# Part III Essay for Mathematical Tripos 2009/2010
# Blocks With Cyclic Defect Groups

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**Blocks with Cyclic Defect Groups**

# Preface

In the 1930s and 40s, Brauer introduced the defect groups [Br35] (to be defined later) associated with a block in order to investigate representation of finite groups in a more general setting via block and character theory, i.e. the ordinary representation as well as the modular representation. One of the results he obtained [Br41] while studying the defect groups is that for defect group of prime order, we can construct a graph, called the Brauer tree, such that the edges of the graph correspond to the modular irreducible characters and the vertices correspond to the ordinary irreducible character. This was further investigated by Thompson [Th] in 1960s and leads to the generalisation to all cyclic defect group done by Dade [Da] in 1966. In practice, to construct the Brauer tree via following Dade's proof is almost sure to be an unfeasible choice. Few years late, Green gave a construction of the Brauer tree [Gr] by quoting some results from a section of Dade's work as well as make use of his famous module correspondence (the Green's correspondence, see later) extensively. Green's approach also avoided the investigation into the generalised decomposition number and many of the character theory arithmetic involved in Dade's work. Moreover, most of the final result he acquire is module theoretic, character plays no part in those. This sees the trend of that time that representation theorists were shifting their emphasis on character theoretic approach to module theoretic approach.

In this paper, I am going to explore this approach done by Green. Chapter 1 will be dedicated to quoting most of the tools that will be used in the construction of the Brauer tree. I will begin Chapter 2 with some description of Brauer tree, then move on to show some of the results that were used by Green but quoted or derived from Dade's work. These results give the structure of the indecomposable $kH$-modules, where $H$ is a $p$-local subgroup of the group of interest, and $p$ divides the group order. The remaining section of Chapter 2 will be main content of how Green has construct the Brauer tree. The last chapter will be dedicated to compute the Brauer tree for the principal 5-block of $S_5$ via rather elementary and character theoretic method, and use this example to verify some other results that will be shown throughout this essay.

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# Contents

**1 Settings and Tools** **3**

1.1 Notation and Terminology . . . 3

1.2 Relative Projectivity . . . 3

1.3 Tools from homological algebra . . . 5

1.4 Vertex . . . 7

1.5 Defect groups . . . 7

1.6 Green's Correspondence . . . 9

1.7 Brauer's Correspondence . . . 10

**2 Construction of Brauer Tree** **13**

2.1 Prerequisites: Characters afforded by modules . . . 13

2.2 Introduction to the Brauer tree . . . 13

2.3 Results and consequences from Dade's original work . . . 16

2.4 Walking around the Brauer tree . . . 21

2.5 Permutation $\delta$ and sequences over $kG$ . . . 23

2.6 Lifting results from $kG$ to $\mathcal{O}G$ . . . 27

**3 Example** **30**

3.1 Brauer tree of the principal block of $S_5$ . . . 30

3.2 The walk around Brauer tree . . . 33

3.3 Module diagram of indecomposables . . . 34

3.4 Brauer tree of principal block of $H = N_G(P)$ . . . 35

# Chapter 1

# Settings and Tools

## 1.1 Notation and Terminology

In this essay, $G$ will always denote a finite group, with prime $p$ dividing $|G|$. For $H, K$ subgroups of $G$, the notation $\leq_G H$ means there exist a $G$-conjugate of $K$ which is a subgroup of $H$; $K =_G H$ mean there exist a $G$-conjugate of $K$ which is equal to $H$.

We will be working over a $p$-modular system $(K, \mathcal{O}, k)$, where $\mathcal{O}$ is a characteristic 0 complete discrete valuation ring with unique maximal ideal $\mathfrak{m}$, $K$ being the field of fraction of $\mathcal{O}$, and $k = \mathcal{O} \,/\, \mathfrak{m}$ the residue field of characteristic $p$. We require that $k$ is the splitting field of $x^{|G|} - 1$.

$RG$ will denote the group algebra with coefficient ring $R$. All the modules in the essay will be finitely generated. An $RG$-lattice is a finitely generated $RG$-module and free as an $R$-module.

If $M, N$ are $RG$-module, then we use the standard notation $M|N$ to denote $M$ is a direct summand of $N$. For $H \leq G$, $M$ an $RG$-module and $N$ and $RH$-module, $N \uparrow^G$ is the induction of $N$ to $G$, and $M \downarrow_H$ is the restriction of $M$ to $H$. For given $RG$-module $U, V$, we shorten the notation $\mathrm{Hom}_{RG}(U, V)$ and denote as $(U, V)^G$

## 1.2 Relative Projectivity

One of the importance of studying modular representation theory is that we can study the representation of a group $G$ by studying the representation of its $p$-subgroup, or even some subgroups $H$ of $G$ with "nice" properties. They are usually simpler to understand and to play with. Having obtained results related to the subgroup, we want as many tools as possible to help us investigate the representation of $G$. In other words, we need to exploit the relation between $RH$ and $RG$.

Recall projectivity of $M$ is that for given surjective module map $\phi : U \to V$, and module map $\theta : M \to V$, there exists $\psi : M \to U$ such that $\theta = \phi\psi$, i.e. $\psi$ completes the diagram:

$$\begin{tikzcd}
& M \arrow[dl, dashed, "\exists\psi"'] \arrow[d, "\theta"] & \\
U \arrow[r, "\phi"] & V \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is an exact sequence. The following definition gives a generalisation of projectivity:

**Definition 1.2.1**
Let $H \leq G$. $\theta \in (M,V)^G$, $\phi \in (U,V)^G$ with $\phi$ surjective. An $RG$-module is *$H$-projective* or *projective relative to $H$* if the following condition is satisfied:

If there exists $h \in (M,U)^H$ such that $\phi \circ g = \theta$ as $RH$-module map, then there exists $g \in (M,U)^G$ such that $\phi \circ h = \theta$ as $RG$-module map

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists h \Rightarrow \exists g"'] \arrow[d, "\theta"] & \\ U \arrow[r, "\phi"] & V \arrow[r] & 0 \end{tikzcd}$$

$M$ is *$H$-projective-free* if there is no $H$-projective module being a direct summand of $M$

Note that when $H = \{1\}$, this coincides with the usual above notion of projective. We now give introduce a map that plays a central role in representation theory of finite groups, which relates $RG$ and $RH$ homomorphisms.

**Definition 1.2.2**
Let $H \leq G$. $U, V$ be $RG$-modules. We define the *trace map* or the *transfer map* as follows

$$\begin{array}{rcl} \mathrm{Tr}_{H,G} : (U\downarrow_H, V\downarrow_H)^H & \to & (U,V)^G \\ \phi & \mapsto & \sum \phi^g \end{array}$$

where the sum is over coset representatives of $H$ in $G$. Note that $\phi^g$ denotes the map acted by conjugation action of $g \in G$, i.e. $\phi^g(u) = g(\phi(g^{-1}u))$ for $u \in U$. Also note that $(U\downarrow_H, V\downarrow_H)^H$ is the same as the set of $RH$-homomorphisms, i.e. equal to $(U,V)^H = \{\theta \in \mathrm{Hom}_k(U,V) | \theta^h = \theta \ \ \forall h \in H\}$.

The image of the trace map is denoted as $(U,V)^{G,H}$
The cokernel of the trace map is denoted as $(U,V)^G_H$

We say that the map $\theta \in (U,V)^{G,H}$ is *$H$-projective*

Also recall that the notion of an $RG$-module $M$ being projective can be thought of as a generalisation of free modules. For $M$ projective is equivalent to $M$ being a direct summand of a free $RG$-module. The following theorem sees that relative projective is a notion that is 'compatible' with this notion and gives us other equivalence notions of relative projectivity:

**Theorem 1.2.3**
$H$ is a subgroup of $G$. $M$ is an $RG$-module, the following are equivalent:

(1) $M$ is $H$-projective

(2) $\lambda \in (U,M)^G$ and split as $RH$-module, then $\lambda$ split as $RG$-module

(3) $M | M\downarrow_H\uparrow^G$

(4) $M$ is a direct summand of the induction of some $RH$-module $N$, i.e. $M | N\uparrow^G$

(5) (*Higman's Criterion*) $\mathrm{id}_M \in (M,M)^{G,H}$, i.e. $(M,M)^G = (M,M)^{G,H}$ hence $(M,M)^G_H = 0$

The Higman's Criterion in the above list is the most useful of all and is used to prove some results in the following sections. Also note that the second point resembles the equivalence definition of projective module which relates to module map that splits. For the proof the theorem, the reader can refer to [Bn].

Instead of projective relative to a subgroup $H$, we can further generalise the notion to a collection of subgroup of $G$, say $\mathfrak{X}$. In fact the above properties works in this general setting, by replacing $H$ as $\mathfrak{X}$ or replace $H$ by $H \in \mathfrak{X}$. We also give a notion for modules that fails to be relative $\mathfrak{X}$-projective , i.e. a more general notion of $H$-projective-free this will help us determine $(M,N)_1^G$

**Definition 1.2.4**
Given $\mathfrak{X}$ is a collection of subgroup of $G$, an $RG$-module $M$ is $\mathfrak{X}$*-projective* or *projective relative to* $\mathfrak{X}$ if $M$ is direct sum of modules with each summand projective relative to $H \in \mathfrak{X}$

When $R = k$ a field, an $kG$-module is $\mathfrak{X}$*-projective-free* if there is non-zero $\mathfrak{X}$-projective direct summand. In the special case of $\mathfrak{X} = \{1\}$, we can omit $\mathfrak{X}$ and use the term *projective* (in the previous definitions), *projective-free* instead

We now give some result that help us understand and calculate $(M,N)^{G,1}$ and $(M,N)_1^G$. For the proofs, reader can refer to [Gr]. Alternatively, [Al] section 20 and 21, which uses quite different notation.

**Lemma 1.2.5**
Let $M, N$ be $kG$-modules, $\theta \in (M,N)^G$

(1) Let $kG$-map $\pi : Q \twoheadrightarrow N$ be surjective and $Q$ projective.
Then, $\theta \in (M,N)^{G,1}$ (i.e. projective) if and only if $\theta$ can be factored through $Q$

(2) Dual to the above, let $kG$-map $\iota : M \hookrightarrow Q$ be injective and $Q$ injective (hence projective).
Then, $\theta \in (M,N)^{G,1}$ (i.e. projective) if and only if $\theta$ can be factored through $Q$

(3) If $\theta$ injective with $N$ projective-free, then $\theta$ is not projective unless $\theta = 0$

(4) Dual to the above, if $\theta$ surjective with $M$ projective-free, then $\theta$ is not projective unless $\theta = 0$

Point (1), (2) also explain the reason why we call a $kG$-map to be projective if it is in the image of the transfer map $(M,N)^{G,1}$. Point (3) and (4) also gives a corollary which plays an important role in later part of this essay, as for indecomposable modules, projective-free is the same as being non-projective and for a block with non-trivial defect (see later), we are interested in its indecomposables, and they are categorised into the projectives and the non-projectives.

**Corollary 1.2.6**

(1) $M$ projective-free and $N$ simple $\Rightarrow (M,N)^{G,1} = 0$ (hence $(M,N)^G \cong (M,N)_1^G$)

(2) $M$ simple and $N$ projective-free $\Rightarrow (M,N)^{G,1} = 0$ (hence $(M,N)^G \cong (M,N)_1^G$)

## 1.3 Tools from homological algebra

To relate any module with projective module, we use the notion of *projective cover*:

**Definition 1.3.1**
Let $M$ be an $A$-module, where $A$ is a ring. If there exists an projective $A$-module $P$ such that $f : P \twoheadrightarrow M$ is the minimal presentation of $M$, i.e. $P$ minimal in the sense of direct sum decomposition, then $P$ is called the *projective cover* of $M$

The *Heller operator* takes an $A$-module $M$ to an $A$-module $\Omega(M) := \ker f$, i.e. we have the short exact sequence:

$$0 \to \Omega(M) \to P \xrightarrow{f} M \to 0$$

The dual notion of projective cover is *injective hull*, i.e. a minimal embedding of an $A$-module $M$. We can define the inverse Heller operator on $M$ using

$$0 \to M \to I \to \Omega^{-1}(M) \to 0$$

where $I$ is the injective hull of $M$. Injective hull always exists (as oppose to projective cover, but for $A$ as an group algebra, both of these objects exists for all $A$-module). Moreover, it can be shown that

$$\Omega^{-1}\,\Omega\, M \cong \Omega\,\Omega^{-1}\, M \cong M$$

the reason why we can regard $\Omega^{-1}$ as an inverse to $\Omega$.

*Remark.* When we want to determine whether a $kG$-map $\theta \in (M, N)^G$ is projective, it suffices to check whether it factor through the projective cover of $N$ by Lemma 1.2.5 (1),(2). i.e. the projective cover is a canonical choice of such projective modules.

**Lemma 1.3.2 (Schanuel)**
If there exists two short exact sequence of $A$-module

$$\begin{aligned} 0 \to \quad N \to \quad P \to M \to 0 \\ 0 \to \quad N' \to \quad P' \to M \to 0 \end{aligned}$$

where $P$ and $P'$ are projective, then $N' \oplus P \cong N \oplus P'$

*Remark.* There is a dual form of Schanuel Lemma for injective modules.

This gives an immediate corollary that Heller operator is defined up to isomorphism when projective cover exists, since $f$ (as in the definition) is the minimal projective presentation of $M$

Heller operator allow us to transfer indecomposability and non-projectivity of a module:

**Lemma 1.3.3**
Let $M$ be an $kG$ or $\mathcal{O}G$-module. If $M$ is an indecomposable non-projective, then so is $\Omega\, M$

A $p$-modular system provide us a way to switch between different coefficient ring in the following way:

$$\begin{tikzcd} K \arrow[r, squiggly, shift left, "\substack{\text{intersecting with}\\ \mathcal{O}\,G\text{-modules}}"] & \mathcal{O} \arrow[l, squiggly, shift left, "\substack{\text{extension}\\ \text{by scalar}}"] \arrow[r, squiggly, shift left, "\substack{\text{reduction}\\ \text{mod } \mathfrak{m}}"] & k \arrow[l, squiggly, shift left, "\text{lifting}"] \end{tikzcd}$$

However, this switching is not always possible in the sense that these operations does not necessarily give the correct ‘inverse’. In particular, we usually are interested in knowing when we can lift an $kG$-module to an $\mathcal{O}G$-module. Basic representation theory tells us that projective $kG$-module can be lifted by lifting the corresponding primitive idempotent, and hence, we can lift the blocks (corresponding to central primitive idempotent).

One of the application of the Heller operator is that we can lift some $kG$-short exact sequences:

**Theorem 1.3.4**
Given a $kG$-short exact sequence:

$$0 \to V \to Q \to U \to 0$$

with $Q$ projective, and $Q, U$ liftable to $P, M$ respectively (i.e. $P/\mathfrak{m}P \cong Q, M/\mathfrak{m}M \cong U$)
Then $V$ can be lifted to an $RG$-lattice $N$ and we have a short exact sequence

$$0 \to N \to P \to M \to 0$$

We will see that this theorem is one of the main tools used in the construction of Brauer tree.

## 1.4 Vertex

We are now going to define an important object in the study of representation theory introduced by Green ([Gr59]). It is going to capture the information about which subgroup is an indecomposable module is relative projective to.

**Definition 1.4.1**
For an indecomposable $RG$-module $M$, the *vertex* of $M$ is a subgroup $P$ of $G$ such that, if $M$ is $P$-projective, then $M$ is not $Q$-projective for all $Q \lneqq P$. i.e. this is equivalent to, if $M$ is $H$-projective, then $P \leq_G H$

**Proposition 1.4.2 (Green)**

(1) If $H \leq G$ contains a Sylow $p$-subgroup of $G$, then every module is $H$-projective

(2) If $p'$-part of $|G|$ is a unit of $R$, then the vertex of $M$ is a $p$-subgroup
In particular, in our setting, as $k$ is splitting field of $G$, vertex of $M$ is a $p$-subgroup

(3) Vertex of $M$ is unique up to $G$-conjugacy

The reader can read [Bn], Proposition 3.10.2, for the proof.

Point (2) implies that with our choice of $k$, the vertex of a $kG$-module is a $p$-subgroup. Point (1) says that if the vertex is the trivial subgroup $\{1\}$, then the module is projective. Therefore, vertex tells us how far away our module is from being projective.

## 1.5 Defect groups

Brauer introduced the notion of defect group associated to a block ([Br35]), which allows us to measure how far away the block is from being semisimple, hence the name defect.

The original definition requires other definitions and set up which will serve no further application in the essay, so I will introduced the result proved by Green ([Gr62]), which is easier to understand and provides one of the many application of vertex of indecomposable modules.

We first observe that the group algebra $RG$ can be regarded as an $R(G \times G)$-module in a canonical way, with the action of $G \times G$ on $RG$ defined by

$$(g_1, g_2)g \mapsto g_1 g g_2^{-1} \tag{1.1}$$

An other way to think of this is that the group algebra $RG$ is regarded as an $RG - (RG)^{op}$ bimodule with action of $G$ as above.

Consider the diagonal map:

$$\begin{array}{rcl} \Delta : G & \to & G \times G \\ g & \mapsto & (g, g) \end{array}$$

and $\Delta G$ is a subgroup of $G$. So the action (1.1) permutes the cosets of $\Delta G$. Thus, $RG \cong R \uparrow^{G\times G}$ where $R$ is regarded as trivial $R(\Delta G)$-module. So $RG$ is $\Delta(G)$-projective by Theorem 1.2.3. Moreover, we have the following:

**Definition 1.5.1**
Take an indecomposable summand $B$ of $RG$. Then $B$ is an $RG$-block as it is indecomposable $RG - RG$ bimodule. The vertex of $B$ is then of the form $\Delta(D)$. We now call $D$ the *defect group of block $B$*. $D$ is a $p$-subgroup of $G$ and unique up to $G$-conjugacy, by Proposition 1.4.2. Hence, $|D| = p^d$, and we called $d$ the *defect of block $B$*

From now on, we fix the notation of $B$, $D$, $d$ as above.

*Remark.* Green, by making use of vertex theory, discovered that we can relate the defect group with the Sylow $p$-subgroup of $G$ by $D = S \cap S^g$ where $S$ is a Sylow $p$-subgroup of $G$ ([Gr62]) In fact, Green further showed in [Gr68] that there exists $x \in C_G(D)$ in place of $g$.

The following results are also done by Green, the reader can refer to [Bn] for the proof, which require Brauer correspondence covered in the next section.

**Proposition 1.5.2 (Green)**

(1) Let $e \in RG$ be the idempotent associated with $B$, then $e \in RG^{\Delta G}_{\Delta H}$ if and only if $D \leq_G H$

(2) Every $RG$-module lying in $B$ is $D$-projective, hence some $G$-conjugate of the vertex of indecomposable module lying in $B$ is subgroup of $D$

**Corollary 1.5.3**
Let $B$ be an $kG$-block and $\widehat{B}$ its corresponding $\mathcal{O}G$-block, then $B$ and $\widehat{B}$ has the same defect group

The following theorem tells us how defect group can be used to measure the 'semisimplicity' of a block, and hence the name defect. Again, this rely on the Brauer Main Theorems which we will see later.

**Theorem 1.5.4 (Blocks of defect zero)**
Let $B$ be a block of $G$, with defect group $D$, then the following are equivalent

(1) $\mathrm{Rad}(B) = 0$ (equivalently, $B$ semisimple; equivalently, $B$ is a matrix algebra over a division ring; equivalently, every module in $B$ is projective)

(2) $D = \{1\}$

(3) $B$ contains projective simple modules

## 1.6 Green's Correspondence

As mentioned before, modular representation helps us discover structure of $RG$ by studying the $RH$ for some $H$ subgroup of $G$. In particular, we are usually interested in the indecomposable $RG$-modules along with their vertices, as this is how $RG$-module relates to some ($p$-)subgroup of $G$. It is then natural to take $H$ such that it carries as much information as possible about $p$-subgroup $P$ of $G$ (however, at the same time, we want $H$ to be as small as possible). Two natural choices would be taking $H \geq C_G(P)$ or $N_G(P)$, for the later case, we sometimes term them *p-local subgroup*. In order to discover the relations between the $RG$ and $RH$-modules, we introduce the Green's correspondence.

We first fix our notation for the current section:

(1) $P$ is a $p$-subgroup of $G$

(2) $H$ is a subgroup of $G$ containing $N_G(P)$

(3) $\mathfrak{X} = \{X \leq G | X \leq P \cap P^g \text{ for some } g \in G - H\}$

(4) $\mathfrak{Y} = \{Y \leq G | Y \leq H \cap P^g \text{ for some } g \in G - H\}$

(5) $\mathfrak{Z} = \{K \leq G | K \leq P, K \notin_G \mathfrak{X}\}$

**Theorem 1.6.1 (Green's Correspondence)**
With set up as above, there is a bijection, depends on $G, H, P$:

$$\left\{ \begin{array}{c} \text{indecomposable } RG\text{-module} \\ \text{with vertex in } \mathfrak{Z} \end{array} \right\} \begin{array}{c} \xrightarrow{f} \\ \xleftarrow[g]{} \end{array} \left\{ \begin{array}{c} \text{indecomposable } RH\text{-module} \\ \text{with vertex in } \mathfrak{Z} \end{array} \right\}$$

such that

(1) $M$ indecomposable $RG$-module with vertex in $\mathfrak{Z}$, then

$$M \downarrow_H \cong f(M) \oplus M_0$$

with $M_0$ projective relative to $\mathfrak{Y}$

(2) $N$ indecomposable $RH$-module with vertex in $\mathfrak{Z}$, then

$$N \uparrow^G \cong g(N) \oplus N_0$$

with $N_0$ projective relative to $\mathfrak{X}$

(3)

$$\begin{aligned} gfM &\cong M \\ fgN &\cong N \end{aligned}$$

Note that $f$ and $g$ are well-defined by Krull-Schmidt Theorem. The proof of this can be found in most literature on representation theory, the reader can refer to [Bn].

Due to its close connection with relative projectivity, we aim to apply Green's correspondence not only on modules but also to maps. In other words, we want to turn $f$ and $g$ into functor. Our aim is, with given $\theta : U \to V$ where $U, V$ are indecomposable $RG$-module with vertex in $\mathfrak{Z}$, we get a module map $f\theta : fU \to fV$, and similarly for $g$. We do this for $f$ and $RG$-modules, result for $g$ and $RH$-module will be similar.

By Green's correspondence, there is natural projection and inclusion:

$$\begin{aligned} \pi_V : V \downarrow_H &\twoheadrightarrow fV \\ \iota_U : fU &\hookrightarrow U \downarrow_H \end{aligned}$$

We can then define

$$f\theta := \iota_U \circ \theta_H \circ \pi_v$$

where $\theta_H$ is $\theta$ regarded as $RH$-map. Therefore, $f\theta$ is now a $kH$-map $fU \to fV$

**Lemma 1.6.2**
$U, V$ are $P$-projective $RG$-module. $\theta \in (U, V)^G$, then

(1) $f(\iota_U) = \iota_{fU}$

(2) $(U, V)^G_{\mathfrak{X}} \cong (fU, fV)^H_{\mathfrak{X}}$ (as $R$-module) via $\theta \mapsto f\theta$
In particular, when $U = V$, this is a $k$-algebra isomorphism

One useful application of 'functorising' $f$ and $g$ is that we can show $f, g$ commutes with the Heller operator:

**Theorem 1.6.3**
Let $N$ be $RH$-module and $M$ be $RG$-module, both of the projective relative to $P$, then

$$\begin{aligned} g\,\Omega\, N &\cong \Omega\, gN \\ f\,\Omega\, M &\cong \Omega\, fM \end{aligned}$$

## 1.7 Brauer's Correspondence

Another structure of the group algebra which we are interested in is the blocks and their defect group. Recall, $k$ is the residue field of characteristic $p$ which also is the splitting field for $G$. We now exploit the relation of these structure in $kH$ and those in $kG$, where $H \geq N_G(D)$ (hence a $p$-local subgroup again). The main tool we use for this is Brauer correspondence.

There are three main theorem related to this correspondence, termed as the Brauer First, Second, Third Main Theorem. I will briefly talk about each of them here and their application, no proof will be given, the reader can refer to [Bn],[Al] and [Na] for more detailed description and for proofs. Another point to mention is that Brauer correspondence works on the $k$-representation (modules) but not necessary on $\mathcal{O}$-representation (modules). This also demonstrates why it is convenient to work in a $p$-modular system, as then we can study the structure of modules in the modular representation (those in $k$), and then we can try to lift certain modules that lies in the block to the corresponding block in ordinary representation.

**Theorem 1.7.1 (Brauer First Main Theorem)**
Let $D$ be a $p$-subgroup of $G$, define the *Brauer map* (or *Brauer homomorphism*) as the well-defined $k$-algebra homomorphism

$$\begin{aligned} \mathrm{Br}_D : Z(kG) &\rightarrow Z(kC_G(D)) \\ \sum_{g\in G} a_g g &\mapsto \sum_{x\in C_G(D)} a_x x \end{aligned}$$

This map sets up a one-to-one correspondence between the idempotents associated to $kG$-block with defect group $D$ and idempotents associated to $kN_G(D)$-block with defect group $D$.

For $H$ a subgroup of $G$ containing $N_G(D)$, we notice $N_H(D) = N_G(D)$, so we can extend the Brauer correspondence:

**Corollary 1.7.2**
Let $H$ be subgroup of $G$ containing $N_G(D)$, there is a one-to-one correspondence (Brauer correspondence) between the $kG$-block with defect group $D$ and $kN_G(D)$-block with defect group $D$.

Let $b$ be a $kN_G(D)$-block, we denote $b^G$ to be the corresponding $kG$-block under the Brauer correspondence.

As mentioned before, we usually want to generalise our result as much as possible by making $H$ as small as possible. In fact, there is a more general form of the First Main Theorem:

**Theorem 1.7.3**
Let $H$ be a subgroup of $G$ containing $DC_G(D)$, then the Brauer map defines a surjection from the set of $kG$-block with defect groups containing $D$ to the set of $kH$-blocks with defect group containing $D$
Moreover, if $b_1, b_2$ are the $kH$-blocks in the former set, then $b_1^G = b_2^G$, if and only if, $b_1 =_G b_2$

This general form tells us that correspondence exists, but given a block, we do not exactly know what the corresponding block is. So the next thing we are interested in is, what criteria will be sufficient to help us determine whether two blocks corresponds under the Brauer map. This is what the Second Main Theorem tells us. Instead of the original version by Brauer, which uses generalised decomposition number, we give the modular version of it, originated from Nagao.

**Theorem 1.7.4 (Second Main Theorem, Nagao's modular version)**
Let $D$ be a $p$-subgroup of $G$. Let $M$ be an indecomposable $kG$-module lying in $B$, block of $kG$. Let $N$ be an indecomposable $kH$-module lying in $b$, block of $kH$, with $H$ containing $C_G(D)$ and vertex of $N$ is $D$.
If $N$ is a direct summand of $M\downarrow_H$, then $b^G = B$

The Second Main Theorem gives a connection of Brauer's and Green's correspondence as follows:

**Corollary 1.7.5**
Let $M$ be indecomposable $kG$-module lying in $kG$-block $B$ with vertex $D$
Consider the map $f$ as Green's correspondence depends on $G$, $H = N_G(D)$, $P = D$ (see Theorem 1.6.1).
If $f(M)$ lies in $kH$-block $b$, then $b^G = B$

The following is also a corollary of the Second Main Theorem, which is an interesting result about indecomposable modules lying in $B$ with defect group $D$

**Corollary 1.7.6**
If $B$ is a $kG$-block with defect group $D$, then there is an indecomposable $kG$-block in $B$ with vertex being $D$

**Theorem 1.7.7 (Brauer Third Main Theorem)**
Let $H$ be a subgroup of $G$ containing $DC_G(D)$, and $B_0(G)$ denote the *principal block* of $kG$, i.e. the block which the trivial module $k$ lies. Then $b = B_0(H)$ (principal $kH$-block), if and only if, $b^G = B_0(G)$

Principal block is usually the block with the most complex structure in the group algebra (which means it contains more information). So the Third Main Theorem helps us in the way that, we can study the principal block of $kH$, rather than the more complicated $kG$, and then transfer the results back using Brauer correspondence.

The interested reader should note that the Theorem on blocks of defect zero (Theorem 1.5.4) is an application of the Brauer's three main theorems.

We conclude this chapter by connecting the two important correspondence. In the defect group section, we see that blocks are indecomposable summand of $kG$ regarded as $k(G \times G)$-module. So we see a connection of the Green's and Brauer's correspondence as follows. If $b$ a $kH$-block and $B$ a $kG$-block, both have defect group $D$ and correspond to each other (under Brauer's correspondence). We then set $P$ in the Green's correspondence as the defect group $D$ (see Theorem 1.6.1). Then $b$ is the indecomposable $k(H \times H)$-module with vertex $\Delta D$, and its Green's correspondent is $b^G$ with vertex $\Delta D$, as $b | (b^G) \downarrow_{H \times H}$. Another connection is the following proposition, which essentially addressed that studying the $p$-local subgroup helps the study of the original group as we can categorised the $kG$-modules in the same way as $kH$-modules.

**Proposition 1.7.8 (Alperin)**
Let $H$ be a subgroup of $G$ containing $N_G(D)$; $M$ be indecomposable $kG$-module and $N$ be indecomposable $kH$-module.
Let $B$ be a $kG$-block with defect $D$ and $b$ be $kH$-block with defect $D$ such that $B$ is the Brauer correspondent of $b$. Then

$$\begin{aligned} M \text{ lies in } B &\Leftrightarrow fM \text{ lies in } b \\ gN \text{ lies in } B &\Leftrightarrow N \text{ lies in } b \end{aligned}$$

where $f, g$ are the Green correspondence depending on $G, H, D$, i.e. every indecomposable modules lying in a block has its Green's correspondent lying inside the Brauer's correspondent.

# Chapter 2

# Construction of Brauer Tree

## 2.1 Prerequisites: Characters afforded by modules

An ordinary character $\chi$ is *afforded by finite dimensional $KG$-module $V$* has the same meaning as ordinary character over $\mathbb{C}$.

It can be shown that ([PO], Theorem 3.3) for such $V$, there exists and $\mathcal{O}G$-lattice $M$ such that $V = K \otimes_{\mathcal{O}} M$ (c.f. Section 1.3). We say $\chi$ is *afforded by $M$*.

It can also be shown that ([PO], Theorem 3.5), for a group $L$, prime $p \nmid |L|$, and $kL$-module $W$, there is a $\mathcal{O}L$-lattice $M$ such that its reduction $\overline{M} := M/\mathfrak{m}M \cong W$. Moreover, $V := K \otimes_{\mathcal{O}} M$ is a $KL$-module determined uniquely up to isomorphism by $W$.

Therefore, for all $p'$-element $x \in G$, and $U$ an $kG$-module, set $L = \langle x \rangle$, $W = U\downarrow_L$, we get a $KL$-module $V = K \otimes_{\mathcal{O}} M$.

A *Brauer character* is a function $\phi : \{p'\text{-element of } G\} \to K$ such that $\phi(x)$ takes the character value of the ordinary character afforded by $KL$-module $V$ (or $\mathcal{O}L$-lattice $M$). Note that these values only depends on $U$. We say that $\phi$ is *afforded by $kG$-module $U$*. An *irreducible Brauer character* is the Brauer character afforded by an irreducible $kG$-module.

A *projective indecomposable character* $\eta$ is the Brauer character afforded by the projective indecomposable $kG$-module.

As in ordinary character theory, $\chi$ uniquely determine the (isomorphism class of) $KG$-module, and vice versa. Note that this is not true for $\mathcal{O}G$-lattice. Brauer character $\phi$ also uniquely determine the (isomorphism class of) $kG$-module, and vice versa (see [PO], section 3.6).

## 2.2 Introduction to the Brauer tree

As its name suggests, Brauer tree is a tree (graph with no cycle). A vertex $P$ on the Brauer tree corresponds to either an irreducible ordinary (i.e. $\mathcal{O}$-representation) character $\chi_P$ lying in $B$ or a so-called *exceptional character* which is the sum of finitely many irreducible ordinary character. These character summands are denoted as $\chi_\lambda$ with $\lambda \in \Lambda$ an indexing set. The vertex corresponding to the exceptional character $\chi_P = \sum_{\lambda\in\Lambda} \chi_\lambda$ is called *exceptional vertex*. The edges $E$ correspond to irreducible modular (i.e. $k$-representation) characters (the Brauer characters) $\phi_E$ lying in $B$.

Note that the Brauer character arises from simple $kG$-modules, $S$ say, which correspond to a projective indecomposable $kG$-module $P$ such that $P/\operatorname{Rad}(P) \cong S$, we denote $\eta_E$ as the character arises from the projective indecomposable module corresponding to the simple module, which correspond to Brauer character $\phi_E$. Brauer showed in [Br41] that there is a relation for the characters corresponding to the two vertices $i, j$ of an edge $E$:

$$\eta_E = \chi_P + \chi_Q \tag{2.1}$$

Moreover, for block $B$ with defect group $D$ and $|D| = p$, it is possible to put all the irreducible ordinary and Brauer character into the Brauer tree.

This result did not get improved for a quarter of a century, until Dade shows that it can be generalised to $D$ being a cyclic group. First, Dade showed ([Da]) that there exists a positive integer $e$ such that $e$ divides $p^d - 1$ (Recall we fixed the notation $|D| = p^d$). Then the Brauer tree has $e$ edges and $e+1$ vertices, where $e$ of the vertices represents the irreducible ordinary character and the remaining one is the exceptional character with *multiplicity* $(p^d-1)/e$, this is equal to the number of exceptional characters (i.e. $|\Lambda| = (p^d-1)/e$). Dade showed that relation (2.1) holds for such block, i.e. it is possible to draw the Brauer tree for blocks with cyclic defect groups.

The most useful information a Brauer tree gives is the composition factor of each of the projective indecomposable lying inside $B$. This also implies that it can tell us what the Cartan matrix and decomposition matrix associated to $B$ is. Hence we can draw the module diagram for the projective indecomposable module, using procedure as follows.

First, take an edge $E$ from the Brauer true. It corresponds to an irreducible Brauer character $\eta$, afforded by a simple $kG$-module $S$, and hence correspond to a projective indecomposable $kG$-module $P$ such that $P/\operatorname{Rad}(P) \cong S$. Since $kG$ is a symmetric algebra, we have

$$S \cong P/\operatorname{Rad}(S) \cong \operatorname{Soc}(P) \tag{2.2}$$

Suppose for simplicity that neither of the two ends of the edge, is the exceptional vertex, i.e. $i, j \in \{1, \ldots, e\}$. Then relation (2.1) tells us that the multiplicity of the $\mathcal{O}G$-module affording $\chi_i$ (respectively $\chi_j$) as a composition factor in the lift of the $kG$-module affording $\eta$ is 1, this is the *decomposition number* correspond to $\chi_i$ and $\eta$, labelled $d_{ia}$ (respectively $d_{jb}$). Equivalently, we can say that multiplicity of the simple $kG$-module $S$ (defined above) as a composition factor of the reduction of $\mathcal{O}G$-module affording $\chi_i$ is 1.

Using the relation $C = D^\top D$ where $C$ is the Cartan matrix associated to block $B$ and $D$ is the decomposition associated to block $B$ (*Remark* this is true on group algebra $kG$ but not all algebra), by $E, F$ for two edges on the Brauer tree, we have

$$c_{EF} = \begin{cases} 2 & \text{if } E = F \text{and none of its ends is exceptional} \\ 1 & \text{if } E, F \text{ has a common (non-exceptional) vertex} \\ 0 & \text{if } E, F \text{ has no common vertex} \end{cases}$$

where $c_{EF}$ is the Cartan invariant.

When one of the vertex is exceptional, correspond to the family of exceptional irreducible characters $\chi_{\lambda_1}, \ldots, \chi_{\lambda_r}$ (note $r = (p^n-1)/e$), the decomposition matrix would looks like

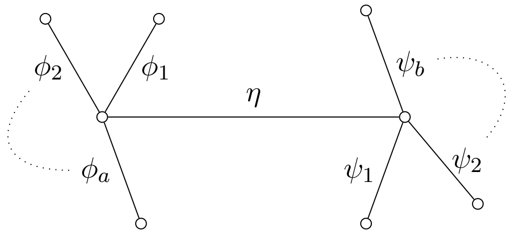
$$
\begin{array}{c|ccc|ccc}
 & \eta_1 & \dots & \eta_i & \eta_{i+1} & \dots & \eta_e \\
\hline
\chi_1 & & & & & & \\
\vdots & & * & & & * & \\
\chi_e & & & & & & \\
\hline
\chi_{\lambda_1} & & & & 1 & 1 & 1 \\
\vdots & & 0 & & \vdots & \vdots & \vdots \\
\chi_{\lambda_r} & & & & 1 & 1 & 1
\end{array}
$$

where $\eta_{i+1}, \dots, \eta_e$ correspond to edges with one end being the exceptional vertex. Now we have the relation of edges and Cartan invariants similar as previous case,

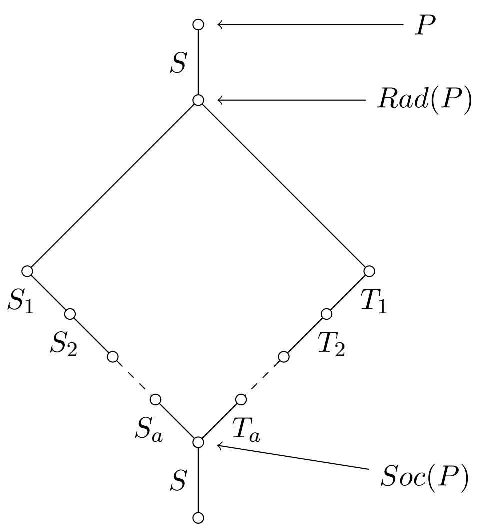
$$
c_{EF} = \begin{cases} 0, 1, 2 & \text{as above} \\ m & \text{if } E, F \text{ has a common exceptional vertex} \\ m+1 & \text{if } E = F \text{ with one end being exceptional vertex} \end{cases}
$$

[Jn] provides a detailed explanation to this. Combining these relations with the fact (2.2), and reorder the edges connected to the two ends of $E$, we summarises the above using the following diagrams:

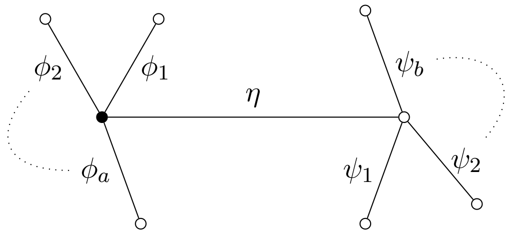
**Case I: No exceptional vertices**



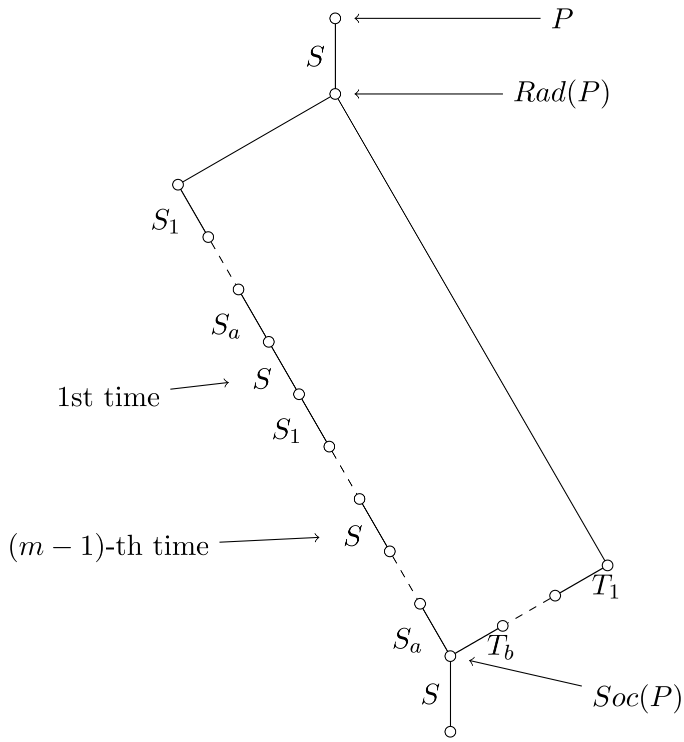

Label on the edges are corresponding Brauer character
$\eta$ is the projective indecomposable character afforded by projective indecomposable $kG$-module, which correspond to a simple $kG$-module $S$.
$\phi_1, \dots, \phi_a$ Brauer characters correspond to simple $kG$-modules $S_1, \dots, S_a$
$\psi_1, \dots, \psi_b$ Brauer characters correspond to simple $kG$-modules $T_1, \dots, T_b$
Then the module diagram of $P$ is



**Case II: With exceptional vertices**




Label on the edges are corresponding Brauer character.
• denotes exceptional vertex with multiplicity $m$
$\eta$ is the projective indecomposable character afforded by projective indecomposable $kG$-module, which correspond to a simple $kG$-module $S$.
$\phi_1, \ldots, \phi_a$ projective indecomposable character correspond to simple $kG$-modules $S_1, \ldots, S_a$
$\psi_1, \ldots, \psi_b$ projective indecomposable character correspond to simple $kG$-modules $T_1, \ldots, T_b$
Then the module diagram of $P$ is




Knowing that such meaningful graph exists for block with cyclic defect, Green aimed to produce some algorithm that would provide a way to construct such tree in [Gr]. The rest of this chapter will be to present and explain the proofs for such approach.

## 2.3 Results and consequences from Dade's original work

Our construction of the Brauer tree associated with a block with cyclic defect will be based on the approach done by Green in 1974 [Gr], which can provide an alternative way to construct a Brauer tree instead of using the method presented in Dade's work. However, Green cannot avoid using many results obtained from Dade's paper. Most notable of all is the analysis on the indecomposable and simple $kH$-module, where $H = N_G(P)$ with $P$ the order $p$ subgroup of the defect group. In this section, I will quote and explain the results of Dade that Green has used in his construction of the Brauer tree.

The set up we need is as follows:

(1) $P$ is the unique order $p$ subgroup of $D$

(2) $H = N_G(P)$

(3) $C = C_G(P)$

(4) $b$ a block of $kH$

(5) $B$ (Brauer) corresponding block of $kG$ (i.e. $B = b^G$), Brauer correspondence works as $N_G(P) \geq N_G(D)$

(6) $\beta$ a block of $kC$ such that $\beta^H = b$, this block exists and unique up to $H$-conjugacy, by Brauer First Main Theorem 1.7.3

The following groups play important role in Dade's paper as they give the value of $e$, i.e. the number of $kG$-simples, or equivalently, the number of edges in the Brauer tree.

$$\begin{aligned} T(\beta) &= \text{inertia group of } kC\text{-block } \beta \\ &= \{h \in H | h\beta h^{-1} = \beta\} \end{aligned}$$

**Fact**:

(1) $T(\beta) = EC$ where $E \leq N_G(D)$

(2) $EC/C$ is a subgroup of $H/C (\cong \mathrm{Aut}(P))$ and is cyclic of order $e$, thus $e$ divides $|H/C| = p-1$ and hence $e$ divides $p^d - 1$

Using Green's approach, we can avoid dealing with block covering, Clifford theory, and the complicated extended version of Brauer First Main Theorem, (see [Bn] 6.4) by just quoting these result and aim to exploit what is more important to us (i.e. how to use modules to determine the Brauer tree). The relation between these subgroup can be visualise as follows. Although this visualisation would not help us too much on our analysis and construction, it will show us the various relations and correspondence interplaying in the theory that has been discussed and will be used.

Now we start our analysis on the $kH$-modules

**Theorem 2.3.1**

(1) $b$ contains $e$ non-isomorphic simple $kH$-modules $S_i$, $i = 0, \dots, e-1$

(2) There exists a multiplicative isomorphism between $D$ and $Z(kC)$

$$\begin{array}{ccc} D & \xrightarrow{\sim} & Z(kC) \\ \sigma & \mapsto & \overline{\sigma} \end{array}$$

such that, if $D = \langle \alpha \rangle$, then for $i = 0, \dots, e-1$, we have a unique composition series for projective indecomposable $kH$-modules $T_i$ corresponding to the simple $kH$-module $S_i$

$$T_i > T_i(\overline{\alpha} - 1) > T_i(\overline{\alpha} - 1)^2 > \cdots > T_i(\overline{\alpha} - 1)^{p^d - 1} > T_i(\overline{\alpha} - 1)^{p^d} = 0$$

(3) Let $M$ be an indecomposable $kH$-module lying in $b$, then

$$M \cong T_{i,v} := T_i / T_i(\overline{\alpha} - 1)^v \quad \text{some } i \in \{0, \dots, e-1\}, v \in \{1, \dots, p^d\}$$

In particular, $S_i \cong T_{i,1}, T_i \cong T_{i,p^d} \ \ \forall i \in \{0, \dots, e-1\}$

The significance on part (1) is obvious since, the next goal (which is done later) we want is to prove going up from $kH$-module in $b$ to $kG$-module in $b^G$ preserves the number of simple modules (i.e. there is a bijection between them). In fact, this exactly is what Green's correspondence allows us to do. Part (2) of the theorem tells us that the projective indecomposable $kH$-modules are uniserial. And as before, we would like to pass this nice property to the $kG$-module. Significance of part (3) is obvious as it tells us that we have determined all the indecomposable and the simple $kH$-module.

The complication of this theorem is the multiplicative isomorphism given in part (2). In brief, $\overline{\sigma} = (\widetilde{\sigma} \mod \mathfrak{m})kC$ (recall $\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}$), where $\widetilde{\sigma}$ has an explicit formulation in Dade's paper ([Da], Section 5, (5.3)). It is enough to serve our purpose that one such isomorphism exists and so we can investigate the composition series of $T_i$. Also note that uniqueness of the composition series can be proved using more modern technique, see [Bn] 6.5.2.

The next thing we need is the information of the composition factor of $T_i$, i.e. $T_i(\overline{\alpha}-1)^v / T_i(\overline{\alpha}-1)^{v+1}$. Denote these composition factor as $S_{i,v}$. From the theorem, we know that $S_{i,v} \cong S_j$ for some $j \in \{0, \dots, e-1\}$. We now explore the relation between each of them. In particular, we will find out that we can order the simple in a very nice manner.

First we need a theorem to help us investigate these $kH$-module by studying the action of $T(b) = EC$ on them (instead of action of $H$), and a theorem giving the criteria of different simple modules lying in the same block.

**Theorem 2.3.2**
Let $S, S'$ be simple $kH$-module lying in $b$ such that $S \downarrow_{EC} \cong S' \downarrow_{EC}$. Then $S \cong S'$

**Theorem 2.3.3**
$S, S'$ simple $kH$-module. Then $S, S'$ lie in the same block if and only if there exists a sequence of simple $kH$-module:

$$S = M_1, \dots, M_l = S'$$

such that each pair $M_i, M_{i+1}$ are composition factors of the same indecomposable projective $kG$-module

**Lemma 2.3.4**
There is a one-dimensional simple $kH$-module $W$, affording character $\psi$, satisfy the following

(1) $S_{i,v} \cong \underbrace{W \otimes_k \cdots \otimes_k W}_{v \text{ times}} \otimes_k S_i$

(2) Set $S_0$ be the simple module containing the trivial module $k$. Set $S_n = \underbrace{W \otimes_k \cdots \otimes_k W}_{n \text{ times}} \otimes_k S_0$, then the set of non-isomorphic simple $kH$-modules are $\{S_0, \ldots, S_{e-1}\}$

(3) Composition factor of $T_i$ are $S_i, S_{i+1}, \ldots, S_{i+q-1} \cong S_i$

**Proof**
First let $D = \langle\alpha\rangle, P = \langle\alpha_1\rangle$

(1) We start by defining the character $\psi$ as follows

$$\begin{array}{rcl} \psi : H & \to & k^\times \\ h & \mapsto & n_h \end{array}$$

where $n_h$ is defined uniquely up to mod $p$ such that

$$\text{Conjugation of } \alpha_1 \text{ by } h\text{:} \quad \alpha_1^h = \alpha_1^{n_h}$$

Also note that $\psi(c) = 1 \ \ \forall c \in C$, hence $\psi^{[H:C]} = 1_H$ the trivial representation of $H$.
Let $z \in E \leq H = N_G(P)$
$\Rightarrow \quad \alpha^z \in D \quad \Rightarrow \quad \alpha^z = \alpha^{n_z}$ for some $n_z \in \mathbb{Z}$, unique up to mod $p^d$
$\Rightarrow \quad \alpha_1^z = \alpha_1^{n_z}$ and $\psi(z) = n_z$
We now quote another result obtained from Dade so that we can study the action of $E$ and $C$ (hence the action of $T(b) = EC$) on the composition factors $S_{i,v}$:

$$\forall \sigma \in D \qquad (\overline{\sigma})^z = \overline{\sigma^z} \tag{2.3}$$

In particular, we have

$$\overline{\alpha}^z = \overline{\alpha^z} = \overline{\alpha^{n_z}} = \overline{\alpha}^{n_z} \quad \forall z \in E \tag{2.4}$$

Now consider $t \in T_i, z \in E$, i.e. $t(\overline{\alpha} - 1)^v \in T_i(\overline{\alpha} - 1)^v$

$$\begin{array}{rcl} t(\overline{\alpha} - 1)^v z & = & t\,\overbrace{(\overline{\alpha} - 1) \cdots (\overline{\alpha} - 1)}^{v \text{ times}}\,z \\ & = & t(\overline{\alpha} - 1)z(\overline{\alpha}^{n_z} - 1) \\ & = & tz(\overline{\alpha}^{n_z} - 1)^v \\ & = & tz(\overline{\alpha} - 1)^v(n_z + (\overline{\alpha} - 1) + \cdots + (\overline{\alpha}^{n_z - 1} - 1))^v \\ & = & tz(\overline{\alpha} - 1)^v n_z^v + tz(\overline{\alpha} - 1)^{v+1} y' \quad \text{some } y' \\ & = & tz(\overline{\alpha} - 1)^v n_z^v \mod T_i(\overline{\alpha} - 1)^{v+1} \\ & = & \psi^v(z) tz(\overline{\alpha} - 1)^v \mod T_i(\overline{\alpha} - 1)^{v+1} \end{array} \tag{2.5}$$

the last line (2.5) is true for all $z \in E$, but as $C \leq E$, it is also true for all $z \in C$. Hence (2.5) true for all $z \in T(b) = EC$
$\Rightarrow \quad S_{i,v} \downarrow_{EC} \cong (\underbrace{W \otimes_k \cdots \otimes_k W}_{v \text{ times}} \otimes_k S_i) \downarrow_{EC}$
$\Rightarrow \quad S_{i,v} \cong \underbrace{W \otimes_k \cdots \otimes_k W}_{v \text{ times}} \otimes_k S_i$ by Theorem 2.3.2

(2) As noted before, we have $\psi^{[H:C]} = 1_H$, i.e. $\underbrace{W \otimes \cdots \otimes W}_{[H:C] \text{ times}} = S_0$

So if $m \equiv n \mod [H:C]$, then $S_m \cong S_n$

Also as $H/C \cong \mathrm{Aut}(P)$, we have $[H:C] | p-1$

$\Rightarrow$ $[H:C] < p^d$

$\Rightarrow$ $T_0$ has composition factor $S_0, S_1, \ldots$ by point (1)

$\Rightarrow$ $S_0, S_1, \ldots$ all lie in $b$

Conversely, if $S$ is a simple $kH$-module lying in $b$, using Theorem 2.3.3, there exists sequence of $kH$-module:

$$S_0 = S_{i_0}, \ldots, S_{i_r} = S$$

such that $S_{i_j}$ are composition factor of $T_{i_{j-1}}$ $\forall j = 1, \ldots, r$

$\Rightarrow$ (1) tells us that

$$\begin{aligned} S_{i_j} &\cong \underbrace{W \otimes \cdots \otimes W}_{n_j \text{ times}} \otimes S_{i_{j-1}} \quad \text{some } n_j \in \mathbb{Z}, \forall j = 1, \ldots, r \\ \Rightarrow \quad S &\cong \underbrace{W \otimes \cdots \otimes W}_{n \text{ times}} \otimes S_0 \quad \text{some } n \in \mathbb{Z} \\ \Rightarrow \quad S &\cong S_i \quad \text{some } i \in \mathbb{Z} \end{aligned}$$

Therefore, by Theorem 2.3.1, $b$ has $e$ non-isomorphic simple module $S_0, \ldots, S_{e-1}$

(3) Follows from (1), (2) and definition of $T_i, S_{i,v}$

□

Summarising the results, and by considering the projective indecomposables, we have:

(1) $b$ has $e$ non-isomorphic simple $kH$-modules $S_0, \ldots, S_{e-1}$, with corresponding projective indecomposable $T_0, \ldots, T_1$

(2) There is a one-dimensional $kH$-module $W$, such that we can set

$$S_n := \underbrace{W \otimes \cdots \otimes W}_{n \text{ times}} \otimes S_0$$

so that the subscript $n$ can be taken modulo mod $e$

(3) $T_i$ is uniserial, with composition factor

$$S_i, S_{i+1}, \ldots, S_{i+1-p^d} (\cong S_i)$$

$$S_i = T_i / T_i(\overline{\alpha} - 1) = T_i / \mathrm{Rad}(T_i) \quad \text{some } \overline{\alpha} \in Z(kC)$$

Also by (2), we get $T_i \cong T_{i+e}$

(4) All indecomposable $kH$-module arise form $T_{i,v} = T_i / T_i(\overline{\alpha} - 1)^v$ with $v = 1, \ldots, p^d$.
In another words, all indecomposables are quotient of $T_i$ and of form $T_i / \mathrm{Rad}^v(T_i)$
In particular, $T_{i,v}$ simple $\Leftrightarrow$ $v = 1$ and $T_{i,v}$ projective $\Leftrightarrow$ $v = p^d$

(5) There exists $kH$-short exact sequence:

$$0 \to T_{i+v, p^d - v} \to T_{i,p^d} \to T_{i,v} \to 0 \tag{2.6}$$

for each $v = 1, \ldots, p^d - 1$

(6) $\Omega T_{i,v} \cong T_{i+v,p^d-v}$ for each $v = 1, \dots, p^d - 1$
(since, by indecomposability and projectivity of $T_{i,p^d} = T_i$, the short exact sequence (2.6) is the minimal presentation of $T_{i,v}$)

(7) $\Omega^2 S_i \cong S_{i+1}$ (as $S_i \cong T_{i,1}\ \ \forall i$)

We now finish our investigation of the $kH$-module. The proof of this section comes from [Gr]. We notice that there is a heavy use of character theory. To avoid using character theory, the reader can refer to [Bn] for a concise investigation using more complicated algebra machinery or [Al] for lengthy investigation which is easier to understand. Both proof started by proving the $kC$-block covered by $b$ has a unique simple module with its corresponding projective indecomposable module being uniserial of length $p^d$. Also note that this is essentially saying that such $kC$-block has a Brauer tree with one edge ($e = 1$) and the exceptional vertex has multiplicity $p^d - 1$. The next step is then to show that there are $e$ different extensions of this simple module, i.e. the $S_0, \dots, S_{e-1}$ in our notation, with each of them being uniserial of length $p^d$ and multiplicity $(p^d - 1)/e$. The approach in [Al] also proved the uniseriality (and its length) of $kH$-indecomposables.

## 2.4 Walking around the Brauer tree

The idea that Green uses in his paper [Gr] is the following main theorem:

**Theorem 2.4.1**
We can construct a collection of $\mathcal{O}G$-short exact sequences:

$$\mathbf{E}_{2i} \quad : \qquad 0 \longrightarrow A_{2i+1} \longrightarrow W_{\delta(i)} \longrightarrow A_{2i} \longrightarrow 0$$

$$\mathbf{E}_{2i+1} \quad : \qquad 0 \longrightarrow A_{2i+2} \longrightarrow W_{i+1} \longrightarrow A_{2i+1} \longrightarrow 0$$

where $\delta$ is a permutation on the set $\{0, \dots, e-1\}$ (i.e., $\delta$ permutes the $e$ projecitve indecomposable modules), also $A_{n+2e} \cong A_n$ and $W_{n+e} \cong W_n$.

Once we have the above theorem, we get:

**Theorem 2.4.2**
There is a projective $\mathcal{O}G$-resolution of $A_0$:

$$\cdots \to W_0 \to W_{\delta(e-1)} \to W_{e-1} \to \cdots \to W_1 \to W_{\delta(0)} \to A_0 \to 0$$

**Proof**
First consider

$$\mathbf{E}_0 \quad : \qquad 0 \longrightarrow A_1 \longrightarrow W_{\delta(0)} \longrightarrow A_0 \longrightarrow 0$$

$$\mathbf{E}_1 \quad : \qquad 0 \longrightarrow A_2 \longrightarrow W_1 \longrightarrow A_1 \longrightarrow 0$$

$$\mathbf{E}_2 \quad : \qquad 0 \longrightarrow A_3 \longrightarrow W_{\delta(1)} \longrightarrow A_2 \longrightarrow 0$$

We form the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & & \\
0 \arrow[r] & A_3 \arrow[r] & W_{\delta(1)} \arrow[r] \arrow[dr, dashed, "\partial_2"'] & A_2 \arrow[r] \arrow[d] & 0 & & \\
 & & & W_1 \arrow[d, "g"'] \arrow[dr, dashed, "\partial_1 = f\circ g"] & & & \\
 & & 0 \arrow[r] & A_1 \arrow[r, "f"'] \arrow[d] & W_{\delta(0)} \arrow[r] & A_0 \arrow[r, "\partial_0"] & 0 \\
 & & & 0 & & &
\end{tikzcd}$$

By exactness at $A_n$, the $\partial_n$ are exact for all $n$. Iterating this process, form the desired long exact sequence. □

*Remark.* If the block $B$ is the principal block of $G$, then we can take $A_0 = \mathcal{O}$ the trivial $\mathcal{O}G$-module

We then think of these modules in the above resolution being directed edges, labelled by the subscripts of $W_{i+1}$ (or $W_{\delta(i)}$), going from vertex $A_{2i}$ (resp. $A_{2i+1}$) to vertex $A_{2i+1}$ (resp. $A_{2i+2}$). The graph formed using this view satisfies:

**Theorem 2.4.3**

(1) The graph is a cycle of $2e$ directed edges

(2) Character $\psi_n$ afforded by $A_n$ are vertices of the Brauer tree

(3) In each cycle of the walk, we visit each of $W_i$ twice

(4) We can identify the edges with the same module and opposite edges, the resulting graph will be the Brauer tree.

**Proof**

(1) Follows from Theorem 2.4.1

(2) See later, section 2.6

(3) The walk is a cycle of $2e$ edges, and $\delta$ is a permutation, hence a bijective mapping of set $\{0, \ldots, e-1\}$

(4) As the graph of the 'walk' is connected, such identification will get a connected graph. This new graph has $e$ edges by (3), and $e+1$ vertices by (2) and (3) together. Therefore it is a tree.

This is indeed the Brauer tree because

(a) By (2), the vertices correspond to module affording ordinary irreducible character

(b) Since $W_n$ are projective $\mathcal{O}G$-modules. The sequence $\mathbf{E}_n$ in theorem 2.4.1 splits. Hence,

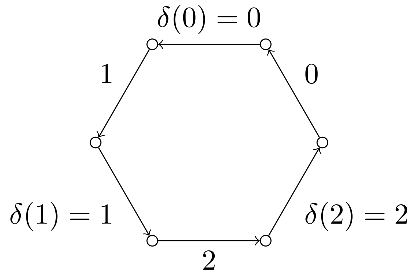
$$\begin{aligned} W_{\delta(i)} &= A_{2i+1} \oplus A_{2i} \\ W_{i+1} &= A_{2i+2} \oplus A_{2i+1} \end{aligned}$$

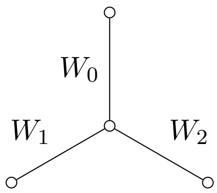
Satisfying relation (2.1)of being a Brauer tree.

□

*Remark.* The permutation $\delta$ plays an important role. As this determines the Brauer graph as a tree. Also note that not every permutation on $e$ letters can produce a same tree. See the following example.

**Example 2.4.4**
We now give an easy example. Suppose we get three projective indecomposables lying in block $B$. And permutation $\delta$ is trivial, i.e. fixes every point. Then the cyclic graph is

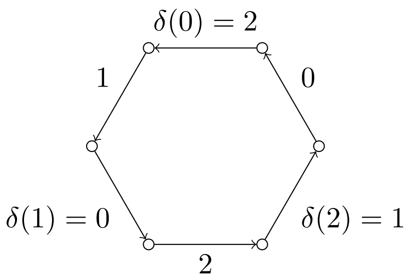


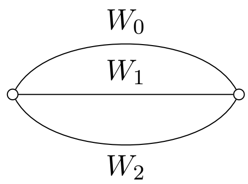

And the Brauer tree is the star: 


We now give an example when the permutation cannot form a tree out of the cyclic graph. Suppose the permutation $\delta$ is defined as

$$0 \mapsto 2 \quad , \quad 1 \mapsto 0 \quad , \quad 2 \mapsto 1$$

Then the graphs are:




The remaining of this chapter will be to prove Theorem 2.4.1. This will be done in a two stages:

(1) See how the permutation $\delta$ arises and construct similar sequence over $kG$

(2) Lifts the sequences to $\mathcal{O}G$

## 2.5 Permutation $\delta$ and sequences over $kG$

The set up of this section is as follows:

(1) $B$ a $kG$-block, with cyclic defect group $D$ of order $p^d$

(2) $P$ is the subgroup of order $p$ of $D$

(3) $H = N_G(P)$ $(\leq N_G(D))$ a subgroup of $G$

(4) $b$ is a $kH$-block

(5) Indecomposable $kH$-modules lying in $b$ are of form $T_{i,j} = T_i/\operatorname{Rad}^j(T_i)$, where $T_i$ is the projective indecomposable corresponding to the simple $S_i$, $i = 0, \dots, e-1$ (see Section 2.3)

Recall that, we have our analysis of $kH$-modules. We also have the Brauer's correspondence on $kG$-blocks and $kH$-blocks; the Green's correspondent on indecomposable $kG$- and $kH$-modules. All the above useful information and techniques over the residue field $k$. We also have lifting which allows us the lift projective indecomposables in $kG$ to $\mathcal{O}G$. Therefore, we now try to find the permutation $\delta$ via studying the $kG$-modules, and derive a similar result of Theorem 2.4.1, i.e. constructing some $kG$-sequences, and attempt to lift them to $\mathcal{O}G$-sequence.

The aim of this section is to show:

**Theorem 2.5.1**

(1) $B$ has $e$ non-isomorphic simple $kG$-modules $V_0, \dots, V_{e-1}$ We let $\overline{W_0}, \dots, \overline{W_{e-1}}$ be the corresponding projective indecomposable $kG$-module, i.e. $\overline{W_i}/\operatorname{Rad}(\overline{W_i}) \cong V_i$ $\forall i = 0, \dots, e-1$

(2) There is an ordering of these $kG$-simples and a permutation $\delta$ on the set $\{0, \dots, e-1\}$ such that:

$$\operatorname{Soc}(gS_i) \cong V_i \tag{2.7}$$
$$gS_i/\operatorname{Rad}(gS_i) \cong V_{\delta(i)} \tag{2.8}$$

where $f$ and $g$ are the Green's correspondence as in Theorem 1.6.1

(3) For each $i = 0, \dots, e-1$, there is an $kG$-short exact sequence

$$\mathbf{F}_{2i} : \quad 0 \longrightarrow \Omega g S_i \longrightarrow \overline{W_{\delta(i)}} \longrightarrow gS_i \longrightarrow 0$$

$$\mathbf{F}_{2i+1} : \quad 0 \longrightarrow gS_{i+1} \longrightarrow \overline{W_{i+1}} \longrightarrow \Omega g S_i \longrightarrow 0$$

where $\Omega$ denotes the Heller operator.
In fact, as the ordering of simple $kH$-modules $S_i$ satisfies $S_{n+e} \cong S_n$ $\forall n \in \mathbb{Z}$, we can extend the definition of $\mathbf{F}_n$ to all $n \in \mathbb{Z}$. Therefore, $\overline{W_{n+e}} \cong \overline{W_n}$ and $\mathbf{F}_n \cong \mathbf{F}_{n+2e}$, for all $n \in \mathbb{Z}$

We first exploit some useful result which help us proving this.

Let $I$ be the indexing set of the $kH$-simples and take $S_i$, $i \in I$
Let $J$ be the indexing set of the $kG$-simples and take $V_j$, $j \in J$

Since $b$ is the Brauer corresponding block of $B$, they have the same defect group, and therefore, $S_i, V_j$ are $D$-projective.

Also recall Theorem 1.5.4 that a block contains simple projective modules if and only if it has defect zero, so all of $S_i$ and $V_j$ are non-projective. Hence vertex of $S_i$ and vertex of $V_j$ are both in $\mathfrak{Z} = \{Q \leq D | Q \neq \{1\}\}$
Applying Green's correspondence and Proposition 1.7.8:

- $fV_j$ are non-projective indecomposable $kH$-modules lying in $b$
- $gS_i$ are non-projective indecomposable $kG$-modules lying in $B$

Now combine this with the following results:

- Lemma 1.6.2: $(U, V)^G_{\{1\}} \cong (fU, fV)^H_{\{1\}}$ where $U, V$ are $kG$-modules
- Theorem 1.6.1 (3): $fgU = U, gfV = V$ for any $kG$-modules $U, V$
- Corollary 1.2.6: $M$ projective-free (resp. simple), $N$ simple (resp. projective-free), then $(M, N)^G \cong (M, N)^G_1$

we get:

$$(S_i, fV_j)^H \cong (gS_i, V_j)^G \tag{2.9}$$
$$(fV_j, S_i)^H \cong (V_j, gS_i)^G \tag{2.10}$$

The above result allows us to transfer the maps of $kH$-modules to $kG$-modules, which is essentially what we need to use to construct the sequences $\mathbf{F}_n$. We now use the result of our analysis of the $kH$-modules to get, for each $j \in J$,

$$fV_j \cong T_{h(j),v(j)} \quad \text{some } h(j) \in I, v(j) \in \{1, \ldots, p^d - 1\} \tag{2.11}$$

and $fV_j$ is uniserial with $fV_j/\operatorname{Rad}(fV_j) \cong S_{h(j)}$ and $\operatorname{Soc}(fV_j) \cong S_{h'(j)}$ for some $h'(j) \in I$.

### Proof of Theorem 2.5.1 (1)

To prove point (1) of the theorem, we want a bijection between $kH$-simples in $b$ and $kG$-simples in $B$. The above results give us an idea to consider the two following maps:

$$\begin{array}{rrcl} & \{kG\text{-simples in } B\} & \to & \{kH\text{-simples in } b\} \\ \alpha: & V_j & \mapsto & fV_j/\operatorname{Rad}(fV_j) \\ \beta: & V_j & \mapsto & \operatorname{Soc}(fV_j) \end{array}$$

**Claim:** $\alpha$ and $\beta$ are bijective. In particular, Theorem 2.5.1 (1) follows immediately

**Proof of Claim:**
Injective:
Given distinct $V_{j_1}, V_{j_2}$ simple $kG$-modules in $B$. (2.11) says that they have composition length $v(j_1)$ and $v(j_2)$ respectively, and have 'top' $S_{h(j_1)}, S_{h(j_2)}$ respectively.

The analysis of indecomposable $kH$-modules in section 2.3 implies that the 'top' and the composition length determines the module uniquely, hence, $fV_{j_1}$ is a quotient of $fV_{j_2}$ (swap the two if necessary)
$\Rightarrow \quad (fV_{j_1}, fV_{j_2})^H \neq 0$
$\Rightarrow \quad (fV_{j_1}, fV_{j_2})^H_1 \neq 0$ (by Corollary 1.2.6)
$\Rightarrow \quad (V_{j_1}, V_{j_2})^G_1 \neq 0$ (by Lemma 1.6.2)

$\Rightarrow \quad (V_{j_1}, V_{j_2})^H \neq 0$ which contradict Schur's Lemma
$\Rightarrow \quad \alpha$ injective.

Similarly, the 'bottom' and the composition length determines the modules uniquely. Suppose $fV_{j_1}$ and $fV_{j_2}$ have the same 'bottom', then $fV_{j_1}$ is a submodule of $fV_{j_2}$ (swap if necessary), by a dual argument as for $\alpha$, we get $\beta$ is injective as well.

Surjective:
We want every $kH$-simple $S_i$ can be expressed as form $fV_j/\operatorname{Rad}(V_j)$ some $j$, and $\operatorname{Soc}(V_{j'})$ some $j'$
Again, Green's correspondence is vital. We first take the indecomposable $kG$-module $gS_i$, it is a homomorphic image of some simple $kG$-module $V_j$ some $j$
$\Rightarrow \quad (V_j, gS_i)^G \neq 0$
$\Rightarrow \quad (fV_j, S_i)^H \neq 0$ by (2.10)
$\Rightarrow \quad S_i$ is a homomorphic image of $fV_j$ (by simplicity of $S_i$)
$\Rightarrow \quad S_i \cong fV_j/\operatorname{Rad}(fV_j)$ as $fV_j$ uniserial with each composition factor being distinct
$\Rightarrow \quad \alpha$ surjective

Similarly, there is a $kG$-module $V_{j'}$ which is a homomorphic image of $gS_i$
$\Rightarrow \quad (gS_i, V_j)^G \neq 0$
$\Rightarrow \quad (S_i, fV_j)^H \neq 0$ by (2.9)
$\Rightarrow \quad S_i \cong \operatorname{Soc}(fV_j)$ by simplicity of $S_i$ and as $fV_j$ uniserial with all composition factor distinct
$\Rightarrow \quad \beta$ surjective □

### Proof of Theorem 2.5.1 (2)

Consider this map:

$$\begin{array}{rcll} \{kG\text{-simples in } B\} & \leftrightarrow & \{kH\text{-simples in } b\} & \\ gS_i/\operatorname{Rad}(gS_i) & \leftarrow\!\shortmid & S_i & :\alpha' \\ \operatorname{Soc}(gS_i) & \leftarrow\!\shortmid & S_i & :\beta' \end{array}$$

However, we do not need to go through the same process as above again, instead, as we already have bijectivity of the two sets (which implies $I = J$), it suffice to show:

**Claim:** For each $j \in I$, there is a unique $i$ and $\delta(i)$ such that

$$\begin{array}{rcll} \dim_k(V_j, gS_k)^G & = & 1 \text{ if } k = i, & 0 \text{ otherwise} \\ \dim_k(gS_k, V_j)^G & = & 1 \text{ if } k = \delta(i), & 0 \text{ otherwise} \end{array}$$

**Proof of Claim:**
This can be easily done using the relations (2.9), (2.10):

$$\dim_k(V_j, gS_k)^G = \dim_k(fV_j, S_k)^H$$

Structure of indecomposable $kH$-modules in $b$ implies that the later is 1 for when $S_k \cong fV_j/\operatorname{Rad}(fV_j)$, and zero otherwise. Now take $i$ as such $k$

Similarly,

$$\dim_k(gS_k, V_j)^G = \dim_k(S_k, fV_j)^H = \begin{cases} 1 \text{ if } S_k \cong \operatorname{Soc}(fV_j) \\ 0 \text{ otherwise} \end{cases}$$

Take $\delta(i)$ as such $k$ □

Now, reorder $V_i$ such that

$$\begin{aligned} V_i &\cong \operatorname{Soc}(gS_i) \\ V_{\delta(i)} &\cong gS_i/\operatorname{Rad}(gS_i) \end{aligned}$$

Moreover, $\delta$ is now a one-to-one bijection of the set $I$, hence a permutation. Now we have proved Theorem 2.5.1 (2).

### Proof of Theorem 2.5.1 (3)

We now aim to construct the short exact sequence as stated. First we fix an $i \in I$

From the last result $gS_i/\operatorname{Rad}(gS_i) \cong V_{\delta(i)}$ and the projectivity of $\overline{W_{\delta(i)}}$, we see that there is a surjective map $\overline{W_{\delta(i)}} \twoheadrightarrow gS_i$. By indecomposability of $\overline{W_{\delta(i)}}$, $\overline{W_{\delta(i)}}$ is the projective cover of $gS_i$, and hence we get short exact sequence $\mathbf{F}_{2i}$ as required. (Recall $\Omega gS_i = \ker(\overline{W_{\delta(i)}} \to gS_i)$)

Now as $\overline{W_{i+e}} \cong \overline{W_i}$ for all $i \in \mathbb{Z}$ (see statement of Theorem 2.5.1 (3)) using $\operatorname{Soc}(gS_{i+1}) \cong V_{i+1} \cong \overline{W_{i+1}}/\operatorname{Rad}(\overline{W_{i+1}})$, and since $\overline{W_{i+1}}$ projective implies it is injective as well, there is an injective map $gS_{i+1} \hookrightarrow \overline{W_{i+1}}$, (i.e. $\overline{W_{i+1}}$ is an injective hull of $gS_{i+1}$), so we get an exact sequence

$$0 \to gS_{i+1} \to \overline{W_{i+1}} \to \Omega^{-1}\, gS_{i+1} \to 0$$

Recall at the end of section 2.3 that we deduced $\Omega^2\, S_i \cong S_{i+1}$, and Theorem 1.6.3 which says Heller operator commutes with Green's correspondence ($f$ and $g$)

$\Rightarrow \quad gS_{i+1} \cong g(\Omega^2\, S_i) \cong \Omega^2(gS_i)$

$$0 \to gS_{i+1} \to \overline{W_{i+1}} \to \underbrace{\Omega\, gS_i}_{\cong \Omega^{-1}(\Omega^2\, gS_i)} \to 0$$

This is the sequence $\mathbf{F}_{2i+1}$ as required. We now completed the proofs of this section. The above proof comes partly from [Al] and partly from [Gr].

## 2.6 Lifting results from $kG$ to $\mathcal{O}\,G$

In this section, we finish all the statements that were left unproven in section 2.4. The first step is to construct the sequences $\mathbf{E}_{2i}$ and $\mathbf{E}_{2i+1}$, by lifting the $kG$-short exact sequences $\mathbf{F}_{2i}$ and $\mathbf{F}_{2i+1}$. Most of the proofs in this section originate from [Gr].

From basic representation theory, there is a unique (up to isomorphism) lift of projective $kG$-modules. Therefore, we first set $W_n$, as appear in sequences $\mathbf{E}_n$, being the lift of $\overline{W_n}$ as appear in sequences $\mathbf{F}_n$.

To simplify notation, we set $B_{2i} := gS_i$ and $B_{2i+1} := \Omega\, gS_i$, for all $i \in \mathbb{Z}$

The strategy we take to proof the theorems is as follows

**Step 1**: If some $B_m$ can be 'lifted' to an $\mathcal{O}\,G$-module, then all the sequences $\mathbf{F}_n$ can be lifted to $\mathbf{E}_n$.

**Step 2**: Given the condition in Step 1, let $\psi_n$ denote character afforded by $A_n$ (as appear in sequence $\mathbf{E}_n$). First show that $\psi_{2n+e} = \psi_e$, then deduce $A_{n+2e} \cong A_n$ for all $n \in \mathbb{Z}$

**Step 3**: Prove that it is possible to lift the $B_m$.

We then complete the proof for the construction of the Brauer tree.

**Lemma 2.6.1 (Step 1)**
If $M$ is an $RG$-lattice such that $\overline{M} := M/\mathfrak{m}\, M \cong B_m$ for some fixed $m$. Then we can construct $RG$-lattices $A_n$ and sequences $\mathbf{E}_n$ with $A_m = M$ and $\overline{A} \cong B_n \;\; \forall n \in \mathbb{Z}$.
Hence the sequences $\mathbf{F}_n$ are lifted to $\mathbf{E}_n$, for all $n \in \mathbb{Z}$

**Proof**
This is where Theorem 1.3.4 comes into play. This theorem says that we can lift the sequence $\mathbf{F}_m$:

$$\mathbf{F}_m : \qquad 0 \to B_{m+1} \to \overline{W_x} \to B_m \to 0$$

(Note $x \in \mathbb{Z}/e\,\mathbb{Z}$, depends on $m$ and the permutation $\delta$) to the short exact sequence:

$$0 \to N \to W_x \to M \to 0$$

So by setting $A_m := M$ and $A_{m+1} := N$, we get the sequence $\mathbf{E}_m$. Hence $\mathbf{F}_m$ has lifted to $\mathbf{E}_m$. Now as $B_{m+1}$ is liftable, we can then invoke Theorem 1.3.4 repeatedly to get $\mathbf{F}_n$ for all $n > m$.

For $n < m$, we use the dual of $\mathbf{F}_n$. As dualising preserve exactness, we have dual of $\mathbf{F}_{m-1}$:

$$0 \to B_{m-1}^* \to \overline{W_y}^* \to B_m^* \to 0$$

Invoke Theorem 1.3.4 again and dualise the resulting short exact sequence, we get $\mathbf{E}_{m-1}$ (the lift of $\mathbf{F}_{m-1}$). Repeat this process and we can get $\mathbf{E}_n$ being the lift of $\mathbf{F}_n$ for all $n < m$. $\square$

To get step 2, we make use of the character, and investigate its relation with the Brauer tree.

**Lemma 2.6.2 (Step 2a)**
Suppose character $\psi_m$ of $A_m$ is a vertex in the Brauer tree. Then character $\psi_n$ of $A_n$ are all in the Brauer tree and $\psi_{n+2e} = \psi_n$

**Proof**
By projectivity of $W_n$ (for all $n \in \mathbb{Z}$), the sequences $\mathbf{E}_n$ splits. Let $\eta_n$ denote character of $W_n$, then we have

$$\eta_{\delta(i)} = \psi_{2i} + \psi_{2i+1} \quad , \quad \eta_{i+1} = \psi_{2i+1} + \psi_{2i+2} \tag{2.12}$$

and since $\psi_m$ is a vertex of the Brauer tree, $\psi_{m+1}, \psi_{m-1}$ are also as well, repeating this we get the first statement.

To see that $\psi_{n+2e} = \psi_n$, suppose $n = 2i$, by the condition on Brauer tree, we have $2e$ equations:

$$\begin{aligned}
\eta_{\delta(i)} &= \psi_n + \psi_{n+1} \\
\eta_{i+1} &= \psi_{n+1} + \psi_{n+2} \\
&\vdots \\
\eta_{\delta(i+e-1)} &= \psi_{n+2e-2} + \psi_{n+2e-1} \\
\eta_{i+e} &= \psi_{n+2e-1} + \psi_{n+2e}
\end{aligned}$$

$$\Rightarrow \qquad \sum_{j=0}^{e-1} \eta_{\delta(i+j)} - \eta_{i+j+1} = \psi_n - \psi_{n+2e}$$

But left hand side is 0. Hence our statement. For $n = 2i+1$, proceed similarly. $\square$

**Lemma 2.6.3 (Step 2b)**
Given condition as in above lemma, $A_n \cong A_{n+2e}\ \ \forall n \in \mathbb{Z}$

**Proof**
The relation (2.1), of edge and the two vertices at its ends, says that the $KG$-module $K \otimes_{\mathcal{O}} W_i$ (correspond to $\eta_i$) has unique submodules $Y_{i_1}, Y_{i_2}$ affording characters $\chi_{i_1}, \chi_{i_2}$ respectively, and they are the only two affording characters being vertices of the Brauer tree.

This implies that $W_i \cap Y_{i_1}, W_i \cap Y_{i_2}$ (c.f. Section 1.3, the way to switch from $KG$-world to $\mathcal{O}G$-world is via intersection) are the only $\mathcal{O}$-pure submodules of $W_i$ (see remark) which affords character in the Brauer tree.

Now fix an $n \in \mathbb{Z}$, $\mathbf{E}_{n-1}$ implies that $A_n$ is isomorphic to a $\mathcal{O}$-pure submodule of $W_i$ for some $i$ (symbolically, $A_n \cong W_i \cap Y_{i_j} \gneqq W_i$ some $i \in \mathbb{Z}/e\mathbb{Z}, j \in \{1,2\}$)
$\mathbf{E}_{n+2e-1}$ implies $A_{n+2e}$ is isomorphic to a $\mathcal{O}$-pure submodule of $W_{i+e} = W_i$
(symbolically, $A_n \cong W_i \cap Y_{i_l} \gneqq W_i$ some $i \in \mathbb{Z}/e\mathbb{Z}, l \in \{1,2\}$) But we also showed in the last lemma that $A_n$ and $A_{n+2e}$ affords the same character, so $W_i \cap Y_{i_j} \cong W_i \cap Y_{i_l}$, hence $A_n \cong A_{n+2e}$.
□

*Remark.*

(1) An $\mathcal{O}G$-module $M$ is an *$\mathcal{O}$-pure submodule of* $\mathcal{O}G$-module $L$ if, for all $r \in \mathcal{O}$, $rM = M \cap rL$.

(2) The two main structure that make this proof works is $W_i$ is a projective indecomposable, and $W_i \cong W_{i+e}$. In general, as noted in section 2.1, $RG$-lattice affording the same character does not necessarily imply isomorphism.

**Lemma 2.6.4 (Step 3)**
There is an indecomposable $\mathcal{O}G$-lattice $M$ lying in $B$ such that $\overline{M} = B_m$ for some $m$

**Proof**
If $B = B_0(G)$ the principal block of $G$. We can choose $M = A_0 = \mathcal{O}$ the trivial module.

For general $B$, we first take $M$ as an $\mathcal{O}G$-lattice in $B$ affording character $\chi_1$ corresponding to an vertex of the Brauer tree, its reduction $\overline{M}$ is indecomposable. This is possible by taking $M$ as a quotient of the projective indecomposable $W_i$, some $i \in \{0, \ldots, e-1\}$ such that $\chi_1$ is a constituent of $\eta_i$. (see [Th] Theorem 1 and Corollary, or [PO] Lemma 8.4A, proof is similar to the proof in Step 2b above)

Using a corollary of the Green's correspondence (depending on $G, H, D$ with $H = N_G(P)$), we get

$$M \downarrow_H = L \oplus (\text{projectives}) \oplus (\text{modules not in } b)$$

(see [Bn] Lemma 6.5.1), where $L = fM$ the Green's correspondent of $M$. Both indecomposable $\mathcal{O}G$-modules. Their reduction $\overline{L}, \overline{M}$ are also Green's correspondent (over $kG$) to each other, i.e. $\overline{L} = f\overline{M}, \overline{M} = g\overline{L}$.

Using Proposition 1.7.8, and the fact that $L$ lies in the block which has a Brauer tree (see section 2.3), these imply that $L$ has a character corresponding to a vertex of the Brauer tee. Combining with the results on the structure of $kH$-indecomposables, we have

$$\overline{L} \cong S_i \quad \text{or} \quad \Omega\, S_i \ \text{ for some } i$$

$\Rightarrow\quad \overline{M} \cong gS_i$ or $g\,\Omega\, S_i$ $(= \Omega\, gS_i)$
$\Rightarrow\quad \cong B_m$ some $m$, as required □

# Chapter 3

# Example

## 3.1 Brauer tree of the principal block of $S_5$

In this section, we will compute a Brauer tree when $G = S_5$ ($|S_5| = 2^3 \cdot 3 \cdot 5$) with $p = 5$ the prime of interest. Basic results of character theory are assumed. Few other results from character theory will be quoted, for details and proofs of those results, the reader can refer [Nav]. By abusing notation, for an ordinary character $\chi$, we also use $\chi$ to denote the $KG$-module (sometimes $\mathcal{O}G$) affording it. For modular (Brauer) character $\phi$, we use $\phi$ to the corresponding $kG$-module. Hence, 'a character lies in a $p$-block $B$' means that the module affording it lies in $B$.

Let $k$ be the splitting field of the cyclotomic polynomial $\Phi_{24}(x)$ splits (i.e. Splitting field which contains the 24-th root of unity; the number 24 comes from $2^3 \cdot 3$)

Since 5 is the highest power of 5 dividing the group order, by Sylow Theorem, there is a 5-subgroup of $S_5$, which is a group of order 5, hence it is the cyclic group $C_5$.

Our first goal is to find the blocks of the group algebra which has defect group $C_5$. To do this we use the *central character*.

**Lemma 3.1.1**
The central character associated to an irreducible ordinary character $\chi$, denote $\omega_\chi$ can be computed using the formula:

$$\omega_\chi(\widehat{\mathcal{C}}) = \frac{|\mathcal{C}|\chi(g)}{\deg \chi}$$

where $\mathcal{C}$ is the conjugacy class, $g \in \mathcal{C}$, $\widehat{\mathcal{C}} = \sum_{g' \in \mathcal{C}} g' \in Z(kG)$

The following lemma helps us determine the $kG$-blocks for (arbitrary) $G$

**Lemma 3.1.2**
Let $\omega_\chi$ and $\omega_\psi$ be central character associated to irreducible ordinary characters $\chi$ and $\psi$. Then $\chi$ and $\psi$ belongs to the same $p$-block (i.e. the module affording $\chi$ and module affording $\psi$ has its reduction mod $\mathfrak{m}$ lying in the same block of $kG$ with char $k = p$) if and only if we have:

$$\omega_\chi(\widehat{\mathcal{C}}) \equiv \omega_\psi(\widehat{\mathcal{C}}) \mod p$$

for all $p$-regular classes $\mathcal{C}$

Back to our example $G = S_5$, first look at the ordinary character table and calculate the $p$-blocks using the above lemmas:

Ordinary character table of $S_5$

| $\mathcal{C}$ | 1 | (12)(34) | (123) | (12345) | (12) | (1234) | (12)(345) |
|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_a$ | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | 2 | 0 | $-1$ |
| $\chi_{4a} = \chi_4 \otimes \chi_a$ | 4 | 0 | 1 | $-1$ | $-2$ | 0 | 1 |
| $\chi_5$ | 5 | 1 | $-1$ | 0 | $-1$ | 1 | $-1$ |
| $\chi_{5a} = \chi_5 \otimes \chi_a$ | 5 | 1 | $-1$ | 0 | 1 | $-1$ | 1 |
| $\chi_6$ | 6 | $-2$ | 0 | 1 | 0 | 0 | 0 |

*Remark.* The arrangement of the table is such that the left 4 columns correspond to the the character value of $A_5$, also note that $\chi_6$ splits into two characters (direct sum of modules) of degree 3 in $A_5$.

Central characters of $S_5$ and the $p$-blocks

| $\mathcal{C}$ | 1 | (12)(34) | (123) | (12345) | (12) | (1234) | (12)(345) | mod 5 |
|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 15 | 20 | 24 | 10 | 30 | 20 | $B_0$ |
| $\chi_a$ | 1 | 15 | 20 | 24 | $-10$ | $-30$ | $-20$ | $B_0$ |
| $\chi_4$ | 1 | 0 | 5 | $-6$ | 5 | 0 | $-5$ | $B_0$ |
| $\chi_{4a}$ | 1 | 0 | 5 | $-6$ | $-5$ | 0 | 5 | $B_0$ |
| $\chi_5$ | 1 | 3 | $-4$ | 0 | $-2$ | 6 | $-4$ | $B_1$ |
| $\chi_{5a}$ | 1 | 3 | $-4$ | 0 | 2 | $-4$ | 4 | $B_2$ |
| $\chi_6$ | 1 | $-5$ | 0 | 4 | 0 | 0 | 0 | $B_0$ |

In fact, $B_1$ and $B_2$ are distinct block and both of them are of defect zero, due to the following lemma:

**Lemma 3.1.3**
Let (arbitrary) group $G$ has order $p^a b$ such that $(p, b) = 1$. Ordinary character $\chi$ lies in block $B$ of defect $d$ if $p^{a-d} | \deg \chi$

$B_0$ is the principal block of $kG$ as $B_0$ contains the trivial character (hence trivial module), and this has defect group as the Sylow $p$-subgroup, i.e. $C_5$.

Our next goal is to determine the Brauer characters, this is equivalent to determining the $kG$-simples.

By deleting the 5-singular class of $\chi_1$ and $\chi_a$, we get two irreducible Brauer character of $S_5$ as both of them are of degree 1 and the values of them on classes (12), (1234), (12)(345) are different. Denote them as $\phi_1, \phi_a$ respectively.

The next step is to take the ordinary character table of $A_5 \leq S_5$, then delete the columns of 5-singular classes and identified the characters with the same character values on the 5-regular classes. The restriction of irreducible Brauer character of $S_5$ will have the same value on irreducible Brauer character of $A_5$ (this is due to the structure of $S_5$ and $A_5$ that the conjugacy classes of $S_5$ does not split in $A_5$ unless for the 5-singular class of $S_5$). So determining irreducible Brauer characters of $A_5$ helps us to determine the irreducible Brauer characters of $S_5$

'Reduced' ordinary character table of $A_5$

| $\mathcal{C}$ | 1 | (12)(34) | (123) |
|---|---|---|---|
| $\chi'_1$ | 1 | 1 | 1 |
| $\chi'_3$ | 3 | $-1$ | 0 |
| $\chi'_4$ | 4 | 0 | 1 |
| $\chi'_5$ | 5 | 1 | $-1$ |

We can see easily that $\chi'_1 + \chi'_3 = \chi'_4$. So $\chi'_4$ is not an irreducible Brauer character.
$\chi'_3$ and $\chi'_5$ can not be expressed as an linear combination of ordinary characters of smaller degree (on the 5-regular classes). Hence $\chi'_3$ and $\chi'_5$ are the irreducible ordinary Brauer character of $A_5$.

As a result, there are two degree 5 irreducible Brauer character, which comes from $\chi_5$ and $\chi_{5a}$ by deleting the (12345) column (the 5-singular class), call them $\phi_5, \phi_{5a}$. Also note that as mentioned above, these two Brauer characters are not the same $p$-blocks as the other.

$\chi_4$ should be the direct sum of a degree 1 character and a degree 3 character, and these two characters are irreducible. The degree 3 character takes value 3, $-1$, 0 on classes 1, (12)(34), (123) respectively. This can be deduced using the fact that $\chi'_4 = \chi'_1 + \chi'_3$ on the 5-regular classes, and the fact that number of irreducible Brauer character (which equals to number of $kG$-simples) is equal to number of the 5-regular classes. We remain to determine its value on classes (12), (1234), (12)(345).

Now we consider the permutation representation of $S_5$ on 5 points over $k$. This is not irreducible and contains the trivial representation as constituent, we now attempt to find a 3-dimensional constituent of this permutation representation.

Let $V$ be the vector space associated to this permutation representation, hence $V = \langle v_1, v_2, v_3, v_4, v_5 \rangle$. It has a 4-dimensional subspace $W = \{\sum_{i=1}^5 \lambda_i v_i \in V | \sum_{i=1}^5 \lambda_i = 0\}$. This in fact gives the same character value as the 4-dimensional space affording $\chi_4$ on the 5-regular classes.

From above discussion, we guessed that $W$ has a 1-dimensional and 3-dimensional constituent. Indeed, $U = \{\sum_{i=1}^5 \lambda_i v_i \in W | \lambda_i = \lambda_j \ \forall i, j\}$ is another 1-dimensional subspace of $W$ as $U$ is the vector space $\langle (1,1,1,1,1) \rangle$, this is indeed a subspace of $W$ as $5\lambda_i \equiv 0 \mod 5 \ \forall \lambda_i \in k$. Now we have the 3-dimensional representation $W/U$.

Moreover, it is straightforward that $W$ is isomorphic to the trivial representation (over $k$). Hence we have a degree 3 irreducible Brauer character which comes from $\chi_4 - \chi_1$ on the 5-regular classes. Call this $\phi_3$ and its value on classes (12), (1234), (12)(345) are 1, $-1$, $-2$. The remaining degree 3 irreducible Brauer character is then just $\phi_{3a} = \phi_3 \otimes \phi_a$. Hence we deduced the Brauer character table of $S_5$:

Brauer character table of $S_5$

| $\mathcal{C}$ | 1 | (12)(34) | (123) | (12) | (1234) | (12)(345) |
|---|---|---|---|---|---|---|
| $\phi_1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\phi_a$ | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| $\phi_3$ | 3 | $-1$ | 0 | 1 | $-1$ | $-2$ |
| $\phi_{3a}$ | 3 | $-1$ | 0 | $-1$ | 1 | 2 |
| $\phi_5$ | 5 | 1 | $-1$ | $-1$ | 1 | $-1$ |
| $\phi_{5a}$ | 5 | 1 | $-1$ | 1 | $-1$ | 1 |

To determine the Brauer tree, we first compute the decomposition matrix $D$, then use the formula $D^\top D = C$ to compute the Cartan matrix $C$. In section 2.2, we saw how a Brauer tree can be determined using the Cartan matrix (or vice versa).

Using the Brauer character table and ordinary character, we get the decomposition matrix $D = (d_{\chi\phi})$, where $\chi$ is an irreducible ordinary character and $\phi$ irreducible Brauer character such that

$$\phi = d_{\chi\phi}\chi + \cdots$$

Decomposition Matrix of $S_5$ thus is as follows:

| | $\phi_1$ | $\phi_a$ | $\phi_3$ | $\phi_{3a}$ |
|---|---|---|---|---|
| $\chi_1$ | 1 | 0 | 0 | 0 |
| $\chi_a$ | 0 | 1 | 0 | 0 |
| $\chi_4$ | 1 | 0 | 1 | 0 |
| $\chi_{4a}$ | 0 | 1 | 0 | 1 |
| $\chi_6$ | 0 | 0 | 1 | 1 |

$$\Rightarrow \qquad C = D^\top D = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

This gives the Brauer tree of the principal block of $S_5$ as follows:

$$\begin{tikzcd} \chi_1 \arrow[r, dash, "\phi_1"'] & \chi_4 \arrow[r, dash, "\phi_3"'] & \chi_6 \arrow[r, dash, "\phi_{3a}"'] & \chi_{4a} \arrow[r, dash, "\phi_a"'] & \chi_a \end{tikzcd}$$

We now extend this example to various techniques and results that has appeared in this essay.

## 3.2 The walk around Brauer tree

We now construct the $\mathcal{O}G$-resolution

$$\cdots \to W_0 \to W_{\delta(e-1)} \to W_{e-1} \to \cdots \to W_1 \to W_{\delta(0)} \to A_0 \to 0$$

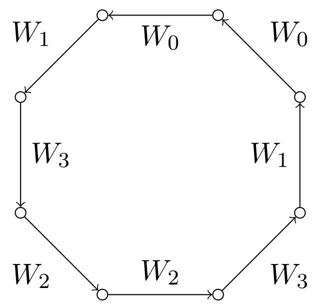
as described in Theorem 2.4.2. As noted under the remark of the theorem, we are in the principal block, we can simply take $A_0 = \mathcal{O}$ (where $\mathcal{O}\,/\,\mathfrak{m} = k$, which we have already set the constraint at the start of the chapter). So we get a resolution for the trivial module. As projective indecomposable module correspond to simple module, the $W_n$ corresponds to irreducible Brauer character, we list the correspondence in the following table, projective indecomposable modules are listed in order they appear in the resolution above.

| projective indecomposable $\mathcal{O}G$-module | $W_{\delta(0)}$ | $W_1$ | $W_{\delta(1)}$ | $W_2$ | $W_{\delta(2)}$ | $W_3$ | $W_{\delta(3)}$ | $W_0$ |
|---|---|---|---|---|---|---|---|---|
| irreducible Brauer character | $\phi_1$ | $\phi_3$ | $\phi_{3a}$ | $\phi_a$ | $\phi_a$ | $\phi_{3a}$ | $\phi_3$ | $\phi_1$ |

Hence we have the permutation $\delta$ as follows:

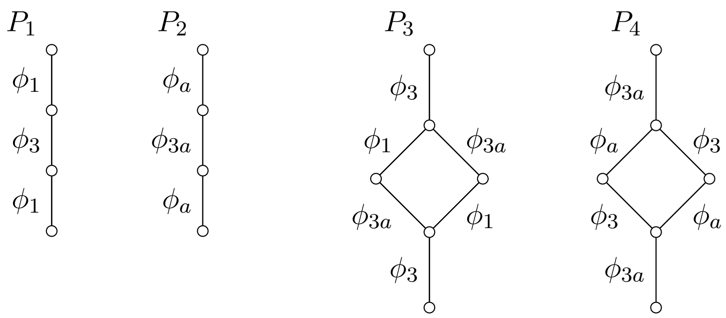
$$\delta : \begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 1 \end{array}$$

And the following is the graph of the cyclic walk: (the modules labelled inside the cycle are the $W_{\delta(i)}$; those labelled outside the cycle are the $W_i$, $i = 0, 1, 2, 3$)

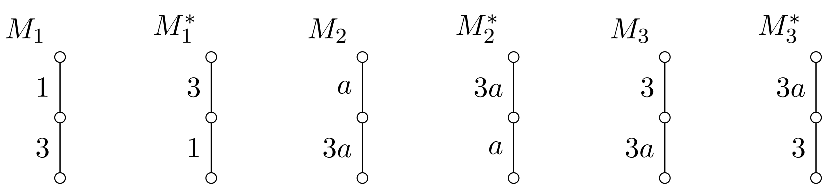


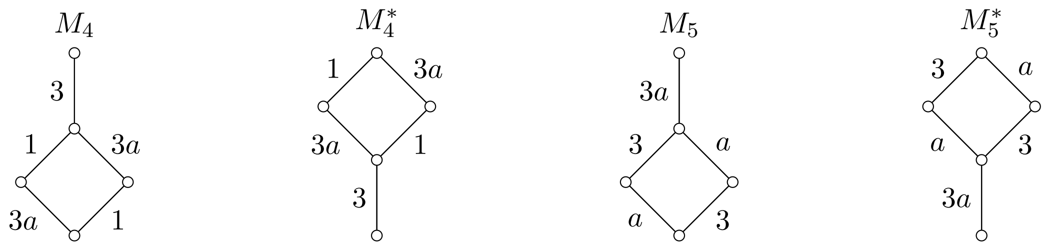

## 3.3 Module diagram of indecomposables

As mentioned in section 2.2, the Brauer tree allows us to draw out the module diagrams for the projective indecomposables:




In fact, in this particular example, we can deduce the module diagram of *all* the (non-projective) $kG$-indecomposables using the above four diagrams. Most of these comes from taking quotients and modules of the projective indecomposables: (we now abbreviate the modules by the subscript of their corresponding Brauer character)






There are in fact two more non-projective indecomposables:


A non-rigorous way to see how this two modules arises is that we attempt to 'glue' $M_1$ with other non-projective indecomposables, and the only possible one is to 'glue' $M_1$ and $M_2$. The other non-projective indecomposable comes from taking the dual of this new one. The details are omitted here. We can in fact find out these are all the indecomposable $kG$-modules by using the Green's correspondent, by further looking at the Brauer tree of $kH$-principal block.

## 3.4 Brauer tree of principal block of $H = N_G(P)$

In section 2.3, the results we obtained implies the Brauer corresponding block of $B$ in $kH$ (denoted as $B'$) has a Brauer tree with the number of $kH$-simples equal to number of $kG$-simples (i.e. the number $e$). It is a star shaped Brauer tree, with exceptional vertex in the centre with multiplicity $(p^d - 1)/e$.

In our example, $G = S_5$, $P = D = C_5$, $H = N_G(P) = G_{20}$ (The Frobenius group of order 20). The Brauer corresponding block is the principal block, by Brauer's Third Main Theorem, call this $b_0$. We now investigate the Brauer tree of $b_0$

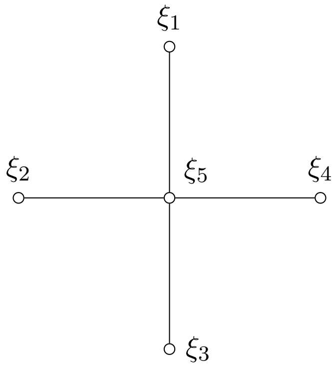
Let $H = \langle a, b | a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$ $(a = (12345), b = (2453))$. The ordinary character table is as follows:

Brauer character table of $G_{20}$

| $\mathcal{C}$ | 1 | $a$ | $b$ | $b^2$ | $b^3$ |
|---|---|---|---|---|---|
| $\xi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\xi_2$ | 1 | 1 | $i$ | $-1$ | $-i$ |
| $\xi_3$ | 1 | 1 | $-1$ | 1 | $-1$ |
| $\xi_4$ | 1 | 1 | $-i$ | $-1$ | $i$ |
| $\xi_5$ | 4 | -1 | 0 | 0 | 0 |

(Note: $i = \sqrt{-1}$)

It is easy to see that $\xi_5 = \sum_{i=1}^{4} \xi_i \mod 5$ on the 5-regular classes (representatives: $1, b, b^2, b^3$). And since number of 5-regular classes equal to number of irreducible Brauer character, can see that all the irreducible Brauer character are 1-dimensional, arise from $\xi_1, \ldots, \xi_4$ by deleting the column $a$. This also shows that Brauer tree is, as showed in section 2.3, star shaped. The vertex at the centre, correspond to $\xi_4$, has multiplicity $(p^d - 1)/e = 1$, as shown in the following figure.




We know, again from section 2.3, that there are $p^d - 1 = 4$ indecomposable $kH$-module lying in $b_0$ arising from submodules of each projective $kH$-indecomposable, making a total of 16 $kH$-indecomposables in $b_0$. By Green's correspondence, we then know there are 16 indecomposable $kG$-module in $B_0$, hence the modules $M_i, M_i^*$ $(i = 1, \ldots, 5)$, and $P_1, \ldots, P_4$ appeared in section 3.3 are all the $kG$-indecomposables.

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