Cyclic polytopes and higher Auslander–Reiten theory III

The higher Stasheff–Tamari orders in representation theory

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Plan

1. Even-dimensional cyclic polytopes in representation theory
   Lecture 1: Cyclic polytopes, their triangulations, and the higher Stasheff–Tamari orders
   Lecture 2: Higher Auslander–Reiten theory
2. Even-dimensional HST orders in representation theory
3. Odd-dimensional cyclic polytopes in representation theory
4. Equivalence of maximal green sequences
5. Odd-dimensional HST orders in representation theory

1. Even-dimensional cyclic polytopes in representation theory

1.1. Lecture 1: Cyclic polytopes, their triangulations, and the higher Stasheff–Tamari orders

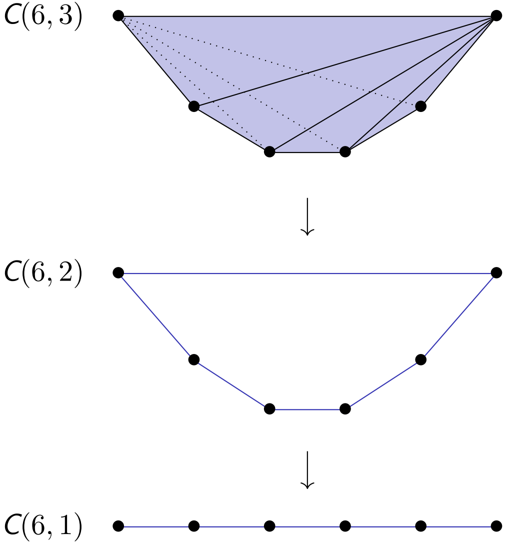
Cyclic polytopes

The *cyclic polytope* $C(m, \delta)$ is the convex hull of m points on the curve

$$p(t) = (t, t^2, \ldots, t^\delta).$$

It has the largest possible number of faces of every given dimension, as well as other nice combinatorial properties.

Its facets are described by Gale's Evenness Criterion and its circuits consist of intertwining subsets of vertices.



Description of triangulations of even-dimensional cyclic polytopes

Theorem ([OT12])

There is a bijection between triangulations of $C(m, 2d)$ and sets of non-intertwining $(d+1)$-subsets from ${}^{\circlearrowleft}\mathbf{I}^d_m$ of size $\binom{m-d-2}{d}$ given by sending a triangulation $\mathcal{T}$ to its set of internal d-simplices $\mathring{e}(\mathcal{T})$.

$$A \in {}^{\circlearrowleft}\mathbf{I}^d_m := \{\{a_0, \dots, a_d\} \subseteq [m] \mid a_{i+1} \geqslant a_i + 2 \mod m\}.$$

$A = \{a_0, a_1, \dots, a_d\}$ intertwines $B = \{b_0, b_1, \dots, b_d\}$ if

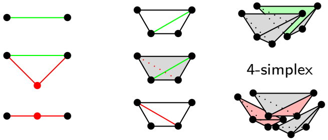
$$a_0 < b_0 < a_1 < b_1 < \cdots < a_d < b_d.$$

Triangulations of $(2d+1)$-dimensional cyclic polytopes may also be described in terms of internal d-simplices.

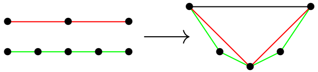
The higher Stasheff–Tamari orders

Two partial orders on the set of triangulations of a cyclic polytope $C(m, \delta)$.

$\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$, the higher-dimensional analogue of flipping a diagonal inside a quadrilateral.



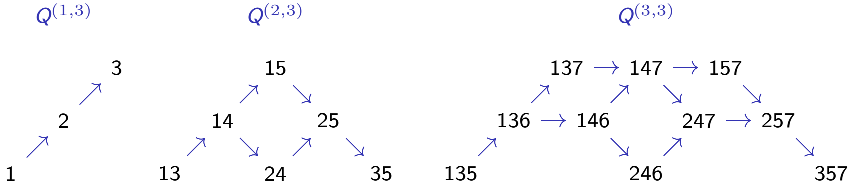
$\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if the section of $\mathcal{T}'$ lies entirely above the section of $\mathcal{T}$ with respect to the $(\delta + 1)$-th coordinate.



1.2. Lecture 2: Higher Auslander–Reiten theory

The higher Auslander algebras of type A

These are defined using the quivers $Q^{(d,n)}$ using the mesh relations.



These algebras are d-representation-finite d-hereditary: they have global dimension d and a unique basic d-cluster-tilting module $M^{(d,n)}$, where $\mathrm{End}\, M^{(d,n)} \cong A_n^{d+1}$.

$\mathrm{add}\{M^{(d,n)}[id] : i \in \mathbb{Z}\}$ is a d-cluster-tilting subcategory of $D^b(\mathrm{mod}\, A_n^d)$. This allows us to define a $(d+2)$-angulated cluster category $\mathcal{O}_{A_n^d}$ and a d-exangulated d-almost positive category $\mathcal{U}_{A_n^d}^{\{-d,0\}}$.

Three different settings

The indecomposables in $\operatorname{add} M^{(d,n+1)}$ are in bijection with

$$\mathbf{I}^d_{n+2d+1} := \left\{ A \in \binom{[n+2d+1]}{d+1} : \forall i \in [d], a_i \geqslant a_{i-1} + 2 \right\}.$$

There are bijections between

- non-projective-injective indecomposables in $\operatorname{add} M^{(d,n+1)}$,
- indecomposables in $\mathcal{O}_{A^d_n}$,
- indecomposables in $\mathcal{U}^{\{-d,0\}}_{A^d_n}$,
- elements of ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$.

These induce bijections between

- tilting modules in $\operatorname{add} M^{(d,n+1)}$,
- cluster-tilting objects in $\mathcal{O}_{A^d_n}$,
- silting complexes in $\mathcal{U}^{\{-d,0\}}_{A^d_n}$,
- non-intertwining subsets of ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$.

The relation between these categories can be explained more formally using d-exangulated categories.

Putting the algebra and combinatorics together

Theorem ([OT12; Wila])

There is a bijection between

- *indecomposables in* $\operatorname{add} M^{(d,n+1)}$,
- *simplices in* $C(n+2d+1,2d)$ *not lying in a lower facet.*

There are further bijections between

- *non-projective-injective indecomposables in* $\operatorname{add} M^{(d,n+1)}$,
- *indecomposables in* $\mathcal{O}_{A_n^d}$,
- *indecomposables in* $\mathcal{U}_{A_n^d}^{\{-d,0\}}$,
- *internal d-simplices in* $C(n+2d+1,2d)$.

inducing bijections between

- *tilting modules in* $\operatorname{add} M^{(d,n+1)}$,
- *cluster-tilting objects in* $\mathcal{O}_{A_n^d}$,
- *silting complexes in* $\mathcal{U}_{A_n^d}^{\{-d,0\}}$,
- *triangulations of* $C(n+2d+1,2d)$.

Numbers of summands

From now on, we will only use the framework of d-silting complexes in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$.

As for the theorem on the previous slide, we have a bijection

$$\left\{\begin{matrix}\text{Triangulations of} \\ C(n+2d+1,2d)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Silting complexes} \\ \text{in } \mathcal{U}_{A_n^d}^{\{-d,0\}}\end{matrix}\right\}.$$

But so far we have only seen justification for a bijection

$$\left\{\begin{matrix}\text{Triangulations of} \\ C(n+2d+1,2d)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Presilting complexes in } \mathcal{U}_{A_n^d}^{\{-d,0\}} \\ \text{with } \binom{n+d-2}{d} \text{ summands}\end{matrix}\right\}.$$

We will prove the first bijection from the second using facts about triangulations.

Reminder on the silting framework

The results in the silting framework will also have analogues in the tilting framework. However, they will not have analogues in the cluster-tilting framework due to the $2d$-Calabi–Yau property, as we will explain.

The key facts about the silting framework that we will use are the following.

Internal simplices in $C(n+2d+1, 2d)$ and indecomposable complexes in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ are in bijection via $B \mapsto U_B$.

We have that $\operatorname{Hom}_{D^b(\operatorname{mod} A_n^d)}(U_A, U_B[d]) \neq 0$ if and only if $B \wr A$.

2. Even-dimensional HST orders in representation theory

Combinatorial interpretation of even-dim HST orders

Recall that, given a triangulation $\mathcal{T}$ of $C(m, 2d)$ or $C(m, 2d+1)$, we denote its set of internal d-simplices by $\mathring{e}(\mathcal{T})$.

Theorem ([Wilb])

Given triangulations $\mathcal{T}, \mathcal{T}'$ of $C(m, 2d)$, we have that

1. $\mathcal{T} \lessdot_1 \mathcal{T}'$ *if and only if* $\mathring{e}(\mathcal{T}) = \mathcal{R} \cup \{A\}$ *and* $\mathring{e}(\mathcal{T}') = \mathcal{R} \cup \{B\}$, *where* $A \wr B$, *for some subset* $\mathcal{R} \subseteq {}^{\circlearrowleft}\mathbf{I}^d_m$;
2. *an internal d-simplex A is submerged by a triangulation $\mathcal{T}$ if and only if there is no $B \in \mathring{e}(\mathcal{T})$ such that $B \wr A$.*

Recall that a simplex A is *submerged* by a triangulation $\mathcal{T}$ if

$$s_A(x)_{\delta+1} \leqslant s_{\mathcal{T}}(x)_{\delta+1} \quad \forall x \in |A|.$$

Sketch proof for even-dimensional first order

We claim that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) = \mathcal{R} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{R} \cup \{B\}$, where $A \wr B$.

Suppose $\mathcal{T} \lessdot_1 \mathcal{T}'$.

Let the bistellar flip happen inside the subpolytope $C(A \cup B, 2d)$, where $A \wr B$.

The lower triangulation of $C(A \cup B, 2d)$ has A has its only internal d-simplex and the upper triangulation has B as its only internal d-simplex by Gale's Eveness Criterion.

Hence, $\mathring{e}(\mathcal{T}) = \mathcal{R} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{R} \cup \{B\}$, where $A \wr B$.

Conversely, one can show that if $\mathring{e}(\mathcal{T}) = \mathcal{R} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{R} \cup \{B\}$, where $A \wr B$, then $\mathcal{T}$ and $\mathcal{T}'$ can only differ inside the subpolytope $C(A \cup B, 2d)$.

Sketch proof for submersion interpretation

Claim: an internal d-simplex A is submerged by a triangulation $\mathcal{T}$ if and only if there is no $B \in \mathring{e}(\mathcal{T})$ such that $B \wr A$.

It is clear that if there exists $B \in \mathring{e}(\mathcal{T})$ such that $B \wr A$, then A cannot be submerged by $\mathcal{T}$, since A lies above the d-simplex B in $C(m, 2d+1)$, by Gale's Evenness Criterion.

Conversely, suppose that A is not submerged by $\mathcal{T}$. There are two cases: either A lies entirely below the section of $\mathcal{T}$, or A intersects the section of $\mathcal{T}$.

If A lies entirely below the section of $\mathcal{T}$, then it cannot be in $\mathcal{T}$. Hence, there is $B \in \mathcal{T}$ such that $A \wr B$ or $B \wr A$. Since A lies below $\mathcal{T}$, we must have $A \wr B$ by GEC.

If A intersects the section of $\mathcal{T}$, then by the description of the circuits of $C(m, 2d+1)$, there is a $(d+1)$-simplex B of $\mathcal{T}$ such that $A \wr B$. Then $\{b_1, b_2, \dots, b_{d+1}\}$ is the d-simplex we need.

Algebraic interpretation of even-dimensional HST orders

Theorem ([Wilb])

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(n+2d+1, 2d)$ corresponding to d-silting complexes T and T' for A_n^d. We then have that

1. $\mathcal{T} \lessdot_1 \mathcal{T}'$ *if and only if T' is a left mutation of T; and*
2. $\mathcal{T} \leqslant_2 \mathcal{T}'$ *if and only if* ${}^\perp T \subseteq {}^\perp T'$.

Left mutation: $T = E \oplus X$, $T' = E \oplus Y$, with $\mathrm{Hom}_{\mathcal{U}_{A_n^d}^{\{-d,0\}}}(Y, X[d]) \neq 0$.

$$
\begin{aligned}
{}^\perp T &= \{X \in \mathcal{U}_{A_n^d}^{\{-d,0\}} \mid \mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(X, T[i]) = 0,\ \forall i > 0\} \\
&= \{X \in \mathcal{U}_{A_n^d}^{\{-d,0\}} \mid \mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(X, T[d]) = 0\}.
\end{aligned}
$$

Sketch proof for first HST order

We know that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) = \mathcal{R} \cup \{A\}$ and $\mathring{e}(\mathcal{T}') = \mathcal{R} \cup \{B\}$, where $A \wr B$.

Hence, if T and T' are the corresponding d-silting complexes, then we have $T = E \oplus U_A$, $T' = E \oplus U_B$ with $A \wr B$.

Since we know that $\mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(U_B, U_A[d]) \neq 0$ if and only if $A \wr B$, we obtain that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if T' is a left mutation of T.

Sketch proof for second HST order

We know that an internal d-simplex A is submerged by a triangulation $\mathcal{T}$ if and only if there is no $B \in \mathring{e}(\mathcal{T})$ such that $B \wr A$.

By the interpretation of extensions $\mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(U_A, U_B[d]) \neq 0$ if and only if $B \wr A$, we obtain that $U_A \in {}^{\perp}T$ if and only if A is submerged by $\mathcal{T}$.

Since we know that the second higher Stasheff–Tamari order is equivalent to inclusion of submersion sets, we obtain that $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if ${}^{\perp}T \subseteq {}^{\perp}T'$.

The difficulty with using the cluster-tilting framework

The cluster category $\mathcal{O}_{A_n^d}$ is $2d$-Calabi–Yau, meaning that

$$\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(X, Y[d]) \cong D \operatorname{Hom}_{\mathcal{O}_{A_n^d}}(Y, X[d]).$$

Hence, given a mutation from $E \oplus X$ to $E \oplus Y$ in $\mathcal{O}_{A_n^d}$, with $X \ncong Y$, we must in fact have both $\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(X, Y[d]) \neq 0$ and $\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(Y, X[d]) \neq 0$.

Hence, left mutation cannot be distinguished from right mutation and we cannot distinguish between increasing and decreasing bistellar flips.

Similarly, for T a cluster-tilting object in $\mathcal{O}_{A_n^d}$, we have that ${}^{\perp}T = T^{\perp} = \operatorname{add} T$.

Hence, ${}^{\perp}T \subseteq {}^{\perp}T'$ if and only if $T \cong T'$, assuming both are basic. Hence we cannot encode the second order either.

Sufficient to have maximal number of summands

We have seen that there is a bijection

$$\left\{\begin{matrix}\text{Triangulations of} \\ C(n+2d+1, 2d)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Presilting complexes in } \mathcal{U}_{A_n^d}^{\{-d,0\}} \\ \text{with } \binom{n+d-2}{d} \text{ summands}\end{matrix}\right\}.$$

But we have also seen that left mutations correspond to increasing bistellar flips.

We have that if $E \oplus X$ is a silting complex and $E \oplus Y$ is a presilting complex which is a left mutation of $E \oplus X$, then $E \oplus Y$ is also a silting complex [AI12].

It is clear that the projectives give a silting complex. This corresponds to the lower triangulation.

Since all triangulations are connected by bistellar flips, all the presilting complexes with $\binom{n+d-2}{d}$ summands are connected by left mutations, and so all are silting.

Implications of equality of orders

The orders on d-silting complexes we obtain are the higher-dimensional analogues of orders on silting complexes studied in [AI12].

In the $d = 1$ case these orders are known to be equal by arguments using homological algebra [AI12].

In general it is an open problem whether the two orders are generally equal in higher Auslander–Reiten theory, but our result on the equality of the higher Stasheff–Tamari orders shows that they are equal for the higher Auslander algebras of type A.

Theorem ([Wila])

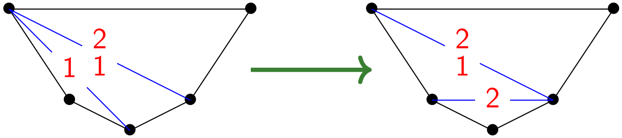
Given two d-silting complexes T and T' for A_n^d, we have that ${}^{\perp}T \subseteq {}^{\perp}T'$ if and only if there is a sequence of left mutations from T to T'.

The orders on silting complexes arise from earlier orders on tilting modules defined in [RS91].

Illustration of even dimensions

We consider the case of $\mathcal{U}_{A_2^1}^{\{-1,0\}}$.

There is a left mutation from $T = 1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$ to $T' = 2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$ since $\mathrm{Hom}_{D^b(\mathrm{mod}\,A_2)}(2, 1[1]) \neq 0$, corresponding to the bistellar flip



Then ${}^{\perp}T = \mathrm{add}\{1, \begin{smallmatrix}2\\1\end{smallmatrix}\} \subseteq \mathrm{add}\{1, \begin{smallmatrix}2\\1\end{smallmatrix}, 2\} = {}^{\perp}T'$, corresponding to the fact the second higher Stasheff–Tamari order holds.

3. Odd-dimensional cyclic polytopes in representation theory

d-maximal green sequences

We know from Rambau's theorem that triangulations of $C(n+2d+1, 2d+1)$ are given by equivalence classes of maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$.

We know from our algebraic interpretation of the higher Stasheff–Tamari orders in dimension $2d$ that maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$ correspond to sequences of left mutations from A_n^d to $A_n^d[d]$ in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$.

For $d = 1$, a sequence of left mutations from the projectives to the shifted projectives is a *maximal green sequence*.

Hence, we define a *d-maximal green sequence* of a d-representation-finite d-hereditary algebra Λ as a sequence of left mutations from Λ to $\Lambda[d]$ in $\mathcal{U}_{\Lambda}^{\{-d,0\}}$.

Equivalence of d-maximal green sequences

In Rambau's theorem, we have that *equivalence classes* of maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$ correspond to triangulations of $C(n+2d+1, 2d+1)$.

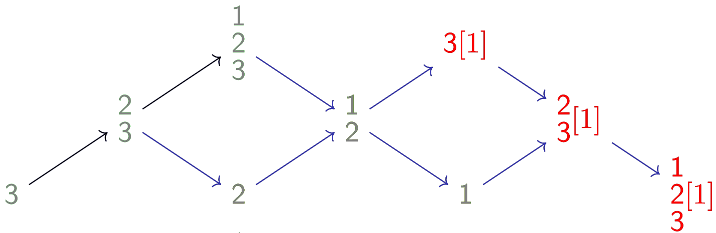
Hence, in order to get a bijection with odd-dimensional triangulations, we put an equivalence relation on d-maximal green sequences, which is as follows.

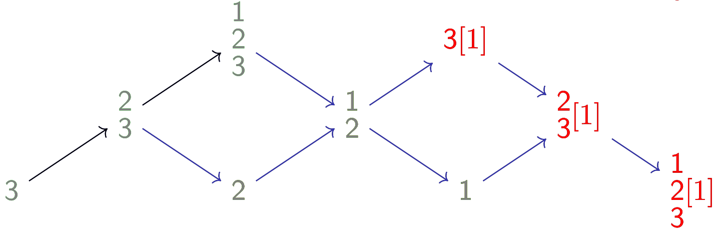
Given a d-maximal green sequence G, we write $\mathsf{S}(G)$ for the set of indecomposable summands of objects occurring in G.

We write $G \sim G'$ if $\mathsf{S}(G) = \mathsf{S}(G')$ and write $\widetilde{\mathcal{MG}}_d(A_n^d)$ for the set of $\sim$-equivalence classes of d-maximal green sequences of A_n^d.

Equivalence of *d*-maximal green sequences: example

For example, for the algebra A_3, the following two maximal green sequences are equivalent:





Algebraic bijection for odd-dimensional triangulations

Theorem ([Wilb])

There is a bijection between triangulations $C(n+2d+1, 2d+1)$ *and* $\widetilde{\mathcal{MG}}_d(A_n^d)$*. Moreover, if a triangulation* $\mathcal{T}$ *of* $C(n+2d+1, 2d+1)$ *corresponds to an equivalence class of d-maximal green sequences* $[G] \in \widetilde{\mathcal{MG}}_d(A_n^d)$*, then*

1. *there is a bijection between mutations in G and* $(2d+1)$*-simplices of* $\mathcal{T}$*; and*
2. *there is a bijection between the internal d-simplices of* $\mathcal{T}$ *and elements of* $\mathsf{S}(G)$ *which are neither projectives nor shifted projectives.*

Our description of triangulations of $C(n+2d+1, 2d+1)$ as supporting and bridging subsets of $\mathbf{J}^d_{n+2d+1}$ therefore gives us a classification of equivalence classes of d-maximal green sequences of A_n^d.

Algebraic bijection for odd-dimensional triangulations: sketch proof

Recall from lecture 1 that the internal d-simplices of $C(n+2d+1, 2d+1)$ correspond to elements of

$$\mathbf{J}^{d}_{n+2d+1} = \{A \in {}^{\circlearrowleft}\mathbf{I}^{d}_{n+2d+1} : a_0 \neq 1, a_d \neq n+2d+1\}.$$

It follows from lecture 2 that in $\mathcal{U}^{\{-d,0\}}_{A^d_n}$, U_A is projective if $a_0 = 1$ and shifted projective if $a_d = n+2d+1$.

Hence, the internal d-simplices in $C(n+2d+1, 2d+1)$ are in bijection with indecomposable complexes in $\mathcal{U}^{\{-d,0\}}_{A^d_n}$ which are neither projectives or shifted projectives.

The projectives and shifted projectives turn into simplices in the lower and upper facets of $C(n+2d+1, 2d+1)$, respectively.

Algebraic bijection for odd-dimensional triangulations: sketch proof

By the interpretation of the even-dimensional HST orders, we get that d-maximal green sequences of A_n^d correspond to maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$, where the mutations correspond to bistellar flips.

By Rambau's theorem, we have that maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$ give triangulations of $C(n+2d+1, 2d+1)$, with the $(2d+1)$-simplices of the triangulation correspondings to the bistellar flips.

Hence, d-maximal green sequences of A_n^d give triangulations of $C(n+2d+1, 2d+1)$ with mutations from $E \oplus U_A$ to $E \oplus U_B$ corresponding to $(2d+1)$-simplices $A \cup B$.

Since internal d-simplices determine the triangulation, we get a bijection between $\widetilde{\mathcal{MG}}_d(A_n^d)$ and triangulations of $C(n+2d+1, 2d+1)$.

4. Equivalence of maximal green sequences

General results on equivalence of maximal green sequences

Theorem (Gorsky–W)

Let Λ be a finite-dimensional algebra over a field K, with G and G' maximal green sequences of Λ. Then the following are equivalent.

1. *G and G' can be deformed into each other across squares.*
2. $\mathsf{S}(G) = \mathsf{S}(G')$.
3. *G and G' have the same set of exchange pairs.*
4. *Any Λ-module M has the same stable Harder–Narasimhan factors under G as under G'.*

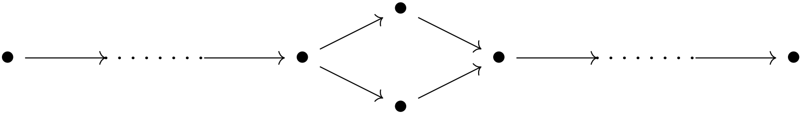
However, the following is not equivalent.

5. *G and G' have the same set of bricks.*

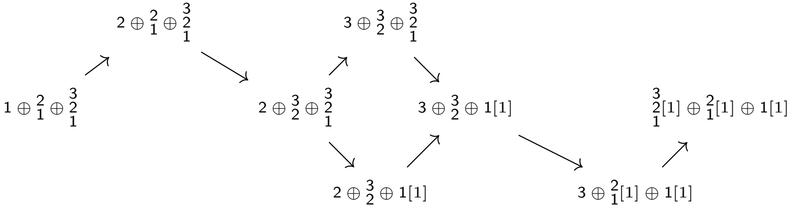
The *exchange pair* of a left mutation from $E \oplus X$ to $E \oplus Y$ is (X, Y).

Deformation across squares

We say that two maximal green sequences are related by deformation across a square if the following occurs.



For example, the following two maximal green sequences of A_3 are related by deformation across a square.



Torsion pairs

Definition ([Dic66])

A *torsion pair* in $\operatorname{mod}\Lambda$ is a pair of full subcategories $(\mathcal{T},\mathcal{F})$ such that

1. $\operatorname{Hom}_\Lambda(\mathcal{T},\mathcal{F})=0$;
2. if $\operatorname{Hom}_\Lambda(T,\mathcal{F})=0$, then $T\in\mathcal{T}$;
3. if $\operatorname{Hom}_\Lambda(\mathcal{T},F)=0$, then $F\in\mathcal{F}$.

Here $\mathcal{T}$ is called the *torsion class* and $\mathcal{F}$ is called the *torsion-free class*. More generally, a full subcategory $\mathcal{T}$ is called a torsion class if it is a torsion class in some torsion pair, and likewise for torsion-free classes.

Fact

A full subcategory $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is a torsion class if and only if it is closed under quotients and extensions.

Torsion classes and two-term silting complexes

Theorem ([AIR14])

There is a bijection

$$\left\{\begin{array}{l}\textit{Two-term}\\ \textit{silting complexes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{Functorially finite}\\ \textit{torsion classes}\end{array}\right\}$$

induced by

$$T \mapsto \mathrm{Fac}\, H^0(T),$$

where $\mathrm{Fac}\, M$ *is the category of factor modules of* $M^{\oplus k}$*.*

Torsion classes and maximal green sequences

Torsion classes of $\operatorname{mod}\Lambda$ form a poset $\operatorname{Tors}\Lambda$ under inclusion.

Theorem ([DIJ19; BST19])

The bijection between two-term silting complexes and functorially finite torsion classes induces a bijection

$$\left\{\begin{matrix}\textit{Maximal} \\ \textit{green sequences}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Maximal} \\ \textit{chains in } \operatorname{Tors}\Lambda\end{matrix}\right\}.$$

This gives an alternative way of viewing maximal green sequences.

Bricks of a maximal green sequence

An inclusion of torsion classes $\mathcal{T} \supset \mathcal{T}'$ is a *minimal inclusion of torsion classes* if whenever $\mathcal{T} \supseteq \mathcal{T}'' \supseteq \mathcal{T}'$, we must have either $\mathcal{T}'' = \mathcal{T}$ or $\mathcal{T}'' = \mathcal{T}'$.

These correspond to the covering relations in $\operatorname{Tors} \Lambda$.

Theorem ([BCZ19; Dem+18])

We have that $\mathcal{T} \supset \mathcal{T}'$ is a minimal inclusion of torsion classes if and only if $\mathcal{T} \cap \mathcal{T}'^{\perp_0} = \operatorname{Filt} B$ for some brick B.

Here

$$\mathcal{T}'^{\perp_0} = \{M \in \operatorname{mod} \Lambda : \operatorname{Hom}_\Lambda(\mathcal{T}', M) = 0\}.$$

A Λ-module B is a *brick* if $\operatorname{End}_\Lambda B$ is a division ring.

Hence, we can label the arrows of a maximal green sequence by bricks.

Brick labelling, example

Consider the following maximal green sequence of A_3.

$$1 \oplus \substack{2\\1} \oplus \substack{3\\2\\1} \to 1 \oplus 3 \oplus \substack{3\\2\\1} \to 1 \oplus 3 \oplus \substack{2\\1}[1] \to 1 \oplus \substack{2\\1}[1] \oplus \substack{3\\2\\1}[1] \to 1[1] \oplus \substack{2\\1}[1] \oplus \substack{3\\2\\1}[1].$$

This corresponds to the maximal chain of torsion classes

$$\operatorname{mod} A_3 \xrightarrow{2} \operatorname{add}\left\{1, \substack{3\\2\\1}, \substack{3\\2}, 3\right\} \xrightarrow{\substack{3\\2\\1}} \operatorname{add}\{1, 3\} \xrightarrow{3} \operatorname{add}\{1\} \xrightarrow{1} \{0\},$$

which has the brick labels shown.

Stable Harder–Narasimhan filtrations given by a maximal green sequence

Let

$$B_1, B_2, \ldots, B_r$$

be the bricks of a maximal green sequence, labelled in order.

By [Tre18], it follows that every Λ-module M has a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_{l-1} \supset M_l = 0$$

such that $M_{j-1}/M_j = B_{i_j}$ for some i_j, with

$$i_1 \leqslant i_2 \leqslant \cdots \leqslant i_l.$$

Moreover, this filtration is essentially unique.

We call this the *stable Harder–Narasimhan filtration* of M given by the maximal green sequence.

Stable Harder–Narasimhan filtrations: example

We take the maximal green sequence we had before.

$$\operatorname{mod} A_3 \xrightarrow{2} \operatorname{add}\left\{1, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, 3\right\} \xrightarrow{\begin{smallmatrix}3\\2\end{smallmatrix}} \operatorname{add}\{1, 3\} \xrightarrow{3} \operatorname{add}\{1\} \xrightarrow{1} \{0\}.$$

Here are the factors of the stable Harder–Narasimhan filtrations given by this maximal green sequence for various modules.

- $\begin{smallmatrix}3\\2\\1\end{smallmatrix}$ has a filtration with factors $\begin{smallmatrix}3\\2\end{smallmatrix}$ and 1.
- $\begin{smallmatrix}3\\2\end{smallmatrix}$ has a filtration with factors $\begin{smallmatrix}3\\2\end{smallmatrix}$.

General results on equivalence of maximal green sequences

Theorem (Gorsky–W)

Let Λ be a finite-dimensional algebra over a field K, with G and G' maximal green sequences of Λ. Then the following are equivalent.

1. *G and G' can be deformed into each other across squares.*
2. $\mathsf{S}(G) = \mathsf{S}(G')$.
3. *G and G' have the same set of exchange pairs.*
4. *Any Λ-module M has the same stable Harder–Narasimhan factors under G as under G'.*

However, the following is not equivalent.

5. *G and G' have the same set of bricks.*

5. Odd-dimensional HST orders in representation theory

Combinatorial interpretation of odd-dimensional HST orders

Theorem ([Wilb])

Given triangulations $\mathcal{T}, \mathcal{T}'$ of $C(m, 2d+1)$, we have that

1. $\mathcal{T} \lessdot_1 \mathcal{T}'$ *if and only if* $\mathring{e}(\mathcal{T}) = \mathring{e}(\mathcal{T}') \cup \{A\}$ *for some* $A \in \mathbf{J}^d_m \setminus \mathring{e}(\mathcal{T}')$;
2. $\mathcal{T} \leqslant_2 \mathcal{T}'$ *if and only if* $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$.

Combinatorial interpretation of odd-dimensional first HST order: sketch proof

Claim: $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) = \mathring{e}(\mathcal{T}') \cup \{A\}$ for some $A \in \mathbf{J}_m^d \setminus \mathring{e}(\mathcal{T}')$.

The increasing bistellar flip from $\mathcal{T}$ to $\mathcal{T}'$ happens inside some $C(2d+3, 2d+1)$ subpolytope.

We may label the vertices of this subpolytope $A \cup B$ where where A is a d-simplex and B is a $(d+1)$-simplex with $A \wr B$.

The lower triangulation of $C(A \cup B, 2d+1)$ has A as its only internal d-simplex, whereas the upper triangulation has no internal d-simplices, by Gale's Evenness Criterion.

Hence increasing bistellar flips correspond to removing internal d-simplices.

Combinatorial interpretation of odd-dimensional second HST order: sketch proof

$\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$.

Recall that $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathrm{sub}_{d+1}(\mathcal{T}) \subseteq \mathrm{sub}_{d+1}(\mathcal{T}')$.

One can show that $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathrm{sup}_d(\mathcal{T}) \supseteq \mathrm{sup}_d(\mathcal{T}')$, where submersion is defined dually to supermersion.

It turns out that the d-supermersion set of a triangulation of $C(m, 2d+1)$ is equal to its set of d-simplices.

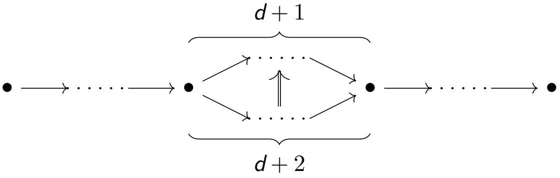
Algebraic interpretation of odd-dimensional HST orders

Theorem ([Wilb])

Let $\mathcal{T}, \mathcal{T}'$ be triangulations of $C(n+2d+1, 2d+1)$ corresponding to equivalence classes of d-maximal green sequences $[G], [G'] \in \widetilde{\mathcal{MG}}_d(A_n^d)$. We then have that

1. *$\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if there are equivalence class representatives $\widehat{G} \in [G]$ and $\widehat{G}' \in [G']$ such that $\widehat{G}'$ is an increasing elementary polygonal deformation of $\widehat{G}$; and*
2. *$\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathsf{S}(G) \supseteq \mathsf{S}(G')$.*

Increasing elementary polygonal deformations:



Odd dimensions: sketch of proof for first order

Claim: $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if there are equivalence class representatives $\widehat{G} \in [G]$ and $\widehat{G}' \in [G']$ such that $\widehat{G}'$ is an increasing elementary polygonal deformation of $\widehat{G}$.

The $(2d+2)$-simplex inducing a bistellar flip has $d+1$ $(2d+1)$-simplices as its upper facets and $d+2$ $(2d+1)$-simplices as its lower facets.

Each of these $(2d+1)$-simplices corresponds to a bistellar flip in the maximal chain in $\mathcal{S}_1(n+2d+1, 2d)$, and so a left mutation in the d-maximal green sequence.

We can find a chain in the equivalence class such that the left mutations corresponding to the $d+2$ lower facets all occur in a row.

The increasing bistellar flip then replaces these with $d+1$ left mutations occurring in a row.

Hence, we get an increasing elementary polygonal deformation as described.

Odd dimensions: sketch of proof for second order

Claim: $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathsf{S}(G) \supseteq \mathsf{S}(G')$.

We use the fact that $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$.

We know that $\mathring{e}(\mathcal{T})$ corresponds to the summands in $[G]$ which are neither projective nor shifted projective.

Since the projectives and shifted projectives are in every d-maximal green sequence, we have that $\mathring{e}(\mathcal{T}) \supseteq \mathring{e}(\mathcal{T}')$ if and only if $\mathsf{S}(G) \supseteq \mathsf{S}(G')$.

The "no-gap" conjecture

In [BDP14], Brüstle, Dupont, and Perotin conjectured that there was no gap in the set of lengths of maximal green sequences of a hereditary algebra over an algebraically closed field.

This conjecture was proved in some types by Garver and McConville [GM19] and for all tame types by Hermes and Igusa [HI19].

If the two orders on equivalence classes of d-maximal green sequences from the theorem are equal, then whenever $\mathsf{S}(G) \supseteq \mathsf{S}(G')$ we have a series of increasing elementary polygonal deformations from G to G' (up to equivalence).

Since an increasing elementary polygonal deformation changes the length of the d-maximal green sequence by 1, there are therefore no gaps in the lengths of maximal green sequences between G and G'.

Consequences of the algebraic interpretation

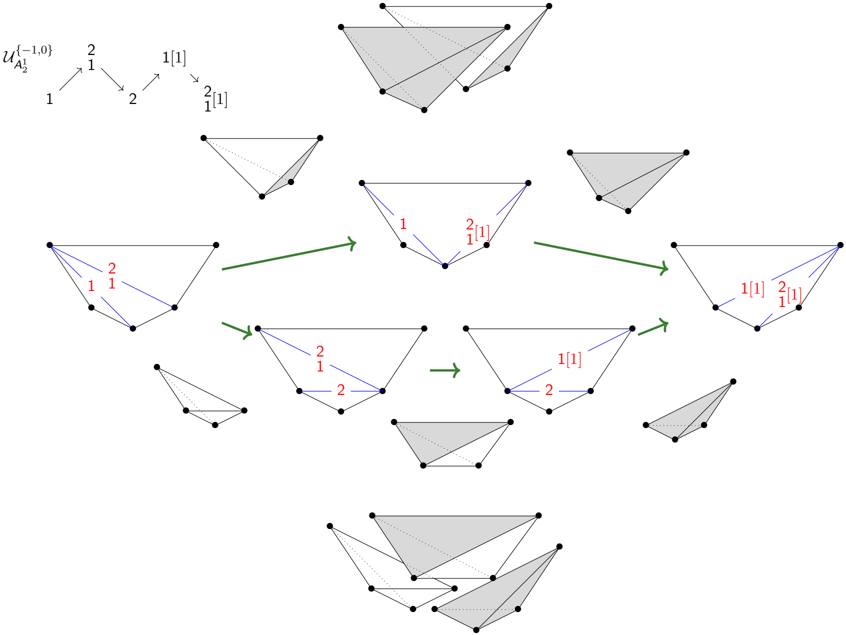
Because we know from Edelman and Reiner that the higher Stasheff–Tamari orders are equal and are lattices for $\delta \leqslant 3$, we obtain the following result.

Corollary ([Wilb])

The two orders on $\widetilde{\mathcal{MG}}_1(A_n)$ are equal and are lattices.

Computer calculations reveal that the poset of equivalence classes of maximal green sequences is not a lattice in type D, however.

Illustration in odd dimensions



Implications of equality of orders

Due to the fact that we know that the higher Stasheff–Tamari orders are equal, we obtain the following theorem.

Theorem ([Wila])

Given $[G], [G'] \in \widetilde{\mathcal{MG}}_d(A_n^d)$*, we have that* $\mathsf{S}(G) \supseteq \mathsf{S}(G')$ *if and only if there is a series of increasing elementary polygonal deformations from* $[G]$ *to* $[G']$*.*

In particular, the "no-gap" conjecture holds for A_n^d.

Whether or not this holds for all d-representation-finite d-hereditary algebras is an open question.

ありがとうございました!

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