Exact-categorical properties of subcategories of abelian categories Part II

Haruhisa Enomoto (Osaka Pref. Univ.)

Last time

- exact cat
- proj, inj, simple obj, (JHP)
- $K_0(\mathcal{E})$: Grothendieck grp
  $M(\mathcal{E})$: ——— monoid
- $\mathcal{E}$: (JHP) $\iff$ $M(\mathcal{E})$: free

Today (I) Classify subcat!
(II) Study properties of subcat!

Setting k: field
Λ: f.-d. k-alg
$\operatorname{mod}\Lambda$: the cat of f.g. right Λ-mods
($\rightsquigarrow$ abelian)

For $\mathcal{C} \subseteq \operatorname{mod}\Lambda$: subcat,
$\operatorname{ind}\mathcal{C} := \{ C \in \mathcal{C} \mid \text{indecomp} \} / \cong$

Aim
Study "good" subcats of $\operatorname{mod}\Lambda$ using exact cat.

(I) Classify subcats using invariants of exact cat.
(proj, inj, sim)

(II) Study invariants & properties for a given subcat.

Def $\mathcal{C} \subseteq \operatorname{mod}\Lambda$: ext-closed subcat is

(1) wide if closed under ker, coker.
(i.e., $\forall\, C_1 \xrightarrow{f} C_2$ with $C_1, C_2 \in \mathcal{C}$, $\operatorname{Ker} f$, $\operatorname{Cok} f \in \mathcal{C}$,)

(2) torsion class (tors) if closed under quotient modules.
torsion-free (torf) if ——— submodules.

① Classifying tors

via proj

Thm (Adachi-Iyama-Reiten)

$\underset{\text{tors}}{\mathcal{T}} \longmapsto \{\text{proj obj in } \mathcal{T}\}$ gives

a bijection between

- $\{$tors in $\mathrm{mod}\Lambda$ with enough proj$\}$
- $\{$support τ-tilting Λ-modules$\}$ $\lrcorner$

In general, tors may not have non-zero proj objs.

via simple

Thm ([E, Monobrick, ...])

$\underset{\text{tors}}{\mathcal{T}} \longmapsto \operatorname{sim} \mathcal{T}$ gives

a bij between

- $\{$tors in $\mathrm{mod}\Lambda\}$ and
- $\{\underset{\mathcal{B}}{\text{set}}$ of Λ-modules satisfying $\underset{(1)\ (2)}{}\}$

where

(1) $\forall B_1, B_2 \in \mathcal{B}$,

$\forall f: B_1 \to B_2$ is either 0 or surj.

(2) $\forall B \in \mathcal{B}$.

if $\exists\, B \twoheadrightarrow C$: surj,

s.t. $C \notin \mathcal{B}$,

C : brick ($:\Leftrightarrow$ $\mathrm{End}_\Lambda(C)$ is a division ring)

then $\exists\, C \to B' \in \mathcal{B}$.

non-zero, not surjective. $\lrcorner$

Classifying wide

Thm (Ringel)

$\underset{\text{wide}}{\mathcal{W}} \longmapsto \operatorname{sim} \mathcal{W}$ gives

a bij between

- $\{$wide subcats$\}$ and
- semibrick $\to$ $\{\underset{\mathcal{S}}{\text{set}}$ of Λ-modules satisfying $\underset{(3)}{}\}$.

where (3): $\forall S_1, S_2 \in \mathcal{S}$,

$\forall f: S_1 \to S_2$ is either 0 or isom. $\lrcorner$

Problem (with enough proj)

Classify wide subcat using projectives!

c.f. If Λ is hereditary, then

wide (enough proj) $\longleftrightarrow$ $\{ M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(M,M)=0$ and Fac-minimal $\}$

Problem

Classify more classes of subcat!

c.f. subcats closed under Images-Cok-Ext (ICE-closed)

$\xrightarrow[\sim]{\text{proj}}$ $\{$ wide τ-tilt. $\}$ [E-Sakai]

$\xrightarrow{\text{sim}}$?

How about closed under Image-Ext?

Ext & summand.

ICE

$\cup$ $\cup$

tors wide

II Subcats arising from submonoid of $\mathbb{N}^n$

Def Λ : f.d. alg with $\mathrm{sim}(\mathrm{mod}\,\Lambda) = \{S_1, \cdots, S_n\}$

Define $\underline{\dim} : \mathrm{mod}\,\Lambda \to \mathbb{N}^n$ by

$X \longmapsto (m_1, \cdots, m_n)$

(m_i : number of S_i appearing in comp. factors of X)

Def $\mathcal{M} \le \mathbb{N}^n$: submonoid

$\rightsquigarrow \mathcal{E}_{\mathcal{M}} := \{ X \in \mathrm{mod}\,\Lambda \mid \underline{\dim}\, X \in \mathcal{M} \}$

$\subseteq \mathrm{mod}\,\Lambda$: ext-closed.

(but not closed under summands!)

Problem

Compute $M(\mathcal{E}_{\mathcal{M}})$!

Ex $\Lambda = k^n$, then

$M(\mathcal{E}_{\mathcal{M}}) \xrightarrow[\underline{\dim}]{\sim} \mathcal{M}$: isom.

☺ $\exists\ M(\mathcal{E}_m) \xrightarrow{\dim} m$: monoid hom. $\underline{\text{surj}}$

This is inj by Λ : semisimple.

$(\because \dim X = \dim Y \Rightarrow X \cong Y)$ $\square$

Ex $\Lambda = k(1 \leftarrow 2)$

$m := \mathbb{N}(1,1) \subseteq \mathbb{N}^2$

$(\ \mathcal{E}_m = \{ X \mid \dim X = (i,i)\ \exists i \}\ )$

Then

- $\operatorname{sim}(\mathcal{E}_m) = \{ \underset{\displaystyle (k \xleftarrow{0} k}{\overset{a_0}{\overset{\shortparallel}{S_1 \oplus S_2}}},\ \underset{\displaystyle k \xleftarrow{1} k)}{\overset{{2 \atop 1}}{\overset{\shortparallel}{\overset{a_1}{\overset{\shortparallel}{P(2)}}}}} \}$

- $M(\mathcal{E}_m)$ looks like

$$\begin{tikzcd}
 & a_0 \arrow[r, shift left, "+a_0"] \arrow[r, shift right, "+a_1"'] & 2a_0 \arrow[r, shift left, "+a_0"] \arrow[r, shift right] & 3a_0 \arrow[r, shift left] \arrow[r, shift right] & \cdots \\
0 \arrow[ur, "+a_0"] \arrow[dr, "+a_1"'] & & & & \\
 & a_1 \arrow[r, "+a_1"'] \arrow[uur, "+a_0"] & 2a_1 \arrow[r, "+a_1"'] \arrow[uur, "+a_0"] & 3a_1 \arrow[r, "+a_1"'] & \cdots
\end{tikzcd}$$

$$\downarrow \dim$$

$$0 \to (1,1) \to (2,2) \to \cdots$$

$a_0 + a_1 = 2a_0$ by

$$0 \to \begin{matrix} 1 \\ \oplus \\ 2 \end{matrix} \to \begin{matrix} 1 \\ \oplus \\ {2 \atop 1} \\ \oplus \\ 2 \end{matrix} \to \begin{matrix} 1 \\ \oplus \\ 2 \end{matrix} \to 0 \quad \text{in } \mathcal{E}_m$$

$$a_0 \qquad a_0 + a_1 \qquad a_0$$

Def $\mathcal{A}$: abelian cat

$\mathcal{E} \subseteq \mathcal{A}$: $\underline{\text{finitely resolving}}$

$:\Leftrightarrow$
- $\mathcal{E}$: ext.-closed
- $\forall\ 0 \to X \to Y \to Z \to 0$: ex in $\mathcal{A}$, $Y, Z \in \mathcal{E} \Rightarrow X \in \mathcal{E}$
- $\forall A \in \mathcal{A}$, $\exists$ ex seq $0 \to E_n \to \cdots \to E_0 \to A \to 0$ with $E_i \in \mathcal{E}$.

Thm (Quillen's resolution theorem)

In the above,

$K_0(\mathcal{E}) \xrightarrow{\sim} K_0(\mathcal{A})$: isom. ┘

Rem In the above,

$$\left(\begin{array}{l} D^b(\mathcal{E}) \simeq D^b(\mathcal{A}) \quad : \text{tri.} \\ K_0(\mathcal{E}) \quad K_0(\mathcal{E}) \simeq K_0(\mathcal{A}) \end{array} \right)$$

Cor

Consider

(FR) : $\mathcal{E}$ is equiv to fin. resolving subcat of $\operatorname{mod}\Gamma$ $\exists\ \Gamma$: f.d. alg.

If $\mathcal{E}$: (FR), then

$\mathcal{E}$: (JHP) $\iff$ $\#\operatorname{sim}\mathcal{E} = \#\{\text{indec proj in } \mathcal{E}\}$ $\lrcorner$

☺ $\mathcal{E}$: (FR).

$\rightsquigarrow$ $K_0(\mathcal{E}) \xrightarrow{\sim} K_0(\operatorname{mod}\Gamma) \cong \mathbb{Z}^n$

where $n = \#\{\text{indec proj } \Gamma\text{-mod}\}$
$\qquad\qquad = \#\{\text{———— obj in } \mathcal{E}\}$.

Since $\{[S] \mid S \in \operatorname{sim}\mathcal{E}\}$ generates $K_0(\mathcal{E})$

they are linearly indep

$\iff$ $\#\operatorname{sim}\mathcal{E} = n$. $\square$

Ex

- $\mathcal{F} \subseteq \operatorname{mod}\Lambda$: torf

$\Rightarrow$ $\mathcal{F}$: (FR)

by $\mathcal{F} \subseteq \operatorname{mod} \Lambda/\underline{\operatorname{ann}\mathcal{F}}$.

- $\mathcal{T} \subseteq \operatorname{mod}\Lambda$: tors. functorially finite

$\not\Rightarrow$ $\mathcal{T}$: (FR)

Torf over path alg

Q : Dynkin quiver. (ADE)

Φ : the corresponding root system

W : Weyl grp of Φ.

$\{\alpha_i \mid i \in Q\}$: set of simple roots. $\lrcorner$

Def For $w \in W$,

$\operatorname{inv}(w) = \{\beta \in \Phi^+ \mid w^{-1}(\beta) \in \Phi^-\}$ $\lrcorner$

Ex Q : type A_n

$w \in W = S_{n+1}$

$\Phi^+ = \{\beta_{[i,j)} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \mid 1 \le i < j \le n+1\}$

$\beta_{[i,j)} \in \operatorname{inv}(w)$

$\iff$ $w^{-1}(i) > w^{-1}(j)$ $\quad\left(\ w = \cdots j \cdots i \cdots\right.$

Thm (Gabriel)

$$\underline{\dim} : \mathrm{mod}\, kQ \longrightarrow \mathbb{Z}^n \xrightarrow{\sim} \mathbb{Z}\Phi, \quad e_i \longmapsto \alpha_i$$

induces a bij

$$\mathrm{ind}(\mathrm{mod}\, kQ) \simeq \Phi^+$$

Def For $w \in W$

$$\mathcal{F}(w) := \mathrm{add}\left\{ M \in \mathrm{mod}\, kQ \;\middle|\; \begin{array}{l} M : \text{indec} \\ \underline{\dim}\, M \in \mathrm{inv}(w) \end{array} \right\}$$

Thm [Ingalls–Thomas]

$w \longmapsto \mathcal{F}(w)$ gives

a bij

- $\{ c_Q\text{-sortable elements in } W \}$ and
- $\{ \text{torf in } \mathrm{mod}\, kQ \}$

Thm (E)

For $w \in W$: c_Q-sortable,

$\underline{\dim} : \mathrm{ind}\, \mathcal{F}(w) \xrightarrow{\sim} \mathrm{inv}(w)$ restricts to

$$\mathrm{sim}\, \mathcal{F}(w) \xleftrightarrow{\sim} \{ \text{Bruhat inversions of } w \}$$

where $\beta \in \mathrm{inv}(w)$: Bruhat

$:\Longleftrightarrow \beta \neq \gamma + \delta \quad (\gamma, \delta \in \mathrm{inv}(w))$

Cor

$\exists$ characterization of (JHP) in $\mathcal{F}(w)$ using root system.

Rem Similar results hold for preproj alg Π_Q

$(W \simeq \mathrm{torf}\, \Pi_Q)$

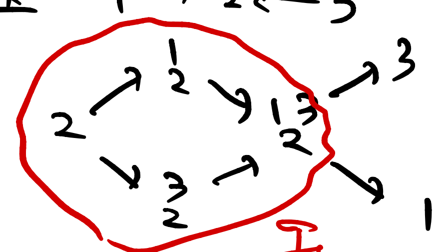
Problem

- Compute $M(\mathcal{F}(w))$!

($\mathcal{F}(w)$: JHP $\Longleftrightarrow$ $M(\mathcal{F}(w))$: free)

What if not JHP ?

Ex $1 \rightarrow 2 \leftarrow 3$



$\mathcal{F}$: torf.

$M(\mathcal{F}) = 2.$

$$M(\mathcal{F}) \overset{\underline{\dim}}{\twoheadrightarrow} \bigoplus \mathbb{Z}\, \mathrm{inv}(w) \subseteq \mathbb{Z}^n$$

injective ?

What if: Q: non-Dynkin (Λ: species)

- Geiss-Leclerc-Schröer's generalized preproj & path alg.

Extriangulated cat

$\mathcal{D}$: tri cat.

$\mathcal{E} \subseteq \mathcal{D}$: ext-closed subcat

($\forall\ X \to Y \to Z \to X[1]$: tri, $X, Z \in \mathcal{E} \Rightarrow Y \in \mathcal{E}$)

Then

$$\mathbb{E} := \{ X \to Y \to Z \mid \text{tri},\ X, Y, Z \in \mathcal{E} \}$$

∵ "conflation"

$\rightsquigarrow$ $(\mathcal{E}, \mathbb{E})$: extriangulated cat introduced by Nakaoka-Palu.

$\rightsquigarrow$ $M(\mathcal{E})$ can be defined similarly.

Ex $\mathcal{E} = \mathcal{D}$: tri cat

$\Rightarrow$ $M(\mathcal{E}) = K_0(\mathcal{E})$ ┘

☺ $\forall D \in \mathcal{D}$,

$D \to 0 \to D[1]$: confl.

$\rightsquigarrow$ $[D] + [D[1]] = 0$

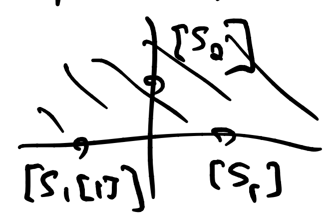
$\therefore$ $\forall$ elem in $M(\mathcal{D})$ is invertible

$\therefore$ $M(\mathcal{D})$: abelian grp $= K_0(\mathcal{D})$ ┘

Ex $\mathcal{D} = D^b(K(1 \leftarrow 2))$ $\quad$ $\mathcal{E}$: ext-closed.

$$\cdots \quad S_1 \to P_2 \to S_2 \to S_1[1] \to \quad \cdots$$

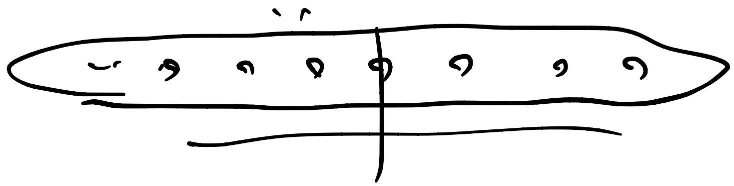
$\rightsquigarrow$ $M(\mathcal{E}) \overset{\dim}{\cong} \{ (m, n) \mid m \in \mathbb{Z},\ n \in \mathbb{N} \}$



$\Big($ $a \in M(\mathcal{E})$: atom

$:\Longleftrightarrow$ $a = b + c \Rightarrow b$ or c is invertible. $\Big)$

$\rightsquigarrow$ atom in $M(\mathcal{E})$



$\longleftrightarrow$? in $\mathcal{E}$

<u>Obs</u> If $\exists\, X \in \mathcal{E}$, $X \neq 0$, $X[1] \in \mathcal{E}$,

then $\operatorname{sim} \mathcal{E} = \emptyset$.

☹ $\forall\, 0 \neq M \in \mathcal{E}$

$$\left(\begin{array}{ccccc} X & \to & 0 & \to & X[1] \\ \oplus & & \oplus & & \\ M & = & M & \to & 0 \end{array}\right.$$

conflation. $\therefore$ M : not simple.

<u>Problem.</u>

Characterize

- invertible elem
- atom

in $M(\mathcal{E})$

categorically.