Update on singularity categories

based on arXiv: 2108.03292
(cf. also 2103.06584)

Martin Kalck, Freiburg

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Tokyo-Nagoya Algebra Seminar

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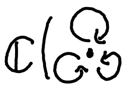
$\mathbb{C}($  $)/(\text{arrow ideal})^2$ ("wild")

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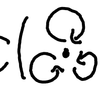
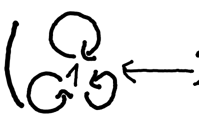
$\cong$ $\mathbb{C}($$)/(\text{arrow ideal})^2$ ("wild")

(b') $\Omega^\infty\left(\mathbb{C}(\right.$$\left.)/(\text{arrow ideal})^2\right)$

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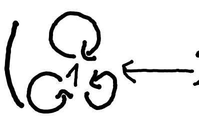
(b') $\Omega^\infty(\mathbb{C}($  $)/(\text{arrowideal})^2) \cong \underline{\mathrm{add}}\text{-}S_1$

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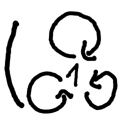
(b') $\Omega^\infty(\mathbb{C}($  $\longleftarrow 2)/(\text{arrow ideal})^2) \cong \underline{\text{add}}\text{-}S_1$

(c) R selfinjective $\Rightarrow \Omega^\infty(R) \cong R\text{-}\underline{\text{mod}}$.

(d) R Gorenstein, i.e. $\operatorname{inj.dim}_R R, R_R < \infty$

$\Rightarrow \Omega^\infty(R) \cong \underline{GP}(R) := \{M \in R\text{-mod} \mid \operatorname{Ext}^{>0}_R(M,R) = 0\}$

In particular, in (c) & (d) $\Omega^\infty(R)$ admits triang. struct.

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In analogy with homotopy categories of complexes one can define the homotopy category of MFs $[MF(f)]$ with morphism spaces

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More precisely, MFs (A,B) of f yield 2-periodic diagrams

$$\cdots \xrightarrow{A} S^n \xrightarrow{B} S^n \xrightarrow{A} S^n \xrightarrow{B} \cdots$$

which (analogous to chain complexes) form a differential $\mathbb{Z}$-graded category $MF(f)$

with degree d maps $m \in \mathrm{Hom}^d_{MF(f)}((A,B),(A',B'))$

(e.g. for $d=-2$: $m = (m_i)_i$

$$\begin{tikzcd}
\cdots \arrow[r, "A"] & S^n \arrow[r, "B"] & S^n \arrow[r, "A"] \arrow[dll, "m_i"'] & S^n \arrow[r, "B"] \arrow[dll, "m_{i+1}"'] & S^n \arrow[r, "A"] \arrow[dll, "m_{i+2}"'] & S^n \arrow[r] \arrow[dll, "m_{i+3}"'] & \cdots \arrow[dll, "m_{i+4}"'] \\
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Knörrer's equivalences are the only known non-trivial Gorenstein examples.

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Explain this observation (to some extent), i.e. explain why we (maybe) shouldn't expect to find (many) more equivalences.

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Thm [cf. K. '21 and also Puthenpurakal '21]

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Step 1: R_1 Gorenstein isolated singularity

$\xRightarrow{\text{Auslander}}$ $D_{sg}(R_1)$ Hom-finite

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Auslander, Boudal & Kapranov

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The End.

Thank you very much!

if you have comments or questions later, you are very welcome to send me an email:

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