

Grothendieck enriched categories

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§0. Introduction

Def

A is a Grothendieck category

- \Leftrightarrow (i) A : cocomplete abelian cat.
(ii) filtered colimits are exact
(iii) A has a generator G

Example

- (1) Ab = cat. of abelian groups
- (2) $\text{Mod}(R)$ = cat. of right R -modules
(R = non-commutative ring)
- (3) $\text{Qcoh}(X)$ = cat. of quasi-coherent sheaves on a scheme X
- (4) $\text{Ch}(\text{Ab})$ = cat. of complexes of abelian groups

次のような良い性質を持つ。

Prop

For a Grothendieck cat. A , the following holds:

- (1) A : complete
- (2) A has an injective cogenerator
- (3) A has enough injectives
- (4) functor representability theorem
- (5) adjoint functor theorem \square

Thm (Gabriel-Popescu)

\mathcal{A} : Grothendieck cat. with a generator G

$R := \mathcal{A}(G, G)$: ring of endomorphisms

Then

(1) $T := \mathcal{A}(G, -) : \mathcal{A} \rightarrow \text{Mod}(R)$
admits a left adjoint S

(2) T : fully faithful

(3) S : (left) exact

$$\text{Mod}(R) \begin{array}{c} \xrightarrow{S: \text{lex}} \\ \perp \\ \xleftarrow{T} \end{array} \mathcal{A}$$

□

Def

$\mathcal{C} \subseteq \mathcal{D}$: full subcategory

is lex-reflective

\Leftrightarrow incl. $\mathcal{C} \hookrightarrow \mathcal{D}$ admits
a left exact left adjoint

\mathcal{A} is a lex-reflective
full subcat. of $\text{Mod}(R)$

The Gabriel-Popescu theorem asserts:

Grothendieck categories = lex-reflective subcategories
of some module categories.

&
intrinsic properties

&
extrinsic characterization

⊙ similar result for Grothendieck topoi

Def

A cat. \mathcal{E} satisfies the Giraud condition

- \Leftrightarrow
- (1) \mathcal{E} : finitely complete
 - (2) \mathcal{E} has coproducts, which are disjoint and universal
 - (3) every epi. is a coequalizer
 - (4) every equiv. rel. is effective
 - (5) every equiv. rel. is universal
 - (6) \mathcal{E} has a generating set

Thm (Giraud thm)

For a cat. \mathcal{E} , TFAE: intrinsic

- (1) \mathcal{E} satisfies the Giraud condition
- (2) \mathcal{E} is equivalent to a lex-reflective full subcat. of the presheaf cat.

$[\mathcal{C}^{\text{op}}, \mathcal{S}et]$ of some small cat \mathcal{C} .

- (3) \mathcal{E} is a Grothendieck topos.
(i.e. $\mathcal{E} \simeq \mathcal{S}h(\mathcal{C}, \mathcal{J})$ for some site) \square

extrinsic

\Downarrow

The G-P thm can be seen as
an additive version of this thm.
Ab-enriched

Ab-enriched cat.

\equiv cat. whose Hom's are
abelian groups (objects of Ab)

Note: every abelian cat. has
the canonical Ab -enriched str.

- R : (non-commutative) ring
= Ab -enriched cat.
with a single object
- right R -module
= Ab -functor $R^{\text{op}} \rightarrow \text{Ab}$
- $\text{Mod}(R)$
= Ab -enriched "presheaf" cat.
 $[R^{\text{op}}, \text{Ab}]$

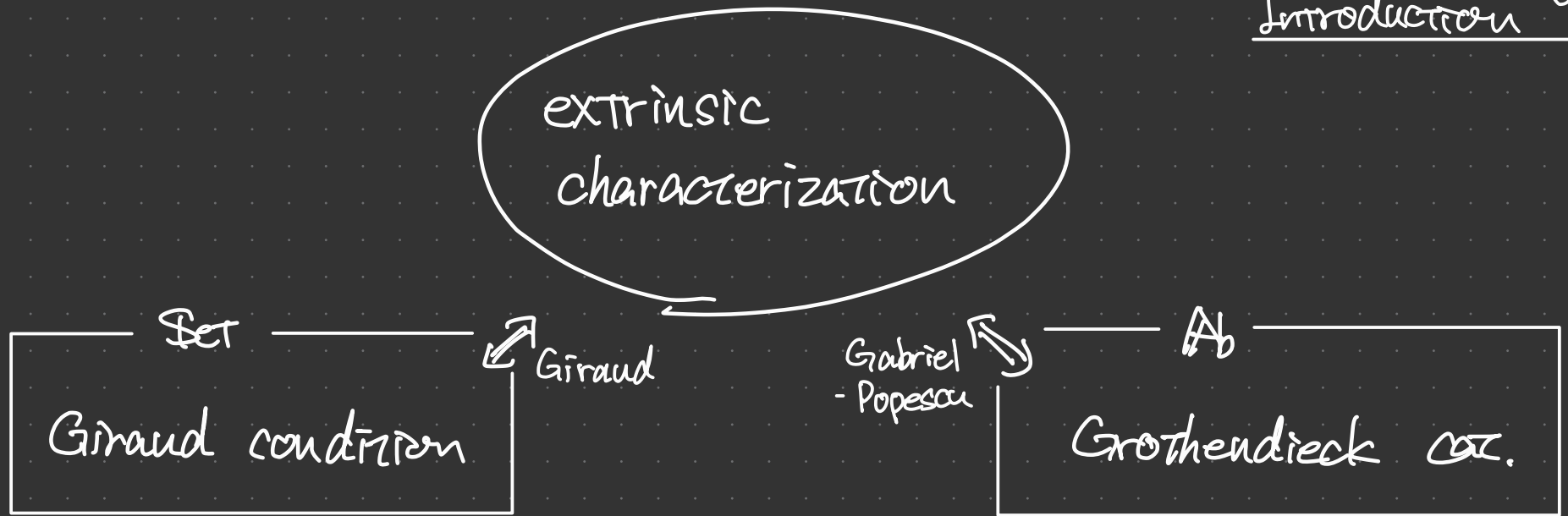
Thm

For an Ab -enriched cat. \mathcal{A} , TFAE:

- (1) \mathcal{A} is a Grothendieck cat.
- (2) \mathcal{A} is equivalent to a lex-reflective full subcat. of $\text{Mod}(R) = [R^{\text{op}}, \text{Ab}]$ of some ring R .
- (3) \mathcal{A} is an "additive" topos \square

Rmk

- | Set | | Ab |
|-------------------------|-------------------|--------------------------------|
| • Grothendieck topology | \leftrightarrow | Gabriel topology |
| • sheaf | \leftrightarrow | closed module (additive sheaf) |



Q. $\mathcal{C}h := \mathcal{C}h(\mathbf{Ab})$ -enriched (可富とどうなる?)

Ans.

\mathcal{V} : "nice" Grothendieck monoidal cat. ($\mathcal{C}h$ 含む)

$\mathcal{C}h$ enriched cat. (可富). 類似が成り立つ. [Imamura 2022]

§1. Preliminaries

Fix a complete and cocomplete symmetric monoidal closed cat. $\mathcal{V} = (\mathcal{V}, \otimes, I, [-, -])$.
($=$: cosmos)

Def

A \mathcal{V} -enriched category (or \mathcal{V} -category) \mathcal{C}

consists of the following data

- $\text{obj}(\mathcal{C})$: collection of objects
- $\mathcal{C}(X, Y) \in \mathcal{V}$ for $X, Y \in \mathcal{C}$
- $m: \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ in \mathcal{V}
for $X, Y, Z \in \mathcal{C}$

- $i_X: I \rightarrow \mathcal{C}(X, X)$ in \mathcal{V} for $X \in \mathcal{C}$

satisfying the associativity and unit
axioms.

Example

- $\mathcal{V} = \text{Set} \rightsquigarrow$ (locally small) category
- $\mathcal{V} = \text{Ab} \rightsquigarrow$ preadditive category
- $\mathcal{V} = \text{Ch} \rightsquigarrow$ dg category
- $\mathcal{V} = \text{Cat} \rightsquigarrow$ (strict) 2-category

Rmk

A cosmos \mathcal{V} itself becomes
a \mathcal{V} -enriched category
with $\mathcal{V}(X, Y) := [X, Y] \in \mathcal{V}$

Def

A unit \mathcal{V} -cat. \mathcal{I} is the \mathcal{V} -cat.
with a single obj $*$ and $\mathcal{I}(*, *) = I$.

As in the usual category theory,
we have:

- \mathcal{V} -functor $F: \mathcal{C} \rightarrow \mathcal{D}$
 - \mathcal{V} -nat. trans. $\alpha: F \Rightarrow G$
 - opposite \mathcal{V} -cat. \mathcal{C}^{op}
 - functor \mathcal{V} -cat. $[\mathcal{C}, \mathcal{D}]$
 - yoneda embedding $\gamma: \mathcal{C} \hookrightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}]$
 - enriched yoneda lemma
 - \mathcal{V} -adjunction,
- etc.

Def

For a \mathcal{V} -cat. \mathcal{C} , the underlying category
is the ordinary category \mathcal{C}_0

$$\text{s.t. } \begin{cases} \text{obj}(\mathcal{C}_0) := \text{obj}(\mathcal{C}) \\ \text{Hom}_{\mathcal{C}_0}(X, Y) := \text{Hom}_{\mathcal{V}}(\mathbb{I}, \mathcal{C}(X, Y)) \end{cases}$$

$$\rightsquigarrow U = (-)_0: \mathcal{V}\text{-Cat} \longrightarrow \text{Cat}$$

Example

$$\begin{aligned} \mathcal{A} = \text{dg cat.} &\rightsquigarrow \mathcal{A}_0 = \mathbb{Z}^0(\mathcal{A}) \\ &(\text{Hom}_{\text{ch}}(k, -) = \mathbb{Z}^0(-)) \end{aligned}$$

$(-)_0$ has a left adjoint

$$(-)_{\mathcal{V}}: \text{Cat} \longrightarrow \mathcal{V}\text{-Cat}$$

$$\mathcal{I} \longmapsto \mathcal{L}_{\mathcal{V}} = \underline{\text{free } \mathcal{V}\text{-category}}$$

(co)limits in enriched categories

Def

(1) For $F: J \rightarrow \mathcal{C}$ and $W: J \rightarrow \mathcal{V}$,

the limit of F weighted by W is

an object $\{W, F\} \in \mathcal{C}$ s.t.

$$\mathcal{C}(C, \{W, F\}) \cong [J, \mathcal{V}](W, \mathcal{C}(C, F-))$$

: natural iso. in \mathcal{C} .

(2) For $F: J \rightarrow \mathcal{C}$ and $W: J^{\text{op}} \rightarrow \mathcal{V}$,

the colimit of F weighted by W is

an object $W \star F \in \mathcal{C}$ s.t.

$$\mathcal{C}(W \star F, C) \cong [J^{\text{op}}, \mathcal{V}](W, \mathcal{C}(F-, C))$$

: natural iso. in \mathcal{C} .

As a special case,

Def ($J = I$)

(1) For $X \in \mathcal{V}$ and $D \in \mathcal{C}$, the cotensor product is an object $X \pitchfork D \in \mathcal{C}$ s.t.

$$\mathcal{C}(C, X \pitchfork D) \cong \mathcal{V}(X, \mathcal{C}(C, D))$$

(2) For $X \in \mathcal{V}$ and $D \in \mathcal{C}$, the tensor product is an object $X \otimes D \in \mathcal{C}$ s.t.

$$\mathcal{C}(X \otimes D, C) \cong \mathcal{V}(X, \mathcal{C}(D, C))$$

Def (J : free \mathcal{V} -cat, $W = \Delta I$)

\mathcal{C} : \mathcal{V} -cat.

$F: J \rightarrow \mathcal{C}_0$: ordinary functor

$\overset{\text{adj.}}{\leftarrow} F: J_{\mathcal{V}} \rightarrow \mathcal{C}$: \mathcal{V} -functor

(1) the conical limit of F is $\text{lim} F \in \mathcal{C}$ s.t. $\mathcal{C}(C, \text{lim} F) \cong \text{lim} \mathcal{V}(C, F-)$.

(2) the conical colimit of F is $\text{colim} F \in \mathcal{C}$ s.t. $\mathcal{C}(\text{colim} F, C) \cong \text{lim} \mathcal{V}(F-, C)$.

Example

$$S(\mathbb{Z}): \dots \rightarrow 0 \rightarrow \overset{-1 \times}{0} \rightarrow \overset{0 \times}{\mathbb{Z}} \rightarrow 0 \rightarrow \dots \in \text{Ch}$$

$$D(\mathbb{Z}): \dots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \rightarrow 0 \rightarrow \dots \in \text{Ch}$$

A : dg cat.

(i) $A \in A$ の shift $A[1]$ とは、tensor product

$$A[1] = S(\mathbb{Z})[-1] \otimes A$$

の \cong .

(ii) $f \in Z^0(A)(A, B)$ の cone とは、

$$W: [1]^{\text{op}} \rightarrow \text{Ch} \quad ; \quad \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array} \mapsto \begin{array}{c} D(\mathbb{Z}) \\ \downarrow \\ S(\mathbb{Z}) \end{array}$$

\cong する

$$F: [1] \rightarrow Z^0 A \quad ; \quad \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array} \mapsto \begin{array}{c} A \\ \downarrow f \\ B \end{array}$$

の weighted colimit

$$\text{Cone}(f) = W \star F$$

の \cong .

Def

\mathcal{C} is (co)complete

$\Leftrightarrow \mathcal{C}$ has all weighted (co)limits on small categories.

Prop

\mathcal{C} is (co)complete

$\Leftrightarrow \mathcal{C}$ has all conical (co)limits and all cotensor (tensor) products.

Prop

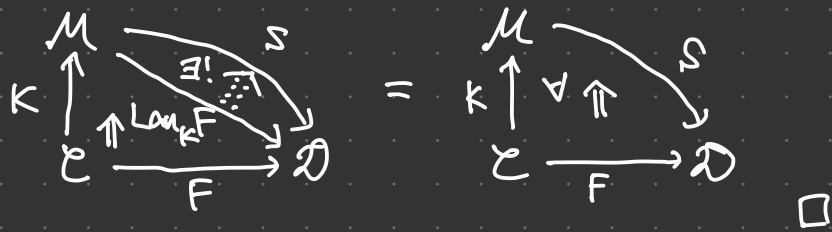
The \mathcal{V} -cat. \mathcal{V} is complete and cocomplete and so is $[\mathcal{C}^{\text{op}}, \mathcal{V}]$. □

Def

For $F: \mathcal{C} \rightarrow \mathcal{D}$ and $K: \mathcal{C} \rightarrow \mathcal{M}$, the left Kan extension of F along K is

$\text{Lan}_K F: \mathcal{M} \rightarrow \mathcal{D}$ s.t.

$$\text{Hom}_{\text{Fun}(\mathcal{M}, \mathcal{D})}(\text{Lan}_K F, S) \cong \text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, S \circ K).$$



Def

For $F: \mathcal{C} \rightarrow \mathcal{D}$ and $K: \mathcal{C} \rightarrow \mathcal{M}$, the pointwise left Kan extension of F along K is

$T: \mathcal{M} \rightarrow \mathcal{D}$ s.t.

$$\mathcal{D}(Tm, d) \cong [\mathcal{C}^{\text{op}}, \mathcal{V}](\mathcal{M}(K-, m), \mathcal{D}(F-, d)).$$

Prop

(1) Any pointwise left Kan extension is a left Kan extension.

(2) If \mathcal{C} is small and \mathcal{D} is cocomplete, then p.w. left Kan ext. always exist. \square

Prop

$F: \mathcal{C} \rightarrow \mathcal{D}$, \mathcal{C} small.

Then $\text{Lan}_F \gamma: \mathcal{D} \rightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}]$ exists and $\text{Lan}_F \gamma(d) \cong \mathcal{D}(F-, d)$.

proof

$$[\mathcal{C}^{\text{op}}, \mathcal{V}](\text{Lan}_F \gamma(d), P) \cong [\mathcal{C}^{\text{op}}, \mathcal{V}](\mathcal{D}(F-, d), \underbrace{[\mathcal{C}^{\text{op}}, \mathcal{V}](\gamma(-), P)}_{\text{Sif 未田}})$$

$$\cong [\mathcal{C}^{\text{op}}, \mathcal{V}](\mathcal{D}(F-, d), P).$$

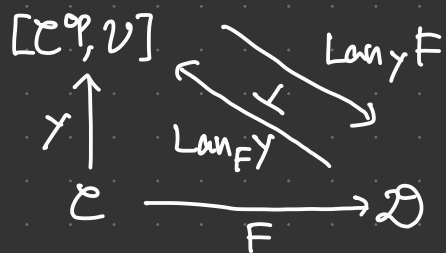
$$\therefore \text{Lan}_F \gamma(d) \cong \mathcal{D}(F-, d) \quad \square$$

普遍函子

Thm (nerve-realization adj.)

$F: \mathcal{C} \rightarrow \mathcal{D}$, \mathcal{C} : small, \mathcal{D} : cocomplete

Then, $\text{Lan}_Y F \dashv \text{Lan}_F Y$.



proof

$$\mathcal{D}(\text{Lan}_Y F(P), d)$$

$$\cong [\mathcal{C}^{\text{op}}, \mathcal{V}] \left(\underbrace{[\mathcal{C}^{\text{op}}, \mathcal{V}](\gamma(-), P)}_{\cong \text{ *田}}, \mathcal{D}(F-, d) \right)$$

$$\cong [\mathcal{C}^{\text{op}}, \mathcal{V}](P, \mathcal{D}(F-, d))$$

$$\cong [\mathcal{C}^{\text{op}}, \mathcal{V}](P, \text{Lan}_F Y(d)).$$

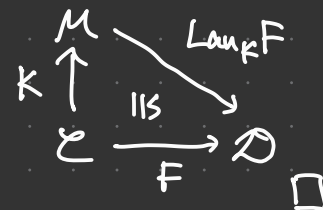
$$\therefore \text{Lan}_Y F \dashv \text{Lan}_F Y. \quad \square$$

Prop

$\text{Lan}_K F$: pointwise Kan ext.

K : fully faithful

$$\Rightarrow \text{Lan}_K F \circ K \cong F$$



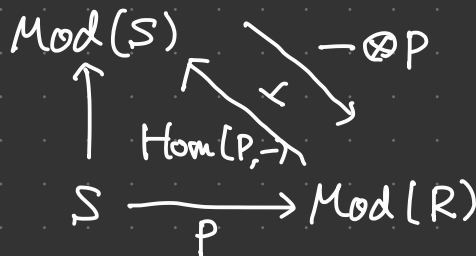
Thm

\mathcal{C} : small, \mathcal{D} : cocomplete

Then

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \mathcal{V}\text{-functors} \\ F: \mathcal{C} \rightarrow \mathcal{D} \end{array} \right\} & \xleftrightarrow{1:1} & \left\{ \begin{array}{l} \text{adjunction} \\ S: [\mathcal{C}^{\text{op}}, \mathcal{V}] \rightleftarrows \mathcal{D}: T \end{array} \right\} \\
 F & \longmapsto & \text{Lan}_Y F \dashv \text{Lan}_F Y \\
 S \circ \gamma & \longleftarrow & S \dashv T
 \end{array}$$

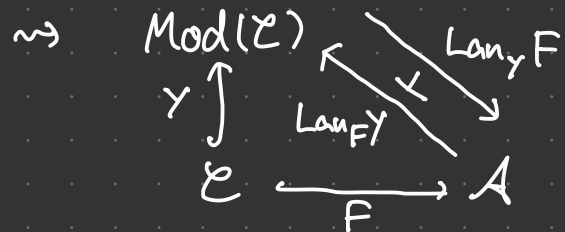
Example (Eilenberg-Watts)



Thm (G-P thm)

A : Grothendieck cat.

\mathcal{C} : generating set, $F: \mathcal{C} \hookrightarrow A$



Then, (1) $\text{Lan}_F \gamma$: fully faithful

(2) $\text{Lan}_Y F$: left exact \square

Def

$\mathcal{C} \subseteq A$: small full subcat.

\mathcal{C} is a generating set

$\Leftrightarrow \text{Lan}_F \gamma$: faithful.

\mathcal{C} is a strongly generating set

$\Leftrightarrow \text{Lan}_F \gamma$: faithful and conservative.

Prop

A : cocomplete abelian cat.

$\mathcal{C} \subseteq A$: small full subcat.

Then TFAE:

(i) \mathcal{C} : a generating set

(ii) \mathcal{C} : a strongly generating set

(iii) $\text{Lan}_F \gamma$: conservative

(iv) $\forall A \in A, \exists s_i, s_j \in \mathcal{C},$

$$\bigoplus_i s_i \rightarrow \bigoplus_j s_j \rightarrow A \rightarrow 0 = \text{exact} \quad \square$$

① finiteness of objects

Def

In \mathcal{C} : cocomplete ordinary cat.,

$C \in \mathcal{C}$ is finitely presentable

$\Leftrightarrow \text{Hom}_{\mathcal{C}}(C, -)$ preserves filtered colimits.

Example

- $\mathcal{C} = \text{Set} \rightsquigarrow$ finite sets
- $\mathcal{C} = \text{Mod}(R) \rightsquigarrow$ finitely presented modules
- $\mathcal{C} = \text{Ch} \rightsquigarrow$ bounded complexes of finitely presented modules

Def

A cosmos \mathcal{V} is a l.f.p. base

- \Leftrightarrow (i) \mathcal{V} : locally finitely presentable
(i.e. has a strongly generating set of finitely presentable objects)
- (ii) $I \in \mathcal{V}$ is finitely presentable
- (iii) $X, Y \in \mathcal{V}: \text{f.p.} \Rightarrow X \otimes Y: \text{f.p.}$

Def

Let \mathcal{V} be a l.f.p. base.

(i) $J: \mathcal{V}\text{-cat.}$ is finite
 $\Leftrightarrow \text{obj}(J): \text{finite set}$
 and $\forall j, k \in J, J(j, k) \in \mathcal{V}: \text{f.p.}$

(ii) $W: J \rightarrow \mathcal{V}$ is finite
 $\Leftrightarrow J: \text{finite}$
 and $\forall j \in J, W(j) \in \mathcal{V}: \text{f.p.}$

(iii) finite limit
 $=$ limit weighted by finite functors.

(iv) \mathcal{C} is finitely complete
 $\Leftrightarrow \mathcal{C}$ admits all finite limits.

(v) F is left exact
 $\Leftrightarrow F$ preserves finite limits.

Prop

\mathcal{V} : l.f.p. base.

\mathcal{C} : \mathcal{V} -cat. admits finite limits

$\Leftrightarrow \mathcal{C}$ has finite conical limits

and cotensor with X : f.p. \square

Prop

\mathcal{V} : l.f.p. base.

$F: \mathcal{C} \rightarrow \mathcal{D}$ is left exact

$\Leftrightarrow F$ preserves finite conical limits

and cotensor with X : f.p. \square

Def

In \mathcal{V} : cosmos,

$X \in \mathcal{V}$ is dualizable

$\Leftrightarrow [X, I] \otimes - \cong [X, -]$

Example

$\mathcal{V} = \text{Mod}(k) \rightsquigarrow$ finitely generated
and projective modules

$\mathcal{V} = \text{Ch} \rightsquigarrow$ bounded complexes of
f.g. and proj. modules

Prop

\mathcal{V} : cosmos, $X \in \mathcal{V}$

TFAE:

(i) X : dualizable

(ii) $\forall \mathcal{C}: \mathcal{V}\text{-cat}, \forall C \in \mathcal{C},$

cotensor $X \pitchfork C$ is absolute

(i.e. preserved by any functor)

(iii) $\forall \mathcal{C}: \mathcal{V}\text{-cat}, \forall C \in \mathcal{C},$

tensor $X \otimes C$ is absolute \square

§2. Main Results

A Grothendieck cosmos is a cosmos which is also a Grothendieck cat.

For a Grothendieck cosmos \mathcal{V} , we consider the following finiteness conditions:

- (c) $\left\{ \begin{array}{l} \text{(c1) unit obj. } I \text{ is finitely presentable,} \\ \text{(c2) } \mathcal{V} \text{ has a generating set } \{g_i\}_{i \in \mathbb{N}} \\ \text{of dualizable objects.} \end{array} \right.$

(from [Holm-Odabashi])

Example

$\mathcal{V} = \text{Mod}(k)$ ($k = \text{commutative ring}$)

is a Grothendieck cosmos in which

- $k \in \text{Mod}(k)$

is the finitely presentable unit object and a dualizable generator.

Main Results 1/5

Example

$\mathcal{V} = \text{Ch}(k) := \text{Ch}(\text{Mod}(k))$ ($k = \text{comm. ring}$)

is a Grothendieck cosmos in which

- $S(k) \in \text{Ch}(k)$ is the finitely presentable unit object and
- $\{D(k)[n] \mid n \in \mathbb{Z}\}$ is a generating set of dualizable objects,

where $S(k): \dots \rightarrow 0 \rightarrow 0 \rightarrow k \rightarrow 0 \rightarrow \dots$

$$D(k): \dots \rightarrow 0 \rightarrow k \xrightarrow[-i \times]{\text{id}} k \xrightarrow[0 \times]{\text{id}} 0 \rightarrow \dots$$

Rmk

More generally,

$\mathcal{V} : \text{Groth. cosmos} \Rightarrow \text{Ch}(\mathcal{V}) : \text{so}$

$\mathcal{V} \text{ satisfies (c)} \Rightarrow \text{Ch}(\mathcal{V}) = \text{so}.$

Prop ([Holm-Odabashi])

\mathcal{V} : Grothendieck cosmos + (c)

$\Rightarrow \mathcal{V}$: l.f.p. base \square

Def

\mathcal{V} : l.f.p. base

A \mathcal{V} -cat. \mathcal{A} is a Grothendieck \mathcal{V} -category

$\Leftrightarrow \exists \mathcal{C}$: small \mathcal{V} -category,

$\exists S: [\mathcal{C}^{\text{op}}, \mathcal{V}] \xrightleftharpoons{\perp} \mathcal{A} = T : \text{adj.}$

s.t. (i) T : fully faithful

(ii) S : left exact. \square

This is called a \mathcal{V} -topos in [Garner-Lack].

Thm (I)

\mathcal{V} : Grothendieck cosmos + (c)

Then \mathcal{A} is a Grothendieck \mathcal{V} -category

\Leftrightarrow (1) \mathcal{A} : cocomplete

(2) \mathcal{A} : finitely complete

(3) homomorphism thm holds in \mathcal{A}

(4) conical filtered colimits are left exact

(5) \mathcal{A} has a \mathcal{V} -generating set. \square

Def

\mathcal{V} : cosmos, \mathcal{A} : \mathcal{V} -cat.

$\mathcal{C} \subseteq \mathcal{A}$: small full subcat, $F: \mathcal{C} \hookrightarrow \mathcal{A}$

Then \mathcal{C} is a \mathcal{V} -generating set

$\Leftrightarrow (Lan_F \gamma)_0$: faithful.

Prop

$\mathcal{C} \subseteq \mathcal{A}$: small full \mathcal{V} -subcat.

Then $\mathcal{C}_0 \subseteq \mathcal{A}_0$: generating set

$\Rightarrow \mathcal{C} \subseteq \mathcal{A}$: \mathcal{V} -generating set

proof

$$\begin{array}{ccc}
 (\text{Lan}_{\mathcal{F}}\gamma)_0 : \mathcal{A}_0 & \longrightarrow & \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) := [\mathcal{C}^{\text{op}}, \mathcal{V}]_0 \\
 A & \longmapsto & A(-, A)|_{\mathcal{C}} \\
 f \downarrow & & \downarrow f_0 - \\
 B & \longmapsto & A(-, B)|_{\mathcal{C}}
 \end{array}$$

For each $C \in \mathcal{C}$,

$$A_0(C, f) = \text{Hom}_{\mathcal{V}}(I, A(C, f)) = \text{Hom}_{\mathcal{V}}(I, (f_0 -)_C)$$

Hence, if $(\text{Lan}_{\mathcal{F}}\gamma)_0(f) = (\text{Lan}_{\mathcal{F}}\gamma)_0(g)$,

then $\forall C \in \mathcal{C}, A_0(C, f) = A_0(C, g)$,

which shows $f = g$

since $\mathcal{C}_0 \subseteq \mathcal{A}_0$ is a generating set. ▣

Prop

\mathcal{V} : Grothendieck Cosmos + (C)

$S : \mathcal{C} \rightarrow \mathcal{D}$, \mathcal{C}, \mathcal{D} : finitely complete.

Then S : left exact $\Leftrightarrow S_0$: left exact.

proof

(\Leftarrow): It suffices to show S preserves cotensor $X \pitchfork C$ ($X \in \mathcal{V}$: f.p., $C \in \mathcal{C}$).

For $X \in \mathcal{V}$: f.p., there is

$$P_1 \rightarrow P_0 \rightarrow X \rightarrow 0 : \text{exact}$$

with P_1, P_0 : dualizable.

$$\therefore 0 \rightarrow X \pitchfork C \rightarrow P_0 \pitchfork C \rightarrow P_1 \pitchfork C : \text{exact}$$

Since S_0 : left exact,

$$\begin{array}{ccccc}
 0 & \rightarrow & S(X \pitchfork C) & \rightarrow & S(P_0 \pitchfork C) & \rightarrow & S(P_1 \pitchfork C) & : \text{ex.} \\
 & & & & \parallel \cong & & \parallel \cong & \\
 & & & & P_0 \pitchfork S(C) & & P_1 \pitchfork S(C) &
 \end{array}$$

Thus we have $S(X \pitchfork C) \cong X \pitchfork S(C)$. ▣

Prop ([A] Hweer - Garkusha)

\mathcal{V} : Groth. cosmos with $\{g_j | j \in J\}$: generating set

\mathcal{C} : small \mathcal{V} -cat.

$\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) := [\mathcal{C}^{\text{op}}, \mathcal{V}]_0$

Then $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V})$ is a Grothendieck cat.

with $\{g_j \otimes \mathcal{C}(-, c) | j \in J, c \in \mathcal{C}\}$: generating set. \square

Proof of main thm

Suppose

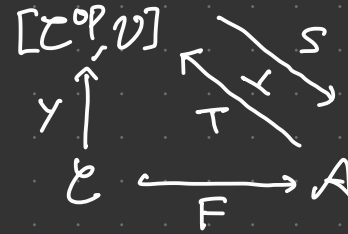
- (1) A : cocomplete
- (2) A : finitely complete
- (3) homomorphism thm holds in A
- (4) conical filtered colimits are left exact
- (5) A has a \mathcal{V} -generating set \mathcal{C} .

Then A_0 : (AB5)-abelian category.

Let $F: \mathcal{C} \hookrightarrow A$ be the inclusion.

Since A : cocomplete, we have

Main Results 4/5



$S := \text{Lan}_\gamma F$

$T := \text{Lan}_F \gamma$

We want to show

S : left exact, T : fully faithful.

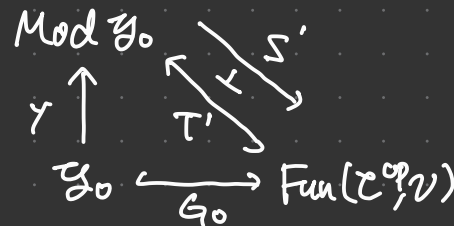
It suffices to show

S_0 : left exact, T_0 : fully faithful.

Now $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V})$ is a Groth. cat. with

$\mathcal{G}_0 := \{g_j \otimes \mathcal{C}(-, c) | j \in J, c \in \mathcal{C}\}$

a generating set, so by GP thm we have



S' : left exact

s.t.

T' : fully faithful.

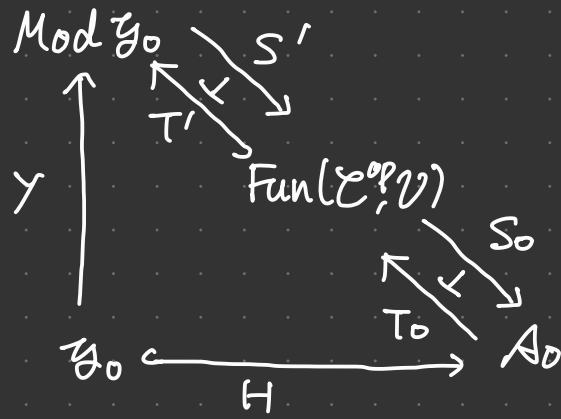
Note

$S_0(g_j \otimes \mathcal{C}(-, c)) \cong g_j \otimes S_0(\mathcal{C}(-, c)) \cong g_j \otimes c$,

and $H := S_0 \circ G_0$ can be seen as incl.

$\{g_j \otimes c | j \in J, c \in \mathcal{C}\} \hookrightarrow A_0$.

Therefore we have



and

$$S_0 \circ S' = \text{Lan}_Y H, \quad T'_0 \circ T_0 = \text{Lan}_H Y.$$

Here since T_0 is faithful,

$$\text{Lan}_H Y \cong T'_0 \circ T_0 \text{ is faithful}$$

and so $Y_0 \subseteq A_0$ is a generating set.

Hence by G-P thm, we have

$$S_0 \circ S' : \text{left exact}, \quad T'_0 \circ T_0 : \text{fully faithful.}$$

$$\therefore S_0 : \text{left exact}, \quad T_0 : \text{fully faithful.}$$

⊙ When $\lim_p P_p$: finite limit in $\text{Fun}(C^{\text{op}}, U)$,

$$\lim_p P_p \cong S'_0 T'_0(\lim_p P_p) \cong S'_0(\lim_p T'_0(P_p))$$

$$\begin{aligned} \therefore S_0(\lim_p P_p) &\cong S_0 \circ S'_0(\lim_p T'_0(P_p)) \cong \lim_p S_0 S'_0 T'_0(P_p) \\ &\cong \lim_p S_0(P_p) \end{aligned}$$



Prop (I)

U, W : Grothendieck l.f.p. base

$$F : U \xrightleftharpoons{\perp} W : G, \quad F : \text{monoidal}$$

Assume U satisfies (C). Then

B : Grothendieck W -cat.

$$\Rightarrow G(B) : \text{Grothendieck } U\text{-cat.} \quad \square$$

Example

$$X \xrightarrow{f} \text{Spec}(k) : \text{qcqs scheme } (k = \text{comm. ring})$$

$$\rightsquigarrow f^* : \text{Mod}(k) \xrightleftharpoons{\perp} \text{Qcoh}(X) : f_*$$

$$\rightsquigarrow f^* : \text{Ch}(k) \xrightleftharpoons{\perp} \text{Ch}(\text{Qcoh}(X)) : f_*$$

$\begin{matrix} \parallel \\ U \end{matrix} \qquad \qquad \qquad \begin{matrix} \parallel \\ W \end{matrix}$

Since W itself is a Groth. W -cat.,

$$f_*(\text{Ch}(\text{Qcoh}(X))) : \text{Groth. } U\text{-cat.}$$

This is the dg cat. A over k

$$\text{s.t. } \begin{cases} \text{Obj} : \text{cpxes of quasi-coherent sheaves} \\ \text{mor} : A(\mathcal{F}, \mathcal{G})^n = \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}^i, \mathcal{G}^{i+n}) \end{cases}$$

§. References

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