講演ノート

Tilting ideals of deformed preprojective algebras

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§0 Introduction

Notation K : field

$Q = (Q_0, Q_1, t(a) \xrightarrow{a} h(a))$: finite connected quiver

Def • $\overline{Q}$: double quiver of Q

(i.e. $\overline{Q}_0 = Q_0$

$\overline{Q}_1 = Q_1 \sqcup \underbrace{\{a^* : h(a) \to t(a) \mid a \in Q_1\}}_{Q_1^*}$)

• $\lambda = (\lambda_i)_{i \in Q_0} \in K^{Q_0}$ weigh

• $\Pi^\lambda(Q) := K\overline{Q} \Big/ \Big\langle \sum_{a \in \overline{Q}_1} \varepsilon(a) a a^* - \sum_{i \in Q_0} \lambda_i e_i \Big\rangle$

deformed preprojective algebra

where $\varepsilon(a) = \begin{cases} +1 & a \in Q_1 \\ -1 & a \in Q_1^* \end{cases}$ $\quad (a^*)^* := a \quad a \in Q_1$

Rmk $\lambda = 0 \Rightarrow \Pi^0(Q)$: (classical) preprojective algebra by Gelfand-Ponomarev

Properties of $\Pi^\lambda(Q)$

• Q : Dynkin quiver $\Leftrightarrow \dim_K \Pi^\lambda(Q) < \infty$

In this case, $\exists\, Q' \subset Q$: full subquiver

s.t. $\Pi^\lambda(Q) \underset{\text{Morita}}{\sim} \Pi^0(Q')$

• Q : ext Dynkin $\longrightarrow \Gamma \subset SL_2(K)$: finite subgp

$\mathrm{ch}\, K = 0$, $K = \overline{K}$

deformation of $K[x, y]^\Gamma$

$e := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g \in K\Gamma \subset K\langle x, y \rangle * \Gamma$

$\lambda \in Z(K\Gamma)$

$$S^\lambda := \frac{K\langle x, y\rangle * \Gamma}{\langle xy - yx - \lambda\rangle}$$

$\mathcal{O}^\lambda := e S^\lambda e$

$$\begin{array}{ccc} Z(K\Gamma) & \simeq & K^{Q_0} \\ \lambda & \longmapsto & \lambda \end{array}$$ by [Maschke + McKay]

<u>Thm</u> [Crawley-Boevey - Holland] $e_0 \in \Pi^\lambda(Q)$ excep vertex

$\mathcal{O}^\lambda \simeq e_0 \Pi^\lambda(Q) e_0$, $S^\lambda \underset{\text{Morita}}{\sim} \Pi^\lambda(Q)$ //

$\lambda = 0 \Rightarrow S^0 = K[x, y] * \Gamma \underset{M}{\sim} \Pi^0(Q)$ //

$\boxed{\S 1 \text{ Reflection functors}}$

$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}$ Ringel form

$$\langle \alpha, \beta \rangle := \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}$$

$(-, -) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}$

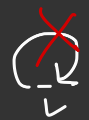
$(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ $\qquad (0 \cdots \overset{i}{1} \cdots 0) \in \mathbb{Z}^{Q_0}$

$s_i : \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}^{Q_0}$ $\quad s_i(\alpha) = \alpha - (\alpha, \varepsilon_i)\varepsilon_i$

$r_i : K^{Q_0} \longrightarrow K^{Q_0}$ $\quad (r_i \lambda)_j = \lambda_j - (\varepsilon_i, \varepsilon_j)\lambda_i$

- $\lambda \cdot (s_i \alpha) = (r_i \lambda) \cdot \alpha$ $\qquad \lambda \in K^{Q_0}$ $\quad \alpha \in \mathbb{Z}^{Q_0}$

$\lambda \cdot \alpha := \sum_{i \in Q_0} \lambda_i \alpha_i$



<u>Def</u> $i \in Q_0$: loop-free, define a functor

$$C_i^\lambda : \mathrm{Mod}\,\Pi^\lambda(Q) \longrightarrow \mathrm{Mod}\,\Pi^{r_i\lambda}(Q)$$

$M = (M_i, M_a)_{i \in Q_0,\, a \in \overline{Q}_1}$: quiver rep. of $\Pi^\lambda(Q)$-mod

$$\begin{tikzcd}
M_i \arrow[r, shift left, "\mu"] & \bigoplus\limits_{\substack{a \in \overline{Q}_1 \\ h(a)=i}} M_{t(a)} \arrow[l, shift left, "\pi"] \arrow[r, "c"] \arrow[d, "\mu\pi - \lambda_i \mathrm{Id}"'] \arrow[dr, phantom, "\circlearrowleft"] & C_i^\lambda(M)_i \arrow[dl, dashed, "\gamma"] \\
& \bigoplus\limits_{\substack{a \in \overline{Q}_1 \\ h(a)=i}} M_{t(a)} &
\end{tikzcd}$$

$$C_i^\lambda(M)_j = \begin{cases} M_j & j \neq i \\ \mathrm{Cok}(\mu) & j = i \end{cases}$$

$C_i^\lambda(M)_a$ are defined by using γ and c //

$\boxed{\text{Def}}$ $i \in Q_0$. $I_i = \Pi^\lambda(1-e_i)\Pi^\lambda$

two-sided ideal of $\Pi^\lambda(Q)$

$\boxed{\text{Thm}}$ $i, j \in Q_0$: loop-free

(1) [CB-H] $\lambda_i \neq 0$ then C_i is an equiv.

$$\mathrm{Mod}\,{}^\lambda(Q) \xrightarrow{\sim} \mathrm{Mod}\,\Pi^{r_i\lambda}(Q)$$

s.t. $M \in$ f.d. $\Pi^\lambda(Q)$, $\underline{\dim}\, C_i(M) = s_i(\underline{\dim}\, M)$

(2) $\lambda_i = 0 \Rightarrow C_i^{\lambda} \simeq I_i \otimes_{\Pi^\lambda} (-)$

(3) $C_i^{r_j\lambda} C_j^{\lambda} \simeq C_j^{r_i\lambda} C_i^{\lambda}$ $\quad i \overset{\#}{\cancel{\quad}} j$ in Q

$C_i^{r_j r_i \lambda} C_j^{r_i\lambda} C_i^{\lambda} \simeq C_j^{r_i r_j \lambda} C_i^{r_j\lambda} C_j^{\lambda}$ $\quad i \overset{1}{\text{———}} j$ in Q

$\boxed{\S 2 \quad \text{2-CY property}}$

A : K-alg $\quad A^e = A \otimes_K A^{op}$

<u>Def</u> $\quad A$: (bimodule) d-CY

$:\Leftrightarrow \quad A \in \operatorname{per} A^e \subset D(\operatorname{Mod} A^e)$

$\quad R\operatorname{Hom}_{A^e}(A, A^e) \simeq A[-d]$ in $D(\operatorname{Mod} A^e)$

[Keller] A : 2-CY, then

$D\operatorname{Hom}_{\mathcal{D}}(X, Y) \simeq \operatorname{Hom}_{\mathcal{D}}(Y, X[d])$

for $X, Y \in \mathcal{D} = D(\operatorname{Mod} A)$ $\quad \sum_{i \in \mathbb{Z}} \dim_K H^i X < \infty$

- $\Pi = \Pi^\lambda(Q)$

$P_0 = \bigoplus_{i \in Q_0} \Pi e_i \otimes_K e_i \Pi \qquad P_1 = \bigoplus_{a \in \overline{Q}_1} \Pi e_{h(a)} \otimes_K e_{t(a)} \Pi$

Prop $\overbrace{P_0 \xrightarrow{f} P_1 \xrightarrow{g} P_0}^{P_\bullet} \longrightarrow \Pi \to 0$ ex

$$f(e_i \otimes e_i) = \sum_{\substack{a \in \bar{Q}_1 \\ h(a) = i}} \varepsilon(a) \left(e_{h(a)} \otimes e_{t(a)} a^* + a e_{h(a)} \otimes e_{t(a)} \right)$$

$$g(e_{h(a)} \otimes e_{t(a)}) = a e_{t(a)} \otimes e_{t(a)} - e_{h(a)} \otimes e_{h(a)} a$$

Prop (1) $\mathrm{Hom}_{\Pi^e}(P_\bullet, \Pi^e) \simeq P_\bullet[-2]$

(2) Q is non-Dynkin $\Rightarrow$ f is injective

Thm [CB-K, Kaplan-Schedler]

$\Pi^\lambda(Q)$ is 2-CY if Q is non-Dynkin

§3 Tilting ideals of 2-CY algebras

A : (bimodule) 2-CY

Def $T \in \mathrm{Mod}\, A$ is tilting if

- $\exists\, 0 \to P_1 \to P_0 \to T \to 0$ ex, $P_i \in \mathrm{proj}\, A$
- $\mathrm{Ext}^1_A(T, T) = 0$
- $\exists\, 0 \to A \to T_0 \to T_1 \to 0$ ex, $T_i \in \mathrm{add}\, T$

Prop $S \in$ f.d. A simple, rigid (i.e. $\mathrm{Ext}^1_A(S, S) = 0$)

$I_S := \mathrm{Ann}_A(S) = \{ a \in A \mid aS = 0 \}$ tilting ideal

$\Rightarrow$ • ${}_A(I_S)$, $(I_S)_A$ are tilting module

• $\mathrm{End}_A({}_A I_S) \simeq A^{op}$ $(\cdot a) \leftarrow\!\!\mapsto a$

$\mathcal{S} \subset \text{rsim}(A) := \{ \text{f.d. rigid simple } A\text{-mod} \}/\simeq$

For $S_1, \cdots, S_r \in \mathcal{S}$

$I_{S_1 \cdots S_r} := I_{S_1} \cdots I_{S_r}$

$$I(\mathcal{S}) := \{ I_{S_1 \cdots S_r} \mid r \geq 0,\ S_i \in \mathcal{S} \}, \quad \overset{r=0}{I_\emptyset = A}$$

$\mathcal{E}(\mathcal{S})$: Serre subcat of f.d. A generated by $\mathcal{S}$

Thm [CB-K]

(1) $\forall I \in I(\mathcal{S})$ is a tilting ideal, $A/I \in \mathcal{E}(\mathcal{S})$ and $\text{End}_A(I) \simeq A^{op}$

(2) ${}_A I \subset A$: partilting left ideal, $A/I \in \mathcal{E}(\mathcal{S})$ then $I \in I(\mathcal{S})$

Rmk [Buan-Iyama-Reiten-Scott]

$A = \widehat{\Pi^0}, \quad \mathcal{S} = \{ S_i \mid i \in Q_0 \}$

$I_{S_i} = \text{Ann}_A(S_i) = A(1-e_i)A$

Prop $S, T \in \mathcal{S}$

(1) $I_S I_S = I_S$

(2) $I_S I_T = I_T I_S$ if $\text{Ext}^1_A(S, T) = 0$

(3) $I_S I_T I_S = I_T I_S I_T$ if $\dim_{\text{End}(S)} \text{Ext}^1_A(S, T) = 1$

$\dim \text{End}(T) \to \text{Ext}^1_A(S, T) = 1$

<u>Proof</u> $\operatorname{ind}\mathcal{E}(\{S,T\}) = \{S, T, \frac{S}{T}, \frac{T}{S}\}$ by assumption

$\Rightarrow \operatorname{add}S * \operatorname{add}T * \operatorname{add}S = \operatorname{add}T * \operatorname{add}S * \operatorname{add}T$

For $M \in$ f.d. A, $\quad I_{STS} M = 0 \Leftrightarrow M \in \operatorname{add}S * \operatorname{add}T * \operatorname{add}S$

So $M = A/I_{STS}, A/I_{TST}$. $\square$

$\boxed{\text{Def}}$ Assume that $\forall S \in \mathcal{S}$, $\operatorname{End}_A(S) \simeq K$

A Coxeter group $W(\mathcal{S})$

generater $\{\sigma_S \mid S \in \mathcal{S}\}$

relations $\sigma_S{}^2 = 1$

$\sigma_S\sigma_T = \sigma_T\sigma_S$ if $\operatorname{Ext}^2_A(S,T) = 0$

$\sigma_S\sigma_T\sigma_S = \sigma_T\sigma_S\sigma_T$ if $\dim_K \operatorname{Ext}^2_A(S,T) = 1$

<u>Thm [CB-K]</u> Assume $\forall S \in \mathcal{S}$, $\operatorname{End}_A(S) \simeq K$

then $W(\mathcal{S}) \longrightarrow I(\mathcal{S})$ $\quad \underbrace{w = \sigma_{S_1}\cdots\sigma_{S_r}}_{\text{reduced exp}} \longmapsto I_{S_1,\dots,S_r}$

is bijection

$\boxed{\S 4 \quad \text{Rigid simples of } \Pi^\lambda(Q)}$

$(\alpha, \beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$ sym. bilinear form of Q

$\Rightarrow \{\text{real root}\} \sqcup \{\text{imag. root}\} \subset \mathbb{Z}^{Q_0}$

<u>Lem</u> $M \in$ f.d. $\Pi^\lambda(Q) \Rightarrow \lambda \cdot \underline{\dim} M = 0$

$$\Sigma^\lambda_{re} := \left\{ \alpha \;\middle|\; \begin{array}{l} \alpha \text{ is a positive real root}, \quad \lambda\cdot\alpha = 0 \\ \text{s.t. } \nexists\, \beta^{(i)} : \text{positive roots}, \ i = 1, \dots, r \ (r \geq 2) \\ \qquad \alpha = \beta^{(1)} + \cdots + \beta^{(r)}, \quad \lambda \cdot \beta^{(i)} = 0 \end{array} \right\}$$

<u>Prop</u> [CB]

(1) $\operatorname{rsim}(\Pi^\lambda(Q)) \longrightarrow \Sigma^\lambda_{re}$ is bijective

$\qquad S \longmapsto \underline{\dim}\, S$

(2) $\forall S \in \mathrm{rsim}(\Pi^\lambda(Q))$, $\mathrm{End}_{\Pi^\lambda}(S) \simeq K$

(3) Q : ext Dynkin $\Rightarrow$ $\# \Sigma^\lambda_{re} < \infty$

Proof (2) $M, N \in$ f.d. $\Pi^\lambda(Q)$

$$\dim \mathrm{Ext}^1(M,N) = \dim \mathrm{Hom}(M,N) + \dim \mathrm{Hom}(N,M) - (\underline{\dim} M, \underline{\dim} N)$$

So $M=N=S$, $0 = 2 \dim \mathrm{End}(S) - \underbrace{(\underline{\dim} S, \underline{\dim} S)}_{=2}$ $\square$

Thm [CB-K] Q : ext-Dynkin

$\mathcal{R} := \{ \text{f.d. rigid simple } \Pi^\lambda \}/\simeq$

Then

$$I(\mathcal{R}) = \{ I \subset \Pi^\lambda(Q) \mid I \text{ is a tilting ideal}, \ \dim_K \Pi^\lambda/I < \infty \}$$

$\Pi^\lambda/I \in \mathcal{E}(\mathcal{R})$

Q : ext Dynkin

$\boxed{\text{Def}}$ The Ext-quiver $Q(\mathcal{R})$

$Q(\mathcal{R})_0 = \mathcal{R}$

draw $\dim_K \mathrm{Ext}^1_A(S,T)$ - arrows from S to T

- $\exists \Gamma$: quiver s.t. $\overline{\Gamma} = Q(\mathcal{R})$ by 2-CY duality

$\boxed{\text{Prop}}$ Γ is a disjoint union of ext Dynkin quivers

$\boxed{\text{Thm}}$ [CB-K] Q : ext Dynkin δ : min imag root of Q

assume $\lambda \cdot \delta = 0$, $K = \overline{K}$, $\mathrm{ch} K = 0$

Then $\exists$ bijection between

(1) the connected component of $Q(\mathcal{R})$

(2) the singular points of $\mathrm{Rep}(\Pi^\lambda(Q), \delta) /\!/ GL(\delta)$

$$\mathrm{Rep}(\Pi^\lambda(Q), \delta) /\!/ GL(\delta) \overset{1-1}{\longleftrightarrow} \left\{ S \in \text{f.d. } \Pi^\lambda(Q) \;\middle|\; \begin{array}{l} S \text{ is semisimple} \\ \underline{\dim}\, S = \delta \end{array} \right\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\{\text{non-singular point}\} \underset{[\text{le Bruyn}]}{\overset{1-1}{\longleftrightarrow}} \{\text{simple module}\}$$

$\Gamma' \subset \Gamma$ connected component, δ' : minimal imag root of Γ'

$\Gamma'_0 = \{ S_1, \dots, S_r \} \subset R$

$$\Rightarrow \underline{\dim}\left(\oplus_{i=1}^{r} S_i^{\oplus \delta'_i} \right) = \delta$$

We have a map $(1) \longrightarrow (2)$ by

$$\Gamma' \longmapsto \bigoplus_{i=1}^{r} S_i^{\oplus \delta'_i}$$

//