in H_n(S^n)$

$$\begin{array}{ccc} f_* : H_n(S^n) & \longrightarrow & H_n(S^n) \\ \| & & \| \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

Remark.
- (Exercise)$\deg(fg) = \deg f \times \deg g$
- If f is homeomorphism, then f_* is invertible, $\deg f = \pm 1$
  $\deg f = +1$ if f is orientation preserving
  $\deg f = -1$ if f is orientation reversing

If f is a smooth map (a diffeomorphism), then $df|_x$ is the derivative
$x \in S^n = \mathbb{R}^n \cup \{\infty\}$

$$\begin{array}{rcl} df|_x : TS^n|_x & \longrightarrow & TS^n|_{f(x)} \\ \| & & \| \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

f is orientation preserving if $\det df > 0$
f is orientation reversing if $\det df < 0$

Example: If $O : S^n \to S^n$ is multiplication by an orthogonal map, $\deg O = \det O$

Proposition 19.2
Suppose $f : S^n \to S^n$, $x \in S^n$ is a regular value for f
i.e. $\exists$ open ball $U \ni x$ with $f^{-1}(U) = \bigsqcup_{i=1}^{m} U_i$ $\quad f|_{U_i} : U_i \to U_i$ is a homeomorphism. Then

$$\deg f = \sum_{i=1}^{m} \deg f|_{U_i} \quad , \quad \deg f|_{U_i} = \begin{cases} +1 & f \text{ orientation preserving} \\ -1 & f \text{ orientation reversing} \end{cases}$$

Proof

Claim 1: $\alpha_*(1) = 1 \oplus \ldots \oplus 1$ in $H_n(\bigvee S^n)$

(see notes)

Claim 2: $\beta_*(x_1 \oplus \ldots \oplus x_n) = \sum f|_{U_i}(X_i)$

So $\deg f = \beta_*(\alpha_*(1)) = \beta_*(1 \oplus \cdots \oplus 1) = \sum (f|_{U_i})_*(1) = \sum \deg f|_{U_i}$ $\square$

Example 19.3
$H^{\text{cell}}(\mathbb{R}P^n), \mathbb{R}P^n$ has 1 cell of dimension 0,1,...,n

$$C_*^{\text{cell}}(\mathbb{R}P^n) = \begin{array}{ccccccc} \langle e_n \rangle & & \langle e_{n-1} \rangle & & & & \langle e_0 \rangle \\ \mathbb{Z} & \to & \mathbb{Z} & \to & \cdots & \to & \mathbb{Z} \\ C_n & & C_{n-1} & & & & C_0 \end{array}$$

$d(e_1) = f_*(1)$ where $f : S^{i-1} \to (\mathbb{R}P^{i-1} \to \mathbb{R}P^{i-1}/\mathbb{R}P^{i-2} =)S^{i-1}$
Now we want $\deg f$
Pick $x \in S^{i-1}$ that is in the interior of the $i-1$ cell, i.e. it is in the interior of the $i-1$ cell in $\mathbb{R}P^n$
$f^{-1}(x)$ is 2 points, y and $-y \in S^{i-1}$
So if $f|_y$ has degree 1, $f|_{-y}$ has degree $= \deg A = (-1)^i$
($f|_{-y} = f|_y \circ A$, where $A : S^{i-1} \to S^{i-1}$ is the antipodal map)
So $\deg f = \deg f|_y + \deg f|_{-y} = 1 + (-1)^i$

$$C_*^{\text{cell}}(\mathbb{R}P^n) : C_n \to \cdots C_3 \stackrel{1-1}{\to} C_2(=\mathbb{Z}) \stackrel{1+1}{\to} C_1(=\mathbb{Z}) \stackrel{1-1}{\to} C_0(=\mathbb{Z})$$

Example:
$n=2$: $\mathbb{Z} \stackrel{2}{\to} \mathbb{Z} \stackrel{0}{\to} \mathbb{Z}$
$n=3$: $\mathbb{Z} \stackrel{0}{\to} \mathbb{Z} \stackrel{2}{\to} \mathbb{Z} \stackrel{0}{\to} \mathbb{Z}$

$$H_*(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases} \qquad n \text{ even}$$

$$H_*(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ \mathbb{Z}/2 & * = 1, 3, \ldots, n-2 \\ 0 & \text{otherwise} \end{cases} \qquad n \text{ odd}$$

20 Euler Characteristic

Lemma 20.1
Suppose that C_* is a finitely generated chain complex over $\mathbb{Z}$. Then

$$\chi(C_*) := \sum (-1)^i \operatorname{rk} C_i = \sum (-1)^i \operatorname{rk} H_i(C_i)$$

Proof

$$\begin{array}{rrcl} & \operatorname{rk} C_i & = & \operatorname{rk}\ker d_i + \operatorname{rk}\operatorname{Im} d_i \\ & & = & (\operatorname{rk}\operatorname{Im} d_{i+1} + \operatorname{rk} H_i) + \operatorname{rk}\operatorname{Im} d_i \\ \Rightarrow & \sum (-1)^i \operatorname{rk} C_i & = & \sum (-1)^i \operatorname{rk} H_i \qquad (\operatorname{rk}\operatorname{Im} d_* \text{ cancels}) \end{array}$$

$\square$

Corollary 20.2
$\chi(X) = \chi(H_*(X))$

Corollary 20.3
Suppose X has a cell decomposition with K_n n-cells. Then $\chi(X) = \sum(-1)^n K_n$

<u>Exercise</u>: If triangulation of S^2 has v vertices, e edges, f faces, then $v - e + f = 2$

21 Cellular Homology

<u>Goal</u>: To show that $H_*^{\text{cell}}(X) \cong H_*(X)$

Lemma 21.1 (Dimension Axiom)
Suppose X is a cell complex of dimension n. Then $H_*(X) = 0 \ \forall * > n$

Proof
Induction on n.
<u>n=0</u>: $X = \sqcup \text{points} \quad \Rightarrow \quad H_*(X) = \bigoplus H_*(\text{point}) = 0 \ \forall * > 0$

Given X of dimension n, consider l.e.s. of $(X, X_{(n-1)})$

$$\cdots \to H_*(X_{(n-1)}) \to H_*(X) \to H_*(X, X_{(n-1)}) \to \cdots$$

for $* > n$, $H_*(X_{(n-1)}) = 0$ by induction
$H_*(X, X_{(n-1)}) = H_*(X/X_{(n-1)}) = H_*(\bigvee S^n) = 0$ for $* > n$
$\Rightarrow \quad H_*(X) = 0$ for $* > n$ $\square$

Lemma 21.2 (Naturality of ∂)
Suppose $f : (X, A) \to (Y, B)$

$$\begin{array}{ccccccc} H_*(A) & \longrightarrow & H_*(X) & \longrightarrow & H_*(X,A) & \longrightarrow & H_{*-1}(A) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_*(B) & \longrightarrow & H_*(Y) & \longrightarrow & H_*(Y,B) & \longrightarrow & H_{*-1}(B) \end{array}$$

All squares commute

<u>Exercise</u> Proof the lemma.

Theorem 21.3

$$H_*^{\text{cell}}(X) \cong H_*(X)$$

(We will drop parenthesis on skeleton from now on)
(See notes for all the diagram and details)

Proof
<u>Step 1</u>:
$d = \pi_* \partial$
$e_\tau = i_*(1)$
Cell: $(D^n, S^{n-1}) \to (X_n, X_{n-1})$
$d(e_\tau) = \pi_* f_{\tau *}(1) = \alpha_*(1) = \beta_*(1) = \pi_* \partial(e_\tau)$

Step 2:
$d^2 = \gamma = \pi^1_* \circ \partial_1 \circ \pi^2_* \circ \partial_2 = pi^1_* \circ 0 \circ \partial_2 = 0$

Step 3:
Consider l.e.s. of (X_{n+1}, X_n, X_{n-1}) and $(X_{n+1}, X_{n-1}, X_{n-2})$

If $x \in \ker d_n^{\text{cell}}$, then $\partial(i_*(X)) = 0 \Rightarrow i_*(X) = \operatorname{Im} j_*$
For $x \in \ker d_n^{\text{cell}}$, let $\phi(x) = j_*^{-1} i_*(x)$

Claim that $\ker i_* = \ker \phi = \operatorname{Im} d_{n+1}^{\text{cell}}$

$$\phi : \frac{\ker d_n}{\operatorname{Im} d_{n+1}} \longrightarrow H_n(X_{n+1}, X_{n-2})$$

i_* is surjective $\quad \Rightarrow \quad \phi$ is surjective
$\Rightarrow \quad \phi : H_n^{\text{cell}}(X) \xrightarrow{\sim} H_n(X_{n+1}, X_{n-2})$

Step 4:

$$0 = H_n(X_{n-2}) \longrightarrow H_n(X_{n+1}) \longrightarrow H_n(X_{n+1}, X_{n-2}) \longrightarrow H_{n-1}(X_{n-2}) = 0$$

$\Rightarrow \quad H_n(X_{n+1}, X_{n-2}) \cong H_n(X_{n+1})$
(Exercise) Check that $H_n(X_{n+1}) \cong H_n(X)$ to complete the proof $\square$

22 Uniqueness of Ordinary Homology

Theorem 22.1 (Eilenberg-Steenrod)
Suppose

$$\mathcal{H} : \left\{ \begin{array}{c} \text{pairs of squares} \\ \text{maps of pairs} \end{array} \right\} \to \left\{ \begin{array}{c} \text{graded abelian group} \\ \text{homomorphism} \end{array} \right\}$$

is a functor satisfying:
(1) Homotopy Invariance

$$\begin{gathered} f, g : (X, A) \to (Y, B) \\ f_*, g_* : \mathcal{H}_*(X, A) \to \mathcal{H}_*(Y, B) \\ f \sim g \quad \Rightarrow \quad f_* = g_* \end{gathered}$$

(2) Excision

$$\overline{B} \subseteq \text{Int} A \qquad \iota_* : \mathcal{H}_*(X - B, A - B) \xrightarrow{\sim} \mathcal{H}_*(X, A)$$

(3) Dimension Axiom

$$\mathcal{H}_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

(4) Exact Sequence of a Pair
Define $\mathcal{H}_n(X) = \mathcal{H}_n(X, \emptyset)$ $\quad f : (X, A) \to (Y, B)$ map of pairs

$$\begin{array}{ccccccccc} > \mathcal{H}_n(A) & \longrightarrow & \mathcal{H}_n(X) & \longrightarrow & \mathcal{H}_n(X, A) & \longrightarrow & \mathcal{H}_{n-1}(A) & \longrightarrow \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ > \mathcal{H}_n(B) & \longrightarrow & \mathcal{H}_n(Y) & \longrightarrow & \mathcal{H}_n(Y, B) & \longrightarrow & \mathcal{H}_{n-1}(B) & \longrightarrow \end{array}$$

Then $\mathcal{H}_*(X) \cong \mathcal{H}_*(X)$ for any finite cell complex X

Proof
Step 1:
Exact sequence of a pair
$\Rightarrow$ Exact sequence of a triple

Step 2:
Excision + Homotopy Invariance + Exact sequence of a triple
$\Rightarrow$ Collapsing a good pair

Step 3:
Excision to compute $\mathcal{H}_*(S^0)(= \mathcal{H}_*(\{p, q\})) \cong \mathcal{H}_*(S^0)$

$$\mathcal{H}_1(S^0, p) \to \mathcal{H}_0(p) \to \mathcal{H}_0(S^0) \to \mathcal{H}_0(S^0, p)$$

We have $\mathcal{H}_1(S^0, p) \cong \mathcal{H}_1(q)$ (by excision) $= 0$ (by dimension axiom)
And $\mathcal{H}_0(S^0, p) \cong \mathcal{H}_9 = 0(q) = \mathbb{Z}$

Step 4:
Use exact sequence of (D^n, S^{n-1}) to prove $\mathcal{H}_*(S^n) \cong \mathcal{H}_*(S^n)$ by induction

Step 5:
Define cell complex $\mathcal{C}_*^{\text{cell}}(X)$ for X
Prove that $\mathcal{H}_*^{\text{cell}}(X) \cong \mathcal{H}_*(X)$ (goes as before) once you compute $\mathcal{H}_*(\bigvee S^n)$ (by excision)

Step 6:
Show that $\mathcal{C}_*^{\text{cell}}(X) \cong C_*^{\text{cell}}(X)$
As a group: $\mathcal{C}_*^{\text{cell}}(X) = \mathcal{H}_*(X_n, X_{n-1}) \cong \mathcal{H}_*(\bigvee S^n) = H_*(\bigvee S^n)$

Remains to check:

Want to know that the map $\gamma_* : \mathcal{H}_{n-1}(S^{n-1}) \to \mathcal{H}_{n-2}(S^{n-1})$ (hence $\gamma_* : H_{n-1}(S^{n-1}) \to H_{n-2}(S^{n-1})$) commutes for all γ

Fact: $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$ generated by $\text{id}_{S^{n-1}}$
True for $\gamma = $id since $\mathcal{H}$ is a functor $\square$

23 Homology with Coefficients

23.1 Motivation

Consider $C_*^{\text{cell}}(\mathbb{R}P^3)$

$$\begin{aligned} C_* :&\quad \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \\ H_* :&\quad \mathbb{Z} \to 0 \to \mathbb{Z}/2 \to \mathbb{Z} \end{aligned}$$

Example 1: Replace $\mathbb{Z}$ by $\mathbb{Q}$

$$\begin{aligned} C_* :&\quad \mathbb{Q} \xrightarrow{\times 0} \mathbb{Q} \xrightarrow{\times 2} \mathbb{Q} \xrightarrow{\times 0} \mathbb{Q} \\ H_* :&\quad \mathbb{Q} \to 0 \to 0 \to \mathbb{Q} \end{aligned}$$

First motivation: Does the process of going from a ring to a field make life easier?

Example 2: Replace by $\mathbb{Z}/2$

$$\begin{aligned} C_* :&\quad \mathbb{Z}/2 \xrightarrow{\times 0} \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/2 \xrightarrow{\times 0} \mathbb{Z}/2 \\ H_* :&\quad \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \end{aligned}$$

Example 3 Replace by $\mathbb{Z}/3$

$$\begin{aligned} C_* :&\quad \mathbb{Z}/3 \xrightarrow{\times 0} \mathbb{Z}/3 \xrightarrow{\times 2} \mathbb{Z}/3 \xrightarrow{\times 0} \mathbb{Z}/3 \\ H_* :&\quad \mathbb{Z}/3 \to 0 \to 0 \to \mathbb{Z}/3 \end{aligned}$$

23.2 Tensor Product

(For definitions, see Commutative Algebra)

Examples:

1. $M \otimes_R R \cong M$
   $m \otimes r \mapsto rm$
2. $R = K$ a field, V, W vector space over K with basis $\{e_i\}, \{f_j\}$, then $V \otimes W$ has basis $\{e_i \otimes f_j\}$
3. $R = \mathbb{Z} \quad \Rightarrow \quad \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2 = 0$
   Since $a \otimes b = 2(a/2) \otimes b = (a/2) \otimes 2b = 0$
4. $\mathbb{Z}/3 \otimes \mathbb{Z}/2 = 0$
5. $\mathbb{Z}/2 \otimes \mathbb{Z}/2 = \mathbb{Z}/2$
6. R is a PID $\quad \Rightarrow \quad R/(a) \otimes R/(b) \cong R/(\gcd(a,b))(= R/(a) + (b))$
7. $R = K$ field $\quad \Rightarrow \quad K[X] \otimes K[Y] = K[X, Y]$

Observation:
If (C_*, d) is a chain complex over R and M is an R-module
then $(C_* \otimes M, d \otimes 1)$ is also a chain complex
i.e. $d(x \otimes a) = dx \otimes a$ and $d^2(x \otimes a) = d^2 x \otimes a = 0$
Exercise: $(C, d) \sim (C', d') \Rightarrow (C \otimes M, d \otimes 1) \sim (C' \otimes M, d' \otimes 1)$

Lemma 23.1
There is a natural map

$$\begin{array}{rcl} H_*(C) \otimes M & \to & H_*(C \otimes M) \\ [x] \otimes m & \mapsto & [x \otimes m] \end{array}$$

Proof
Exercise

$$dx = 0 \quad \Rightarrow \quad \begin{cases} d(x \otimes m) = dx \otimes m = 0 \\ [dx] \otimes m \mapsto dx \otimes m = d[x \otimes m] = 0 \end{cases}$$

□

23.3 Homology with Coefficients

Definition 23.2
If G a $\mathbb{Z}$-module (i.e. an abelian group). X a space then define homology with coefficients in G as

$$C_*(X; G) = C_*(X) \otimes_{\mathbb{Z}} G$$

Still have:

- Exact sequence of a pair: $0 \to C_*(A; G) \to C_*(X; G) \to C_*(X, A; G) \to 0$
- Mayer-Vietoris sequence: $0 \to C_*(A \cap B; G) \to C_*(A; G) \oplus C_*(B; G) \to C_*(X; G) \to 0$ if A, B open cover of X

In the start of the section, we actually computed $H_*(\mathbb{R}P^3; \mathbb{Q}), H_*(\mathbb{R}P^3; \mathbb{Z}/2), H_*(\mathbb{R}P^3; \mathbb{Z}/3)$, because of this:

Proposition 23.3
$H_*(X;G) \cong H_*(C_*^{\text{cell}}(X);G)$

Proof
Step 1:

$$\begin{array}{rl} C_*(\text{point}) & \cdots \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} = C_0 \\ C_*(\text{point};G) & \cdots \xrightarrow{\times 0} G \xrightarrow{\times 1} G \xrightarrow{\times 0} G \end{array}$$

$$\Rightarrow \qquad H_*(\text{point};G) = \begin{cases} G & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Step 2:
Now do everything we did to show $H_*^{\text{cell}}(X) \cong H_*(X)$
Use exact sequence of a pair (D^n, S^{n-1}) to show

$$H_*(S^n;G) \cong \begin{cases} G & * = 0, n \\ 0 & \text{otherwise} \end{cases}$$

And then compute

$$\widetilde{H}_*(\bigvee^k S^n;G) \cong \begin{cases} G^k & * = n \\ 0 & \text{otherwise} \end{cases}$$

Show that the matrix entries in $d^{\text{cell}}(X;G)$ agree with entries in $d^{\text{cell}}(X)$. Also check this:

$$\begin{array}{ccc} \overbrace{H_n(S^n)}^{\mathbb{Z}} \otimes G & \xrightarrow{\beta_*} & H_n(S^n) \otimes G \\ \text{isom} \downarrow & & \downarrow \text{isom} \\ \underbrace{H_n(S^n;G)}_{G} & \xrightarrow{\beta_*} & H_n(S^n;G) \end{array}$$

□

23.4 Universal Coefficient Theorem

Theorem 23.4
If M is a module over PID R, m is torsion-free (i.e. $rm = 0 \Rightarrow m = 0$ or $r = 0$), then $M \cong R^n$ is free

(Proof omitted)

Corollary 23.5
$A \subseteq R^n \Rightarrow A$ free

Corollary 23.6
R^n/A torsion free $\Rightarrow R^n = A \oplus B$ some B

Proof
R^n/A torsion free $\Rightarrow R^n/A$ is free
Pick a basis $\{e_i\}$ for R^n/A
Choose $e_i' \in R^n$ s.t. $\pi(e_i') = e_i \Rightarrow \langle e_i' \rangle = B$ □

Definition 23.7
Very short chain complex (v.s.cx) is a chain with only one non-zero component
Short chain complex (s.c.cx) is a chain in form of $0 \to R \xrightarrow{\times a} R \to 0 \quad a \neq 0$

Theorem 23.8 (Universal Coefficient Theorem)
Suppose R is PID and (C_*, d) is a free finitely generated chain complex over R (i.e. $C_n = R^k$ some k). Then C_* is the direct sum of very short chain complex and short chain complex

Example:
$R = \mathbb{C}[X, Y]$
Chain complex that is not a sum of v.s.cx

$$\begin{array}{ccccccccc} 0 & \to & R & \to & R^2 & \to & R & \to & 0 \\ & & 1 & \mapsto & (x,y) & & & & \\ & & & & (a,b) & \mapsto & ay - bx & & \end{array}$$

Exercise:

$$H_*(C) = \begin{cases} R/(x,y) & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

The ideal $(X, Y)$is not principle $\Rightarrow$ C_* is not a sum

Proof
Suppose C_* free finitely generated chain complex on $\mathbb{Q}$
Let $K_n = \ker d_n \subseteq C_n$
$C_n/K_n \cong \operatorname{Im} d_n \subseteq C_{n-1}$
$\Rightarrow$ C_n/K_n free by Corollary 23.5
$\Rightarrow$ $C_n/K_n \oplus A_n$ for some A_n free by Corollary 23.6
Also have $d^2 = 0 \Rightarrow d(A_n) \subseteq K_n \Rightarrow \quad C_* \cong \bigoplus \left(0 \to A_n \xrightarrow{d} K_{n-1} \to 0\right)$ To finish the proof, need to show that we can pick basis for A_n, K_{n-1} s.t. matrices of d_n looks like

$$\left(\begin{array}{ccc|c} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_k & \\ \hline & 0 & & 0 \end{array}\right) \qquad a_i \neq 0 \qquad R \xrightarrow{a_i} R \tag{23.1}$$

(This is the Smith Normal Form) □

Theorem 23.9 (Smith Normal Form)
$L : \mathbb{Z}^m = M \to N = \mathbb{Z}^n$ with right choice of basis on M and N, then L has matrix as in equation 23.1

Sketch Proof
Start with any matrix
Elementary basis change includes (1) swapping 2 rows (or columns) and (2) Add a multiple of 1 row (or column) to another
WLOG, $|a_{11}| > 0$ and is minimal among $|a_{ij}| > 0$
Subtract first row from other rows to make
either $\qquad |a_{i1}| < |a_{11}| \quad i > 1$
or $\qquad |a_{1i}| < |a_{11}| \quad i > 1$
So we get

$$\left(\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & L' \end{array}\right)$$

Repeat for L' □

23.5 Torsion and Computing $H_*(X;G)$

Definition 23.10
M,N are R-modules
Say a chain complex (F_*,d) over R is a free resolution of M if

$$(1) \qquad F_* \text{ is free over } R, \quad F_*=0 \quad \forall * < 0$$
$$(2) \qquad H_*(F) = \begin{cases} M & *=0 \\ 0 & *>0 \end{cases}$$

Definition 23.11
If (F,d) is a free resolution of M over R

$$\mathrm{Tor}_*^R(M,N) = H_*(F\otimes N)$$

Fact: This does not depend on the choice of free resolution
Exercise: $\mathrm{Tor}_0^R(M,N) = M\otimes N$

Examples

1. $M=R \qquad 0\to 0\to R(=F_0)\to 0$

$$\mathrm{Tor}_*(R,N) = \begin{cases} N & \text{if } *=0 \\ 0 & *>0 \end{cases}$$

2. $R=\mathbb{Z} \quad M=\mathbb{Z}/a \qquad 0\to \mathbb{Z} \xrightarrow{\times a} \mathbb{Z}\to 0$
   Take $N=\mathbb{Z}/(b)$, get $F\otimes \mathbb{Z}/(b) = 0\to \mathbb{Z}/b \xrightarrow{\times a} \mathbb{Z}/b\to 0$

$$\mathrm{Tor}_*(\mathbb{Z}/a,\mathbb{Z}/b) = \begin{cases} \mathbb{Z}/\gcd(a,b) & *=0,1 \\ 0 & \text{otherwise} \end{cases}$$

   If we take $N=\mathbb{Q} \Rightarrow F\otimes\mathbb{Q} = \mathbb{Q}\xrightarrow{\times a}\mathbb{Q}$

$$\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a,\mathbb{Q}) = 0$$

3. $R=\mathbb{C}[X,Y] \quad M=\mathbb{C}[X,Y]/(X,Y)$ as in Example under Theorem 23.8

$$\begin{aligned} F\otimes M &= \mathbb{C}\xrightarrow{0}\mathbb{C}^2\xrightarrow{0}\mathbb{C} \\ \mathrm{Tor}_*(M,N) &= \begin{cases} \mathbb{C} & *=0,2 \\ \mathbb{C}^2 & *=1 \\ 0 & *>2 \end{cases} \end{aligned}$$

4. $M=M_1\otimes M_2 \quad F=F(1)\otimes F(2)$ where $F(i)$ is a free resolution of M_i
   $\Rightarrow \quad \mathrm{Tor}_*(M_1\otimes M_2,N) = \mathrm{Tor}_*(M_1,N)\otimes\mathrm{Tor}_*(M_2,N)$

Exercise: If R is a PID, $\mathrm{Tor}_*^R(M,N)=0$ for $*>1$

Proposition 23.12
Suppose C_* is free finitely generated chain complex over a PID R. Then

$$\begin{aligned} H_*(C_*\otimes N) &= \mathrm{Tor}_0^R(H_*(C),N)\oplus\mathrm{Tor}_1^R(H_{*-1}(C),N) \\ &= \big(H_*(C)\otimes N\big)\oplus\big(\mathrm{Tor}_1^R(H_{*-1}(C),N)\big) \end{aligned}$$

Proof
Suppose C_* is a v.s.cx or s.c.cx
Then C_* is a free resolution of $H_*(C)$ (up to shift in grading)
So in this case, it is immediate from the definition
In general, it follows from the Universal Coefficient Theorem 23.8
and the fact that Tor is additive under $\oplus$ □

Remark.

1. Recall we had a natural map

$$\begin{array}{rcl} H_*(C) \otimes N & \to & H_*(C_* \times N) \\ [x] \otimes m & \mapsto & [x \otimes m] \end{array}$$

2. There is no canonical map

$$\mathrm{Tor}_1^R(H_{*-1}(C), N) \to H_*(C_* \otimes N)$$

3. There is an obvious thing you could try to do for complexes over an arbitrary R. But it is NOT true!

4. Given $H_*(X)$, we can now compute $H_*(X; G)$ for any G

Corollary 23.13
$H_*(X; \mathbb{Q}) \cong H_*(X) \otimes \mathbb{Q}$

Proof
$\mathrm{Tor}_*^{\mathbb{Z}}(M, \mathbb{Q}) = 0 \qquad \text{for } * > 0$ □

Corollary 23.14

$$H_*(X; \mathbb{Z}/p) = \big(H_*(X) \otimes \mathbb{Z}/p\big) \bigoplus \big(\mathrm{Torsion}(H_{*-1}(X)) \otimes \mathbb{Z}/p\big) \quad \text{(prime } p\text{)}$$

(Thus explaining the use of symbol Tor)

24 Cohomology

Definition 24.1
If (C_*, d) is a chain complex over R

$$\begin{array}{rcl} \mathrm{Hom}(C_*, M) & = & ({C_n}', d') \\ {C_n}' = \mathrm{Hom}(C_n, M) & , & d' : C'_n \to C'_{n+1} \\ (d'\alpha)(\sigma) & = & \alpha(d\sigma) \quad (\sigma \in C_{n+1}) \text{ is a } \underline{\text{cochain complex}} \end{array}$$

Special case $M = R$

$$C^* := \mathrm{Hom}(C_*, R) \quad \text{is } \underline{\text{dual cochain complex}} \text{ of } C$$

i.e. if $C_n = R^k \quad C^n = (R^k)^* \cong R^k$
then $d^n : C^n \to C^{n+1}$ is the transpose of d_{n+1}

Example: $C_* = C_*^{\text{cell}}(\mathbb{R}P^3)$

$$\begin{array}{rcccccccc} C_*: & \mathbb{Z} & \stackrel{\times 0}{\longrightarrow} & \mathbb{Z} & \stackrel{\times 2}{\longrightarrow} & \mathbb{Z} & \stackrel{\times 0}{\longrightarrow} & \mathbb{Z} \\ H_*: & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z} \end{array}$$

$$\begin{array}{rcccccccc} C^*: & \mathbb{Z} & \stackrel{\times 0}{\longleftarrow} & \mathbb{Z} & \stackrel{\times 2}{\longleftarrow} & \mathbb{Z} & \stackrel{\times 0}{\longleftarrow} & \mathbb{Z} \\ H^*: & \mathbb{Z} & \longleftarrow & \mathbb{Z}/2 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}$$

Definition 24.2
X is a space

$$\begin{aligned} C^*(X) &= \operatorname{Hom}(C_*(X), \mathbb{Z}) \quad H^*(X) = \underline{\text{singular cohomology}} = \frac{\ker d'_{*+1}}{\operatorname{Im} d'_*} \\ C^*(X;G) &= \operatorname{Hom}(C_*(X), G) \quad H^*(X;G) \end{aligned}$$

Suppose $f : X \to Y$

$$f^* : H^*(Y) \to H^*(X)$$

is induced by

$$f^\# : C^*(Y) \to C^*(X)$$

where $f^\#(\alpha(x)) = \alpha(f_\#(X)) \quad (x \in C_*(X))$ Exercise: This is a chain map.

$H_*(X)$ is a contravariant functor

$$(fg)^* = g^* f^*$$

Lemma 24.3
There is a natural bilinear pairing

$$\begin{aligned} H^n(X) \times H_n(X) &\to \mathbb{Z} \\ \langle [\alpha], [x] \rangle &= \alpha(x) \end{aligned}$$

Check: $\langle [\alpha], [x + dy] \rangle = \langle [\alpha], [x] \rangle$

$$\begin{aligned} \alpha(x + dy) &= \alpha(x) + \alpha(dy) \\ &= \alpha(x) + d\alpha(y) \quad \text{since } dx = 0 \\ &= \alpha(x) \end{aligned}$$

Exercise: Check that $f : X \to Y \quad f^* : H^*(Y) \to H^*(X)$
have $\langle f^*([\alpha]), [x] \rangle = \langle [\alpha], f_*([x]) \rangle$

Exact sequence of a pair:

$$C^*(X, A) = \langle \alpha \in C^*(X) | \alpha(x) = 0 \; \forall x \in C_*(A) \rangle$$

$$0 \to C^*(X, A) \to C^*(X) \stackrel{\iota^\#}{\longrightarrow} C^*(A) \to 0$$

and we get:

Mayer-Vietoris sequence:

$$0 \to C^*(X) \to C^*(A) \oplus C^*(B) \to C^*(A \cap B) \to 0$$

$f : S^n \to S^n \qquad \deg f = k$

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{\times k} & H_n(S^n) & & \langle \alpha, f_*(X) \rangle = \langle \alpha, kX \rangle \\ H^n(S^n) & \xleftarrow{\times k} & H^n(S^n) & & \langle f^*(\alpha), X \rangle = \langle f^*(\alpha), X \rangle \end{array}$$

25 Computing $H^*(X)$ from $H_*(X)$

M, N are R-modules

Definition 25.1
If (F_*, d) is a free resolution of M

$$\mathrm{Ext}^*_R(M, N) = H^*(\mathrm{Hom}(F_*, N))$$

(Think what the RHS is)

Example:

1. $\mathrm{Ext}^*_R(R, N) = H^*(0 \leftarrow \mathrm{Hom}(R, N) \leftarrow 0) = \begin{cases} \mathrm{Hom}(R, N) & * = 0 \\ 0 & \text{otherwise} \end{cases}$
2. If $R = k$ is a field
$$\mathrm{Ext}^n_k(V, k) = \begin{cases} V^* & n = 0 \\ 0 & n \neq 0 \end{cases}$$
3. $R = \mathbb{Z} \qquad M = \mathbb{Z}/a \qquad \mathbb{Z} \xrightarrow{\times a} \mathbb{Z}$
$$\begin{aligned} \mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}) &= H^*(\mathbb{Z} \xleftarrow{\times a} \mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}/a & * = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$
4. $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Q}) = 0$

Proposition 25.2
If C_* is free finitely generated chain complexes over a PID R

$$\begin{aligned} H^*(\mathrm{Hom}(C_*, N)) &= \mathrm{Ext}^0_R(H_*(C), N) \oplus \mathrm{Ext}^1_R(H_{*-1}(C), N) \\ &= \mathrm{Hom}(H_*(C), N) \oplus \mathrm{Ext}^1_R(H_{*-1}(C), N) \end{aligned}$$

Proof
For short and v.s. chain complexes, it is the definition of Ext.
In general, it follows from Universal Coefficient Theorem 23.8
and $\mathrm{Ext}_R(M_1 \otimes M_2, N) = \mathrm{Ext}_R(M_1, N) \oplus \mathrm{Ext}_R(M_2, N)$ □

Corollary 25.3
If k is a field, $H^*(X;k) \cong (H_*(X,k))^*$ (Exercise: prove this)

Corollary 25.4
$\operatorname{rk} H^*(X) = \operatorname{rk} H_*(X) \qquad (H^*(X) = \mathbb{Z}^k \quad H_*(X) = (\mathbb{Z}^k)^*)$

Corollary 25.5
Torsion $H^*(X) =$ Torsion $H_{*-1}(X)$ (Exercise: prove this)

26 Homology of Products

<u>Notations</u>:
Cell τ_i:

Lemma 26.1
X is a finite cell complex with cell $\tau_i \Leftrightarrow$ These are maps $\tau_i : D^n \to X$ s.t.

1. $\iota_{\tau_i}|_{\mathrm{Int}(D^n)}$ is an injection
2. Every $x \in X$ is in $\mathrm{Int}(\tau_i)$ for a unique i
3. $\iota_{\tau_i}|_{S^{n-1}} : S^{n-1} \to X_{(n-1)} = \{x | x \in \mathrm{Int}(\tau_i) \quad \dim \tau_i < n\}$

Proof
$\Leftarrow$: Induct on n showing that $X_{(n)}$ is a finite cell complex
$n = 0$, $X_{(0)}$ is a union of points
In general, I get a continuous map

$$\begin{array}{rcl} X_{(n-1)} \sqcup_{\dim \tau_i = n} D_i^n & \to & X \\ D_i^n & \xrightarrow{\tau_i} & X \end{array}$$

defines

$$\underbrace{X_{(n-1)} \cup_{f_\tau} (\bigcup D_i^n)}_{\text{Hausdoff compact finite cell complexes}} \qquad \to X_{(n)}$$

This is bijective by (2) and continuous
$\Rightarrow$ it is an homeomorphism onto its image $\square$

Proposition 26.2
If X is a finite cell complex with cells σ_i
Y is finite cell complees with cells τ_j
Then $X \times Y$ is a finite cell complex with cells $\sigma_i \times \tau_j$

Proof
If we have maps

$$\begin{aligned} \iota_{\sigma_i} : D^{m_i} &\to X \\ \iota_{\tau_j} : D^{n_j} &\to Y \end{aligned}$$

Then use $\iota_{\sigma_i} \times \iota_{\tau_j} : D^{m_i} \times D^{n_j} \to X \times Y$
It is easy to check that the items hold: e.g. item (3):

$$\begin{aligned} \partial(D^m \times D^n) &\to X \times Y \\ (\partial D^m \times D^n) \cup (D^m \times \partial D^n) &\to (X_{(m-1)} \times \tau_j) \cup (\sigma_i \times Y_{(n-1)}) \subseteq (X \times Y)_{(n+m-1)} \end{aligned}$$

$$\begin{aligned} C_*^{\text{cell}}(X \times Y) &= \langle e_{\sigma_i \times \tau_j} \rangle \\ C_*(X) \otimes C_*(Y) &\to C_*(X \times Y) \\ e_{\sigma_i} \otimes e_{\tau_j} &\mapsto e_{\sigma_i \times \tau_j} \end{aligned}$$

More precisely,

$$C_n(X \times Y) = \bigoplus_{i+j=n} C_i(X) \otimes C_j(Y)$$

What is d? $d(x \otimes y) = dx \otimes y + (-1)^{\deg x} x \otimes dy$
Check: $d^2(x \otimes y) = (-1)^{\deg x - 1} dx \otimes dy + (-1)^{\deg x} dx \otimes dy = 0$ □

Definition 26.3
If A_* and B_* are chain complexes

$$\begin{aligned} C_* &= A_* \otimes B_* \quad \text{has (by definition)} \\ C_n &= \bigoplus_{i+j=n} A_i \otimes B_j \\ d(a \otimes b) &= (d_A a) \otimes b + (-1)^{\deg a} a \otimes d_B b \end{aligned}$$

Theorem 26.4
If X and Y are finite cell complexes

$$C_*^{\text{cell}}(X \times Y) \cong C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y)$$

Proof
We have already seen this is an isomorphism of groups, so just need to check d

$$\begin{aligned} f_\sigma : \partial D^n &\to X_{(n-1)} \\ S^{n-1} &\to S^{n-1} \end{aligned}$$

Pick a regular value $v \in \operatorname{Int} \sigma$
σ' component of de_σ is $\sum_{v \in f_\sigma^{-1}(v)} \operatorname{sgn}(\det df|_v)$ Now,

$$\begin{array}{ccc} F : \partial(D^m \overset{\sigma\times\tau}{\times} D^n) & \to & X_{(n+m-1)} \\ \| & & \cup| \\ \partial D^m \times D^n \cup D^m \times \partial D^n & & \sigma' \times \tau' \ni (v_1, v_2) \text{ regular value} \end{array}$$

(A) If $v_2 \in \operatorname{Int}\tau = \tau'$, then $F(v_1, v_2) = \{f_\sigma^{-1}(v_1), v_2\}$
(B) If $v_1 \in \operatorname{Int}\sigma = \sigma'$, then $F(v_1, v_2) = \{v_1, f_\tau^{-1}(v_2)\}$
If neither $\sigma = \sigma'$ nor $\tau = \tau'$, then $F^{-1}(v_1, v_2) = \emptyset$

On points of type (A), $dF = \begin{pmatrix} df_\sigma & \\ & I \end{pmatrix}$ (product orientation $\partial D^m \times D^n$) so they are all regular, signs are same
On points of type (B), $dF = \begin{pmatrix} I & \\ & df_\tau \end{pmatrix}$

So it looks like

$$d(e_\sigma \otimes e_\tau) = de_\sigma \otimes e_\tau + (-1)^{\deg \sigma} e_\sigma \otimes de_\tau$$

If $\{v_i\}$ is oriented basis for $T\partial D^n$, $\{w_j\}$ is oriented basis for TD^m
Then $(v_i, \dots, w_j)$ is ordered basis for $\partial D^n \times D^m$

Product orientation $D^{??} \times \partial D^n$ and shadowed orientation on S^{n+m-1} differ by $(-1)^n$
Reason:
If x is an outward normal vector for $D^n \times D^m$, then (x, v_i, w_j) should be oriented basis for $T\,\mathbb{R}^{n+m}$
To get an oriented shadows on B, we need (v_i, x, w_j) is an oriented
on (A) set $x = (x_1, 0)$
on (B) set $x = (0, x_2)$ □

Our goal: write $H_*(A \otimes B)$ in terms of $H_*(A)$ and $H_*(B)$

Exercise: If $A \sim B$, then $A \otimes C \sim B \otimes C$

Corollary 26.5
If A_*, B_* are chain complexes defined over a field, then

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$$

Proof

$$\begin{aligned} (A_*, d_A) &\sim (H_*(A), 0) \\ (B_*, d_B) &\sim (H_*(B), 0) \end{aligned}$$

(since we are working over a field)
Then use Exercise □

In general, we always have natural map ($d_A = d_B = 0$)

$$\begin{aligned} H_*(A) \otimes H_*(B) &\to H_*(A \otimes B) \\ [a] \otimes [b] &\to [a \otimes b] \end{aligned}$$

$d(a \otimes b) = da \otimes b + (-1)^{\deg x} a \otimes db$
Exercise: Check this map is well-defined
But this map does not have to be an isomorphism if R is not a field.

Example:
$X = \mathbb{R}P^2$ $\qquad H_*(\mathbb{R}P^2 \times \mathbb{R}P^3; \mathbb{Z}) = ?$

$$H_*(X) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \end{cases} \qquad H_*(X; \mathbb{Z}/2) = \mathbb{Z}/2 \quad * = 0, 1, 2$$

Definition 26.6
Suppose k is a field, define the Poincaré Polynomial

$$\begin{aligned}\mathcal{P}_k(X) &= \sum t^i \dim_k H_i(X;k)\\ \mathcal{P}_k(X)|_{t=-1} &= \chi(X)\end{aligned}$$

Corollary 26.5 $\Rightarrow$ $\mathcal{P}_k(X\times Y) = \mathcal{P}_k(X)\,\mathcal{P}_k(Y)$

Back to our example $X = \mathbb{R}P^2$
$H_*(X\times X; \mathbb{Z}/2)$
$\mathcal{P}_{\mathbb{Z}/2}(X) = 1+t+t^2$
$\mathcal{P}_{\mathbb{Z}/2}(X\times X) = (1+t+t^2)^2 = 1+2t+3t^2+2t^3+t^4$
(??????)

On the other hand

$$H_*(X)\otimes H_*(X) = \begin{cases}\mathbb{Z}\otimes\mathbb{Z} & *=0\\ \mathbb{Z}\otimes\mathbb{Z}/2 = \mathbb{Z}/2\otimes\mathbb{Z} & *=1\\ \mathbb{Z}/2\otimes\mathbb{Z}/2 & *=2\\ 0 & *>2\end{cases}$$

$\Rightarrow$ if $H_*(X\times X)\cong H_*(X)\otimes H_*(X)$ then $H_*(X\times X;\mathbb{Z}/2) = 0$ #

what went wrong:
$C = \mathbb{Z}\xrightarrow{\times 2}\mathbb{Z}$

$$\begin{aligned}\operatorname{Im} d_2 &= \langle(-2,2)\rangle\\ \ker d_1 &= \langle(-1,1)\rangle\end{aligned} \Rightarrow H_1(C\otimes C) = \mathbb{Z}/2$$

Theorem 26.7 (Künneth Theorem)
Suppose A_*, B_* are free (finitely generated) chain complexes over PID R. Then

$$H_n(A\otimes B) = \left(\bigoplus_{i+j=n} H_i(A)\otimes H_j(B)\right)\oplus\left(\bigoplus_{i+j=n-1}\operatorname{Tor}_1(H_i(A), H_j(B))\right)$$

Proof
Check it for short and v.s. chain complex, then use Universal Coefficient Theorem 23.8 and $(A\oplus B)\otimes C = A\otimes C\oplus B\otimes C$ etc. to conclude for general free finitely generated chain complexes.
□

Most interesting case:
$R = \mathbb{Z}$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times a} & \mathbb{Z} \\ {\scriptstyle -b}\downarrow & & \downarrow{\scriptstyle b} \\ \mathbb{Z} & \xrightarrow{\times a} & \mathbb{Z} \end{array}$$

$$A = \mathbb{Z} \xrightarrow{\times a} \mathbb{Z} \qquad B = \mathbb{Z} \xrightarrow{\times b} \mathbb{Z}$$

$\operatorname{Im} d_1 = \langle (a, -b) \rangle \qquad \ker d_1 = \left\langle \frac{a}{\gcd(a,b)}, \frac{-b}{\gcd(a,b)} \right\rangle$

$\Rightarrow \quad H_1 = \mathbb{Z} / \gcd(a,b) = \operatorname{Tor}_1(\mathbb{Z}/a, \mathbb{Z}/b)$

Remark. $C_*(X \times Y) \sim C_*(X) \otimes C_*(Y)$ (Eilenberg-Zilber Theorem)

27 Cup product

$$C^*_{\text{cell}}(X) \otimes C^*_{\text{cell}}(Y) \cong C^*_{\text{cell}}(X \times Y) \qquad \text{gives} \qquad \iota : H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

Definition 27.1

$$\begin{array}{rcl} \Delta : X & \to & X \times X \\ x & \mapsto & (x, x) \end{array}$$

If $\begin{array}{c} a \in H^k(X) \\ b \in H^l(X) \end{array}$, define $\qquad a \smile b = \Delta^*(\iota(a \otimes b)) \in H^{k+l}(X)$

$$\iota(a \otimes b) \in H^k(X) \otimes H^l(X) \subseteq H^{k \times l}(X \times X)$$

$a \smile b$ is called the cup product of a and b

Properties of cup product:

1. $(a \smile b) \smile c = a \smile (b \smile c)$
2. $a \smile b = (-1)^{\deg a \deg b} b \smile a$ (graded commutative)
3. $f : X \to Y \qquad a, b \in H^*(Y) \Rightarrow f^*(a \smile b) = f^*(a) \smile f^*(b)$
4. $1 \in H^0(X)$ is defined by $1(e_x) = 1$ (check this is closed)
   $1 \smile x = x = x \smile 1 \ \forall x$
5. $\iota(a_1 \times b_1), \iota(a_2 \otimes b_2) \in H^*(X \times Y)$
   $\iota(a_1 \otimes b_1) \smile \iota(a_2 \otimes b_2) = (-1)^{\deg b_1 \deg a_2} \iota(a_1 \smile a_2 \otimes b_1 \smile b_2)$

(1), (2), (4) $\Rightarrow H^*(X)$ is a graded-commutative ring with unit
(3) $\Rightarrow$ if $X \sim Y \quad f : X \to Y$, then induces the equivalence $f^* : H^*(Y) \xrightarrow{\sim} H^*(X)$ as a ring

Example:

1. $X = S^n \qquad \deg 1 = 0 \quad \deg a = n$
   $H^*(S^n) = \langle 1, a\rangle$
   $1 \smile a = a \smile 1 = a$
   $0 = a \smile a \in H^{2n}(S^n) = 0$

2. $S^n \times S^n \qquad \deg a = n, \deg b = m, \deg c = m+n$
   $H^*(S^n \times S^n) = \langle 1, a, b, c\rangle$
   $H^*(S^n) \otimes H^*(S^n) = \langle 1\otimes 1, a_n \otimes 1, 1 \otimes a_m, a_n \otimes a_m\rangle$ Property (5) $\Rightarrow$

$$\begin{aligned} a \smile b &= (a_n \otimes 1 \smile 1 \otimes a_m) \\ &= (a_n \smile 1) \otimes (1 \smile a_m)(-1)^0 \\ &= a_n \otimes a_m = c \end{aligned}$$

   So $\smile$ is non-trivial

Definition 27.2
Suppose $\alpha \in C^*_{\text{cell}}(X)$

$$\text{Supp}(\alpha) = \bigcup_{\alpha(e_\tau)\neq 0} \tau$$

Proposition 27.3
Suppose $\text{Supp}\,\alpha \cap \text{Supp}\,\beta = \emptyset$, then $[\alpha] \smile [\beta] = 0$ in $H^*(X)$

Proof
$\Delta^*(\alpha \otimes \beta) = \alpha \smile \beta$
$\text{Supp}(\alpha\otimes\beta) \subseteq \text{Supp}\,\alpha \times \text{Supp}\,\beta \subseteq X \times X$
$\text{Supp}\,\alpha \cap \text{Supp}\,\beta = \emptyset \Rightarrow \text{Supp}\,\alpha \times \text{Supp}\,\beta \cap \Delta = \emptyset$
$\Rightarrow \quad \Delta^*(\alpha\otimes\beta) = 0$ □

Corollary 27.4
$\widetilde{H}^*(X \vee Y) = \widetilde{H}^*(X) \otimes \widetilde{H}^*(Y)$
$a \in \widetilde{H}^*(X), b \in \widetilde{H}^*(Y)$
$\Rightarrow \quad a \smile b = 0$

Proof

□

Corollary 27.5
$S^1 \times S^1 \neq S^1 \vee S^1 \vee S^2$

Proof
LHS: nontrivial $\smile$
RHS: all nontrivial (not with 1) cup products vanish □

Corollary 27.6
$\pi_3(S^2 \vee S^2) \neq 0$

Proof
$S^2 \times S^2 = S^2 \vee S^2 \cup$ 4-cell τ
(LHS cup product is nontrivial)
$f_\tau : S^3 \to S^2 \vee S^2$
f_τ is homotopic to a constant g
Then $S^2 \times S^2 \sim S^2 \vee S^2 \cup_g D^4 = S^2 \vee S^2 \vee S^4$ cup product is trivial □

28 Manifold

A metric space M is a topological n-manifold if every $x \in M$ has an open neighbourhood U_x and a homeomorphism $f_x : U_x \to \mathbb{R}^n$

M is smooth if $f_y \circ f_x^{-1} : f_x(U_x \cap U_y) \to f_y(U_x \cap U_y)$ is differentiable when it is defined

Manifold with boundary: allow
$$f_x : U_x \to \mathbb{R}^{n-1} \times [0, \infty]$$
$x \in \partial M \leftrightarrow H_n(M, M - X) = 0$
$x \in \operatorname{Int} M = M \cdot \partial M \leftrightarrow H_n(M, M - X) = \mathbb{Z}$

M is closed means M is compact, $\partial M = \emptyset$

f is smooth if $f \in C^\infty$

29 de Rham Cohomology

(c.f. Differential Geometry)
M is smooth n-manifold
$C_k^{\text{smooth}}(M) = \langle e_\sigma | \sigma : \Delta^k \to M, \sigma \text{ smooth map}\rangle$
C_k^{smooth} is a subcomplex of $C_k(M)$
$C^*_{\text{smooth}}(M)$ is the dual cochain complex

There is a natural map
$$\begin{array}{rcl} \text{smooth } k\text{-form } \Omega^k(M) & \to & C^k_{\text{smooth}}(M; \mathbb{R}) \\ w & \mapsto & w(e_\sigma) = \int_{\Delta^k} \sigma^*(w) \end{array}$$

Stokes's Theorem says that this map is a chain map

$$\begin{aligned} d\eta(e_\sigma) &= \int_{\Delta^k} \sigma^*(d\eta) \\ \eta(de_\sigma) &= \int_{\Delta^k} d\sigma^*(\eta) \\ &= \int_{\partial\Delta^k} \sigma^*(\eta) = \eta(de_\sigma) \end{aligned}$$

Theorem 29.1 (de Rham)

$$(\Omega^*(M), d) \to C^*_{\text{smooth}}(M;\mathbb{R}) \to C^*(M;\mathbb{R})$$

the following induced maps on homology are isomorphisms

$$H^*(\Omega^*(M), d) \to H^*_{\text{smooth}}(M;\mathbb{R}) \to H^*(M;\mathbb{R})$$

30 Cup Product II

$[w_1] \smile [w_2] = \Delta^*(\iota([w_1] \otimes [w_2]))$
Diagonal map $\Delta : X \to X \times X$

$$\begin{aligned} \iota : \Omega^*(M) \otimes \Omega^*(N) &\to \Omega^*(M \times N) \\ \omega \otimes \eta &\mapsto \omega \wedge \eta \end{aligned}$$

This is not an isomorphism of chain complexes
$[w_1] \smile [w_2] = \Delta^*(w_1 \wedge w_2) \qquad (w_1 \wedge w_2 \in \Omega^*(X \times X))$
Pulling back by Δ exactly sets $x_i = x_i'$
so $[w_1] \smile [w_2] = [w_1 \wedge w_2]$
Now, all basic properties 1-5 from the last cup product section are obvious properties of forms and exterior algebras
Exercise: Saw that $\operatorname{Supp} \alpha \cup \operatorname{Supp} \beta = \emptyset \Rightarrow [\alpha] \smile [\beta] = 0$
In terms of forms, $\operatorname{Supp} \omega = \{x \in M | w|_x \neq 0\}$
This says that if $\operatorname{Supp} \omega \cap \operatorname{Supp} \eta = 0$, then $\omega \wedge \eta \cong 0$
(End of material with relation to Differential Geometry)

31 Handle Decomposition

Cell complexes: Start with some D^0's, attach D^1's, then attach D^2's, etc.
Manfiolds:

Definition 31.1
An n-dimensional k-handle is $D^k \times D^{n-k} = \mathcal{H}_n^k$
The boundary is

$$\partial(\mathcal{H}_n^k) = \partial_1 \smile \partial_2$$

$$\partial_1(\mathcal{H}_n^k) = S^{k-1} \times D^{n-k} \qquad \partial_2(\mathcal{H}_n^k) = D^k \times S^{n-k-1}$$

If $\iota : \partial_1(\mathcal{H}_n^k) \hookrightarrow \partial M$ is an embedding, then $M' = M \cup_\iota \mathcal{H}_n^k$ is an n-manifold with boundary

Note: $M' \sim M \cup_{\iota|_{\partial D^k \times 0}} D^k$
$\partial M' = \partial M - \operatorname{Im} \iota \cup \partial_2 \mathcal{H}$
is obtained by surgery on ∂M

Note: If we want to have homeomorphism type of M, then all of ιmatters, not just $\iota|_{\partial D^k \times 0}$

Fact: Every compact smooth manifold (with or without boundary) can be built out of finitely many handles

32 Morse Theory

n-dimensional k-handle $\mathcal{H}_n^k = D^k \times D^{n-k}$

Example:
0-handle $\cup$ n-handle $= S^n$ (glue by $\mathrm{id}|_{S^{n-1}}$ or a reflection)

Example 2:
$\mathbb{R}\,P^2 =$ 0-handle $\cup$ 1-handle $\cup$ 2-handle

Recall the fact:

Theorem 32.1
Every compact smooth manifold has a finite handle decomposition

Outline of proof
Pick a smooth $f : M \to [0,1]$, $\quad f|_{\partial M} \cong 1$
Say $x \in M$ is a regular point of f if

$$df|_x : TM_x \to T\,\mathbb{R}\,|_x \qquad , \qquad df|_x \neq 0$$

$a \in [0,1]$ is a regular value at f if x is a regular point $\forall x \in f^{-1}(a)$

Implicit function Theorem:
If a is a regular value of f, $M_a := f^{-1}([0,a])$ is a manifold with boundary $f^{-1}(a)$

Strategy: Study how M_a changes as we increase a

Step 1, Claim: If all $c \in [a,b]$ are regular values. Then $f^{-1}([a,b]) \simeq f^{-1}(a) \times [a,b]$
Proof of Claim:

Pick a Riemannion metric on M
If V has $\langle\ ,\ \rangle$, $\quad TM \to T^*M$
$V \leftrightarrow V^*$, $\qquad x \mapsto \langle x, \cdot \rangle$
$df \in \Omega^1(M) \in$ sections of T^*M
$\updownarrow \qquad\qquad \updownarrow$
vector field $\nabla f \qquad$ sections TM
all values of f are regular $\Leftrightarrow \nabla f \neq 0$ in $f^{-1}([a,b])$
$f^{-1}([a,b]) \to f^{-1}(a) \times [a,b]$
$x \mapsto (p(x), f(x))$
Flow of vector field - Df
x flows down to $p(x)$

$\alpha(0) = x$
$\alpha : \mathbb{R} \to M$
$\frac{d\alpha}{dt} = -Df|_{\alpha(t)}$
Define $g(t) = f(\alpha(t))$

$$\frac{dg}{dt} = \nabla f \frac{d\alpha}{dt} = \nabla f(-\nabla f) = -|\nabla f|^2 < -\epsilon$$

$f^{-1}([a,b]) \cong f^{-1}(a) \times [a,b] \Rightarrow M_a \cong M_b$ ■

Step 2: If f is "generic", then all the critical points of f locally look like

$$f(x) = -x_1^2 - x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

for some choice of coordinates near the critical point (This has critical point of index i).

Step 3: Suppose a is a critical value, one critical point $x \in f^{-1}(a)$ with index i, then

$$M_{a+\epsilon} = M_{a-\epsilon} \cup i\text{-handle}$$

Pictures for $n = 3$

□

33 Intersection Numbers

Suppose M^n is a smooth oriented n-manifold
i.e. if you give me an ordered basis for T_xM, either it is compatible with orientation (+1) or not (-1)

Definition 33.1
Submanifolds M_1^k and $M_2^{n-k} \subset M$ intersect transversely if at every $x \in M_1 \cap M_2$

$$T_xM_1 \oplus T_xM_2 = T_*M$$

In picture:

If

1. the intersection is transverse
2. M_1 and M_2 are also oriented

Then there is an intersection sign: sign x at $x \in M_1 \cap M_2$
ordered basis $\{v_i\}$ of TM_1 and $\{w_j\}$ of TM_2
$\{v_i, \dots, w_j\}$ is an ordered basis of TM

$$\operatorname{sign} x = \begin{cases} +1 & \text{if this is compatible with orientation at } T_xM \\ -1 & \text{if not} \end{cases}$$

Definition 33.2
Intersection number of M_1 and M_2 is

$$M_1 \cdot M_2 = \sum_{x \in M_1 \cap M_2} \operatorname{sign} x$$

Notice: With no orientations, $M_1 \cdot M_2 = |M_1 \cap M_2| \in \mathbb{Z}/2$

34 Handles and $C_*^{\text{cell}}(M)$

Question: How to compute $de_{\mathcal{H}}$?

$$\begin{aligned} \text{Handle decomposition of } M &\to \text{Cell decomposition } X \sim M \\ D^k \times D^{n-k} &\to \text{disk } D^k \\ \mathcal{H}_n^k &\to e_{\mathcal{H}} \in C_k^{\text{cell}}(X) \end{aligned}$$

Intersection number:
$M_1, M_2 \subseteq M^n$ closed oriented
M_1 intersects M_2 transversely

$$(M_1 \cdot M_2)_M = \sum_{x \in M_1 \cap M_2} \operatorname{sign} x$$

with $\operatorname{sign} x = +1$ if {basis for T_xM_1, basis for T_xM_2} is ordered basis for T_xM

$\mathcal{H}_n^k = D^k \times D^{n-k}$
$\mathcal{A}(\mathcal{H}) = S^{k-1} \times 0 \subseteq \partial_1 \mathcal{H}$ attaching sphere
$\mathcal{B}(\mathcal{H}) = 0 \times S^{n-k-1} \subseteq \partial_2 \mathcal{H}$ belt sphere

Lemma 34.1
Suppose $\mathcal{H}_n^{k+1}$ is a $k+1$ handle in M, $\mathcal{H}'$ is a k-hanlde in M. Then the coefficient of $e_{\mathcal{H}'}$ in $de_{\mathcal{H}}$ is $(\mathcal{A}(\mathcal{H}) \cdot \mathcal{B}(\mathcal{H}))_{\partial_2(\mathcal{H}')}$

Proof
Attaching map

$$\begin{array}{rcl} \iota : \mathcal{A}(\mathcal{H}) & \to & \partial M_0 \supset \partial_2 \mathcal{H}' = D^k \times S^{n-k-1} \\ x & \mapsto & (f(x), g(x)) \end{array}$$

Cell complex

$$\begin{array}{rcl} \mathcal{A}(\mathcal{H}) & \to & X_0 \\ x & \mapsto & f(x) \end{array}$$

If $0 \in D^k \leftrightarrow e_{\mathcal{H}}$ is a regular value for f
Coefficient of $e_{\mathcal{H}'}$ in $de_{\mathcal{H}}$ is

$$\sum_{x \in f^{-1}(0)} \operatorname{sign} df|_x = \sum_{x \in \mathcal{A}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})} \operatorname{sign} df|_x$$

$\mathcal{A}(\mathcal{H})$ transverse to $\mathcal{B}(\mathcal{H})$ $\Rightarrow$ 0 is a regular value.

$$d\iota = \begin{pmatrix} df \\ dg \end{pmatrix} \begin{array}{l} T(D^k) \\ T\mathcal{B}(\mathcal{H}') \end{array}$$

$$\operatorname{sign} df|_x = \operatorname{sign} x = \det \begin{pmatrix} df & \\ dg & I \end{pmatrix}$$

$$\Rightarrow \qquad \sum_{x \in f^{-1}(0)} \operatorname{sign} df|_x = \sum_{x \in \mathcal{A}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})} \operatorname{sign} x = \mathcal{A}(\mathcal{H}) \cdot \mathcal{B}(\mathcal{H})$$

Turn a handle decomposition "upside-down"

$$\mathcal{H}_n^k = D^k \times D^{n-k} \simeq D^{n-k} \times D^k = \mathcal{H}_n^{n-k}$$

This reverses roles of $\mathcal{A}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$
In Morse theory, this corresponds to replacing the Morse function

$$\begin{array}{rcl} f & \mapsto & -f \\ -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 & \mapsto & x_1^2 + \cdots x_i^2 - x_{i+1}^2 - \cdots - x_n^2 \\ \text{index } i & \mapsto & \text{index } n-i \end{array}$$

So a handle decomposition of M actually gives me 2 different cell decomposition

$$\begin{array}{ccccc} X & \sim & M & \sim & \overline{X} \\ & & \cup| & & \\ e_{\mathcal{H}} \in C_k^{\text{cell}}(X) & \leftarrow & \mathcal{H}_n^k & \rightarrow & \overline{e}_{\mathcal{H}} \in C_{n-k}^{\text{cell}}(\overline{X}) \end{array}$$

□

Theorem 34.2 (Poincaré Duality version 1)
If M is a close n-manifold,

$$H_*(M; \mathbb{Z}/2) \cong H^{n-*}(M; \mathbb{Z}/2)$$

Proof
Consider $C_*^{\text{cell}}(X)$ and $C_*^{\text{cell}}(\overline{X})$
$\mathcal{H}$ is $k+1$-hanlde, $\mathcal{H}'$ is k-handle

$$\begin{aligned} \text{The coefficients of } e_{\mathcal{H}} \text{ in } de_{\mathcal{H}} &\cong \mathcal{A}(\mathcal{H}) \cdot \mathcal{B}(\mathcal{H}') \\ &\cong \mathcal{B}(\overline{\mathcal{H}}) \cdot \mathcal{A}(\overline{\mathcal{H}'}) \\ &= \text{coefficient of } \overline{e_{\mathcal{H}}} \text{ in } d\overline{e_{\mathcal{H}'}} \\ &= \text{coefficient of } (\overline{e_{\mathcal{H}'}})^* \text{ in } d(\overline{e_{\mathcal{H}}}^*) \in C_{\text{cell}}^{n-k}(\overline{X}; \mathbb{Z}/2) \end{aligned}$$

i.e.

$$\begin{array}{rcl} C_*^{\text{cell}}(X; \mathbb{Z}/2) & \cong & C_{\text{cell}}^{n-*}(\overline{X}) \\ e_{\mathcal{H}} & \rightarrow & (\overline{e_{\mathcal{H}}})^* \end{array}$$

□

Corollary 34.3
If M is closed connected n-manifold, either $H^n(M) = \mathbb{Z}$ or $H^n(M) = 0$

Proof
$H^n(M; \mathbb{Z}/2) \cong H_0(M; \mathbb{Z}/2) = \mathbb{Z}/2$
$H_n(M)$ has no torsion since then $H^{n+1}(M)$ has torsion on X
$\Rightarrow \quad H_n(M) = \mathbb{Z}^k$ some k
$\Rightarrow \quad H^n(M; \mathbb{Z}/2) = (\mathbb{Z}/2)^k \oplus$ stuff from $H_{n-1}(M)$ □

Corollary 34.4
If M is closed n-manifold, n odd, then $\chi(M) = 0$

Proof
By Universal Coefficient Theorem 23.8

$$\begin{aligned} \dim_{\mathbb{Z}/2} H_k(M; \mathbb{Z}/2) &= \dim_{\mathbb{Z}/2} H^k(M; \mathbb{Z}/2) \\ &= \dim_{\mathbb{Z}/2} H_{n-k}(M; \mathbb{Z}/2) \end{aligned}$$

$$\begin{aligned} \chi(M) &= \sum (-1)^k \dim_{\mathbb{Z}/2} H_k(M; \mathbb{Z}/2) \\ &= \sum (-1)^k \dim_{\mathbb{Z}/2} H_{n-k}(M; \mathbb{Z}/2) \\ &= (-1)^n \sum (-1)^{n-k} \dim_{\mathbb{Z}/2} H_{n-k}(M; \mathbb{Z}/2) \\ &= (-1)^n \chi(M) = -\chi(M) \qquad \text{since } n \text{ odd} \end{aligned}$$

□

What happens if M has boundary?
Get a cell complex $X \sim M$ by collapsing $\mathcal{H}_n^k \to D^k$

Turn handle decomposition upside-down
This amounts to starting with $\partial M \times [0, \epsilon]$ and adding handles to get no boundary on top
Duplicate:
Start with $T^2 \times [0, \epsilon]$, add a 2-handle, then add a 3-handle
Dual complex to $C_*^{\text{cell}}(X)$ will compute $C^*_{\text{cell}}(M, \partial M)$

Theorem 34.5 (Poincaré Duality version 2)
If M is a compact manifold with boundary

$$\begin{aligned} H_*(M; \mathbb{Z}/2) &\cong H^{n-*}(M, \partial M; \mathbb{Z}/2) \\ H^*(M; \mathbb{Z}/2) &\cong H_{n-*}(M, \partial M; \mathbb{Z}/2) \end{aligned}$$

Corollary 34.6
If M is an odd dimensional manifold with boundary, $\chi(M) = \frac{1}{2}\chi(\partial M)$

Proof
Form $DM = M \cup_{\partial M} M$ is closed
$\chi(DM) = \chi(M) + \chi(M) - \chi(\partial M) = 0$ □

Corollary 34.7
$\mathbb{R}P^2$ does not bound any compact 3-manifold Y

Proof
Otherwise, we would have $\chi(Y) = \frac{1}{2}\chi(\mathbb{R}P^2) = \frac{1}{2}$ # □

Theorem 34.8 (Poincaré Duality version 3)
If n is a closed orientable n-manifold, then

$$H_*(M) \cong H^{n-*}(M)$$

Proof
This is mostly the same as with $\mathbb{Z}/2$ coefficients, but now we need to keep track of orientation.
$\mathcal{H}$ is a $k+1$-handle $= D^{k+1} \times D^{n-k-1}$
$\mathcal{H}'$ is a k-handle $= D^k \times D^{n-k}$
Coefficient of $e_{\mathcal{H}'}$ in $de_{\mathcal{H}} = (\mathcal{A}(\mathcal{H})\mathcal{B}(\mathcal{H}'))_{\partial_2 \mathcal{H}'}$
Orientations: To define C_*^{cell} we picked orientations on D^{k+1} (orients $\mathcal{A}(\mathcal{H})$) and D^k
Pick an orientation on D^{n-k}
It induces orientations on $\mathcal{H}'_k$ (on $\partial_2 \mathcal{H}'_k$) and on $\mathcal{B}(\mathcal{H}')$
Sign of $(\mathcal{A}(\mathcal{H}) \cdot \mathcal{B}(\mathcal{H}'))_{\partial_2 \mathcal{H}'}$ does not depend on orientation we picked on D^{n-k}
In the dual cellular chain complex, look at $(\mathcal{B}(\mathcal{H}') \cdot \mathcal{A}(\mathcal{H}))_{\partial_1 \mathcal{H}}$
Since M orientable, have

$$\begin{aligned} (\mathcal{B}(\mathcal{H}') \cdot \mathcal{A}(\mathcal{H}))_{\partial_1 \mathcal{H}} &= (\mathcal{B}(\mathcal{H}') \cdot \mathcal{A}(\mathcal{H}))_{\partial_0 \overline{M_0}} \\ &= \pm(\mathcal{B}(\mathcal{H}') \cdot \mathcal{A}(\mathcal{H}))_{\partial M_0} \\ &= \pm(\mathcal{B}(\mathcal{H}') \cdot \mathcal{A}(\mathcal{H}))_{\partial_2 \mathcal{H}'} \end{aligned}$$

where M_0=all handles up to dimension k, $\overline{M_0}$=all handles of dimension k
So now we see the coefficient of $(\overline{e}_{\mathcal{H}'})^*$ in $d(\overline{e}^*_{\mathcal{H}})$ is (sign depends on k) $\pm$ coefficient of $e_{\mathcal{H}'}$ in $de_{\mathcal{H}}$ $\square$

35 Cup Product Pairing

k=field ($\mathbb{Q}$ or $\mathbb{Z}/p$)

Definition 35.1
M is orientable over k if there is a class $[M] \in H_n(M;k)$ s.t.

$$\iota : (M, \emptyset) \rightarrow (M, M-x) \qquad (x \in M)$$

with $\iota_*([M])$ generates $H_n(M, M-x; k) \cong k$

If $k = \mathbb{Z}/2$, M is always orientable over k
If $k \neq \mathbb{Z}/2$, M is oreintable over $k \Leftrightarrow M$ is orientable over $\mathbb{Z}$

The choice of $[M]$ defines an orientation on $H_n(M;k)$, $[M]$ is called fundamental class

Bilinear Pairing:

$$\begin{array}{rcl} H^l(M;k) \times H^{n-l}(M;k) & \rightarrow & k \\ (a,b) & \mapsto & (a \smile b)[M]s \end{array}$$

Theorem 35.2 (Poincaré Duality version 4)
If M is orientable over k. Then

$$\langle \ , \ \rangle : H^l(M;k) \times H^{n-l}(M;k) \rightarrow k$$

is nondegenerate, i.e. if $a \neq 0; a \in H^l(M;k)$, there is some $b \in H^{n-l}(M;k)$ so that $\langle a, b\rangle \neq 0$

Notice: cup product pairing define a mmap

$$\begin{array}{rcl} PD_k : H^l(M;k) & \rightarrow & (H^{n-l}(M;k))^* = H_{n-l}(M;k) \\ a & \mapsto & \begin{array}{rcl} \phi_a : H^{n-l}(M;k) & \rightarrow & k \\ b & \mapsto & \langle a, b\rangle \end{array} \end{array}$$

Non-degeneracy of pairing $\Leftrightarrow PD_k$ is an isomorphism

36 Applications of Poincaré Duality

36.1 Cohomology ring of $\mathbb{C}P^n$

$$\begin{array}{rcl} H^*(\mathbb{C}P^1) & = & H^*(S^2) = \langle 1, x\rangle = \mathbb{Z}[X]/X^2 = 0 \\ H^*(\mathbb{C}P^1) & = & \begin{cases} \mathbb{Z} & * = 0,2,4 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

$$H^2(\mathbb{C}P^2) = \langle x\rangle \cong \mathbb{Z}$$
$$H^4(\mathbb{C}P^2) = \langle a\rangle = \mathbb{Z}$$

Claim: $x \cup x = \pm a$

Proof
$x \cup x = ma$. If $m \neq \pm 1$, take $p|m$
Look at $H^*(\mathbb{C}P^2; \mathbb{Z}/p)$
$H^2(\mathbb{C}P^2; integer/m) = \langle x \rangle = \mathbb{Z}/p$ so pairing is not x
but $x \cup x = ma = 0$ in $\mathbb{Z}/m$ nondegenerate □

Proposition 36.1
$H^*(\mathbb{C}P^n) = \mathbb{Z}[X]/X^{n+1} = 0$ when $\langle x \rangle = H^2(\mathbb{C}P^n)$

Proof
Induct on n. We have already done $n = 1, 2$
$\iota : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^n$
$H^*(\mathbb{C}P^{n-1}) = \mathbb{Z}[Y]/Y^n \quad \langle y \rangle = H^2(\mathbb{C}P^{n-1})$
$\iota^* : H^2(\mathbb{C}P^n) \xrightarrow{\sim} H^2(\mathbb{C}P^{n-1}) \quad \iota^*(x) = y$
$\iota_* : H_2(\mathbb{C}P^{n-1}) \xrightarrow{sim} H_2(\mathbb{C}P^n)$
So $\iota^*(x^{n-1}) = \iota^*(x)^{n-1} = y^{n-1}$ generates $H^{2n-2}(\mathbb{C}P^{n-1})$
But $\iota^* : H^2(\mathbb{C}P^n) \xrightarrow{\sim} H^2(\mathbb{C}P^{n-1})$
$\Rightarrow \quad x^{n-1}$ generates $H^{2n-2}(\mathbb{C}P^n)$
More generally, x^k generates $H^{2k}(\mathbb{C}P^n)$
So we just need to check that $x^n = x \cup x^{n-1}$ generates $H^{2n}(\mathbb{C}P^n)$
This follows exactly as for $\mathbb{C}P^2$ □

Remark. Same argument (with $\mathbb{Z}/2$ coefficients) shows that

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[X]/X^{n+1} \quad \langle x \rangle = H^1(\mathbb{R}P^n; \mathbb{Z}/2)$$

Corollary 36.2
$\pi_3(S^2) \neq 0$

Proof
$\mathbb{C}P^2 = S^2 \cup_f D^4$
$f : S^3 \to S^2$
$(z, w) \mapsto [z, w]$ where $|z|^2 + |w|^2 = 1$, z/w in Riemann sphere
If f is homotopic to a constant map g, then

$$(x \cup x \neq 0) = \mathbb{C}P^2 S^2 \cup_f D^4 \sim S^2 \cup_g D^4 = S^4 \vee S^4$$

nontrivial cup products □

Definition 36.3
Suppose $f : S^{4n-1} \to S^{2n}$
$X = S^{2n} \cup_f D^{4n}$

$$H_*(X) = \begin{cases} \mathbb{Z} & * = 0, 2n, 4n \\ 0 & \text{otherwise} \end{cases}$$

$H^{2n}(X) = \langle x \rangle \qquad H^{4n}(X) = \langle a \rangle$
$x \cup x = ka \quad k \in \mathbb{Z}$
k only depends on homotopy class of $f \in \pi_{4n-1}(S^{2n})$
$k = H(f) = H([f])$ is called the Hopf invariant of $[f]$
If $f : S^3 \to S^2$ is the Hopf map, $H(f) = 1$

Exercise: $H([f] + [g]) = H([f]) + H([g])$, i.e. $H : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$ homomorphism

Corollary 36.4
$\pi_3(S^2)$ is infinite

(Exercise: Prove this)

Question: For which n are there $f \in \pi_{n-1}(S^{2n})$ with (1) $H(f) \neq 0$? (2)$H(f) = 1$? (3) $f : \pi_7(S^4)$ with $H(f) = 1$?

Definition 36.5
S^3 = unit quaterions

$$\mathbb{H}\, P^n = \left\{ \overrightarrow{q} = (q_1, \ldots, q_n) \middle| q_i \in \mathbb{H}\,,\ \sum |q_i|^2 = 1 \right\} / \sim$$

where $\overrightarrow{q} \sim w\overrightarrow{q}$ for w, a unit norm quaterion

this has cells of dimension 0,4,8,...,4n
Attaching map $f : S^7 \to S^4$ that defines

$$\mathbb{H}\, P^2 = S^4 \cup_f D^8$$

36.2 Borsuk-Ulam Theorem

Theorem 36.6 (Borsuk-Ulam Theorem)
Suppose $f : S^n \to \mathbb{R}^n$ Then there is some $p \in S^n$ with $f(p) = f(-p)$

Proof
Suppose not. Then can define

$$\begin{aligned} g : S^n &\to S^{n-1} \\ p &\mapsto \frac{f(p) - f(-p)}{|f(p) - f(-p)|} \end{aligned}$$

$g(p) = g(-p) \Rightarrow g$ induces

$$\begin{aligned} G : \mathbb{R}\, P^n &\to \mathbb{R}\, P^{n-1} \\ \widetilde{x} &\mapsto \widetilde{g(x)} \end{aligned}$$

the is well-defined

Claim: $G_* : H_1(\mathbb{R}\, P^n; \mathbb{Z}/2) \xrightarrow{\sim} H_*(\mathbb{R}\, P^{n-1}; \mathbb{Z}/2)$

Proof of Claim:
$\pi : S^n \to \mathbb{R}\, P^n$
$H_1(\mathbb{R}\, P^n)$ is generated by $[\pi(\gamma)]$ where $\gamma : [0,1] \to S^n$ has $\gamma(0) = p, \gamma(1) = -p$
$G_*([\pi(\gamma)]) = [\pi(g(\gamma))]$
So $g(\gamma(0))$ and $g(\gamma(1))$ are antipedal points in S^{n-1} $\therefore$ $[\pi(g(\gamma))]$ generates $H_1(\mathbb{R}\, P^n)$ ■

$$\begin{array}{ccc} H^*(\mathbb{R}\, P^{n-1}; \mathbb{Z}/2) & \to & H^*(\mathbb{R}\, P^n; \mathbb{Z}/2) \\ \| & & \| \\ \mathbb{Z}/2[X]/X^n = 0 & & \mathbb{Z}/2[Y]/Y^{n+1} = 0 \end{array}$$

$G^*(x) = y$
$\Rightarrow \quad G^*(x^n) = G^*(x)^n = y^n \neq 0$
but $G^*(0) = 0$ #

□

36.3 Intersection Numbers

Defining equation for PD:

$$(a \smile PD(x))[M] = a(x)$$

where $a \in H^l(M;k)$
$x \in H_l(M;k)$
k a field
$PD(x) = H^{n-l}(M;k)$

Note $PD(x) \smile a = (-1)^{l(n-l)} a \smile PD(x)$
$x = [N] \qquad N \subset M$ is closed orientabtle submanifold

de Rham cohomology

$$\begin{array}{rcl} [w] \smile PD(x)[M] & = & [w](x) \\ \| & & \\ \int_M [w] \smile PD(N) & = & \int_N w \end{array}$$

How can this happen?
Easy way: $P(N)$ is represented by a closed $n-l$ form η which supported near N
This actually happens:
In local coordinates near a point of $x_1, \ldots, x_n$

(????) $r = \sqrt{\sum_{i=l+1}^n |x_i|^2}$
Pick η so that $\eta = f(r) dx_{l+1} \wedge \ldots dx_n$
$\int_{\mathbb{R}^{n-l}} f = 1$

Important property:

$$\begin{array}{rcl} \iota : \mathbb{R}^{n-l} & \to & \mathbb{R}^n \\ (y_1 \ldots y_{n-l}) & \mapsto & (x_1, \ldots, x_l, y_1, \ldots y_{n-l}) \end{array}$$

(x_i) =fixed point in N
should have $\iota^*(\eta)$ = volume form on $\mathbb{R}^{n-l}$ supported near 0.
Fact: If $N_1^{l_1}, N_2^{l_2}$ are closed oriented transverse submanifolds of M, $l_1 + l_2 = n$
Then $PD(N_1) \smile PD(N_2)[M] = N_1 \cdot N_2 = \sum_{x \in N_1 \cap N_2} \operatorname{sign} x$

Idea why this is true:
In local cooridnates, $N_1 = \{(x_1, \ldots, x_l, 0, \ldots, 0)\}$, $N_2 = \{(0, \ldots, 0, y_{l+1}, \ldots, y_n)\}$
$PD(N_1) \smile PD(N_2) = f_1(r_1) f_2(r_2) = \operatorname{vol}(N_2) \wedge \operatorname{vol}(N_1)$
r_i = distance from N_i

Examples:
$M = \mathbb{C}P^2$
$[\mathbb{C}P^1]$ generates $H_2(\mathbb{C}P^2)$
$PD([\mathbb{C}P^1]) \smile PD([\mathbb{C}P^1])$ generates $H^4(\mathbb{C}P^2)$
$\Rightarrow [\mathbb{C}P^1] \cdot [\mathbb{C}P^1] = 1$

Note on orientations.
If M is a complex manifold, local coordinate $z_i = x_i + y_i$, it has a canonical orientation coming from the complex structure

$$(dx_1 \wedge dy_1) \wedge (dx_2 \wedge dy_2) \cdots (dx_n \wedge dy_n)$$

<u>Exercise</u>: Two complex submanifolds that intersect transversely have sign $x = +1$ at each intersection point

(the above picture) works for smooth submanifolds, not for complexes.

Similarly, $[\mathbb{C}P^k] \cdot [\mathbb{C}P^{n-k}] = 1$ in $\mathbb{C}P^n \Leftrightarrow (x^{n-k}) \smile x^k = x^n$

Definition 36.7
Suppose X^k is a smooth irreducible variety in $\mathbb{C}P^n$

$$[X^k] = \alpha[\mathbb{C}P^k]$$

since $[\mathbb{C}P^k]$ generates $H_{2k}(\mathbb{C}P^n)$
$\alpha = \deg X$

Theorem 36.8 (Bezout's Theorem)
$\deg(X \cap Y) = \deg X \deg Y$
Topologically, this says that

$$((\deg(X))x^{n-k_1}) \smile ((\deg Y)x^{n-k_2}) = \underbrace{(\deg X \deg Y)}_{\deg(X \cap Y)} \cdot x^{2n-k_1-k_2}$$

Corollary 36.9
There is no homeomorphism $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ with $f([\mathbb{C}P^2]) = -[\mathbb{C}P^2]$
i.e. no orientation reversing homeomorphism of $\mathbb{C}P^2$

Proof
x generates $H^2(\mathbb{C}P^2)$
$f^*(x) = \pm x$
$f^*(x \smile x) = f^*(x) \smile f^*(x) = (\pm x) \smile (\pm x) = x \smile x$
But $f^*(x \smile x)[\mathbb{C}P^2] = (x \smile x) f_*([\mathbb{C}P^2]) = (x \smile x)(-[\mathbb{C}P^2]) = 1$
However $f^*(x \smile x)[\mathbb{C}P^2] \neq -1$ # □

37 Fibrations

Definition 37.1
A <u>locally trivial fibration</u> with fibre F is a map $\pi : E \to B$ s.t. every $b \in B$ has an open neighbourhood U and a homeomorphism

$$f : \pi^{-1}(U) \to U \times F$$

so that the diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{f} & U \times F \\ \pi \downarrow & & \downarrow \pi 1 \\ U & \xrightarrow{\text{id}} & U \end{array}$$

We call B the base space and E the total space

(The idea is it locally looks like a product)

Examples:

1. $B \times F \to B$ trivial fibration
2. Any covering space $\pi : \widetilde{Y} \to Y$
   $F=$ disjoint union of points
3. Mobius band
   $\pi : M \to S^1 \qquad F = [-1, 1]$
4. Hopf Map

$$\begin{array}{rcl} \pi : S^3 & \to & S^2 = \mathbb{C} \cup \{\infty\} \\ (z, w) & \mapsto & z/w \in \mathbb{C} \cup \{\infty\} \end{array}$$

   $U_1 = \mathbb{C} \quad (|z|^2 + |w|^2 = 1)$
   $\pi^{-1}(U_1) = \{(z, w) | w \neq 0\}$

$$\begin{array}{rcl} \pi^{-1}(U_1) & \to & U_1 \times S^1 \\ (z, w) & \mapsto & \left(\dfrac{z}{w}, \dfrac{w}{\|w\|} \right) \end{array}$$

   $U_2 = \mathbb{C} \cup \{\infty\} - 0$

5. More generally, we have fibrations

$$\begin{array}{ccccccccc} S^1 & \longrightarrow & S^{2n+1} & \{(z_0, \dots, z_n)\} & S^3 & \longrightarrow & S^{4n+1} & F \longrightarrow & E \\ & & \downarrow & \downarrow & & & \downarrow & & \downarrow \\ & & \mathbb{C}\,P^n & [z_0 : \dots : z_n] & & & \mathbb{H}\,P^n & & B \end{array} \text{ means}$$

   E fibres over B with fibre F

6. Lots of interesting fibrations can be built using Lie groups

$$\begin{array}{rcl} \pi : SO(n) & \to & S^{n-1} \\ A & \mapsto & Ae_1 \qquad e_1 = (1, 0, \dots, 0)^T \end{array}$$

$$\begin{array}{ccc} SO(n-1) & \longrightarrow & SO(n) \\ & & \big\downarrow \pi \\ & & S^{n-1} \end{array} \quad \text{fibres are cosets of the subgroup } SO(n-1)$$

Definition 37.2
Pullback: If $\pi : E \to B$ and $g : X \to B$
I can build a new fibration

$$\begin{array}{ccc} g^*(E) & = & \{(x, e) \in X \times E | g(x) = \pi(e)\} \\ \pi' \downarrow & & \downarrow \\ X & & x \end{array}$$

check:
If $\pi : E \to B$ is trivial over U
$f : \pi^{-1}(U) \to U \times F$
$\pi' : g^*(E) \to X$ is trivial over $g^{-1}(U)$

$$\begin{array}{rcl} \pi'^{-1}(g^{-1}(U)) & \to & g^{-1}(U) \times F \\ (x, e) & \mapsto & (x, f(e)) \end{array}$$

Transition Functions:
$\pi : E \to B$ is locally trivial over U_1, U_2

$$\begin{array}{rcl} f_1 : \pi^{-1}(U_1) & \to & U_1 \times F \\ f_2 : \pi^{-1}(U_2) & \to & U_2 \times F \end{array}$$

This commutes

$$\begin{array}{ccccc} U_1 \cap U_2 \times F & \xleftarrow{f_2} & \pi^{-1}(U_1 \cap U_2) & \xrightarrow{f_1} & U_1 \cap U_2 \times F \\ \downarrow & & \downarrow \pi & & \downarrow \\ U_1 \cap U_2 & \xleftarrow{\text{id}} & U_1 \cap U_2 & \xrightarrow{\text{id}} & U_1 \cap U_2 \end{array}$$

Get

$$\begin{array}{rcl} f_{12} = f_1 f_2^{-1} : (U_1 \cap U_2) \times F & \xrightarrow{\sim} & U_1 \cap U_2 \times F \\ (b, x) & \mapsto & (b, \overline{f_{12}}(b, x)) \end{array}$$

For fixed b, $x \mapsto f_{12}(bx)$ $(F \xrightarrow{\sim} F)$
In other words f_{12} defines a map (Let $U_{12} = U_1 \cap U_2$)

$$\phi_{12} : U_{12} \to \text{Homeo}(F)$$

ϕ_{12} is a transition function
On $U_1 \cap U_2 \cap U_3$
$f_{12} f_{23} = f_{13}$
$\phi_{12} \phi_{23} = \phi_{13}$ Cocycle condition

Example

1. $\pi : M \to S^1$

(????)

2. $S^3 \to S^2 \qquad (z, w) \mapsto (z/w, w/\|w\|)$
   $U_1 = \{w \neq 0\} \qquad U_2 = \{z \neq 0\}$
   Son $S^1 \subset U_1 \cap U_2$, transition function is $\lambda \mapsto \phi_\lambda$

$$\begin{array}{rcl} \phi_\lambda : S^1 & \to & S^1 \\ z & \mapsto & \lambda z \end{array}$$

Remark:
Often, we have a Lie group G and a homeomorphism $\alpha : G \to \text{Homeo}(F)$
If transition functions are contained in the image of α, we say $\pi : E \to B$ is a bundle with structure group G

Example:
Vector bundles
$F = \mathbb{R}^k$
$GL_k(\mathbb{R}) \to \text{Homeo}(\mathbb{R}^k)$
Vector bundle with metric $O(k)$
Oriented vector bundle $SO(k)$

38 Fibrations and Homotopies

Theorem 38.1
Suppose $\pi : E \to B$ is locally trivial fibration and $f, g : X \to E$ with $f \sim g$
Then $f^*(E) \simeq g^*(E)$ in the sense that diagram commutes:

Corollary 38.2
If B is contractible, $\pi : E \to B$ locally trivial fibration, then $E \simeq B \times F$

Proof
$\text{id}_B \sim g$
$g(b) \cong p$
$\Rightarrow \quad E = (\text{id}_B)^*(E) \simeq g^*(E) = B \times F$ $\square$

Application:
Vector bundles over spheres
$B = S^n = D^n_N \cup D^n_S$ (Nothern and Sourthern hemisphere)
If $V \to B$ is a vector bundle, then

$$\begin{array}{rcl} V|_{D^n_N} & \simeq & D^n_N \times \mathbb{R}^k \\ V|_{D^n_S} & \simeq & D^n_S \times \mathbb{R}^k \end{array}$$

Transition function $f : S^{n-1} \to O(k)$

Given such f, we ca construct a vector bundle V_f as

$$\begin{array}{rcc} (D^n_N \times \mathbb{R}^k \sqcup D^n_S \times \mathbb{R}^k)/\sim \cup & & \cup \\ S^{n-1} \times \mathbb{R}^k & & S^{n-1} \times \mathbb{R}^k \\ & (x,v) \to (x, f(x)v) & \end{array}$$

<u>Example</u>:

1. If $f \sim g$, then $V_f \simeq V_g$
   So vector bundles over S^n are determined by $\pi_{n-1}(O(k)) = \pi_{n-1}(GL(k))$
2. $n = 1$ $\quad \pi_0(O(k)) = \{\pm 1\}$ 2 components of $O(k)$
   2 different $\mathbb{R}^k$ bundles over S^1
   $k = 1$ $\quad$ trivial bundle, Mobius band
   general k $\quad$ trivial bundle, non-orientable bundle $\mathbb{R}^k \times [0,1]/\sim$
   $(v, 0) \sim (Av, 1)$ $\quad \det A = -1$
3. $n = 2, k - 2$
   Real 2-plane bundles over S^2
   $\pi_1(O(2)) = \pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$
   (<u>Exercise</u>: Hopf bundle $\leftrightarrow$ generator of $\pi_1(SO(2))$)
   $n = 2, k = 3$
   $\pi_1(SO(3)) = \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$

39 Spectral Sequences

Lemma 39.1 (Cancellation Lemma)
(see picture)

is chain homotopy equivalent to

Proof

Exercise: $fg = 1_{E'} \qquad gf = 1_E + dH + Hd$ where $H : E \to E$ is 0 except for $H : A \oplus C \to A \oplus B, \ \ (a, c) \to (a, 0)$

Observation: If C_* is a chain complex over a field, then by UCT, C_* has the following form:

By repeatedly cancelling, can see that $(C_*, d) \sim (H_*, 0)$ □

Definition 39.2
(C_*, d) is a filtered chain complex means

$$C_i = \bigoplus_{k \in \mathbb{Z}} C_i(k)$$

for each $k =$ filtration grading s.t. $d(C_*(k)) \in \bigoplus_{j \le k} C_*(j)$
$d = d_0 + d_1 + d_2 + \cdots$
$d_j : C_*(k) \to C_{*-1}(k - j)$

$d^2 = 0 \Rightarrow (d_0 + d_1 + d_2 + \cdots)^2 = 0$

$$\begin{cases} d_0^2 = 0 \\ d_0 d_1 + d_1 d_0 = 0 \\ d_0 d_2 + d_1^2 + d_2 d_0 = 0 \\ \vdots \end{cases}$$

i.e. $(C_*, d_0) \cong \bigoplus (C_*(k), d_0)$ is a chain complex, called the associated graded complex

Notation: $(E_0, d_0) = (C_*, d_0)$

Set $E_1 = H_*(E_0, d_0) = H_*(C_*, d_0)$
In fact

Proposition 39.3
$(C_*, d) \sim (E_1, d(1))$ and E_1 is still filtered, $d(1) = d_1(1) + d_2(1) + \cdots$

Proof
Cancel all the differentials in (C_*, d_0) but do the cancellation in (C_*, d)

At each cancellation, the property of being filtered is preserved cancel until there is no nontrivial d_0 left $\square$

Now look at $(E_1, d(1))$
$d_j(1)$ lower filtration by j
$d(1) = (d_1(1) + d_2(1) + \cdots)^2$

$$\Rightarrow \quad \begin{cases} (d(1))^2 = 0 \\ (d_1(1))^2 = 0 \\ d_1(1)d_2(1) + d_2(1)d_1(1) = 0 \cdots \end{cases}$$

$\Rightarrow \quad (E_1, d_1(1))$ is a chain complex

Now let $E_2 = H_*(E_1, d_1(1))$
Cancel in the chain complex $(E_1, d(1))$ to get $(E_1, d(1)) \sim (E_2, d(2))$
$d(2) = d_2(2) + d_3(2) + \cdots$
and now keep on going
If C_* is finitely generated, we eventually get $(E_N, d(N)) = (E_N, 0) \cong H_*(C)$
say E_* converges to $H_*(C)$ at E_N

Note:
Even if $d = d_0 + d_1$, $d(n)$ can be non-zero for large n

40 Fibrations and Spectral Sequences

Theorem 40.1 (Leray-Serre Spectral Sequence)
Suppose $F \to E \to B$ is locally trivial fibration.
Then there is a spectral sequence $(E^i, d(i))$ with $E^2 = H^*(B; \phi)$
$\phi : \pi_1(B) \to \mathrm{Aut}(H^*(F))$ is monodromy.
In particular, if $\pi_1(B) = 1$, $E^2 = H^*(B) \otimes H^*(F)$

Examples:
$E = S^n \times S^m \qquad (B = S^n, F = S^m)$

Example 2:
$S^1 \to S^3 \to S^2$

Homological grading on $H^*(E)$ is $i+j$
Filtration grading is i

Example 3:
$S^1 \to S^{2n+1} \to \mathbb{C}P^n$

41 Leray-Serre Spectral Sequence

locally trivial fibration $F \to E \xrightarrow{\pi} B$
Monodromy of $E \xrightarrow{\pi} B$:
$\phi : \pi_1(B) \to \mathrm{Aut}(H_*(F))$
$\gamma \mapsto (\phi_\gamma)_*$

$\phi : S^1 \to B$, $S^1 \to \phi^*(E) \to S^1$ monodromy ϕ_γ
$\phi^*(E) = F \times [0,1]/\sim \qquad (f,0) \sim (\phi_\gamma(f),1)$

If $\phi \sim \phi'$, $\phi^*(E) \simeq \phi'^*(E)$

Theorem 41.1 (Leray-Serre)
(note coefficient over a field)
There is a spectral sequence converging to $H_*(E)$
$E^2 = H_*(B;\phi)$
$\pi_1(B) = 1 \qquad E^2 = H_*(B) \otimes H_*(F)$

Idea of Proof
Suppose B and F are finite cell complexes

Claim: E is a finite cell complex

$$\begin{aligned} \text{cells of } E &\leftrightarrow \text{pairs } (\text{cells of } B, \text{cells of } F) \\ \tau_{b,f} &\leftrightarrow (\tau_b, \tau_f) \end{aligned}$$

Proof of Claim:
$\mathrm{Int}\,\tau_b = \mathrm{Int}\,D^n$ is contractible
$\Rightarrow \quad \pi^{-1}(\mathrm{Int}\,\tau_b) = \mathrm{Int}\,\tau_b \times F$
Define $\tau_{b,f}$ by $\mathrm{Int}\,\tau_{b,f} = \mathrm{Int}\,\tau_b \times \mathrm{Int}\,\tau_f$
$\Rightarrow \quad C_*^{\mathrm{cell}}(E) \cong C_*^{\mathrm{cell}}(B) \otimes C_*^{\mathrm{cell}}(F)$ as a group
Filtration:

$C_* = \oplus C_*(k) \qquad d(C_*(k)) \subset \bigoplus_{j\leq k} C_{*-1}(j)$
Define $C_*(k) = C_k^{\text{cell}}(B) \otimes C_*^{\text{cell}}(F)$
Check that if $\tau_b \in C_k^{\text{cell}}(B)$

$$\begin{aligned} d(\tau_b \otimes \tau_f) &\subset \bigoplus_{j\neq k} C_j^{\text{cell}}(B) \otimes C_*^{\text{cell}}(F) \\ &= C_*^{\text{cell}}(B_{(k)}) \otimes C_*^{\text{cell}}(F) \end{aligned}$$

This is true since $\tau_{b,f} (= \tau_b \otimes \tau_f) = \overline{\text{Int}\, \tau_{b,f}} \subset \overline{\pi^{-1}(\text{Int}\, \tau_b)} \subset \pi^{-1}(\tau_b) \subset \pi^{-1}(B_{(k)})$
(Note: $\tau_b \subset B_{(k)}$)
so the term in $d(\tau_{b,f})$ only involves things in the k-skeleton ■

.

Claim

$$\begin{aligned} d_0 : C_k(B) \otimes C_j(F) &\to C_k(B) \otimes C_{j-1}(F) \\ x \otimes y &\mapsto x \otimes d_F y \end{aligned}$$

$\Rightarrow \quad E^1$ term is $H_*(E^0, d_0) = C_*(B) \otimes H_*(F)$
Now $d_1 : C_*(B) \otimes H_*(F)$ is given by monodromy representation $x \otimes [y] \mapsto d_B x \otimes \phi_*(y)$ □

<u>Example:</u>
S^1 bundles over S^2
$\alpha \in \pi_1(SO(2)) = \mathbb{Z}$ determins a bundle
$S^1 \to E_n \to S^2$
$S^1 \to E_\alpha \to S^2$
Transition functions
$E_n = D^2 \times S^1 \sqcup (D^2 \times S^1)/\sim$
for $z \in \partial D^2 \qquad (z, w) \sim (z, z^n w)$
This complex has nontrivial d_2
E_0 term in the sequence $C_*^{\text{cell}}(S^2) \otimes C_*^{\text{cell}}(S^1)$

Claim: d_2 is multiplication by n

Proof of Claim:
Look at ∂(2-cell in $S^2 \otimes$point)

$\partial(D^2 \times \text{point})$ wraps n times around S^1 ■

42 Thom Isomorphism

Definition 42.1
$\mathbb{R}^n \to V \xrightarrow{\pi} B$
A real n-dimensional Riemmanian vector bundle is a fibration whose transition function are in $O(n)$

If $v \in V$, $\|v\|$ makes sense
$\pi(v) = b \in U$
$\pi^{-1}(U) = U \times \mathbb{R}^n$
$v \mapsto (b, v_0)$
Define $\|v\| = \|v_0\|$
This is well-defined; if U' is another such
$pi^{-1}(U') \to U^1 \times \mathbb{R}^n$
$v \to (b, v_0')$
$(b, v_0') = (b, A_b v_0) \quad A_b \in O(n)$
$\Rightarrow \quad \|v_0'\| = \|v_0\|$

Definition 42.2
If $V \to B$ is a n-dimensional real vector bundle
$S(V) = \{v \in V | \|v\| = 1\}$ unit sphere bundle of V
$D(V) = \{v \in V | \|v\| \leq 1\}$ disk bundle of V

$S^{n-1} \to S(V) \to B$
Note: $j : B \to V \qquad b \mapsto (b, 0)$
$pi : V \to B$ define a homotopy equivalence $B \sim V \sim D(V)$
But $S(V) \nsim S^{n-1} \times B$ unless V is trivial bundle

Theorem 42.3 (Thom Isomorphism)
Suppose $V \xrightarrow{\pi} B$ is an oriented n-dimensional real vector bundle, and B connected
Then there is a class $U \in H^n(D(V), S(V))$ s.t.

$$\begin{array}{ccc} H^k(B) & \xrightarrow{\sim} & H^{k+n}(D(V), S(V)) \\ x & \mapsto & \pi^*(x) \smile U \end{array}$$

is an isomorphism $\forall k$
Moreover, $j : (D^n, S^{n-1}) \to (D(V), S(V))$ (inclusion of fibre) with $j^*(U)$ generates $H^n(D^n, S^{n-1})$

Proof
V is oriented $\Leftrightarrow$ monodromy action on $H^n(D^n, S^{n-1})$ is trivial
Use Leray-Serre Spectral Sequence with respect to the pair (D^n, S^{n-1})

E^2 term is

$H_0(B) = k$
Chose U to be the generator $\square$

$\pi : V \to B$ is n-dimensional vector bundle
$D(V) = D$ =disk bundle
$S(V) = S$ =sphere bundle
$j : (D^n, S^{n-1}) \to (D(V), S(V))$ inclusion of fibre
B is path-connected. Then

1. $H^k(B) \cong H^{n+k}(D(V), S(V))$
2. $H^n(D, S) \cong H^0(B) = \mathbb{Z}$ is generated by U_V = Thom class of V, $j^*(U_V)$ generates $H^*(D^n, S^{n-1}) \cong \mathbb{Z}$
3. the isomorphism in (1) is given by $x \mapsto \pi^*(x) \smile U_V$
4. If $f : X \to B$, then $U_{f^{-1}(V)} = \overline{f}^*(U_V)$

Proof

1. follows from L-S spectral sequence

2. In $S.S$ (project on to this chain), j^* is given by

3. Over $\mathbb{R}$ using de Rham cohomology
   $H^k(B) \to H^{k+n}(D,S) \to H^k(B)$
   $x \mapsto x \smile U$ then integrate along fibres

4. $j' : (D^n, S^{n-1}) \to (D(f^*(V)), S(f^*(V)))$
   $\overline{f}j' = j$
   $j'^*(\overline{f}^*(U_V)) = j^*(U_V)$ generates $H^n(D^n, S^{n-1})$
   $\Rightarrow \quad j^*(U_V)$ must $= U_{f^*(V)}$

□

43 Constructing Poincare Duals

Suppose $B = M$ is a m-manifold
$PD(M) \Rightarrow PD(D(V), S(V) = \partial D(V))$
$H^{n-k}(D(V), S(V)) \cong H^k(M) \cong H_{m-k}(M) \cong H_{m-k}(D(V)) \cong H_{m+n-(n+k)}(D(V))$
In particular $PD([M]) = U_V$
Rework $([M] \cdot j_*(x) =) PD([M])(j_*(x)) = U(j_*(x)) = j*(U)(X) = 1$
x generates $H_n(D^n, S^{n-1})$

Now suppose $M \subset N$ (with some Riemanian metric) as a smooth submanifold

Definition 43.1
vM= normal bundle to M in N
$vM \to M = \{v \in T_xN, x \in M | V \perp T_xM\}$

$P : N \to N/(N - \operatorname{Int} U) \cong U/\partial U = D(vM)/S(v(M))$
$U_{vM} \in H^n(D,S) = H^n(D/S)$
$PD([M]) = P*(U_{vM})$
Check if $x \smile PD([M])([N]) = x(M)$:
LHS $= x \smile P^*(U)([N]) = i^*(x) \smile U(P_*[N]) \quad i : D(vM) \to N$
$= (i^*(x) \smile U)([D(vM)]) = x([M])$ by Thom isomorphism

Corollary 43.2
$PD(M_1) \smile PD(M_2)[N] = PD(M_1)[M_2] = \pm PD(M_2)[M_1] = M_1 \cdot M_2$

More generally: M_1 and M_2 intersect transversely

$$PD(M_1) \smile PD(M_2) = PD(M_1 \cap M_2)$$

44 Gysin sequence and Euler class

Gysin sequence is the l.e.s of $(D(V), S(V))$
$\pi : V \to B$, $i : B \to D(V), b \mapsto (b, 0)$
$\cdots \to H^*(D(V), S(V)) \to H^*(D(V)) \to H^*(S(V)) \to H^{*+1}(D,S) \to \cdots$
this is the same as $\cdots \to H^{*-n}(B) \to H^*(B) \xrightarrow{\pi^*} H^*(S(V)) \to H^{*+1-n}(B) \to \cdots$ by Thom isomorphism

What is $\alpha(x)$?
$\alpha(x) = i^*(\pi^*(x) \smile U) = (\pi i)^(x) \smile i^*(U) = x \smile i^*(U_V)$

Definition 44.1
$i^*(U_V) = e(V) =$ <u>Euler class of V</u> $\in H^n(B)$

<u>Gysin sequence</u>:

$$H^{*-n}(B) \xrightarrow{ve(v)} H^*(B) \xrightarrow{\pi^*} H^*(S(V)) \xrightarrow{\text{integrate along fibre}} H^{*+1-n}(B)$$

useful for computing $H^*(S(V))$

<u>Properties of $e(v)$</u>

1. (Natural) $f : X \to B$, $e(f^*(v)) = f^*(e(v))$
   (due to naturality of Thom class)
2. $e(V_1 \oplus V_2) = e(V_1) \smile e(V_2)$
   (Exercise: Proof)
3. $e(\text{trivial bundle}) = 0$
   (since trivial bundle $= f^*(\mathbb{R}^n \to pt.)$)
4. If V admits a non-vanishing section, $s : B \to V$, $\pi s = \mathrm{id}_B$, $s(b) \neq 0 \forall b$
   Then $e(V) = 0$
   (since hypothesis $\Rightarrow V = V' \oplus T$ then use (2) and (3))
5. $PD(e(V)) = S^{-1}(0)$ for any transverse section of V
   (Proof omitted)
6. $e(TM) = \chi(M) \cdot PD(1)$
   (See Example class)