Cyclic polytopes and higher Auslander–Reiten theory II

Higher Auslander–Reiten theory

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Introduction

In their paper [OT12], Oppermann and Thomas introduce two representation-theoretic frameworks relating cyclic polytopes and higher Auslander–Reiten theory:

- the d-cluster-tilting subcategories of the module categories of the higher Auslander algebras of type A,
- the $(d+2)$-angulated cluster categories of the higher Auslander algebras of type A.

In [Wil], a third representation-theoretic framework was introduced:

- the d-almost positive categories of the higher Auslander algebras of type A.

In this talk, we describe these three frameworks, and make precise the relationship between them using d-exangulated categories.

Plan

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Plan

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1. Preliminaries

Approximations

We require subcategories to be full and closed under isomorphism.

Given a subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ and a map $f\colon X \to M$, where $X \in \mathcal{X}$ and $M \in \mathcal{A}$, we say that f is a *right $\mathcal{X}$-approximation* if for any $X' \in \mathcal{X}$, the sequence

$$\mathrm{Hom}_{\mathcal{A}}(X', X) \to \mathrm{Hom}_{\mathcal{A}}(X', M) \to 0$$

is exact.

This means that $\mathrm{Hom}_{\mathcal{A}}(-, M)\colon \mathcal{X} \to \mathbf{Ab}$ is a finitely generated contravariant functor, since there is an epimorphism $\mathrm{Hom}_{\mathcal{X}}(-, X) \to \mathrm{Hom}_{\mathcal{A}}(-, M)$.

Dually, $g\colon M \to X$ is a *left $\mathcal{X}$-approximation* if for any $X' \in \mathcal{X}$, the sequence

$$\mathrm{Hom}_{\mathcal{A}}(X, X') \to \mathrm{Hom}_{\mathcal{A}}(M, X') \to 0$$

is exact.

Functorial finiteness, generation, and cogeneration

The subcategory $\mathcal{X}$ is said to be *contravariantly finite* if every $M \in \mathcal{A}$ admits a right $\mathcal{X}$-approximation, and *covariantly finite* if every $M \in \mathcal{A}$ admits a left $\mathcal{X}$-approximation.

If $\mathcal{X}$ is both contravariantly finite and covariantly finite, then $\mathcal{X}$ is *functorially finite*.

The subcategory $\mathcal{X}$ is *generating* if, for any $M \in A$, there is an epimorphism $p \colon X \to M$, where $X \in \mathcal{X}$.

The definition of *cogenerating* is dual to this.

2. d-abelian categories

2.1. Theory

Higher Auslander–Reiten theory: abelian categories

Introduced by Iyama as a higher-dimensional generalisation of classical Auslander–Reiten theory.

Given an abelian category $\mathcal{A}$, a functorially finite generating–cogenerating subcategory $\mathcal{M}$ of $\mathcal{A}$ is called *d-cluster-tilting* if

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{A} : \forall M \in \mathcal{M}, \mathrm{Ext}_{\mathcal{A}}^{1,\dots,d-1}(X, M) = 0\} \\ &= \{X \in \mathcal{A} : \forall M \in \mathcal{M}, \mathrm{Ext}_{\mathcal{A}}^{1,\dots,d-1}(M, X) = 0\}.\end{aligned}$$

d-abelian categories

Introduced by Jasso as the higher analogue of abelian categories [Jas16].

Theorem ([Jas16; Kva21; EN20])

1. *A d-cluster-tilting subcategory of an abelian category is d-abelian.*
2. *Any d-abelian category is equivalent to a d-cluster-tilting subcategory of an abelian category.*

Higher Auslander–Reiten theory: finite-dimensional algebras

Now, let Λ be a finite-dimensional algebra over a field K.

A d-cluster-tilting subcategory of $\operatorname{mod} \Lambda$ must contain the projectives and injectives, and so is automatically generating and cogenerating. Hence, one can drop this condition.

If $\operatorname{add} M$ is a d-cluster-tilting subcategory of $\operatorname{mod} \Lambda$ for some Λ-module M, then M is called a *d-cluster-tilting module*.

If Λ has a d-cluster-tilting module M, then Λ is called *d-representation-finite* [IO11].

If, furthermore, Λ has $\operatorname{gl.dim} \Lambda \leqslant d$, then Λ is called *d-representation-finite d-hereditary* [HIO14].

Auslander correspondence

An *Auslander algebra* Γ is an algebra with
$\mathrm{gl.\,dim}\,\Gamma \leqslant 2 \leqslant \mathrm{dom.\,dim}\,\Gamma$.

Given a minimal injective resolution

$$0 \to \Gamma \to I_0 \to I_1 \to \cdots \to I_m \to \dots$$

$\mathrm{dom.\,dim}\,\Gamma = \{k : I_i \text{ is projective }, \forall 0 \leqslant i < k\}$.

Theorem (Auslander correspondence, [Aus71])

There is a bijection between Morita-equivalence classes of representation-finite algebras and Morita-equivalence classes of Auslander algebras.

If Λ is a representation-finite algebra with M the sum of the indecomposable Λ-modules, this bijection is given by

$$\Lambda \mapsto \mathrm{End}_\Lambda M.$$

Higher Auslander correspondence

A *d-Auslander algebra* Γ is an algebra with
$\mathrm{gl.dim}\,\Gamma \leqslant d + 1 \leqslant \mathrm{dom.dim}\,\Gamma$.

Theorem (Higher Auslander correspondence, [Iya07a])

There is a bijection between Morita-equivalence classes of d-representation-finite algebras and Morita-equivalence classes of d-Auslander algebras.

If Λ is a d-representation-finite algebra with M a d-cluster-tilting Λ-module, this bijection is given by

$$\Lambda \mapsto \mathrm{End}_\Lambda M.$$

d-Auslander–Reiten formulas

There is a higher Auslander–Reiten translate, defined by $\tau_d = \tau\Omega^{d-1}$.

We get higher Auslander–Reiten formulas, analogous to the classical ones.

Theorem ([Iya07b])

For $M, N \in \mathcal{M}$, a d-cluster-tilting subcategory of $\operatorname{mod}\Lambda$, we have

$$\operatorname{Ext}^d_\Lambda(M,N) \cong D\underline{\operatorname{Hom}}_\Lambda(\tau_d^{-1}N, M) \cong D\overline{\operatorname{Hom}}_\Lambda(N, \tau_d M).$$

2.2. Tilting modules

Tilting modules: definition

Tilting modules of projective dimension one were defined by Brenner and Butler [BB80] as a generalisation of BGP reflection functors [BGP73; APR79].

Miyashita defined tilting modules of higher projective dimension [Miy86].

This was in turn generalised by Cline, Parshall, and Scott, whose definition we use here [CPS86, Definition 2.3].

Given a Λ-module T, we say that T is a *tilting* module if:

1. the projective dimension of T is finite;
2. $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for all $i > 0$; that is, T is *rigid*;
3. there is an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_s \to 0$ with each $T_i \in \mathrm{add}\, T$.

Tilting modules: motivation

The main motivation for studying tilting modules comes from the following theorem.

Theorem ([Hap88, $\operatorname{gl.dim} \Lambda < \infty$][CPS86, $\operatorname{gl.dim} \Lambda = \infty$])
If T is a tilting Λ-module, then there is a derived equivalence

$$D^b(\operatorname{mod} \Lambda) \simeq D^b(\operatorname{mod} \operatorname{End}_\Lambda T).$$

2.3. The higher Auslander algebras of type A

Higher quivers of type A

Following [OT12],

$$\mathbf{I}_m^d := \left\{ \{a_0, \ldots, a_d\} \in \binom{[m]}{d+1} : \forall i \in [d], a_i \geqslant a_{i-1} + 2 \right\}$$

Let $Q^{(d,n)}$ be the quiver with vertices

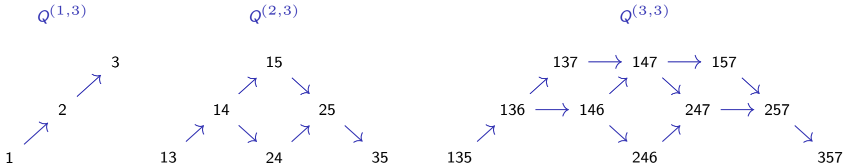
$$Q_0^{(d,n)} := \mathbf{I}_{n+2d-2}^{d-1}$$

and arrows

$$Q_1^{(d,n)} := \{ A \to \sigma_i(A) : A, \sigma_i(A) \in Q_0^{(d,n)} \},$$

where

$$\sigma_i(A) := \{a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_d\}.$$



Higher Auslander algebras of type A

Let A_n^d be the quotient of the path algebra $KQ^{(d,n)}$ by the relations:

$$A \to \sigma_i(A) \to \sigma_j(\sigma_i(A)) = \begin{cases} A \to \sigma_j(A) \to \sigma_j(\sigma_i(A)) & \text{if } \sigma_j(A) \in Q_0^{(d,n)} \\ 0 & \text{otherwise.} \end{cases}$$

We multiply arrows as if we were composing functions, so that $\xrightarrow{\alpha}\xrightarrow{\beta} = \beta\alpha$.

Theorem ([Iya11])

A_n^d is d-representation-finite d-hereditary with unique basic d-cluster-tilting module $M^{(d,n)}$ and

$$\mathrm{End}_{A_n^d} M^{(d,n)} \cong A_n^{d+1}.$$

One can do something similar in other Dynkin types, but it is more complicated.

Higher Auslander algebras of type A: derived equivalence

Theorem ([Bec; DJL21])

The algebras A^{d}_{n+d-1} and A^{n-d}_{d+1} are derived equivalent; that is,

$$D^b(\operatorname{mod} A^{d}_{n+d-1}) \cong D^b(\operatorname{mod} A^{n-d}_{d+1}).$$

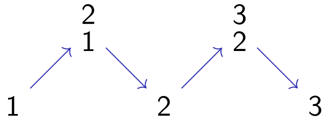
For example, $A^1_3 = A^1_{3+1-1}$ is derived equivalent to $A^2_2 = A^{3-1}_{1+1}$.

$$1 \longrightarrow 2 \longrightarrow 3 \qquad\qquad \begin{array}{ccccc} & & 14 & & \\ & \nearrow & & \searrow & \\ 13 & & & & 24 \end{array}$$

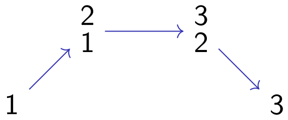
The d-cluster-tilting subcategory of $\operatorname{mod} A_n^d$: example

If we label $Q^{(2,2)}$ as

then this has Auslander–Reiten quiver.



The 2-cluster-tilting subcategory of $\operatorname{mod} A_3^2$ is given by



2.4. Combinatorial description

The d-cluster-tilting subcategory of $\operatorname{mod} A_n^d$

Theorem ([OT12, Theorem 3.6])

There is a bijection $A \mapsto M_A$ between $\mathbf{I}_{n+2d}^d$ and the indecomposables of $\operatorname{add} M^{(d,n)}$ such that:

1. *M_A is projective if and only if $a_0 = 1$.*
2. *M_A is injective if and only if $a_d = n + 2d$.*
3. *$\operatorname{Hom}_{A_n^d}(M_B, M_A) \neq 0$ if and only if $(B - \mathbf{1}) \wr A$, and in this case the Hom-space is one-dimensional;*
4. *$\operatorname{Ext}_{A_n^d}(M_B, M_A) \neq 0$ if and only if $A \wr B$, and in this case the Ext-space is one-dimensional.*
5. *$\tau_d M_B = M_{B-\mathbf{1}}$.*

Here $B - \mathbf{1} = \{b_0 - 1, b_1 - 1, \ldots, b_d - 1\}$.

Recall that $A \wr B$ if and only if $a_0 < b_0 < a_1 < b_1 < \cdots < a_d < b_d$.

The d-cluster-tilting subcategory of $\operatorname{mod} A_n^d$: example

Recall that if we label A_2^2

$$\begin{array}{ccccc} & & 2 & & \\ & \nearrow & & \searrow & \\ 1 & & & & 3, \end{array}$$

then the 2-cluster-tilting subcategory of $\operatorname{mod} A_2^2$ is given by

$$\begin{array}{ccccccc} & & \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} & & \\ & \nearrow & & & & \searrow & \\ 1 & & & & & & 3. \end{array}$$

Combinatorially, we label this

$$\begin{array}{ccccccc} & & M_{136} & \longrightarrow & M_{146} & & \\ & \nearrow & & & & \searrow & \\ M_{135} & & & & & & M_{246}. \end{array}$$

Tilting modules in $\operatorname{add} M^{(d,n)}$

Theorem ([OT12])

A basic module $\bigoplus_{i=1}^{m} M_{B_i}$ in $\operatorname{add} M^{(d,n)}$ is a tilting module if and only if $m = \binom{n+d-1}{d}$ and $\{B_i : i \in [m]\}$ is non-intertwining.

3. $(d + 2)$-angulated categories

3.1. Theory

Derived categories

Given a triangulated category $\mathcal{D}$, a functorially finite subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *d-cluster-tilting* if

$$\begin{aligned}\mathcal{C} &= \{ X \in \mathcal{D} : \forall i \in [d-1], \forall Y \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(X, Y[i]) = 0 \} \\ &= \{ X \in \mathcal{D} : \forall i \in [d-1], \forall Y \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(Y, X[i]) = 0 \}.\end{aligned}$$

Theorem ([Iya11, Theorem 1.23])

Let Λ be a d-representation-finite d-hereditary algebra with unique basic d-cluster-tilting module M. Then

$$\mathcal{U}_\Lambda := \operatorname{add}\{ M[id] : i \in \mathbb{Z} \}$$

is a d-cluster-tilting subcategory of $D^b(\operatorname{mod}\Lambda)$.

The Nakayama functor

We denote by

$$\nu := D\Lambda \otimes^{\mathbf{L}}_{\Lambda} - \cong D\mathbf{R}\operatorname{Hom}_{\Lambda}(-, \Lambda) \colon \mathcal{D}_{\Lambda} \to \mathcal{D}_{\Lambda},$$

the derived Nakayama functor.

Theorem ([IO11])

Let Λ be d-representation-finite d-hereditary. Then ν restricts to a functor $\mathcal{U}_{\Lambda} \to \mathcal{U}_{\Lambda}$.

We write desuspensions of the Nakayama functor with subscripts, so that $\nu_d := \nu[-d]$ is the derived analogue of the d-Auslander–Reiten translate.

$(d+2)$-angulated categories

Geiß, Keller, and Oppermann defined $(d+2)$-angulated categories as the higher-dimensional generalisation of triangulated categories [GKO13].

Theorem ([GKO13])

A d-cluster-tilting subcategory of a triangulated category is $(d+2)$-angulated.

The higher derived category problem

For a d-representation-finite d-hereditary algebra Λ, we have the subcategory $\mathcal{U}_\Lambda$ as the higher analogue of the derived category.

One can ask whether there is a higher analogue of the derived category for all d-representation-finite algebras, not just d-hereditary algebras.

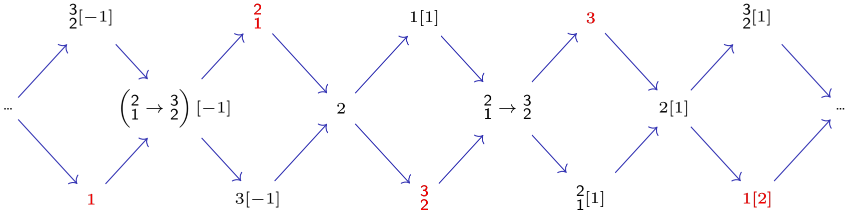
However, given a d-representation-finite algebra Λ with d-cluster-tilting module M, the subcategory

$$\mathcal{U}_\Lambda = \operatorname{add}\{\, M[id] : i \in \mathbb{Z} \,\}$$

is not always d-cluster-tilting and is not always $(d+2)$-angulated.

The d-cluster-tilting subcategory of $D^b(\mathrm{mod}\, A_n^d)$: example

The derived category of A_2^2 is as follows.



The 2-cluster-tilting subcategory $\mathcal{U}_{A_2^2}$ is highlighted in red.

An example of a 4-angle is

$$1 \to \begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 3 \to 1[2].$$

3.2. Combinatorial description

The d-cluster-tilting subcategory of $D^b(\operatorname{mod} A_n^d)$

Theorem ([OT12, Proposition 6.1 and Lemma 6.6])

1. *The indecomposable objects of* $\mathcal{U}_{A_n^d}$ *are in bijection with*

$$\tilde{\mathbf{I}}^d_{n+2d+1} = \left\{ A \in \binom{\mathbb{Z}}{d+1} : \begin{array}{c} \forall i \in \{0,1,\ldots,d-1\}, \\ a_{i+1} \geqslant a_i + 2 \text{ and } a_d + 2 \leqslant a_0 + n + 2d + 1 \end{array} \right\}$$

2. $U_A[d] = U_{\{a_1-1, a_2-1, \ldots, a_d-1, a_0+n+2d\}}$.
3. $\operatorname{Hom}_{\mathcal{D}_{A_n^d}}(U_B, U_A) \neq 0$ *if and only if*

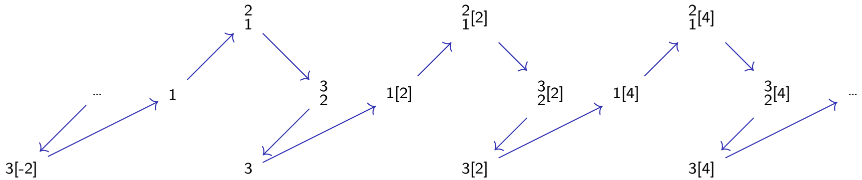
$$b_0 - 1 < a_0 < b_1 - 1 < a_1 < \cdots < b_d - 1 < a_d < b_0 + n + 2d,$$

and in this case the Hom*-space is one-dimensional.*

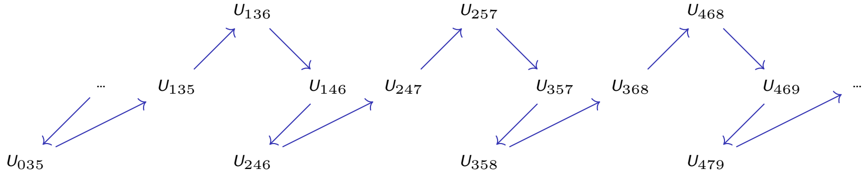
4. $\nu_d U_B = U_{B-\mathbf{1}}$.

The d-cluster-tilting subcategory of $D^b(\mathrm{mod}\, A_n^d)$: example

We consider the example of A_2^2 again. The category $\mathcal{U}_{A_2^2}$ is as follows.



This is then described combinatorially as follows.



4. The $(d+2)$-angulated cluster category

4.1. Theory

Definition of the cluster category

The *cluster category* of Λ is defined to be the orbit category [OT12, Definition 5.22]

$$\mathcal{O}_\Lambda = \frac{\mathcal{U}_\Lambda}{\nu[-2d]}\,.$$

For $d = 1$, this coincides with the classical cluster category of [Bua+06].

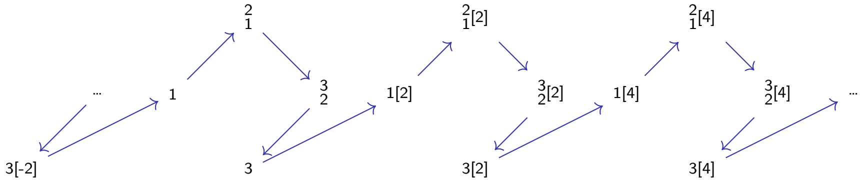
The fundamental domain of $\mathcal{O}_\Lambda$ is $\mathrm{add}(M \oplus \Lambda[d])$, where M is the basic d-cluster-tilting module in $\mathrm{mod}\,\Lambda$.

The category $\mathcal{O}_\Lambda$ is $2d$-Calabi–Yau, that is

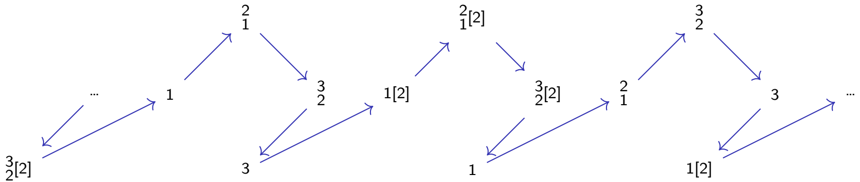
$$\mathrm{Hom}_{\mathcal{O}_\Lambda}(X, Y) \cong D\,\mathrm{Hom}_{\mathcal{O}_\Lambda}(Y, X[2d]).$$

Example: the $(d+2)$-angulated cluster category

We start with the category $\mathcal{U}_{A_2^2}$.



We take the orbit category of this modulo $\nu[-4]$ to obtain the following.



Relation with the $2d$-Amiot cluster category

The $2d$-Amiot cluster category is defined

$$\mathcal{C}_\Lambda^{2d} = \text{triangulated hull}\left(\frac{D^b(\operatorname{mod}\Lambda)}{\nu[-2d]}\right).$$

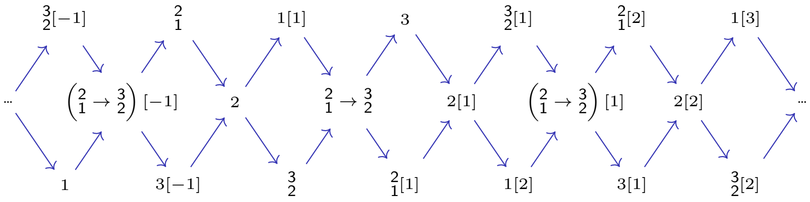
These were introduced in [IO13] based on [Ami09] and [Tho07].

Theorem ([OT12])

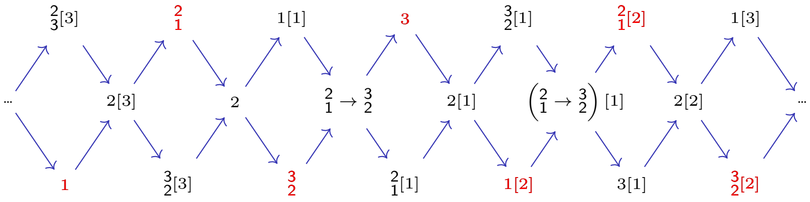
The $(d+2)$-angulated cluster category $\mathcal{O}_\Lambda$ is a d-cluster-tilting subcategory of the $2d$-Amiot cluster category $\mathcal{C}_\Lambda^{2d}$.

Relation with the $2d$-Amiot cluster category: example

We start with the derived category of A_2^2.



The 4-Amiot cluster category $\mathcal{C}^4_{A_2^2}$ is as follows.



The 2-cluster-tilting subcategory $\mathcal{U}_{A_2^2}$ is highlighted in red.

4.2. Cluster-tilting objects

Cluster-tilting objects

Definition ([OT12, Definition 5.3])

An object $T \in \mathcal{O}_\Lambda$ is *cluster-tilting* if

1. $\mathrm{Hom}_{\mathcal{O}_\Lambda}(T, T[d]) = 0$, and
2. any $X \in \mathcal{O}_\Lambda$ occurs in a $(d+2)$-angle

$$X[-d] \to T_d \to T_{d-1} \to \cdots \to T_1 \to T_0 \to X$$

with $T_i \in \mathrm{add}\, T$.

Theorem ([OT12])

An object T of $\mathcal{O}_\Lambda$ is cluster-tilting if and only if it is $2d$-cluster-tilting in $\mathcal{C}_\Lambda^{2d}$.

Higher cluster-tilted algebras

Theorem ([OT12, Theorem 5.6])

Let T be a cluster-tilting object in $\mathcal{O}_\Lambda$ and set $\Gamma := \operatorname{End}_{\mathcal{O}_\Lambda} T$. Then the functor

$$\operatorname{Hom}_{\mathcal{O}_\Lambda}(T,-)\colon \mathcal{O}_\Lambda \to \operatorname{mod}\Gamma$$

induces a fully faithful embedding

$$\mathcal{O}_\Lambda/(T[d]) \hookrightarrow \operatorname{mod}\Gamma.$$

The image of this functor is a d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$.

Remark

The analogous statement for tilting modules is not true! That is, tilted algebras of d-representation-finite algebras do not always have d-cluster-tilting subcategories in their module categories.

4.3. Combinatorial description

Combinatorial description in type A

Theorem ([OT12, Proposition 6.1 and Theorem 5.2(3)])

There is a bijection $A \mapsto O_A$ between ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ and the isomorphism classes of indecomposable objects of $\mathcal{O}_{A^d_n}$ such that the following properties hold.

1. $O_A[d] = O_{A-\mathbf{1}}$.
2. $\mathrm{Hom}_{\mathcal{O}_{A^d_n}}(O_B, O_A) \neq 0$ *if and only if* $(B-\mathbf{1}) \between A$
3. *For indecomposables O_A, O_B of $\mathcal{O}_{A^d_n}$, we have that* $\mathrm{Hom}_{\mathcal{O}_{A^d_n}}(O_B, O_A[d]) \neq 0$ *if and only if $A \between B$.*

$$
{}^{\circlearrowleft}\mathbf{I}^d_m := \left\{ \{a_0, \ldots, a_d\} \in \binom{[m]}{d+1} : \begin{array}{l} \forall i \in [d], a_i \geqslant a_{i-1} + 2, \\ a_d + 2 \leqslant a_0 + m \end{array} \right\}.
$$

Here $A \between B$ if either $A \wr B$ or $B \wr A$.

Combinatorial description in type A: sketch proof

This follows from taking the combinatorial description of $\mathcal{U}_{A_n^d}$ modulo $n+2d+1$.

We obtain $\mathcal{O}_{A_n^d}$ from $\mathcal{U}_{A_n^d}$ by taking the orbit category modulo $\nu[-2d]$.

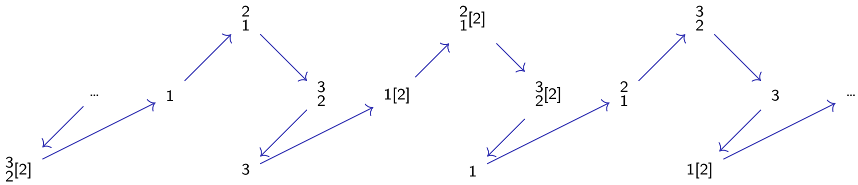
We have that $\nu[-2d]U_{\{a_0,a_1,\dots,a_d\}} = \nu_d[-d]U_{\{a_0,a_1,\dots,a_d\}} = \nu_d U_{\{a_d-(n+2d),a_0+1,\dots,a_{d-1}+1\}} = U_{\{a_d-(n+2d+1),a_0,\dots,a_d\}}$.

Taking $\tilde{\mathbf{I}}^d_{n+2d+1}$ modulo $n+2d+1$ gives ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$.

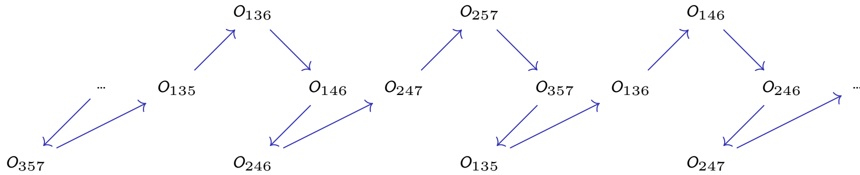
The other parts follow in the natural way.

The d-cluster-tilting subcategory of $D^b(\mathrm{mod}\, A_n^d)$: example

The category $\mathcal{O}_{A_2^2}$ is as follows.



Combinatorially, this is described as follows.



Cluster-tilting objects in $\mathcal{O}_{A_n^d}$

Theorem ([OT12])

A basic object $\bigoplus_{i=1}^{m} O_{B_i}$ in $\mathcal{O}_{A_n^d}$ is a cluster-tilting object if and only if $m = \binom{n+d-2}{d}$ and $\{B_i : i \in [m]\}$ is non-intertwining.

5. The d-almost positive category

5.1. Theory

The d-almost positive category

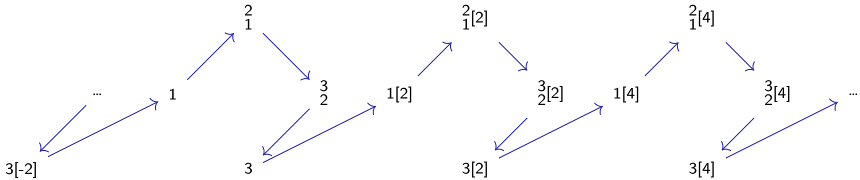
Given a d-representation-finite d-hereditary algebra Λ with d-cluster-tilting module M, define the *d-almost positive* category $\mathcal{U}_{\Lambda}^{\{-d,0\}}$ to be the subcategory $\mathrm{add}(M \oplus \Lambda[d])$ of $D^b(\mathrm{mod}\, A_n^d)$.

For $d = 1$, this coincides with the category of two-term complexes of projectives.

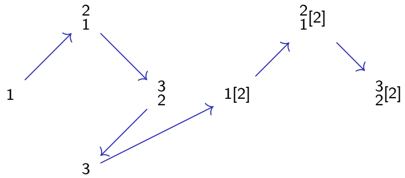
But, for $d > 1$, this category does not contain all $(d+1)$-term complexes of projectives.

The *d*-almost positive category: example

We consider the example of A_2^2 again. The category $\mathcal{U}_{A_2^2}$ is as follows.



The *d*-almost positive category $\mathcal{U}_{A_2^2}^{\{-2,0\}}$.



5.2. d-silting complexes

Silting complexes

A complex T in $D^b(\mathrm{mod}\,\Lambda)$ is called *pre-silting* if $\mathrm{Hom}_{D^b(\mathrm{mod}\,\Lambda)}(T, T[i]) = 0$ for all $i > 0$.

A pre-silting complex T in $D^b(\mathrm{mod}\,\Lambda)$ is called *silting* if, additionally, $\mathrm{thick}\,T = D^b(\mathrm{mod}\,\Lambda)$.

Here $\mathrm{thick}\,T$ denotes the smallest subcategory of $D^b(\mathrm{mod}\,\Lambda)$ which contains T and is closed under cones, $[\pm 1]$ and direct summands.

d-silting complexes

We call a silting object T of $D^b(\operatorname{mod}\Lambda)$ *d-silting* if, additionally, it lies in $\mathcal{U}_\Lambda^{\{-d,0\}}$.

Note that for objects T, T' of $\mathcal{U}_\Lambda^{\{-d,0\}}$ we have $\operatorname{Hom}_{D^b(\operatorname{mod}\Lambda)}(T, T'[i]) = 0$ if $i \notin \{-d, 0, d\}$ due to the d-cluster-tilting condition and the global dimension of Λ.

Hence, for an object T of $\mathcal{U}_\Lambda^{\{-d,0\}}$ with $\operatorname{thick} T = D^b(\operatorname{mod}\Lambda)$ to be d-silting, it suffices that $\operatorname{Hom}_{D^b(\operatorname{mod}\Lambda)}(T, T[d]) = 0$.

5.3. Combinatorial description

The d-AP category for type A

The properties from the combinatorial description of $\mathcal{U}_{A_n^d}$ carry over, and we get the following improved interpretation of extensions.

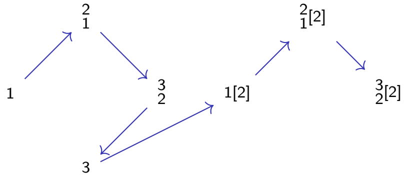
Theorem ([Wil; OT12])

There is a bijection $A \mapsto U_A$ between ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ and the indecomposable objects of $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ such that:

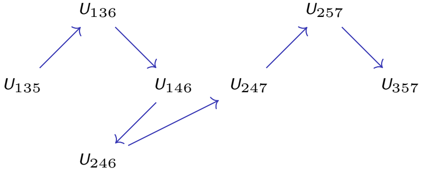
- $\operatorname{Hom}_{D^b(\operatorname{mod} A_n^d)}(U_A, U_B[d]) \neq 0$ *if and only if* $B \wr A$, *and in this case the* Hom*-space is one-dimensional.*

The d-AP category for type A: combinatorial description

We start with the d-almost positive category $\mathcal{U}_{A_2^2}^{\{-2,0\}}$.



This is then labelled as follows.



d-silting complexes in $\mathcal{O}_{A_n^d}$

Theorem ([Wil])

A basic complex $\bigoplus_{i=1}^{m} U_{B_i}$ in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$ is a d-silting complex if and only if $m = \binom{n+d-2}{d}$ and $\{B_i : i \in [m]\}$ is non-intertwining.

Numbers of summands

In general, for tilting modules, cluster-tilting objects, and d-silting complexes, it is not known whether one can replace the generating condition with the condition that the object has as many non-isomorphic indecomposable direct summands as there are indecomposable projectives.

However, this is known for the $d = 1$ cases for

- the tilting modules of projective dimension one from [BB80],
- cluster-tilting objects in the classical cluster category of [Bua+06],
- two-term silting complexes [Aih13; AIR14].

The fact that this is not known to hold for $d > 1$ is one of the things that makes the higher case difficult.

Showing that having the right number of summands is sufficient in the $d > 1$ case for A_n^d actually uses the interpretation in terms of cyclic polytopes.

6. Unifying the settings

d-exangulated categories

Extriangulated categories were introduced in order to axiomatise extension-closed subcategories of triangulated categories and to unify exact categories with triangulated categories [NP19].

d-exangulated categories were introduced as the higher generalisation of extriangulated categories [HLN21]. The d-almost positive category is a d-exangulated category.

Roughly, a d-exangulated category is an additive category with an additive bifunctor to $\mathbf{Ab}$ which represents Ext, and a choice of sequences ("d-exangles") which "realise" the elements of the extension group.

These d-exangles can be $(d+2)$-angles, or exact sequences with d middle terms, for instance.

The set of d-exangles in a d-exangulated structure can be a subset of the $(d+2)$-angles or exact sequences in the underlying category.

Quotienting by projective-injectives

The quotient of an extriangulated category by an additive subcategory consisting of projective-injective modules remains an extriangulated category.

For instance, the quotient of a Frobenius extriangulated category by the subcategory consisting of all projective-injectives gives a triangulated category.

The quotient of a d-exangulated category by a subcategory consisting of projective-injective modules is not always a d-exangulated category, but is in some nice cases [HZZ21].

Relation between the module category and the d-almost positive category

Theorem (W)

Let $\mathcal{J}$ be the category of projective-injective A^d_{n+1}-modules. Then there is an equivalence of d-exangulated categories

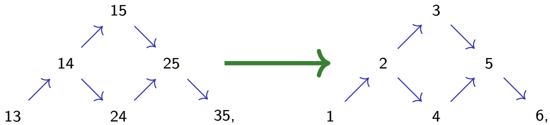
$$\operatorname{add} M^{(d,n+1)}/\mathcal{J} \simeq \mathcal{U}^{\{-d,0\}}_{A^d_n}.$$

This is proved using the combinatorial interpretation.

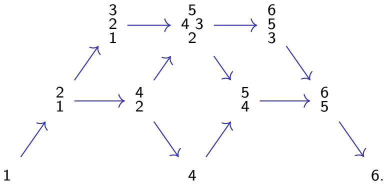
This explains why d-silting for A^d_n behaves the same as tilting inside $\operatorname{add} M^{(d,n+1)}$ for A^d_{n+1}.

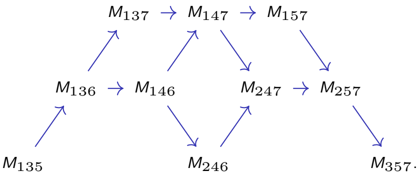
Relation between the module category and the d-almost positive category: example

If we label $Q^{(2,3)}$ as



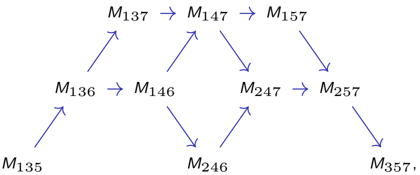
then the 2-cluster-tilting subcategory of $\operatorname{mod} A_3^2$ is given by



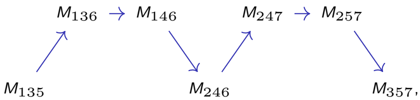


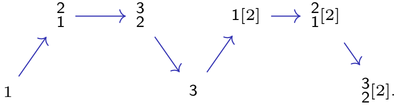
Relation between the module category and the d-almost positive category: example

Taking the category



and quotienting out the projective-injectives, gives the following, which is equivalent to the 2-almost positive category $\mathcal{U}^{\{-2,0\}}_{A_2^2}$.





Relation between the cluster category and the d-almost positive category

Consider the d-exangulated structure on $\mathcal{O}_{A_n^d}$ where the distinguished d-exangles are given by distinguished $(d+2)$-angles

$$O_1 \to G_d \to G_{d-1} \to \cdots \to G_1 \to O_2 \to O_1[d]$$

where $O_2 \to O_1[d]$ factors through $O_3[d]$, where O_3 is the image of a projective A_n^d-module in $\mathcal{O}_{A_n^d}$.

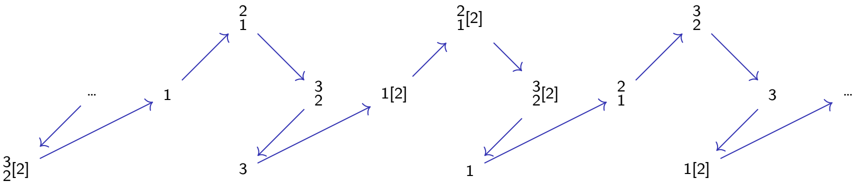
We further quotient by the ideal of morphisms $O_1[d] \to O_2$, where O_1 and O_2 are both images in $\mathcal{O}_{A_n^d}$ of projective A_n^d-modules.

Theorem (W)

The resulting category is equivalent to $\mathcal{U}_{A_n^d}^{\{-d,0\}}$.

Relation between the cluster category and the d-almost positive category: example

We start with the 4-angulated cluster category



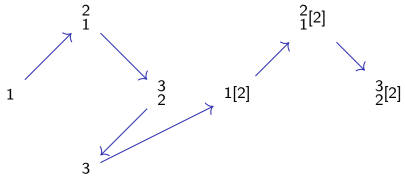
Our new 2-angulated structure consists of those 4-angles

$$G_1 \to G_2 \to G_3 \to G_4 \to G_1[2]$$

where $G_4 \to G_1[2]$ factors through a shifted projective.

Relation between the cluster category and the d-almost positive category: example

Taking the ideal quotient with respect to morphisms which factor through morphisms from shifted projectives to projectives is the same as quotienting out the morphism $\begin{smallmatrix}3\\2\end{smallmatrix}[2] \to 1$.



This is indeed the 2-almost positive category $\mathcal{U}_{A_2^2}^{\{-2,0\}}$.

ありがとうございました!

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