Grothendieck enriched categories

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§0. Introduction

<u>Def</u>

$\mathcal{A}$ is a **Grothendieck category**

$:\Longleftrightarrow$ (i) $\mathcal{A}$: cocomplete abelian cat.

(ii) filtered colimits are exact

(iii) $\mathcal{A}$ has a generator G

<u>Example</u>

(1) Ab = cat. of abelian groups

(2) $\mathrm{Mod}(R)$: cat. of right R-modules (R: non-commutative ring)

(3) $\mathrm{Qcoh}(X)$: cat. of quasi-coherent sheaves on a scheme X

(4) $\mathrm{Ch}(\mathrm{Ab})$: cat. of complexes of abelian groups

次のような良い性質を持つ.

<u>Prop</u>

For a Grothendieck cat. $\mathcal{A}$, the following holds:

(1) $\mathcal{A}$: complete

(2) $\mathcal{A}$ has an injective cogenerator

(3) $\mathcal{A}$ has enough injectives

(4) functor representability theorem

(5) adjoint functor theorem $\square$

Thm (Gabriel-Popescu)

$\mathcal{A}$: Grothendieck cat. with a generator G

$R := \mathcal{A}(G,G)$: ring of endomorphisms

Then

(1) $T := \mathcal{A}(G,-) : \mathcal{A} \longrightarrow \mathrm{Mod}(R)$ admits a left adjoint S

(2) T : fully faithful

(3) S : (left) exact

$\Big\}\Rightarrow$ $\mathcal{A}$ is a lex-reflective full subcat. of $\mathrm{Mod}(R)$

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, shift left=2, "S\,:\,\mathrm{lex}"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left=2, "T"] \end{tikzcd}$$

$\square$

Def

$\mathcal{C} \subseteq \mathcal{D}$: full subcategory is lex-reflective

$:\Longleftrightarrow$ incl. $\mathcal{C} \hookrightarrow \mathcal{D}$ admits a left exact left adjoint

The Gabriel-Popescu theorem asserts :

Grothendieck categories $=$ lex-reflective subcategories of some module categories.

$\&$ intrinsic properties

$\&$ extrinsic characterization

- similar result for Grothendieck topoi

Def

A cat. $\mathcal{E}$ satisfies the Giraud condition

$:\Leftrightarrow$ (1) $\mathcal{E}$: finitely complete

(2) $\mathcal{E}$ has coproducts, which are disjoint and universal

(3) every epi. is a coequalizer

(4) every equiv. rel. is effective

(5) every equiv. rel. is universal

(6) $\mathcal{E}$ has a generating set

Thm (Giraud thm)

For a cat. $\mathcal{E}$, TFAE:

intrinsic

(1) $\mathcal{E}$ satisfies the Giraud condition

(2) $\mathcal{E}$ is equivalent to a lex-reflective full subcat. of the presheaf cat. $[\mathcal{C}^{op}, \mathrm{Set}]$ of some small cat $\mathcal{C}$.

(3) $\mathcal{E}$ is a Grothendieck topos.

(i.e. $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{C}, J)$ for some site) $\square$

extrinsic

$\Large\wr$

The G-P thm can be seen as an additive version of this thm.

Ab-enriched

Ab-enriched cat.

$\fallingdotseq$ cat. whose Hom's are abelian groups (objects of Ab)

Note: every abelian cat. has the canonical Ab-enriched str.

- R: (non-commutative) ring
  $=$ Ab-enriched cat. with a single object
- right R-module
  $=$ Ab-functor $R^{op} \longrightarrow Ab$
- Mod(R)
  $=$ Ab-enriched "presheaf" cat. $[R^{op}, Ab]$

Thm

For an Ab-enriched cat. A, TFAE:

(1) A is a Grothendieck cat.

(2) A is equivalent to a lex-reflective full subcat. of $\mathrm{Mod}(R) = [R^{op}, Ab]$ of some ring R.

((3) A is an "additive" topos) $\square$

Rmk

Set		Ab
· Grothendieck topology	$\longleftrightarrow$	Gabriel topology
· sheaf	$\longleftrightarrow$	closed module (additive sheaf)

extrinsic characterization

Set

Giraud condition

Giraud

Gabriel-Popescu

Ab

Grothendieck cat.

Q. $\mathrm{Ch} := \mathrm{Ch}(\mathrm{Ab})$-enriched にするとどうなる？

Ans.

$\mathcal{V}$: "nice" Grothendieck monoidal cat. (Ch含む)

上の enriched cat. に対し、類似が成り立つ。 [Imamura 2022]

§1. Preliminaries

Fix a complete and cocomplete symmetric monoidal closed cat. $\mathcal{V} = (V, \otimes, I, [-,-])$.
(=: cosmos)

Def

A $\mathcal{V}$-enriched category (or $\mathcal{V}$-category) $\mathcal{C}$ consists of the following data

- $\mathrm{obj}(\mathcal{C})$: collection of objects
- $\mathcal{C}(X,Y) \in \mathcal{V}$ for $X, Y \in \mathcal{C}$
- $m : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$ in $\mathcal{V}$ for $X, Y, Z \in \mathcal{C}$
- $j_X : I \longrightarrow \mathcal{C}(X,X)$ in $\mathcal{V}$ for $X \in \mathcal{C}$

satisfying the associativity and unit axioms.

Example

- $\mathcal{V} = \mathrm{Set}$ ⇝ (locally small) category
- $\mathcal{V} = \mathrm{Ab}$ ⇝ preadditive category
- $\mathcal{V} = \mathrm{Ch}$ ⇝ dg category
- $\mathcal{V} = \mathrm{Cat}$ ⇝ (strict) 2-category

Rmk

A cosmos $\mathcal{V}$ itself becomes a $\mathcal{V}$-enriched category with $\mathcal{V}(X,Y) := [X,Y] \in \mathcal{V}$

Def

A unit $\mathcal{V}$-cat. $\mathcal{I}$ is the $\mathcal{V}$-cat. with a single obj $*$ and $\mathcal{I}(*,*) = I$.

As in the usual category theory, we have :

- $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$
- $\mathcal{V}$-nat. trans. $\alpha : F \Rightarrow G$
- opposite $\mathcal{V}$-cat. $\mathcal{C}^{op}$
- functor $\mathcal{V}$-cat. $[\mathcal{C}, \mathcal{D}]$
- yoneda embedding $y : \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathcal{V}]$
- enriched yoneda lemma
- $\mathcal{V}$-adjunction,

etc.

Def

For a $\mathcal{V}$-cat. $\mathcal{C}$, the underlying category is the ordinary category $\mathcal{C}_0$

s.t. $\begin{cases} obj(\mathcal{C}_0) := obj(\mathcal{C}) \\ Hom_{\mathcal{C}_0}(X, Y) := Hom_{\mathcal{V}}(I, \mathcal{C}(X, Y)) \end{cases}$

$\rightsquigarrow U = (-)_0 : \mathcal{V}\text{-}\mathcal{C}at \longrightarrow \mathcal{C}at$

Example

$\mathcal{A}$: dg cat. $\rightsquigarrow$ $\mathcal{A}_0 = Z^0(\mathcal{A})$

$(Hom_{Ch}(k, -) = Z^0(-))$

$(-)_0$ has a left adjoint

$(-)_{\mathcal{V}} : \mathcal{C}at \longrightarrow \mathcal{V}\text{-}\mathcal{C}at$

$\mathcal{L} \longmapsto \mathcal{L}_{\mathcal{V}}$: free $\mathcal{V}$-category.

(co)limits in enriched categories

Def

(1) For $F: J \longrightarrow \mathcal{C}$ and $W: J \longrightarrow \mathcal{V}$, the limit of F weighted by W is an object $\{W, F\} \in \mathcal{C}$ s.t.

$$\mathcal{C}(C, \{W, F\}) \cong [J, \mathcal{V}](W, \mathcal{C}(C, F-))$$

: natural iso. in C.

(2) For $F: J \longrightarrow \mathcal{C}$ and $W: J^{op} \longrightarrow \mathcal{V}$, the colimit of F weighted by W is an object $W * F \in \mathcal{C}$ s.t.

$$\mathcal{C}(W * F, C) \cong [J^{op}, \mathcal{V}](W, \mathcal{C}(F-, C))$$

: natural iso. in C.

As a special case,

Def ($J = I$)

(1) For $X \in \mathcal{V}$ and $D \in \mathcal{C}$, the cotensor product is an object $X \pitchfork D \in \mathcal{C}$ s.t.

$$\mathcal{C}(C, X \pitchfork D) \cong \mathcal{V}(X, \mathcal{C}(C, D)).$$

(2) For $X \in \mathcal{V}$ and $D \in \mathcal{C}$, the tensor product is an object $X \otimes D \in \mathcal{C}$ s.t.

$$\mathcal{C}(X \otimes D, C) \cong \mathcal{V}(X, \mathcal{C}(D, C)).$$

Def (J: free $\mathcal{V}$-cat, $W = \Delta I$)

$\mathcal{C}$: $\mathcal{V}$-cat.

$F: J \longrightarrow \mathcal{C}_0$: ordinary functor

$\overset{adj.}{\longleftrightarrow}$ $F: J_{\mathcal{V}} \longrightarrow \mathcal{C}$: $\mathcal{V}$-functor

(1) the conical limit of F is $\lim F \in \mathcal{C}$ s.t. $\mathcal{C}(C, \lim F) \cong \lim \mathcal{V}(C, F-)$.

(2) the conical colimit of F is $\operatorname{colim} F \in \mathcal{C}$ s.t. $\mathcal{C}(\operatorname{colim} F, C) \cong \lim \mathcal{V}(F-, C)$.

Example

$$S(\mathbb{Z}) : \cdots \to 0 \to \underset{-1\text{次}}{0} \to \underset{0\text{次}}{\mathbb{Z}} \to 0 \to \cdots \quad \in Ch$$

$$D(\mathbb{Z}) : \cdots \to 0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0 \to \cdots \quad \in Ch$$

$\mathcal{A}$: dg cat.

(i) $A \in \mathcal{A}$ の shift $A[1]$ とは、tensor product

$$A[1] = S(\mathbb{Z})[-1] \otimes A$$

のこと.

(ii) $f \in Z^0(\mathcal{A})(A,B)$ の cone とは、

$$W : [1]^{op} \longrightarrow Ch \;;\; \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array} \mapsto \begin{array}{c} D(\mathbb{Z}) \\ \uparrow \\ S(\mathbb{Z}) \end{array}$$

による

$$F : [1] \longrightarrow Z^0\mathcal{A} \;;\; \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array} \mapsto \begin{array}{c} A \\ \downarrow f \\ B \end{array}$$

の weighted colimit

$$Cone(f) = W \star F$$

のこと.

Def

$\mathcal{C}$ is (co)complete

$:\Leftrightarrow$ $\mathcal{C}$ has all weighted (co)limits on small categories.

Prop

$\mathcal{C}$ is (co) complete

$\Leftrightarrow$ $\mathcal{C}$ has all conical (co)limits and all cotensor (tensor) products. □

Prop

The $\mathcal{V}$-cat. $\mathcal{V}$ is complete and cocomplete and so is $[\mathcal{C}^{op}, \mathcal{V}]$. □

Def

For $F : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{C} \to \mathcal{M}$, the left Kan extension of F along K is $\mathrm{Lan}_K F : \mathcal{M} \longrightarrow \mathcal{D}$ s.t.

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{M},\mathcal{D})}(\mathrm{Lan}_K F, S) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, S \circ K).$$

$$\begin{tikzcd} \mathcal{M} \arrow[dr, "\mathrm{Lan}_K F" description] \arrow[dr, bend left=60, "S"] \arrow[dr, phantom, bend left=30, "\exists !"] & \\ \mathcal{C} \arrow[u, "K"] \arrow[r, "F"'] \arrow[ur, phantom, "\Uparrow"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{M} \arrow[dr, bend left=40, "S"] & \\ \mathcal{C} \arrow[u, "K"] \arrow[r, "F"'] \arrow[ur, phantom, "\forall \Uparrow"] & \mathcal{D} \end{tikzcd}$$

□

Def

For $F : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{C} \to \mathcal{M}$, the pointwise left Kan extension of F along K is $T : \mathcal{M} \longrightarrow \mathcal{D}$ s.t.

$$\mathcal{D}(Tm, d) \cong [\mathcal{C}^{op}, \mathcal{V}](\mathcal{M}(K-, m), \mathcal{D}[F-, d]).$$

Prop

(1) Any pointwise left Kan extension is a left Kan extension.

(2) If $\mathcal{C}$: small and $\mathcal{D}$: cocomplete, then p.w. left Kan ext. always exist. □

Prop

$F : \mathcal{C} \to \mathcal{D}$, $\mathcal{C}$: small.

Then $\mathrm{Lan}_F Y : \mathcal{D} \longrightarrow [\mathcal{C}^{op}, \mathcal{V}]$ exists and

$$\mathrm{Lan}_F Y(d) \cong \mathcal{D}(F-, d).$$

proof

$$\begin{aligned} & [\mathcal{C}^{op}, \mathcal{V}](\mathrm{Lan}_F Y(d), P) \\ \cong\ & [\mathcal{C}^{op}, \mathcal{V}](\mathcal{D}[F-, d], \underbrace{[\mathcal{C}^{op}, \mathcal{V}](y(-), P)}_{\cong P \ \text{米田}}) \\ \cong\ & [\mathcal{C}^{op}, \mathcal{V}](\mathcal{D}(F-, d), P). \end{aligned}$$

$$\therefore \ \mathrm{Lan}_F Y(d) \cong \mathcal{D}(F-, d)$$

▨

普遍随伴

Thm (nerve-and-realization adj.)

$F : \mathcal{C} \to \mathcal{D}$, $\mathcal{C}$: small, $\mathcal{D}$: cocomplete

Then, $\mathrm{Lan}_y F \dashv \mathrm{Lan}_F y$.

$$\begin{tikzcd} {[\mathcal{C}^{op}, \mathcal{V}]} \arrow[dr, shift left, "\mathrm{Lan}_y F"] \arrow[dr, phantom, "\perp"] & \\ \mathcal{C} \arrow[u, "y"] \arrow[r, "F"'] & \mathcal{D} \arrow[ul, shift left, "\mathrm{Lan}_F y"] \end{tikzcd}$$

proof

$$\begin{aligned} &\mathcal{D}(\mathrm{Lan}_y F(P), d) \\ \cong\ & [\mathcal{C}^{op}, \mathcal{V}]\big(\underbrace{[\mathcal{C}^{op}, \mathcal{V}](y(-), P)}_{\substack{\text{\rotatebox{90}{$\cong$}}\ \text{米田} \\ P}}, \mathcal{D}(F-, d)\big) \\ \cong\ & [\mathcal{C}^{op}, \mathcal{V}](P, \mathcal{D}(F-, d)) \\ \cong\ & [\mathcal{C}^{op}, \mathcal{V}](P, \mathrm{Lan}_F y(d)). \end{aligned}$$

$\therefore$ $\mathrm{Lan}_y F \dashv \mathrm{Lan}_F y$. ■

Prop

$\mathrm{Lan}_K F$: pointwise Kan ext.

K : fully faithful

$\Rightarrow \mathrm{Lan}_K F \circ K \cong F$

$$\begin{tikzcd} \mathcal{M} \arrow[dr, "\mathrm{Lan}_K F"] & \\ \mathcal{C} \arrow[u, "K"] \arrow[r, "F"'] \arrow[ur, phantom, "\cong"] & \mathcal{D} \end{tikzcd}$$

□

Thm

$\mathcal{C}$: small, $\mathcal{D}$: cocomplete

Then

$$\left\{ \begin{array}{c} \mathcal{V}\text{-functors} \\ F : \mathcal{C} \to \mathcal{D} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{adjunction} \\ S : [\mathcal{C}^{op}, \mathcal{V}] \underset{\perp}{\rightleftarrows} \mathcal{D} : T \end{array} \right\}$$

$$\begin{array}{ccc} F & \longmapsto & \mathrm{Lan}_y F \dashv \mathrm{Lan}_F y \\ S \circ y & \longleftarrow\!\shortmid & S \dashv T \end{array}$$

□

Example (Eilenberg-Watts)

$$\begin{tikzcd} \mathrm{Mod}(S) \arrow[dr, shift left, "- \otimes P"] \arrow[dr, phantom, "\perp"] & \\ S \arrow[u] \arrow[r, "P"'] & \mathrm{Mod}(R) \arrow[ul, shift left, "\mathrm{Hom}(P, -)"] \end{tikzcd}$$

Thm (G·P thm)

$\mathcal{A}$: Grothendieck cat.

$\mathcal{C}$: generating set, $F : \mathcal{C} \hookrightarrow \mathcal{A}$

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Mod}(\mathcal{C}) \arrow[dr, bend left, "\mathrm{Lan}_Y F"] \arrow[dr, phantom, "\perp"] & \\ \mathcal{C} \arrow[u, hook, "Y"] \arrow[r, "F"'] & \mathcal{A} \arrow[ul, bend left, "\mathrm{Lan}_F Y"] \end{tikzcd}$$

Then, (1) $\mathrm{Lan}_F Y$: fully faithful

(2) $\mathrm{Lan}_Y F$: left exact $\square$

Def

$\mathcal{C} \subseteq \mathcal{A}$: small full subcat.

- $\mathcal{C}$ is a generating set
  $:\Leftrightarrow$ $\mathrm{Lan}_F Y$: faithful.
- $\mathcal{C}$ is a strongly generating set
  $:\Leftrightarrow$ $\mathrm{Lan}_F Y$: faithful and conservative.

Prop

$\mathcal{A}$: cocomplete abelian cat.

$\mathcal{C} \subseteq \mathcal{A}$: small full subcat.

Then TFAE:

(i) $\mathcal{C}$: a generating set

(ii) $\mathcal{C}$: a strongly generating set

(iii) $\mathrm{Lan}_F Y$: conservative

(iv) $\forall A \in \mathcal{A}$, $\exists\, S_i, S_j \in \mathcal{C}$,

$$\bigoplus_i S_i \longrightarrow \bigoplus_j S_j \longrightarrow A \longrightarrow 0 \text{ : exact}$$

$\square$

finiteness of objects

Def

In $\mathcal{C}$: cocomplete ordinary cat.,

$C \in \mathcal{C}$ is finitely presentable

$:\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{C}}(C,-)$ preserves filtered colimits.

Example

$\mathcal{C} = \mathbf{Set}$ $\rightsquigarrow$ finite sets

$\mathcal{C} = \mathrm{Mod}(R)$ $\rightsquigarrow$ finitely presented modules

$\mathcal{C} = \mathrm{Ch}$ $\rightsquigarrow$ bounded complexes of finitely presented modules

Def

A cosmos $\mathcal{V}$ is a l.f.p. base

$:\Leftrightarrow$ (i) $\mathcal{V}$: locally finitely presentable (i.e. has a strongly generating set of finitely presentable objects)

(ii) $I \in \mathcal{V}$ is finitely presentable

(iii) $X, Y \in \mathcal{V}$: f.p. $\Rightarrow$ $X \otimes Y$: f.p.

Def

Let $\mathcal{V}$ be a l.f.p. base.

(i) J: $\mathcal{V}$-cat. is finite

$:\Leftrightarrow$ $\mathrm{obj}(J)$: finite set and $\forall j,k \in J$, $J(j,k) \in \mathcal{V}$: f.p.

(ii) $W : J \longrightarrow \mathcal{V}$ is finite

$:\Leftrightarrow$ J: finite and $\forall j \in J$, $W(j) \in \mathcal{V}$: f.p.

(iii) finite limit

$:=$ limit weighted by finite functors.

(iv) $\mathcal{C}$ is finitely complete

$:\Leftrightarrow$ $\mathcal{C}$ admits all finite limits.

(v) F is left exact

$:\Leftrightarrow$ F preserves finite limits.

Prop

$\mathcal{V}$: l.f.p. base.

$\mathcal{C}$: $\mathcal{V}$-cat. admits finite limits

$\Leftrightarrow$ $\mathcal{C}$ has finite conical limits

and cotensor with X : f.p. $\square$

Prop

$\mathcal{V}$: l.f.p. base.

$F : \mathcal{C} \longrightarrow \mathcal{D}$ is left exact

$\Leftrightarrow$ F preserves finite conical limits

and cotensor with X : f.p. $\square$

Def

In $\mathcal{V}$: cosmos,

$X \in \mathcal{V}$ is dualizable

$:\Leftrightarrow$ $[X, I] \otimes - \cong [X, -]$

Example

$\mathcal{V} = \mathrm{Mod}(k)$ $\rightsquigarrow$ finitely generated and projective modules

$\mathcal{V} = \mathrm{Ch}$ $\rightsquigarrow$ bounded complexes of f.g. and proj. modules

Prop

$\mathcal{V}$: cosmos, $X \in \mathcal{V}$

TFAE:

(i) X : dualizable

(ii) $\forall \mathcal{C}$: $\mathcal{V}$-cat, $\forall C \in \mathcal{C}$,

cotensor $X \pitchfork C$ is absolute

(i.e. preserved by any functor)

(iii) $\forall \mathcal{C}$: $\mathcal{V}$-cat, $\forall C \in \mathcal{C}$,

tensor $X \otimes C$ is absolute $\square$

§2. Main Results

A **Grothendieck cosmos** is a cosmos which is also a Grothendieck cat.

For a Grothendieck cosmos $\mathcal{V}$, we consider the following finiteness conditions:

$$(C)\begin{cases} (C1) & \text{unit obj. } I \text{ is finitely presentable,} \\ (C2) & \mathcal{V} \text{ has a generating set } \{g_j\}_{j\in J} \text{ of dualizable objects.} \end{cases}$$

(from [Holm-Odabashi])

Example

$\mathcal{V} = \mathrm{Mod}(k)$ (k: commutative ring)

is a Grothendieck cosmos in which

- $k \in \mathrm{Mod}(k)$ is the finitely presentable unit object and a dualizable generator.

Example

$\mathcal{V} = \mathrm{Ch}(k) := \mathrm{Ch}(\mathrm{Mod}(k))$ (k: comm. ring)

is a Grothendieck cosmos in which

- $S(k) \in \mathrm{Ch}(k)$ is the finitely presentable unit object and
- $\{D(k)[n] \mid n \in \mathbb{Z}\}$ is a generating set of dualizable objects,

where
$$S(k): \cdots \to 0 \to 0 \to k \to 0 \to \cdots$$
$$D(k): \cdots \to 0 \to \underset{-1次}{k} \xrightarrow{\mathrm{id}} \underset{0次}{k} \to 0 \to \cdots.$$

Rmk

More generally,

$\mathcal{V}$: Groth. cosmos $\Rightarrow$ $\mathrm{Ch}(\mathcal{V})$: so

$\mathcal{V}$ satisfies (C) $\Rightarrow$ $\mathrm{Ch}(\mathcal{V})$: so.

Prop ([Holm-Odabashi])

$\mathcal{V}$: Grothendieck cosmos + (C)

$\Longrightarrow$ $\mathcal{V}$: l.f.p. base $\square$

Def

$\mathcal{V}$: l.f.p. base

A $\mathcal{V}$-cat. $\mathcal{A}$ is a Grothendieck $\mathcal{V}$-category

$:\Longleftrightarrow$ $\exists \mathcal{C}$: small $\mathcal{V}$-category,

$\exists$ $S : [\mathcal{C}^{op}, \mathcal{V}] \rightleftarrows \mathcal{A} : T$: adj. ($S \dashv T$)

s.t. (i) T: fully faithful

(ii) S: left exact. $\square$

This is called a $\mathcal{V}$-topos in [Garner-Lack].

Thm (I)

$\mathcal{V}$: Grothendieck cosmos + (C)

Then $\mathcal{A}$ is a Grothendieck $\mathcal{V}$-category

$\Longleftrightarrow$ (1) $\mathcal{A}$: cocomplete

(2) $\mathcal{A}$: finitely complete

(3) homomorphism thm holds in $\mathcal{A}$

(4) conical filtered colimits are left exact

(5) $\mathcal{A}$ has a $\mathcal{V}$-generating set. $\square$

Def

$\mathcal{V}$: cosmos, $\mathcal{A}$: $\mathcal{V}$-cat.

$\mathcal{C} \subseteq \mathcal{A}$: small full subcat, $F : \mathcal{C} \hookrightarrow \mathcal{A}$

Then $\mathcal{C}$ is a $\mathcal{V}$-generating set

$:\Longleftrightarrow$ $(\mathrm{Lan}_F Y)_0$: faithful.

Prop

$\mathcal{C} \subseteq \mathcal{A}$: small full $\mathcal{V}$-subcat.

Then $\mathcal{C}_0 \subseteq \mathcal{A}_0$: generating set

$\Rightarrow$ $\mathcal{C} \subseteq \mathcal{A}$: $\mathcal{V}$-generating set

proof

$$\begin{array}{ccccc} (\mathrm{Lan}_F Y)_0 : & \mathcal{A}_0 & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V}) := [\mathcal{C}^{op}, \mathcal{V}]_0 \\ & A & \longmapsto & \mathcal{A}(-, A)|_{\mathcal{C}} \\ & f \downarrow & & \downarrow f \circ - \\ & B & \longmapsto & \mathcal{A}(-, B)|_{\mathcal{C}} \end{array}$$

For each $C \in \mathcal{C}$,

$$\mathcal{A}_0(C, f) = \mathrm{Hom}_{\mathcal{V}}(I, \mathcal{A}(C, f)) = \mathrm{Hom}_{\mathcal{V}}(I, (f \circ -)_C).$$

Hence, if $(\mathrm{Lan}_F Y)_0(f) = (\mathrm{Lan}_F Y)_0(g)$,

then $\forall C \in \mathcal{C}$, $\mathcal{A}_0(C, f) = \mathcal{A}_0(C, g)$,

which shows $f = g$

since $\mathcal{C}_0 \subseteq \mathcal{A}_0$ is a generating set. ∎

Prop

$\mathcal{V}$: Grothendieck cosmos + (C)

$S : \mathcal{C} \longrightarrow \mathcal{D}$, $\mathcal{C}, \mathcal{D}$: finitely complete.

Then S : left exact $\Leftrightarrow$ S_0 : left exact.

proof

($\Leftarrow$): It suffices to show S preserves

cotensor $X \pitchfork C$ ($X \in \mathcal{V}$: f.p., $C \in \mathcal{C}$).

For $X \in \mathcal{V}$: f.p., there is

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0 : \text{exact}$$

with P_1, P_0 : dualizable.

$$\therefore\ 0 \longrightarrow X \pitchfork C \longrightarrow P_0 \pitchfork C \longrightarrow P_1 \pitchfork C : \text{exact}$$

Since S_0 : left exact,

$$\begin{array}{ccccccc} 0 \longrightarrow & S(X \pitchfork C) & \longrightarrow & S(P_0 \pitchfork C) & \longrightarrow & S(P_1 \pitchfork C) & : \text{ex.} \\ & & & \wr\wr & & \wr\wr & \\ & & & P_0 \pitchfork S(C) & & P_1 \pitchfork S(C) & \end{array}$$

Thus we have $S(X \pitchfork C) \cong X \pitchfork S(C)$. ∎

<u>Prop</u> ([Al Hwaeer-Garkusha])

$\mathcal{V}$: Groth. cosmos with $\{g_j\}_{j\in J}$: generating set

$\mathcal{C}$: small $\mathcal{V}$-cat.

$\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V}) := [\mathcal{C}^{op}, \mathcal{V}]_0$

Then $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V})$ is a Grothendieck cat.

with $\{g_j \otimes \mathcal{C}(-,c) \mid j \in J, c \in \mathcal{C}\}$: generating set. $\square$

<u>Proof of main thm</u>

Suppose

(1) $\mathcal{A}$: cocomplete

(2) $\mathcal{A}$: finitely complete

(3) homomorphism thm holds in $\mathcal{A}$

(4) conical filtered colimits are left exact

(5) $\mathcal{A}$ has a $\mathcal{V}$-generating set $\mathcal{C}$.

Then $\mathcal{A}_0$: (AB5)-abelian category.

Let $F: \mathcal{C} \hookrightarrow \mathcal{A}$ be the inclusion.

Since $\mathcal{A}$: cocomplete, we have

$$\begin{tikzcd} {[\mathcal{C}^{op}, \mathcal{V}]} \arrow[dr, shift left, "S"] \arrow[dr, phantom, "\perp"] & \\ \mathcal{C} \arrow[u, "Y"] \arrow[r, hook, "F"'] & \mathcal{A} \arrow[ul, shift left, "T"] \end{tikzcd}$$

$S := \mathrm{Lan}_Y F$

$T := \mathrm{Lan}_F Y.$

We want to show

S: left exact, T: fully faithful.

It suffices to show

S_0: left exact, T_0: fully faithful.

Now $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V})$ is a Groth. cat. with

$\mathcal{G}_0 := \{g_j \otimes \mathcal{C}(-,c) \mid j \in J, c \in \mathcal{C}\}$

a generating set, so by G-P thm we have

$$\begin{tikzcd} \mathrm{Mod}\,\mathcal{G}_0 \arrow[dr, shift left, "S'"] \arrow[dr, phantom, "\perp"] & \\ \mathcal{G}_0 \arrow[u, "Y"] \arrow[r, hook, "G_0"'] & \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V}) \arrow[ul, shift left, "T'"] \end{tikzcd}$$

s.t. S': left exact

T': fully faithful.

Note

$S_0(g_j \otimes \mathcal{C}(-,c)) \cong g_j \otimes S_0(\mathcal{C}(-,c)) \cong g_j \otimes c,$

and $H := S_0 \circ G_0$ can be seen as incl.

$\{g_j \otimes c \mid j \in J, c \in \mathcal{C}\} \hookrightarrow \mathcal{A}_0.$

Therefore we have

$$\begin{tikzcd}
\mathrm{Mod}\,\mathcal{Y}_0 \arrow[dr, shift left, "S'"] & & \\
 & \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V}) \arrow[ul, shift left, "T'"] \arrow[dr, shift left, "S_0"] & \\
\mathcal{Y}_0 \arrow[uu, "y"] \arrow[rr, hook, "H"'] & & \mathcal{A}_0 \arrow[ul, shift left, "T_0"]
\end{tikzcd}$$

and

$S_0 \circ S' = \mathrm{Lan}_y H$, $\quad T' \circ T_0 = \mathrm{Lan}_H y$.

Here since T_0 : faithful,

$\mathrm{Lan}_H y \cong T' \circ T_0$ is faithful

and so $\mathcal{Y}_0 \subseteq \mathcal{A}_0$ is a generating set.

Hence by G-P thm, we have

$S_0 \circ S'$: left exact, $\quad T' \circ T_0$: fully faithful.

$\therefore$ S_0 : left exact, $\quad T_0$: fully faithful.

$\because$ When $\lim_k P_k$: finite limit in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V})$,

$\lim_k P_k \cong S' \circ T'(\lim_k P_k) \cong S'(\lim_k T'(P_k))$

$\therefore S_0(\lim_k P_k) \cong S_0 \circ S'(\lim_k T'(P_k)) \cong \lim_k S_0 S' T'(P_k)$

$\cong \lim_k S_0(P_k)$

Prop (I)

$\mathcal{V}, \mathcal{W}$: Grothendieck l.f.p. base

$F : \mathcal{V} \rightleftarrows \mathcal{W} : G$, $\quad F$: monoidal

Assume $\mathcal{V}$ satisfies (C). Then

$\mathcal{B}$: Grothendieck $\mathcal{W}$-cat.

$\Longrightarrow$ $G(\mathcal{B})$: Grothendieck $\mathcal{V}$-cat. $\square$

Example

$X \xrightarrow{f} \mathrm{Spec}(k)$: qcqs scheme (k = comm. ring)

$\rightsquigarrow f^* : \mathrm{Mod}(k) \rightleftarrows \mathrm{Qcoh}(X) : f_*$

$\rightsquigarrow f^* : \underset{\mathcal{V}}{\underset{\parallel}{\mathrm{Ch}(k)}} \rightleftarrows \underset{\mathcal{W}}{\underset{\parallel}{\mathrm{Ch}(\mathrm{Qcoh}(X))}} : f_*$

Since $\mathcal{W}$ itself is a Groth. $\mathcal{W}$-cat.,

$f_*(\mathrm{Ch}(\mathrm{Qcoh}(X)))$: Groth. $\mathcal{V}$-cat.

This is the dg cat. $\mathcal{A}$ over k

s.t. $\begin{cases} \text{obj} : \text{cpxes of quasi-coherent sheaves} \\ \text{mor} : \mathcal{A}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)^n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}^i, \mathcal{G}^{i+n}) \end{cases}$

§. References

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