Cyclic polytopes and higher Auslander–Reiten theory I

Cyclic polytopes, their triangulations, and the higher Stasheff–Tamari orders

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1. Cyclic polytopes

Cyclic polytopes: standard construction

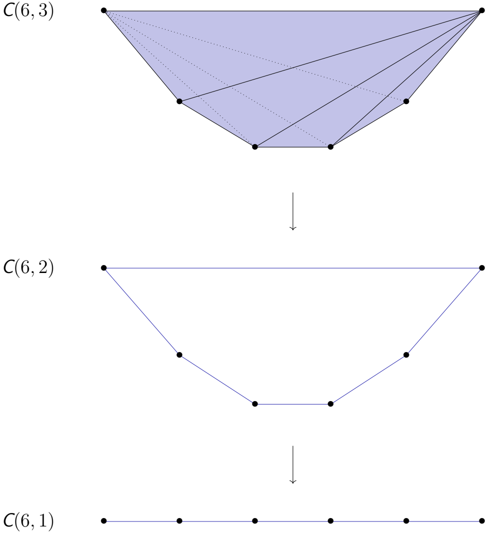
The *cyclic polytope* $C(m, \delta)$ is the convex hull of m points $\{p(t_1), \dots, p(t_m)\} \subset \mathbb{R}^\delta$ on the curve

$$p(t) = (t, t^2, \dots, t^\delta),$$

where $\{t_1, \dots, t_m\} \subset \mathbb{R}$.

This curve is known as the *moment curve*.

To simplify, one can always choose $\{t_1, \dots, t_m\} = \{1, \dots, m\} =: [m]$, without loss of generality.



[ER96, Figure 2]

1.1. Constructing cyclic polytopes

More general cyclic polytopes

Two polytopes are *combinatorially equivalent* if they have isomorphic face lattices.

Here a *subpolytope* is the convex hull of a subset of the vertices.

It follows from the standard construction of cyclic polytopes that any subpolytope of a cyclic polytope must be cyclic too.

More generally, a polytope $C(m, \delta)$ is a cyclic polytope if it is combinatorially equivalent to the polytope $C(m, \delta)$ from the previous slide in such a way that it restricts to combinatorial equivalences between the corresponding subpolytopes.

Other constructions of cyclic polytopes: order δ curves

There are in fact many curves one can choose to define a cyclic polytope.

Definition

A curve is called an *order δ curve* if no affine hyperplane can meet it in strictly more than δ points.

In particular, the moment curve is an order δ curve.

Theorem ([CD00; Stu87])

1. *A curve is an order δ curve if and only if the convex hull of any m points on the curve is a cyclic polytope $C(m, \delta)$.*
2. *Every cyclic polytope $C(m, \delta)$ has an order δ curve passing through its vertices in the right order.*

Other constructions of cyclic polytopes: totally positive matrices

A matrix is *totally positive* if all of its minors are positive.

Embed $\mathbb{R}^{\delta-1}$ as the affine hyperplane in $\mathbb{R}^{\delta}$.

Let $x_1, \dots, x_m$ be the vertices of a cyclic polytope $C(m, \delta-1)$ in this embedded copy of $\mathbb{R}^{\delta-1}$.

Let M be an $(m-\delta) \times \delta$ matrix of homogeneous coordinates of $x_{\delta+1}, \dots, x_m$ in the ordered basis

$$\{(-1)^{\delta+1} x_{\delta}, (-1)^{\delta} x_{\delta-1}, \dots, -x_2, x_1\}$$

of $\mathbb{R}^{\delta}$.

Theorem ([Stu88])

The matrix M is totally positive, and every totally positive matrix arises in this way.

1.2. Properties of cyclic polytopes

Upper Bound Theorem

The following theorem is a key property of cyclic polytopes.

Theorem ([McM70])

Out of all polytopes with m vertices in dimension δ, the cyclic polytope $C(m, \delta)$ has the largest number of k-dimensional faces for every k.

Stanley generalised this theorem to triangulated spheres using Stanley–Reisner theory [Sta75].

In fact, the following result holds for cyclic polytopes.

Theorem ([Gal63])

The cyclic polytope $C(m, \delta)$ is $\lfloor \delta/2 \rfloor$-neighbourly: every set of $\lfloor \delta/2 \rfloor$ vertices spans a face.

Ramsey-theoretic properties

Another remarkable property of cyclic polytopes is the following.

Theorem ([CD00; Bjö+99])

Any sufficiently large collection of points in general position in $\mathbb{R}^\delta$ must contain the set of vertices of a cyclic polytope $C(m, \delta)$. Moreover, any polytope which has this property is a cyclic polytope.

This is a higher-dimensional version of the famous Erdős–Szekeres Theorem, which says that every sufficiently long sequence of real numbers must contain a monotonic sequence of a given length.

1.3. Facets of cyclic polytopes

Upper facets and lower facets

From now on, we restrict our attention to the standard construction of cyclic polytopes.

Recall that a *facet* of a polytope is a face of codimension one.

A facet $|F|$ of $C(m, \delta)$ is an *upper facet* (*lower facet*) if for any $\mathbf{a} = (a_1, \ldots, a_\delta) \in \mathbb{R}^\delta$ such that $\langle \mathbf{a}, - \rangle$ is maximised on $|F|$, we have that $a_\delta > 0$ ($a_\delta < 0$).

More intuitively, a facet is an upper facet if its normal vector pointing out of the polytope points upwards with respect to the δ-th coordinate.

Gale's Evenness Criterion

Definition

Given $F \subset [m]$, then F is an *odd* (*even*) subset if for all $i \in [m] \setminus F$,

$$\#\{j \in F : j > i\}$$

is odd (even).

Theorem ([Gal63; ER96])

Given $F \subset [m]$, we have that $|F|$ is an upper facet (lower facet) of $C(m, \delta)$ if and only if F is an odd subset (even subset).

Proof of Gale's Evenness Criterion

Let $F = \{f_0, \ldots, f_{\delta-1}\} \subset [m]$.

Consider the polynomial $q(t) := \prod_{i=0}^{\delta-1}(t - f_i)$.

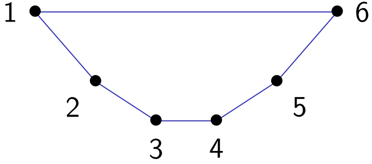
Define β_i such that $q(t) = \sum_{i=0}^{\delta} \beta_i t^i$, and let $\beta = (\beta_1, \ldots, \beta_\delta)$.

The unique hyperplane through $|F|$ is given by $\langle \beta, \mathbf{x} \rangle + \beta_0 = 0$, since $\langle \beta, p(t) \rangle + \beta_0 = q(t)$, which is zero precisely for elements of F (where $p(t) = (t, t^2, \ldots, t^\delta)$ is the moment curve).

$|F|$ is an upper facet if and only if $\langle \beta, p(j) \rangle + \beta_0 = q(j) < 0$ for all $j \in [m] \setminus F$, since $\beta_\delta = 1 > 0$.

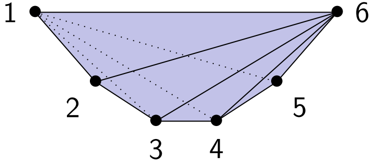
This is equivalent to F's being an odd subset, as desired. The proof for lower facets is similar.

Examples of Gale's Evenness Criterion



Upper $= \{16\}$

Lower $= \{12, 23, 34, 45, 56\}$



Upper $= \{126, 236, 346, 456\}$

Lower $= \{123, 134, 145, 156\}$

1.4. Circuits of cyclic polytopes

Circuits

A *circuit* of a polytope is a pair (A, B) of disjoint sets of vertices such that $\mathrm{conv}(A) \cap \mathrm{conv}(B) \neq \varnothing$ such that A and B are minimal with respect to this property.

Theorem ([Bre73])

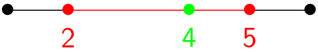
The circuits of $C(m, \delta)$ are the pairs $(Z_-, Z_+), (Z_+, Z_-)$ where $Z_- = \{\dots, z_{\delta-1}, z_{\delta+1}\}, Z_+ = \{\dots, z_\delta, z_{\delta+2}\}$ for $\{z_1, z_2, \dots, z_{\delta+1}, z_{\delta+2}\} \subseteq [m]$.

Here we say that Z_- *intertwines* Z_+ and write $Z_- \wr Z_+$.

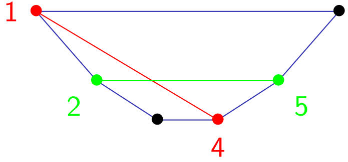
Proof.

The circuits of $C(m, \delta)$ cannot be supported on fewer than $\delta + 2$ vertices, since the moment curve is an order δ curve.
Radon's theorem tells us that $\delta + 2$ vertices in $\mathbb{R}^\delta$ can be partitioned into two halves of a circuit.
No other partition is possible, due to Gale's Evenness Criterion. $\square$

Examples of circuits



(25,4), (4,25)

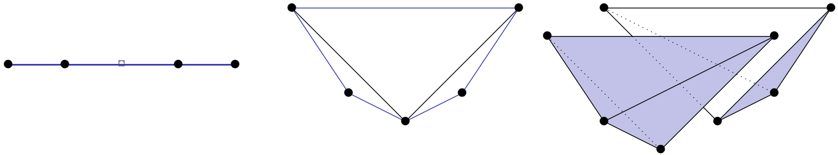


(14,25), (25,14)

2. Triangulations of cyclic polytopes

Triangulations

A *triangulation* of $C(m, \delta)$ is a subdivision of $C(m, \delta)$ into δ-simplices whose vertices are vertices of $C(m, \delta)$.

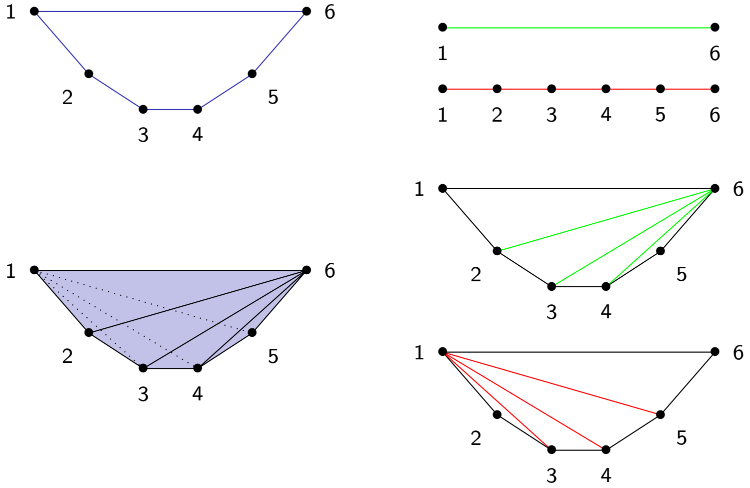


2.1. Fundamental properties of triangulations

The upper and lower triangulations

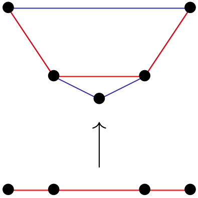
$C(m, \delta)$ possesses two special triangulations called the upper triangulation and the lower triangulation.

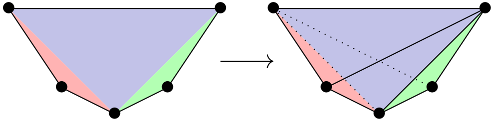
These result respectively from the projections of the upper and lower facets of $C(m, \delta + 1)$.



Sections induced by triangulations

Triangulations $\mathcal{T}$ give sections $s_{\mathcal{T}}\colon C(m,\delta) \to C(m,\delta+1)$. These are composed of simplex-wise maps $s_A\colon |A| \to C(m,\delta+1)$ for simplices $|A|$.





Bistellar flips

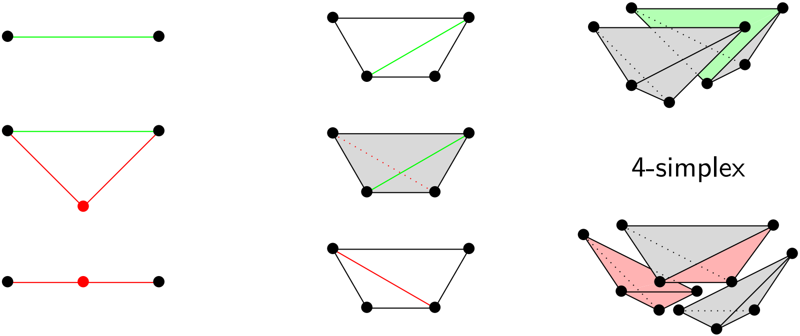
Every triangulation of $C(\delta+2,\delta)$ induces a section of $C(\delta+2,\delta+1)$.

But $C(\delta+2,\delta+1)$ is a simplex, so it only has two sections: the upper facets and the lower facets.

Hence $C(\delta+2,\delta)$ has only two triangulations: the upper triangulation and the lower triangulation.

An *increasing bistellar flip* on a triangulation $\mathcal{T}$ of $C(m,\delta)$ consists of replacing a lower triangulation of a $C(\delta+2,\delta)$ subpolytope with the upper triangulation of this subpolytope.

Examples of bistellar flips

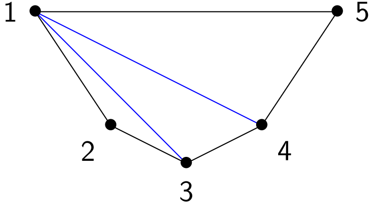


Flipping a diagonal inside a quadrilateral is a well-known operation from cluster algebras. Bistellar flips generalise this to arbitrary dimensions.

2.2. Describing triangulations of cyclic polytopes

Dey's Theorem

When we consider a triangulation of a convex polygon, we think in terms of non-intersecting arcs, not triangles.



One thinks of this as $\{13, 14\}$, not $\{123, 134, 145\}$.

Theorem ([Dey93])

A triangulation of a point configuration in $\mathbb{R}^\delta$ is determined by its $\lfloor \delta/2 \rfloor$-simplices.

For cyclic polytopes, it suffices to consider *internal* $\lfloor \delta/2 \rfloor$-simplices, i.e., ones which are not on the boundary of the polytope.

Description of even-dimensional triangulations

A triangulation of a convex polygon is given by a set of non-crossing arcs of a particular size.

A similar description holds for even-dimensional cyclic polytopes.

Theorem ([OT12])

A triangulation of $C(m, 2d)$ is given by a set of size $\binom{m-d-2}{d}$ of internal d-simplices whose interiors do not intersect.

There are sets of non-intersecting internal d-simplices which are maximal with respect to inclusion but not maximal with respect to size.

Combinatorial description for even dimensions

Theorem ([OT12])

There is a bijection between triangulations of $C(m, 2d)$ and sets of non-intertwining $(d+1)$-subsets from $^{\circlearrowleft}\mathbf{I}_m^d$ of size $\binom{m-d-2}{d}$ given by sending a triangulation $\mathcal{T}$ to its set of internal d-simplices $\mathring{e}(\mathcal{T})$.

A d-simplex $|A|$ in $C(m, 2d)$ is internal if and only if

$$A \in {}^{\circlearrowleft}\mathbf{I}_m^d := \{\{a_0, \ldots, a_d\} \subseteq [m] \mid a_{i+1} \geqslant a_i + 2 \mod m\}.$$

We know from the description of the circuits of $C(m, 2d)$ when the interiors of a pair of d-simplices $|A|$ and $|B|$ intersect, namely when $A \wr B$ or $B \wr A$.

Description of even-dimensional triangulations: example

The cyclic polytope $C(7, 4)$ has the following 7 triangulations, as described by their sets of internal 2-simplices.

$$\{135, 136, 146\}$$
$$\{246, 247, 257\}$$
$$\{135, 136, 357\}$$
$$\{146, 246, 247\}$$
$$\{135, 257, 357\}$$
$$\{136, 146, 246\}$$
$$\{247, 257, 357\}$$

For the first of these triangulations, the full set of 4-simplices is

$$\{12345, 12356, 12367, 13456, 13467, 14567\}.$$

How about odd dimensions?

What would an odd-dimensional analogue of the description of triangulations of even-dimensional cyclic polytopes from [OT12] look like?

In dimension $2d + 1$, the circuits are between a d-simplex and a $(d + 1)$-simplex, not between two simplices of the same dimension.

One could try taking a maximal set of $(d + 1)$-simplices and d-simplices which don't intersect each other.

- In odd dimensions numbers of simplices vary between triangulations, so there is no "maximal size" as in even dimensions.
- Being maximal (with respect to inclusion) doesn't guarantee that one has a triangulation [Ram97].
- The $(d + 1)$-simplices are redundant information here, by Dey's theorem.

Combinatorial description of odd-dimensional triangulations

Instead, we characterise when a set of d-simplices are the internal d-simplices of a triangulation of $C(m, 2d+1)$.

The internal d-simplices of $C(m, 2d+1)$ are those in

$$\mathbf{J}_m^d := \{\{a_0, \dots, a_d\} \in {}^{\circlearrowleft}\mathbf{I}_m^d \mid a_0 \neq 1,\ a_d \neq m\}.$$

Theorem ([Wil21a],[FR21, $d = 1$])

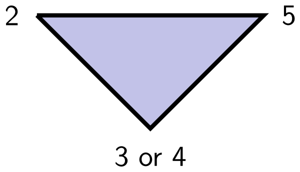
There is a bijection between triangulations of $C(m, 2d+1)$ and sets of $(d+1)$-subsets from $\mathbf{J}_m^d$ which are supporting and bridging, given by sending a triangulation $\mathcal{T}$ to its set of internal d-simplices $\mathring{e}(\mathcal{T})$.

Combinatorial description of odd-dimensional triangulations: supports

Let $\mathbf{X} \subseteq \mathbf{J}_m^d$. We say that $\mathbf{X}$ is *supporting* if for any $(d+1)$-subset $A \in \mathbf{X}$ there is a d-subset A' such that $A' \wr A$ and, for every $(d+1)$-subset $B \subset A \cup A'$ such that $B \in \mathbf{J}_m^d$, we have that $B \in \mathbf{X}$.

$25 \in \mathbf{X} \implies$

- $35 \in \mathbf{X}$, via $3 \wr 25$, or
- $24 \in \mathbf{X}$, via $4 \wr 25$.



$247 \in \mathbf{X} \implies$

- $257, 357 \in \mathbf{X}$, via $35 \wr 247$, or
- $246 \in \mathbf{X}$, via $36 \wr 247$.

Combinatorial description of odd-dimensional triangulations: bridging

Let $\mathbf{X} \subseteq \mathbf{J}^d_m$. We say that $\mathbf{X}$ is *bridging* if whenever

$$A := \{x_0, \ldots, x_{i-1}, a_i, \ldots, a_j, x_{j+1}, \ldots, x_d\},$$
$$B := \{x_0, \ldots, x_{i-1}, b_i, \ldots, b_j, x_{j+1}, \ldots, x_d\} \in \mathbf{X},$$

where possibly $i = 0$ or $j = d$, or both, such that $\{a_i, \ldots, a_j\} \wr \{b_i, \ldots, b_j\}$, we have that

$$S_k := \{x_0, \ldots, x_{i-1}, a_i, \ldots, a_{k-1}, b_k, \ldots, b_j, x_{j+1} \ldots, x_d\} \in \mathbf{X}$$

for all $i \leqslant k \leqslant j+1$.

$246, 357 \in \mathbf{X} \implies 247, 257 \in \mathbf{X}$:

$247 \qquad 257$

$246 \qquad\qquad 357$

Combinatorial description of odd-dimensional triangulations: example

The cyclic polytope $C(6,3)$ has the following triangulations, as described by their sets of internal 1-simplices.

$$\{24, 25, 35\}$$
$$\{24, 25\}$$
$$\{25, 35\}$$
$$\{24\}$$
$$\{35\}$$
$$\varnothing$$

For the last of these triangulations, the full set of 3-simplices is

$$\{1236, 1346, 1456\}.$$

$\{25\}$ is excluded because it is not supporting. $\{24, 35\}$ is excluded because it is not bridging.

3. The higher Stasheff–Tamari orders

The first higher Stasheff–Tamari order $\mathcal{S}_1(m, \delta)$

Defined first by Kapranov and Voevodsky and then by Edelman and Reiner in a different way. Thomas showed the two definitions gave the same order.

We have that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathcal{T}'$ is the result of performing an increasing bistellar flip within $\mathcal{T}$.

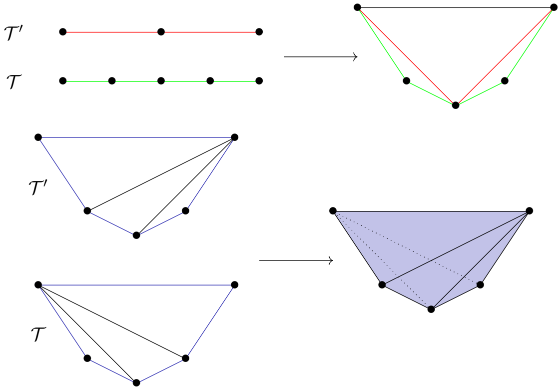
Hence $\mathcal{T} <_1 \mathcal{T}'$ if and only if we have

$$\mathcal{T} = \mathcal{T}_0 \lessdot_1 \mathcal{T}_1 \lessdot_1 \cdots \lessdot_1 \mathcal{T}_r = \mathcal{T}'.$$

The second higher Stasheff–Tamari order $\mathcal{S}_2(m, \delta)$

Defined by Edelman and Reiner [ER96]. Given $\mathcal{T}, \mathcal{T}'$ triangulations of $C(m, \delta)$,

$$\mathcal{T} \leqslant_2 \mathcal{T}' \iff s_{\mathcal{T}}(x)_{\delta+1} \leqslant s_{\mathcal{T}'}(x)_{\delta+1} \quad \forall x \in C(m, \delta).$$



The Edelman–Reiner conjecture

Edelman and Reiner conjectured that $\mathcal{S}_1(m, \delta) = \mathcal{S}_2(m, \delta)$ in their 1996 paper, which they proved for $\delta \leqslant 3$.

It is clear that if $\mathcal{T} \leqslant_1 \mathcal{T}'$, then $\mathcal{T} \leqslant_2 \mathcal{T}'$, since an increasing bistellar flip moves the section upwards.

But it is not clear that we always have $\mathcal{T} \leqslant_1 \mathcal{T}'$ whenever $\mathcal{T} \leqslant_2 \mathcal{T}'$, since it is not obvious how to construct a sequence of increasing bistellar flips from $\mathcal{T}$ to $\mathcal{T}'$.

Submersion sets

Edelman and Reiner give the following alternative characterisation of the second higher Stasheff–Tamari order.

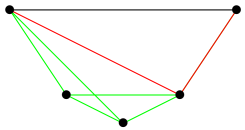
Given a simplex $|A|$ in $C(m, \delta)$, recall the map $s_A \colon |A| \to C(m, \delta + 1)$.

A simplex $|A|$ is *submerged* by a triangulation $\mathcal{T}$ if

$$s_A(x)_{\delta+1} \leqslant s_{\mathcal{T}}(x)_{\delta+1} \quad \forall x \in |A|.$$

The *k-submersion set* $\mathrm{sub}_k \mathcal{T}$ is the set of k-simplices submerged by the triangulation $\mathcal{T}$.

Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathrm{sub}_{\lceil \delta/2 \rceil} \mathcal{T} \subseteq \mathrm{sub}_{\lceil \delta/2 \rceil} \mathcal{T}'$.



3.1. Properties

Motivation for the higher Stasheff–Tamari orders

Generalise the Tamari lattice: both $\mathcal{S}_1(m, 2)$ and $\mathcal{S}_2(m, 2)$ are equal to the Tamari lattice.

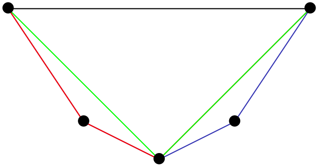
$\mathcal{S}_1(m, \delta)$ is a higher category, which was the original motivation for its introduction by Kapranov and Voevodsky [KV91].

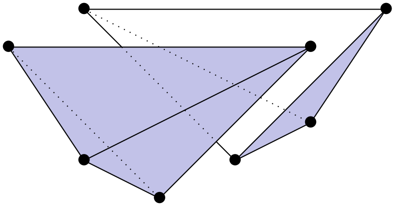
$\mathcal{S}_1(m, \delta)$ describes the evolution of a class of KP solitons [DM12; Wil21b], solutions to a differential equation describing solitary waves.

$\mathcal{S}_2(m, \delta)$ gives a direct way of comparing triangulations in the first order, if indeed the two are equal.

Rambau's Theorem

$$\left\{\begin{array}{l}\text{Triangulations of}\\ C(m,\delta+1)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Maximal chains in}\\ \mathcal{S}_1(m,\delta)\end{array}\right\}\Big/\sim$$





(Lack of) lattice property of the higher Stasheff–Tamari orders

Edelman and Reiner showed that $\mathcal{S}_1(m, \delta)$ and $\mathcal{S}_2(m, \delta)$ are lattices for $\delta \leqslant 3$.

Edelman, Rambau, and Reiner found a counter-example to $\mathcal{S}_2(m, \delta)$ always being a lattice.

The same counter-example was used to show that $\mathcal{S}_1(m, \delta)$ is not always a lattice in [Wil21a].

3.2. Equality of the orders

Result

Theorem ([Wil21c])

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(m, \delta)$. Then $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$.

The first step in proving this theorem is to give new combinatorial interpretations of the orders which make them easier to compare. We will see this later in the talks when we connect the higher Stasheff–Tamari orders with representation theory of algebras.

Introduction to proof of equality

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(m, \delta)$. In order to show that $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$, we need to show that if $\mathcal{T} <_2 \mathcal{T}'$, then there exists an increasing bistellar flip $\mathcal{T}''$ of $\mathcal{T}$ such that $\mathcal{T}'' \leqslant_2 \mathcal{T}'$.

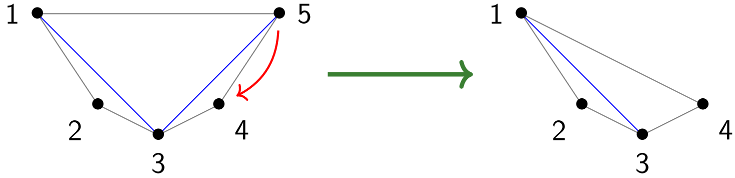
This gives us $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$. Then one can inductively construct a sequence of bistellar flips $\mathcal{T} = \mathcal{T}_0 \lessdot_1 \mathcal{T}'' = \mathcal{T}_1 \lessdot_1 \cdots \lessdot_1 \mathcal{T}_r = \mathcal{T}'$, giving $\mathcal{T} \leqslant_1 \mathcal{T}'$.

The problem is that bistellar flips are quite hard to find.

Our strategy is to use induction on the number of vertices of the cyclic polytope.

Contracting triangulations of cyclic polytopes

We consider the contraction operation $[m-1 \leftarrow m]$. Given a triangulation $\mathcal{T}$ of $C(m,\delta)$, $\mathcal{T}[m-1 \leftarrow m]$ is the triangulation of $C(m-1,\delta)$ which results from moving the vertex m along the moment curve until it coincides with the vertex $m-1$.



Main idea

We begin with two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $C(m, \delta)$ such that $\mathcal{T} <_2 \mathcal{T}'$.

We consider the contractions. We have $\mathcal{T}[m-1 \leftarrow m] \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$.

If $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$, then we need to consider other contractions. Otherwise, the induction hypothesis tells us that there is a triangulation $\mathcal{U}$ of $C(m-1, \delta)$ such that $\mathcal{T}[m-1 \leftarrow m] \lessdot_1 \mathcal{U} \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$.

The increasing bistellar flip from $\mathcal{T}[m-1 \leftarrow m]$ to $\mathcal{U}$ happens inside some $C(\delta + 2, \delta)$ subpolytope.

When we expand back to $\mathcal{T}$, this subpolytope either remains equivalent to $C(\delta + 2, \delta)$, or expands to be equivalent to $C(\delta + 3, \delta)$.

In either case, we look inside this subpolytope to find an increasing bistellar flip $\mathcal{T}''$ of $\mathcal{T}$. It can be shown that $\mathcal{T}'' \leqslant_2 \mathcal{T}'$.

ありがとうございました!

References I

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