Modular Representation Theory of Finite Groups – local vs. global

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§1. Introduction

We are going to discuss local-global conjectures on modular representation theory of finite groups. We assume throughout that:

- p is a (fixed) prime, and a triple $(K, \mathcal{O}, k)$ is a p-modular system, i.e. $\mathcal{O}$ is a complete discrete valuation ring, K is its quotient field with $\mathrm{char}(K) = 0$ and $k := \mathcal{O}/\mathrm{rad}(\mathcal{O})$ is a residue field of $\mathcal{O}$ with $\mathrm{char}(k) = p$ where $\mathrm{rad}(\mathcal{O})$ is the unique maximal subgroup of $\mathcal{O}$.

Compare
$$\begin{array}{ccc} \mathbb{Z} & \subset & \mathbb{Q} \\ \downarrow \text{onto} & & \\ \mathbb{Z}/p\mathbb{Z} & & \end{array} \qquad \begin{array}{ccc} \mathcal{O} & \subset & K \\ \downarrow \text{onto} & & \\ k & & \end{array}$$

- G, H are finite groups, and P, Q, D are finite p-groups.
- We denote $\mathbb{Z}/\mathbb{Z}n$ by C_n, the cyclic group of order n and the symmetric group of degree n by $\mathfrak{S}_n$ for $n \in \mathbb{N}$.

- A module is finitely generated (f.g.) *right* module unless stated otherwise. A is a block (block ideal) of the group algebra kG, i.e. A is an indecomposable direct summand of kG as (kG, kG)-bimodule. Note that a block A is called *principal* if the trivial kG-module $k_G \cong k\cdot(\sum_{g\in G} g) \in kG$ satisfies $k_G \cdot A \neq 0$, and in such a case we denote it by $B_0(kG)$.
- mod-A is the category of all f.g. right A-modules.
- P is a *defect group* of A, i.e., P is a minimal subgroup of G satisfying that, for every indecomposable $X \in$ mod-A, X is a direct summand of $X{\downarrow_P}{\uparrow^G} := X \otimes_{kP} kG$ as right kG-modules (then P must be a p-group by Mascheke's theorem) and P is unique up to G-conjugacy. Especially when $A = B_0(kG)$, the defect groups are nothing but the Sylow p-subgroups of G.

- Slogan 1: Every representation over A has an origin (a source) at the level of kP-modules, that is, the defect group P dominates (controls) mod-A. Namely, we must not forget the influence of what's going on at the level of kP! This is kind of philosophy of *splendid Morita (= Puig) equivalence* instead of just by looking at *Morita equivalence* where one can ignore (forget) the action of P on mod-A. Very important point!!
- For a ring R, 1_R is the unit element of R. Especially for the case $R = A$, 1_A is called the *block idempotent* of A,
- When we discuss modules/representations of G, H, ... (finitely many), we assume that the fields K and k are splitting fields for those representations (in order to have that $\mathrm{End}_{kG}(S) \cong k$ as k-algebras for a simple kG-module S) (Schur's lemma!).

§2. Global vs. local (Richard Brauer's philosophy)

- Consider the following situation:

	global	local
groups	G	$H := N_G(P)$
categories	mod-A	mod-B
defect groups	P	P

  where B is a block of kH such that B is a direct summand of A as (kH, kH)-bimodule (one knows that the multiplicity of B in ${}_{kH}A_{kH}$ is one), B is called *the Brauer correspondent* of A (we then write like $A = B^G$) and P is the unique defect group of B.
- Slogan 2: mod-A and mod-B should be *similar* each other (this is Brauer's philosophy, i.e.*global-local* problems /conjectures)!!

§3. Kiiti Morita (1958) vs. Lluis Puig (1981)

- A finite-dimensional k-algebra $\mathcal{A}$ is called an *interior kG-algebra* if a group homomorpism, say $\phi : G \to \mathcal{A}^\times$ is attached (where $\mathcal{A}^\times$ is the set of all units of $\mathcal{A}$). We write $g{\cdot}a{\cdot}g' := \phi(g)\, a\, \phi(g')$ for $a \in \mathcal{A}$ and $g, g' \in G$.
- Even for such an $\mathcal{A}$ above, a defect group is defined. That is, P is a *defect group* of $\mathcal{A}$ when P is a *maximal* (up to G-conjugacy) p-subgroup of G such that $\mathrm{br}_P^{\mathcal{A}}(1_{\mathcal{A}}) \neq 0$, where $\mathrm{br}_P^{\mathcal{A}}$ is the canonical k-algebra epimorphism (and $\mathcal{A}(P)$ below is called *the Brauer construction* or *Brauer quotient*) defined by $\mathrm{br}_P^{\mathcal{A}} \ : \ \mathcal{A}^P \twoheadrightarrow \mathcal{A}^P/(\sum_{Q \lneqq P} \mathrm{Tr}_Q^P(\mathcal{A}^Q)) := \mathcal{A}(P)$ where $\mathcal{A}^Q := \{a \in \mathcal{A} | au = ua \quad \forall u \in Q\}$ and Tr_Q^P is the trace map. Surely kG is an interior kG-algebra.

- For any p-subgroup $Q \leq G$, $(kG)(Q) \cong kC_G(Q)$ as interior $kC_G(Q)$-algebras.
- A block A of kG is an interior kG-algebra by the group homomorphism $\phi : G \to A^\times$ defined by $\phi(a) := ag$ for $a \in A$, $g \in G$ and also that the defect groups of A defined in the two ways, say as a block and as an interior algebra are the same.
- (L.Puig) For an interior kP-algebra $\mathcal{A}$ and an idempotent $i \in \mathcal{A}^P$, the algebra $\mathfrak{A} := i\mathcal{A}i$ is an interior kP-algebra by $\psi : P \to (i\mathcal{A}i)^\times$ with $\psi(u) := iui$ (that is *monomorphism*!). $\mathfrak{A}$ is called the *source algebra* of an interior kG-algebra $\mathcal{A}$ with respect to the defect group P of $\mathcal{A}$ when $\mathfrak{A} = i\mathcal{A}i$ where i is a primitive idempotent of $\mathcal{A}^P$ with $\mathrm{br}_P^{\mathfrak{A}}(i) \neq 0$. The action of P on $i\mathcal{A}i$ is kept, so the splendid Morita equivalence is splendid from group-representation theoretical point of view!!.

- Observe that if the defect group $P = 1$, then the source algebra $\mathfrak{A}$ of $\mathcal{A}$ is nothing but the basic ring (algebra) of $\mathcal{A}$, especially for the block A of kG if $P = 1$ then $A \cong \mathrm{Mat}_d(k)$ for some $d \in \mathbb{N}$, so that iAi (the source algebra of A) $\cong k$ as k-algebras.
- For two interior kG-algebras $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{A}$ and $\mathcal{B}$ are *splendidly Morita (Puig)* equivalent if, first of all, they have the same defect group P and also that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as interior kG-algebras (i.e. there is a k-algebra isomorphism $f : \mathcal{A} \overset{\approx}{\to} \mathcal{B}$ such that $f(g{\cdot}a{\cdot}g') = g{\cdot}f(a){\cdot}g'$) for $a \in \mathcal{A}$ and $g, g' \in G$ (warning: here possibly $f(1_{\mathcal{A}}) \neq 1_{\mathcal{B}}$). In such a case one knows that $\mathcal{A}$ and $\mathcal{B}$ are at least Morita equivalent, even better is that the action of P on A is still alive, on the other hand the basic algebra eAe has lost the influence by P.

- Observe that for a block A of kG with defect group P, the source algebra iAi of A w.r.t. P and A are splendidly Morita equivalent (just like that the basic algebra, say eAe of A, and A are Morita equivalent where e is an idempotent of A that realizes being a basic algebra).
- More importantly, recall that eAe is the *smallest* k-algebra which keeps being Morita equivalent to A and ovserve that iAi is the *smallest* k-algebra which keeps being *splendid Morita (Puig)* equivalent to A (and hence the important group P's action is still alive, that makes us happy, isn't it?)

§4. Splendid Morita equivalence (= Puig equivalence)

- L. Puig and (independently) L. Scott's theorem: When A and B are block algebras with defect group P and Q of kG and kH, respectively (here H is not necessarily a subgroup of G) there is a splendid Morita equivalence between A and B if and only if first of all, $P \cong Q$ (so we can think $P \leq G \cap H$) and there is an (A, B)-bimodule M such that M is perfect (i.e. ${}_AM$ and M_B are both projective), that $M \otimes_B M^\vee \cong {}_AA_A$ as (A, A)-bimodules and that M is a p-permutation $k(G \times H)$-module, namely $M_{k(G\times H)}$ is a direct summand of the induced module $k_{\Delta P}{\uparrow}^{G\times H}$, where $M^\vee$ is the k-dual of M and the action of $G \times H$ on M is given by $m{\cdot}(g, h) := g^{-1}mh$ for $m \in M, g \in G, h \in H$ and $\Delta P := \{(u, u) \in G \times H | u \in P\}$ (note the algebras are symmetric).

§5. Examples

- Example 1. $kC_p \cong k[x]/(x^p) \cong k\left[\circ \circlearrowright \alpha \,\middle|\, \alpha^p = 0\right]$ (path algebra), as k-algebras.
- Example 2. $p := 2 \mid 6 = |\mathfrak{S}_3|$, then $k\mathfrak{S}_3 \cong k[x]/(x^2) \times \mathrm{Mat}_2(k)$ (the first term of the right hand side is the principal 2-block).
- Example 3: Assume $p := 3 \mid 6 = |\mathfrak{S}_3|$. Then,

  $k\mathfrak{S}_3 \cong k\left[\circ \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} \circ \,\middle|\, \alpha^3 = 0\right]$ (path algebra). Recall that

  $\mathfrak{S}_3 = C_3 \rtimes C_2$ (semi-direct product), and hence

  $\circ \circlearrowright \alpha$ with $\alpha^3 = 0$ $\Rightarrow$ $\circ \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} \circ$ with $\alpha^3 = 0$

  see [Reiten-Riedtmann (1985)].

- Example 4. Set $p = 3$, $G := \mathrm{SL}(2,3) = \mathrm{SL}(2,\mathbb{F}_3)$ so that $G \cong Q_8 \rtimes P = C_2.(C_2 \times C_2).C_3 = C_2.A_4$ where Q_8 is the quaternion group of order 8, $P := C_3$ and A_4 is the alternating group of degree 4. The group algebra kG has three blocks, say $A_0, A_1 := A, A_2$ such that A_0 is the principal block, $A_0 \cong kP$ even as interior kP-algebras, $A \cong \mathrm{Mat}_2(kP)$ and $A_2 \cong \mathrm{Mat}_3(k)$ as k-algebras. This is what's happening at the global level. On the other hand, at the local level, namely set $H := N_G(P)$ so that $H \cong C_2 \times kP$, and hence kH has two blocks, say B_0 (the principal block) and $B := B_1$. We know that A_0 and B_0 are splendid Morita equivalent, but A and B are *not* splendid Morita equivalent although they are Morita equivalent! In fact, the source algebra of A with respect to P is A itself (see [Koshitani-Kunugi (2010)] for detail).

- Theorem 1 (G. Malle - B. Späth, 2016) (McKay Conjecture). If $p = 2$, the number of irreducible ordinary characters of G with odd degree is the same as that of $N_G(P)$ where P is a Sylow 2-subgroup of G. (This is surely global-local conjecture due to J. McKay (1972)).
- There are quite a few very interesting and big local-global conjectures on characters in representation theory of finite groups, essentially due to R. Brauer, such as Brauer's height zero conjecture, Brauer's $k(B)$-conjecture, and so on, see e.g. [G. Malle (2017)].
- There are also another conjectures that are not only counting the number of characters but also more structural one, e.g. Alperin's weight conjecture, Dade's conjecture and Broué's Abelian Defect Group Conjecture (ADGC, for short).

- ADGC claims that if A and B are blocks of kG and $kN_G(P)$ via Brauer correspondence where P is an *ableian* defect groups of A and B, then A and B would/should be derived equivalent, i.e. the bounded derived categories $\mathrm{D}^{\mathrm{b}}(\text{mod-}A)$ and $\mathrm{D}^{\mathrm{b}}(\text{mod-}B)$ would/should be equivalent as triangulated categories. Note that even in Example 5, the two blocks are derived equivalent (there $P \cong C_3$, abelian).
- Theorem 2(G.O.Michler-P.Landrock (1980),T.Okuyama (1997)).

  Suppose that $p = 2$, $G := G(q) := {}^2G_2(q) = \mathrm{Ree}(q)$ where $q := 3^{2n+1}$, $n = 0, 1, 2, \cdots$, P is a Sylow 2-subgroup of G (hence $P = C_2 \times C_2 \times C_2$, independently from q), and set $A := A(q)$ $:= B_0(k\,G(q))$ and $B := B_0(k\,N_G(P)) \cong k(P \rtimes (C_7 \rtimes C_3))$. Then all $A(q)$s are splendidly Morita equivalent to $A(3)$ $= B_0(k(\mathrm{SL}(2,8) \rtimes C_3))$, and ADGC holds for A.

- Theorem 3 (Koshitani - Kunugi (2001)). Assume that $G := G(q) := \mathrm{PSU}(3, q^2) = \mathrm{U}_3(q)$ with $q \equiv 2$ or $5 \pmod 3$, $p = 3$ and P is a Sylow 3-subgroup of $G(q)$ (and hence $P \cong C_3 \times C_3$, $G(2) = N_G(P) \cong P \rtimes Q_8$) and $A := A(q) := B_0(k\,G(q))$. Then all $A(q)$s are splendidly Morita equivalent to $A(2) \cong B_0(k\,N_G(P))$, and hence ADGC holds for A.
- Example 4. If $G := \mathrm{SL}(2, 64)$, $p = 3$, $A := B_0(kG)$, $B := B_0(k\,N_G(P))$ (observe $p \mid q - 1$ where $q := 64$), $A := B_0(kG)$ and $B := B_0(kG) \cong k(P \rtimes C_2)$ where P is a Sylow 3-subgroup of G (and hence $P \cong C_3$). Then A and B are splendidly Morita equivalent, and hence ADGC holds for A.

- Example 5. If $G := \mathrm{SL}(2,8)$, $p = 3$, $A := B_0(kG)$, $B := B_0(k\,N_G(P))$ where P is a Sylow 3-subgroup of G, and hence $P \cong C_3$ (observe $p \mid q+1$ where $q := 8$), then A and B are *not* splendidly Morita equivalent (note $B \cong k(P \rtimes C_2)$, compare to Example 4) but still ADGC holds for A.
- Note that for cyclic defect groups case ADGC holds in general by [Rickard (1989)] and [Linckelmann (1991)].
- Final remark. See the books [Linckelmann (2018)], [Puig (1999)] and [Thévenaz (1995)] for terms, notions, ... whatever, that have not been explained in this small note.
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https://researchseminars.org/seminar/TokyoNagoyaAlgebra

§6. References

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