A surface and a threefold with equivalent singularity categories

---

arXiv:2103.06584

Tokyo-Nagoya Algebra Seminar

Martin Kalck, Freiburg

The Buchweitz – Orlov Singularity Category

X quasi-proj. variety $/\mathbb{C}$

$$\mathrm{Perf}(X) \hookrightarrow D^b(\mathrm{Coh}\,X) \twoheadrightarrow D_{sg}(X) := \frac{D^b(\mathrm{Coh}\,X)}{\mathrm{Perf}(X)}$$

'smooth part' consisting of bounded complexes of vector bundles

Singularity Category measures complexity of singularities of X

Thm (Auslander-Buchsbaum & Serre)

X smooth $\iff D_{sg}(X) = 0$

<u>Thm</u> (Orlov) If X has isolated singularities

$$D_{sg}(X) \cong \bigoplus_{s \in \mathrm{Sing}(X)} D_{sg}(\hat{\mathcal{O}}_s)$$

(up to taking direct summands)

where

$$D_{sg}(R) := \frac{D^b(\text{mod-}R)}{K^b(\text{proj-}R)}$$

<u>singularity category</u> of a noetherian ring R.

<u>Rem:</u>

In other words, in isolated case, suffices to understand $D_{sg}(R)$.

How fine is the invariant $D_{sg}(R)$?

(i.e. when $D_{sg}(R) \cong D_{sg}(S)$?

Call such rings R and S <u>singular equivalent</u>)

Complete list of known singular equivalences between commutative complete local $\mathbb{C}$-algebras:

(0) $D_{sg}(R) = 0 = D_{sg}(S)$, if $\operatorname{gldim} R, S < \infty$

(1) [Knörrer '87] Let $0 \neq f \in \mathbb{C}[[z_0, \ldots, z_d]] =: P_d$

$$D_{sg}\left(\frac{P_d}{(f)}\right) \cong D_{sg}\left(\frac{P_d[[y_1, \ldots, y_{2n}]]}{(f + y_1^2 + \ldots + y_{2n}^2)}\right)$$

Notation:

$$\mathbb{C}[[z_1, \ldots, z_d]]^{\frac{1}{m}(a_1, \ldots, a_d)}$$

invariant ring of group action:

$$z_i \longmapsto \varepsilon_m^{a_i} z_i$$

($\varepsilon_m \in \mathbb{C}$ a primitive m-th root of unity.)

(2) $D_{sg}\left(\mathbb{C}[[z_1, z_2]]^{\frac{1}{m}(1,1)}\right) \simeq D_{sg}\left(\frac{\mathbb{C}[[z_1, \dots, z_{m-1}]]}{(z_1, \dots, z_{m-1})^2}\right)$

[D. Yang, Y. Kawamata, K.–Karmazyn]
all ~ 2015

(3) $D_{sg}\left(\mathbb{C}[[z_1, z_2, z_3]]^{\frac{1}{2}(1,1,1)}\right) \simeq D_{sg}\left(\mathbb{C}[[z_1, z_2]]^{\frac{1}{4}(1,1)}\right)$

[K. 2021]

Remarks:

(a) The Krull dimensions of these invariant rings in (3) are 3 and 2, respectively.
In particular, this singular equivalence does not preserve the parity of Krull dimensions.

b) Knörrer's equivalences are the only known non-trivial Gorenstein examples.

Last time: reported on some evidence for no further singular eq. involving hypersurfaces

Proof

of singular equivalences:

(2) $$D_{sg}\left(\mathbb{C}[|z_1,z_2|]^{\frac{1}{m}(1,1)}\right)\simeq D_{sg}\left(\frac{\mathbb{C}[|z_1,\dots,z_{m-1}|]}{(z_1,\dots,z_{m-1})^2}\right)$$

(3) $$D_{sg}\left(\mathbb{C}[|z_1,z_2,z_3|]^{\frac{1}{2}(1,1,1)}\right)\simeq D_{sg}\left(\mathbb{C}[|z_1,z_2|]^{\frac{1}{4}(1,1)}\right)$$

Def: A noetherian $\mathbb{C}$-algebra R is syzygy simple of order m if there exist $S \in \text{mod-}R$ s.th.

(s1) For every $M \in \text{mod-}R$ there is $n \in \mathbb{N}$ s.th. $\Omega^n(M) \in \underline{\text{add}}\text{-}S$.

(s2) $\underline{\text{add}}\text{-}S \cong \text{mod-}\mathbb{C}$

(s3) $\Omega(S) \cong S^{\oplus m}$ in $\underline{\text{mod}}\text{-}R$.

Prop (cf. X.-W. Chen)
Let R & T be syzygy simple of order m. Then

$$D_{sg}(R) \cong D_{sg}(T).$$

Examples: (a) $K_m := \dfrac{\mathbb{C}[z_1,\dots,z_m]}{(z_1,\dots,z_m)^2}$

is syzygy simple of order m,

with $S = \dfrac{\mathbb{C}[z_1,\dots,z_m]}{(z_1,\dots,z_m)}$ simple

(b) $R_m := \mathbb{C}[[z_1,z_2]]^{\frac{1}{(m+1)}(1,1)}$

is syzygy simple of order m,

where $0 \to S \to R_m^{\oplus 2} \to N \to 0$ is AR-seq.

in $CM(R_m)$.

(c) $R := \mathbb{C}[[z_1,z_2,z_3]]^{\frac{1}{2}(1,1,1)}$

is syzygy simple of order 3.

Here, $S = \Omega(\omega_R)$.

In combination with the Proposition above this shows all known non-trivial singular equivalences between complete commutative rings except for Knörrer's.

⌜Will explain these examples in more detail later.⌟

Proof

of

Proposition

$$
\begin{tikzcd}
\text{mod-}R \arrow[r, hook] \arrow[dr, two heads] & D^b(\text{mod-}R) \arrow[r, two heads] & D_{sg}(R) \\
& \underline{\text{mod}}\text{-}R \arrow[ur, "F"'] &
\end{tikzcd}
$$

By definition, of syzygies

$$0 \longrightarrow \Omega(M) \longrightarrow R^n \longrightarrow M \longrightarrow 0 \quad \text{exact}$$

induces

$$\Omega(M) \longrightarrow 0 \longrightarrow M \xrightarrow{\cong} \Omega(M)[1] \quad \text{triangle in } D_{sg}(R)$$

$$\Longrightarrow \quad F(\Omega(M)) \cong F(M)[-1]$$

$$\begin{tikzcd} \underline{\mathrm{mod}}\text{-}R \arrow[r, "F"] \arrow[loop below, "\Omega"] & D_{sg}(R) \arrow[loop below, "{[-1]}"] \arrow[loop right, phantom, "\cong"] \end{tikzcd}$$

If F is an equivalence

$\implies$ Ω is an equivalence

The converse also holds:

<u>Thm</u> (Buchweitz, Happel, Keller & Vossieck, Rickard)

The following are equivalent:

(a) F is an equivalence.

(b) Ω is an equivalence.

(c) R is self-injective.

Problem:

Our algebras are not self-injective in general

(they are not even Gorenstein!)

Example:

$$K_2 = \frac{\mathbb{C}[z_1, z_2]}{(z_1, z_2)^2} = \mathbb{C}\left(\begin{tikzcd} \bullet \arrow[loop left, "z_1"] \arrow[loop right, "z_2"] \end{tikzcd}\right) \Big/ (\text{arrows})^2$$

Let $S = K_2 / \operatorname{rad} K_2 = \begin{tikzcd} \mathbb{C} \arrow[loop left, "0"] \arrow[loop right, "0"] \end{tikzcd}$

simple K_2-module.

Then $\Omega(S) \cong S \oplus S$.

$\rightarrow$ Ω is <u>not</u> an autoequivalence of <u>mod</u>-K_2 !

e.g. it cannot be full as $0 \neq S \in$ <u>mod</u>-K_2

"<u>Solution</u>" (Heller 1968):

There is a universal category

$St(\underline{\mathrm{mod}}\text{-}R, \Omega) =: \mathcal{C}$

"enlarging" $\underline{\mathrm{mod}}\text{-}R$
so that Ω becomes
an autoequivalence

Universal property implies:

$$\begin{tikzcd} \underline{\text{mod}}\text{-}R \arrow[rr, "F"] \arrow[dr] & \circlearrowleft & D_{sg}(R) \\ & \mathrm{St}(\underline{\text{mod}}\text{-}R, \Omega) \arrow[ur, "\mathrm{St}(F)"'] & \end{tikzcd}$$

<u>Theorem</u> [Keller - Vossieck 1987]

R noetherian. Then

(a) $\mathrm{St}(\underline{\text{mod}}\text{-}R, \Omega)$ is triangulated.

(b) $\mathrm{St}(\underline{\text{mod}}\text{-}R, \Omega) \xrightarrow[\mathrm{St}(F)]{\sim} D_{sg}(R)$

is a Δ-equivalence.

Part (a) holds, since

$\underline{\mathrm{mod}}$-R is a

left triangulated category.

<u>Key idea:</u> "Identify left triang. subcategories"

$$\begin{array}{ccc}
(\underline{\mathrm{mod}}\text{-}R, \Omega_R) & & (\underline{\mathrm{mod}}\text{-}T, \Omega_T) \\
\cup \text{ left triang.} & & \cup \text{ left triang.} \\
(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) & \cong & (\underline{\mathrm{add}}\text{-}S_T, \Omega_T)
\end{array}$$

left triangulated

Where $S_R \in \text{mod-}R$ (respectively, $S_T \in \text{mod-}T$) satisfy

(s1) For every $M \in \text{mod-}R$ there is $n \in \mathbb{N}$ s. th. $\Omega_R^n(M) \in \underline{\text{add}}\text{-}S_R$.

(s2) $\underline{\text{add}}\text{-}S_R \cong \text{mod-}\mathbb{C}$

(s3) $\Omega_R(S_R) \cong S_R^{\oplus m_R}$ in $\underline{\text{mod}}\text{-}R$.

(s3) $\Rightarrow$ $\Omega_R \circlearrowleft \underline{\text{add}}\text{-}S_R$

(s2) $\Rightarrow$ $\underline{\text{add}}\text{-}S_R$ semi-simple abelian

as for Δ-ted categories $\Rightarrow$ left Δ-ted structure on

$(\underline{\mathrm{add}}\text{-}S_R, \Omega_R)$ is trivial,

and completely determined by (s3).

This has two consequences:

(1) $(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) \subset (\underline{\mathrm{mod}}\text{-}R, \Omega_R)$ (*)

left Δ-ted subcategory

(2) If $m_R = m_T$ in (s3) then

$$(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) \overset{\text{left } \Delta}{\cong} (\underline{\mathrm{add}}\text{-}S_R, \Omega_R)$$

Moreover (s1) & (*) imply

$$St(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) \overset{\Delta}{\cong} St(\underline{\mathrm{mod}}\text{-}R, \Omega_R)$$

(1) & (2) show "Key idea":

$$\begin{array}{ccc}
(\underline{\mathrm{mod}}\text{-}R, \Omega_R) & & (\underline{\mathrm{mod}}\text{-}T, \Omega_T) \\
\cup \text{ left triang.} & & \cup \text{ left triang.} \\
(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) & \overset{\text{left } \Delta}{\cong} & (\underline{\mathrm{add}}\text{-}S_T, \Omega_T)
\end{array}$$

$$\Big\downarrow \text{ Apply } St(-)$$

$$\begin{tikzcd}
D_{sg}(R) \arrow[r, dashed, "\overset{\Delta}{\cong}"] \arrow[d, phantom, "\cong {[KV]}"] & D_{sg}(T) \arrow[d, phantom, "\cong {[KV]}"] \\
St(\underline{\mathrm{mod}}\text{-}R, \Omega_R) \arrow[d, phantom, "\cong \Delta"] & St(\underline{\mathrm{mod}}\text{-}T, \Omega_T) \arrow[d, phantom, "\cong \Delta"] \\
St(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) \arrow[r, phantom, "\overset{\Delta}{\cong}"] & St(\underline{\mathrm{add}}\text{-}S_T, \Omega_T)
\end{tikzcd}$$

This finishes the proof of:

<u>Prop</u> (cf. X.-W. Chen)

Let R & T be syzygy simple of order m. Then

$$D_{sg}(R) \stackrel{\Delta}{\cong} D_{sg}(T).$$

The examples of

Syzygy simple algebras

(in more detail)

Preparations

<u>Thm</u> (Hochster-Roberts 1974)

$G \subset GL(d, \mathbb{C})$ finite

Then the invariant ring

$$Q := \mathbb{C}[z_1, \dots, z_d]^G$$

is local Cohen-Macaulay of Krull dimension d.

<u>Lem</u>: R local Cohen-Macaulay. of Krull dim. d.

$M \in \text{mod-}R$.

Then $\Omega^n M \in \mathrm{MCM}(R)$ for all $n \geq d$.

Cor: If R is local Cohen-Macaulay.

Can replace condition

"(s1) For every $M \in \text{mod-}R$ there is
$n \in \mathbb{N}$ s.th. $\Omega_R^n(M) \in \underline{\text{add}}\text{-}S_R$."

by condition

"(s1') For every $X \in \mathrm{MCM}(R)$ there is
$n \in \mathbb{N}$ s.th. $\Omega_R^n(X) \in \underline{\text{add}}\text{-}S_R$."

Thm: R complete local CM.
Then $\exists\, \omega_R \in \mathrm{MCM}(R)$ injective.

<u>Cor:</u> R as above $N \in MCM(R)$

Then $\underline{\mathrm{Hom}}_R(\Omega(N), \omega_R) = 0$

$$\underline{\mathrm{Hom}}_R(\Omega(N), \omega_R) \cong \mathrm{Ext}^1_R(N, \omega_R)$$

injective

⌈Indeed, ⌋

Back to our examples:

<u>Irreducible morphisms</u> in <u>MCM(R)</u>:

[Auslander-Reiten]

$$R = \mathbb{C}[z_1, z_2, z_3]^{\frac{1}{2}(1,1,1)}$$

$$M \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \omega_R$$

<u>stable</u> AR-quiver of $MCM(R)$

In particular,

- M and ω_R are only objects in ind $\underline{\mathrm{MCM}}(R)$.
- $\underline{\mathrm{Hom}}_R(M, \omega_R) = 0$
- $\underline{\mathrm{Hom}}_R(\omega_R, \omega_R) \cong \mathbb{C}$

$\overset{\text{Cor}}{\Longrightarrow}$ $\Omega^n(N) \in \underline{\mathrm{add}}\, M$ for all $N \in \mathrm{MCM}(R)$

In particular, $\Omega(M) = M^{\oplus m_R}$

- $\underline{\mathrm{Hom}}_R(M, M) \cong \mathbb{C} \implies$

$$\underline{\mathrm{add}}\, M \cong \mathrm{mod}\text{-}\mathbb{C}$$

Summing up:

R is syzygy simple of order m_R.

Computing m_R:

Direct approach:

(a) $0 \to M \to R^{\oplus 3} \to \omega_R \to 0$ and

(b) $0 \to R \to \omega_R^{\oplus 3} \to M \to 0$ exact

$$\Omega(M) \overset{(b)}{\cong} \Omega(\omega_R)^{\oplus 3} \overset{(a)}{\cong} M^{\oplus 3}$$

$$\Longrightarrow m_R = 3$$

K-theoretic approach

Proposition (Chen) $\Rightarrow D_{sg}(R) \cong D_{sg}\Big(\underbrace{\frac{\mathbb{C}[z_1,\dots,z_{m_R}]}{(z_1,\dots,z_{m_R})^2}}_{=:K_{m_R}}\Big)$

+ R syzygy simple of order m_R

$$\Rightarrow K_0(D_{sg}(R)) \cong K_0(D_{sg}(K_{m_R})) \cong \mathbb{Z}/_{m_R+1}$$

since K_{m_R} local and $\dim_{\mathbb{C}} K_{m_R} = m_R+1$

Pavic-Shinder '18: $K_0(D_{sg}(R)) \cong \mathbb{Z}/_4$

$\Rightarrow$ Again $m_R = 3$

Surface examples are treated similarly:

$$R_m := \mathbb{C}[[z_1, z_2]]^{\frac{1}{(m+1)}(1,1)}$$

$$\begin{tikzcd}
S & N_1 \arrow[r, shift left] \arrow[r, shift right] & N_2 \arrow[d, shift left] \arrow[d, shift right] \\
\omega_{R_m} \arrow[u, shift left] \arrow[u, shift right] & N_{m-2} \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

stable Auslander-Reiten quiver

$\rightsquigarrow$ $\exists$ non-zero morphisms

$$\left.\begin{array}{c} N_i \longrightarrow \omega_{R_m} \\ \text{and} \\ \omega_{R_m} \xrightarrow{\mathrm{id}} \omega_{R_m} \end{array}\right\} \text{ in } \underline{\mathrm{MCM}}(R_m)$$

Cor. $\rightsquigarrow$ $\Omega(M) \in \underline{\mathrm{add}}\text{-}S$ for $M \in \mathrm{MCM}(R_m)$

Moreover, $\underline{\mathrm{End}}_{R_m}(S) \cong \mathbb{C}$

R_m is syzygy simple.

Again two approaches to determine the order:

(1) Using a "right ladder" for S in $\underline{MCM}(R_m)$ (cf. Iyama)

(2) Again $D_{sg}(R_m) \cong D_{sg}(K_t)$ for some t (Prop (Chen))

$\longrightarrow$

$$\begin{array}{l} Cl(R_m) \cong K_0(D_{sg}(R_m)) \cong K_0(D_{sg}(K_t)) \cong \mathbb{Z}/_{t+1} \\ \quad \rotatebox{90}{$\cong$} \\ \frac{1}{m+1}(1,1) \cong \frac{\mathbb{Z}}{(m+1)} \end{array}$$

$\Rightarrow$ R_m is syzygy simple of order $(m+1)-1 = m$.

Summing up:

$$D_{sg}\left(\mathbb{C}[[z_1,z_2]]^{\frac{1}{m}(1,1)}\right)\simeq D_{sg}\left(\frac{\mathbb{C}[[z_1,\dots,z_{m-1}]]}{(z_1,\dots,z_{m-1})^2}\right)$$

[D. Yang, Y. Kawamata, K.-Karmazyn]
all ~ 2015

$$D_{sg}\left(\mathbb{C}[[z_1,z_2,z_3]]^{\frac{1}{2}(1,1,1)}\right)\simeq D_{sg}\left(\mathbb{C}[[z_1,z_2]]^{\frac{1}{4}(1,1)}\right)$$

[K. 2021]

The End.

Thank you very much!

if you have comments or questions later, you are very welcome to send me an email:

martin.maths@posteo.de