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In this lecture, we consider the following classical result in commutative algebra:

Proposition

- *Let A be a Cohen-Macaulay ring, and let $I \subseteq A$ be an ideal.*
- *Assume that I is generated by an A-regular sequence $a_1, \dots, a_n$.*
- *Then the ring A/I is also Cohen-Macaulay.*

We wish to generalize this result, remove the assumption on I, and prove:

Theorem *Let A be a Cohen-Macaulay ring, and let $I \subseteq A$ be an ideal. Then "A/I" is Cohen-Macaulay.*

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Applications

- A point on an algebraic variety or a scheme can be either nonsingular or singular.
- This geometric situations can be completely understood via the corresponding local rings of these schemes.
- Given a commutative noetherian local ring $(A, \mathfrak{m})$, we have the following coarse classification of singularities:

$$\text{Regular rings} \subsetneq \text{Gorenstein rings} \subsetneq \text{Cohen-Macaulay rings}$$

- Each of these classes has a homological characterization.
  1. A local ring A is regular if all modules over A have a finite injective resolution.
  2. A local ring A is Gorenstein if the A-module A has a finite injective resolution.

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Applications

- The broadest class of well-behaved singular rings is the class of Cohen-Macaulay rings.
- Given a local ring $(A, \mathfrak{m})$, a finitely generated A-module M, and $x \in \mathfrak{m}$, x is called M-regular if the map $x \cdot - : M \to M$ is injective.
- A sequence $x_1, \dots, x_n \in \mathfrak{m}$ is called M-regular if x_1 is M-regular, and $x_2, \dots, x_n$ is M/x_1M-regular.
- A basic example is the sequence $x_1, \dots, x_n$ in $\mathbb{K}[[x_1, \dots, x_n]]$.
- All regular sequences which cannot be extended have the same length.

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Applications

- The length of a maximal regular sequence is called the depth of M.
- It can be calculated using the formula:

$$\mathrm{depth}(M) = \min\{n \mid \mathrm{Ext}^n_A(\Bbbk, M) \neq 0\}$$

where $\Bbbk = A/\mathfrak{m}$ is the residue field of $(A, \mathfrak{m})$.

- More generally, if $I \subseteq A$ is an ideal, the I-depth of M is the longest A-regular sequence inside I.
- There is an equality

$$\mathrm{depth}(I, M) = \min\{n \mid \mathrm{Ext}^n_A(A/I, M) \neq 0\}$$

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Applications

- The Krull dimension of an A-module M is the Krull dimension of the topological space

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}.$$

- It always holds that $\mathrm{depth}(M) \leq \dim(M)$.

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Applications

- The module M is called a Cohen-Macaulay module if $\operatorname{depth}(M) = \dim(M)$.
- A local ring $(A, \mathfrak{m})$ is called a Cohen-Macaulay ring if $\operatorname{depth}(A) = \dim(A)$.
- In that case, for all $\mathfrak{p} \in \operatorname{Spec}(A)$, the localization $A_{\mathfrak{p}}$ is also Cohen-Macaulay.
- If A is a commutative noetherian ring, it is Cohen-Macaulay if for all $\mathfrak{p} \in \operatorname{Spec}(A)$, the local ring $A_{\mathfrak{p}}$ is Cohen-Macaulay.

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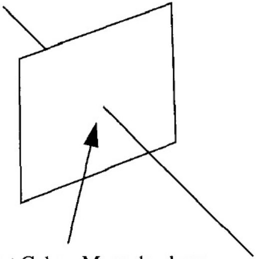
Applications

- Most schemes that arise in practice are Cohen-Macaulay almost everywhere.
- The easiest way to give non-Cohen-Macaulay examples is to mix dimensions. Both depth and Krull dimension count dimension, in two different ways.



Not Cohen-Macaulay here.

FIGURE 18.1.

Figure: Taken from "Commutative Algebra. with a View Toward Algebraic Geometry" (David Eisenbud)

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- The $\mathfrak{m}$-torsion functor is the left exact functor

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^n \cdot x = 0, n \gg 0\} = \varinjlim_{n} \operatorname{Hom}_A(A/\mathfrak{m}^n, M)$$

- Its right derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}$ is called the local cohomology functor.
- Grothendieck proved that for a finitely generated module M:

$$\operatorname{depth}(M) = \min\{n \mid \mathrm{H}^n_{\mathfrak{m}}(M) \neq 0\}$$

$$\dim(M) = \max\{n \mid \mathrm{H}^n_{\mathfrak{m}}(M) \neq 0\}.$$

- It follows that A is a Cohen-Macaulay ring if and only if the complex $\mathrm{R}\Gamma_{\mathfrak{m}}(A)$ is a module concentrated at degree $\dim(A)$.

- A complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is called a dualizing complex if it has a finite injective resolution, and for any finitely generated module M, the canonical map

$$M \to \mathrm{R}\operatorname{Hom}_A(\mathrm{R}\operatorname{Hom}_A(M,R),R)$$

is an isomorphism in the derived category $\mathsf{D}(A)$.

- Grothendieck's local duality theorem implies that $\operatorname{amp}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)) = \operatorname{amp}(R)$.
- It follows that A is a Cohen-Macaulay ring if and only if R is a finitely generated module.
- More generally, a local ring A is Cohen-Macaulay if and only if there exist a non-zero finitely generated module M which has finite injective dimension.

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Applications

The Hironaka's Miracle flatness theorem is one of the following related results:

Theorem *Let R be a regular local ring. Assume $R \subseteq S$, and that S is finitely generated over R. Then S is Cohen-Macaulay if and only if S is free over R.*

Theorem *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism between noetherian local rings. Assume that R is a regular local ring and that S is a Cohen-Macaulay ring. Then φ is flat if and only if* $\dim(S/\mathfrak{m}S) = \dim(R) - \dim(S)$.

Here $S/\mathfrak{m}S = S \otimes_R R/\mathfrak{m}$ is the fiber ring of the homomorphism φ.

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Applications

- Given a commutative ring A and an element $a \in A$, we consider the quotient ring $A/(a)$.
- When a is a regular element, this quotient is well behaved.
- When a is not regular, the quotient loses the data of $\ker(A \xrightarrow{\cdot a} A)$.
- There is an obvious homological way to keep this data: replace $A/(a)$ by the Koszul complex

$$K(A;a) = 0 \to A \xrightarrow{\cdot a} A \to 0$$

Its cohomologies: $\mathrm{H}^{-1}(K(A;a)) = \ker(A \xrightarrow{\cdot a} A)$ and $\mathrm{H}^0(K(A;a)) = A/(a)$.

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Applications

- More generally, if $a_1, \dots, a_n \in A$ is a finite sequence, we define:

$$K(A; a_1, \dots, a_n) = K(A; a_1) \otimes_A K(A; a_2) \otimes_A \cdots \otimes_A K(A; a_n)$$

- This is called the Koszul complex with respect to $a_1, \dots, a_n$ of A.
- If $f : A \to B$ is a ring homomorphism, and if $b_i = f(a_i)$, there is a natural isomorphism $K(A; a_1, \dots, a_n) \otimes_A B \cong K(B; b_1, \dots, b_n)$.

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- Given a commutative ring A and a sequence of elements $a_1, \dots, a_n \in A$, we can realize $A/(a_1, \dots, a_n)$ as follows: make A a $\mathbb{Z}[x_1, \dots, x_n]$-algebra by $x_i \mapsto a_i$, and then

$$A/(a_1, \dots, a_n) \cong A \otimes_{\mathbb{Z}[x_1,\dots,x_n]} \mathbb{Z}.$$

- It follows that we may derive the quotient operation as follows:

$$A \otimes^{\mathrm{L}}_{\mathbb{Z}[x_1,\dots,x_n]} \mathbb{Z} \cong$$
$$A \otimes_{\mathbb{Z}[x_1,\dots,x_n]} K(\mathbb{Z}[x_1, \dots, x_n]; x_1, \dots, x_n) \cong K(A; a_1, \dots, a_n).$$

- This shows that we can consider $K(A; a_1, \dots, a_n)$ as the derived quotient of A with respect to $a_1, \dots, a_n$. We have that $\mathrm{H}^0(K(A; a_1, \dots, a_n)) = A/(a_1, \dots, a_n)$, as one would expect from a derived functor.

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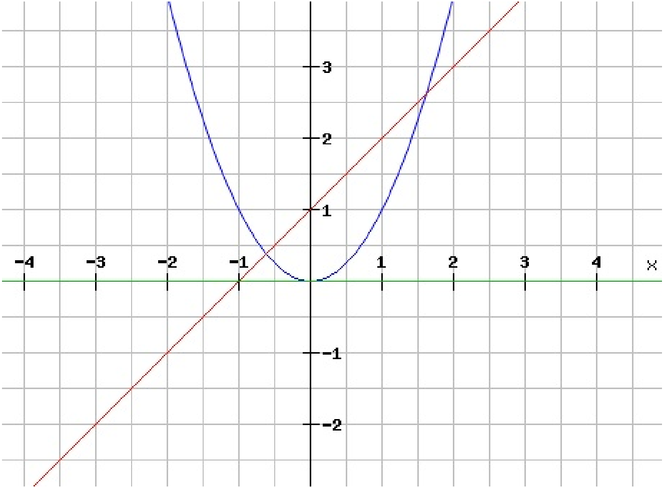
Applications

In the 1950's, Serre initiated the use of commutative homological algebra to study the intersection of algebraic varieties.



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Serre proved that if $V = \operatorname{Spec}(A)$ is a smooth affine variety and $W = \operatorname{Spec}(A/I), U = \operatorname{Spec}(A/J)$ are two closed subvarities that intersect at a point $\mathfrak{p} \in V$, the intersection number is given by:

$$\chi(U, W, p) = \sum_n (-1)^n \operatorname{length}_{A_\mathfrak{p}} \left(\operatorname{Tor}^n_{A_\mathfrak{p}}((A/I)_\mathfrak{p}, (A/J)_\mathfrak{p}) \right).$$

This number is exactly the Euler characteristic of the cochain complex

$$(A/I \otimes^{\mathrm{L}}_A A/J)_\mathfrak{p}$$

Derived algebraic geometry starts with two observations:

1. The intersection of two varieties should be a geometric object, and not just a number.
2. The A-modules A/I and A/J are rings, so they should be resolved as A-algebras and not as A-modules.

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One way to resolve rings as A-algebras leads us to the notion of a commutative DG-algebra. These kind of objects that arise in derived algebraic geometry are of the following form:

$$A = \bigoplus_{n=-\infty}^{0} A^n$$

is a graded commutative ring which is also a complex with a differential of degree $+1$.

$$a \cdot b = (-1)^{|a|\cdot|b|} \cdot b \cdot a, \quad a \cdot a = 0, \text{ if } |a| \text{ is odd.}$$

A Leibniz rule should be satisfied:

$$d(a \cdot b) = d(a) \cdot b + (-1)^{i} \cdot a \cdot d(b).$$

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Applications

- It follows that $\mathrm{H}^0(A)$ is a commutative ring, and for each $i < 0$, $\mathrm{H}^i(A)$ is an $\mathrm{H}^0(A)$-module.
- One should think of such an A as $\mathrm{Spec}(\mathrm{H}^0(A))$, plus some higher order "homotopical" niloptents.
- A is called noetherian if $\mathrm{H}^0(A)$ is a noetherian ring, and each $\mathrm{H}^i(A)$ is a finitely generated $\mathrm{H}^0(A)$-module.
- A is called a local DG-ring if it is noetherian and $\mathrm{H}^0(A)$ is a noetherian local ring.
- We shall now assume all DG-rings are commutative and noetherian, and moreover have bounded cohomology, that is $\mathrm{H}^i(A) = 0$ for $i \ll 0$.

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Applications

- A DG-module over A is a graded A-module $M = \bigoplus_{n=-\infty}^{\infty} M^n$ with a differential which satisfies a graded Leibnitz rule.
- The derived category of DG-modules over A, denoted by $\mathsf{D}(A)$ is a triangulated category.
- For each $M \in \mathsf{D}(A)$ and each $n \in \mathbb{Z}$, the cohomology $\mathrm{H}^n(M)$ is an $\mathrm{H}^0(A)$-module.
- Following Yekutieli, given $\bar{\mathfrak{p}} \in \operatorname{Spec}(\mathrm{H}^0(A))$, letting $\mathfrak{p}$ be its preimage in A^0, the localization of A at $\bar{\mathfrak{p}}$ is defined as: $A_{\bar{\mathfrak{p}}} := A \otimes_{A^0} A^0_{\mathfrak{p}}$.
- Similarly, for $M \in \mathsf{D}(A)$, we have a localization functor $\mathsf{D}(A) \to \mathsf{D}(A_{\bar{\mathfrak{p}}})$ which is a triangulated functor.

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- If A is a local DG-ring, with $\bar{\mathfrak{m}}$ being the maximal ideal of $\mathrm{H}^0(A)$, we denote by $\mathsf{D}(A)_{\bar{\mathfrak{m}}-\mathrm{tor}}$ the full subcategory of $D(A)$ consisting of M such that $\mathrm{H}^n(M)$ is $\bar{\mathfrak{m}}$-torsion for all $n \in \mathbb{Z}$.
- The inclusion functor $\mathsf{D}(A)_{\bar{\mathfrak{m}}-\mathrm{tor}} \hookrightarrow \mathsf{D}(A)$ has a right adjoint.
- The composition of this right adjoint with the inclusion

$$\mathsf{D}(A) \to \mathsf{D}(A)_{\bar{\mathfrak{m}}-\mathrm{tor}} \to \mathsf{D}(A)$$

is called the local cohomology functor of A, denoted by $\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}$.
- If A is an ordinary noetherian local ring, this coincides with the usual local cohomology functor.

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Applications

Theorem *Let $(A, \bar{\mathfrak{m}})$ be a commutative noetherian local DG-ring with bounded cohomology. Then*

$$\operatorname{amp}(\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(A)) \geq \operatorname{amp}(A).$$

The proof is based on proving versions of Grothendieck's vanishing and non-vanishing theorems in the DG setting.

Definition Let $(A, \bar{\mathfrak{m}})$ be a commutative noetherian local DG-ring with bounded cohomology. Then A is called local-Cohen-Macaulay if $\operatorname{amp}(\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(A)) = \operatorname{amp}(A)$.

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A DG-module $R \in \mathsf{D}(A)$ is called a dualizing DG-module if its cohomology is finitely generated over $\mathrm{H}^0(A)$, and for any DG-module M with finitely generated cohomology, the natural map

$$M \to \mathrm{R}\operatorname{Hom}_A(\mathrm{R}\operatorname{Hom}_A(M,R),R)$$

is an isomorphism in $\mathsf{D}(A)$.
This definition is due to Yekutieli, generalizing previous definitions of Hinich and Frankild-Iyengar-Jørgensen.
It follows from a DG version of Grothendieck's local duality:

Theorem *Let A be a commutative noetherian DG-ring with bounded cohomology, and let R be a dualizing DG-module over A. Then* $\operatorname{amp}(R) \geq \operatorname{amp}(A)$. *If A is local then* $\operatorname{amp}(R) = \operatorname{amp}(A)$ *if and only if A local-Cohen-Macaulay.*

A is called a Gorenstein DG-ring if A is a dualizing DG-module over itself. It follows that Gorenstein DG-rings are local-Cohen-Macaulay.

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- There is a theory of regular sequences in the DG setting. It was initiated by Minamoto, following earlier work of Foxby and Christensen.
- If A is a commutative noetherian local DG-ring with bounded cohomology, an element $\bar{x} \in \bar{\mathfrak{m}}$ is called A-regular if it is $\mathrm{H}^{\inf(A)}(A)$-regular.
- More generally, if $M \in \mathsf{D}^{+}(A)$, an element $\bar{x} \in \bar{\mathfrak{m}}$ is called M-regular if it is $\mathrm{H}^{\inf(M)}(M)$-regular.

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- Given $\bar{x} \in \mathrm{H}^0(A)$ it is possible to construct the quotient DG-ring $A//\bar{x}$. The construction is based on a Koszul-complex type construction:
- We take some $x \in A^0$ whose image in $\mathrm{H}^0(A)$ is equal to $\bar{x}$, and make A into a DG-algebra over $\mathbb{Z}[X]$ by $X \mapsto x$.
- Then we set:

$$A//\bar{x} = K(\mathbb{Z}[X];x) \otimes_{\mathbb{Z}[X]} A \cong \mathbb{Z} \otimes^{\mathrm{L}}_{\mathbb{Z}[X]} A.$$

- Up to isomorphism in the homotopy category of DG-rings, the result is independent of the chosen lift of $\bar{x}$.
- It holds that $\mathrm{H}^0(A//\bar{x}) = \mathrm{H}^0(A)/\bar{x}$.

- Another notation for $A//\bar{x}$ is $K(A;\bar{x})$, as it is simply a Koszul complex over A.
- If A is a commutative DG-ring and $\bar{x}_1,\dots,\bar{x}_n$ is a finite sequence of elements, we may simply construct the commutative DG-ring $K(A;\bar{x}_1,\dots,\bar{x}_n)$.
- One may define it inductively as

$$K(A;\bar{x}_1,\dots,\bar{x}_n) = K(K(A;\bar{x}_1);\bar{x}_2\dots,\bar{x}_n).$$

- Alternatively, lifting $\bar{x}_1,\dots,\bar{x}_n \in \mathrm{H}^0(A)$ to elements $x_1,\dots,x_n \in A^0$, and mapping $X_i \mapsto x_i$, we may let

$$K(A;\bar{x}_1,\dots,\bar{x}_n) = A \otimes^{\mathrm{L}}_{\mathbb{Z}[X_1,\dots,X_n]} \mathbb{Z}.$$

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A sequence $\bar{x}_1, \dots, \bar{x}_n \in \bar{\mathfrak{m}}$ is called A-regular if $\bar{x}_1$ is A-regular and $\bar{x}_2, \dots, \bar{x}_n$ is $A//\bar{x}_1$-regular.

Theorem *(Minamoto) Given a noetherian local DG-ring $(A, \bar{\mathfrak{m}})$ with bounded cohomology, all maximal A-regular sequences have the same length, and this length is given by*

$$\inf\left(\mathrm{R}\operatorname{Hom}_A(\mathrm{H}^0(A)/\bar{\mathfrak{m}}, A)\right) - \inf(A) = \operatorname{depth}(A) - \inf(A).$$

We denote this number by $\operatorname{seq-depth}(A)$ and refer to it as the sequential depth of A.

Theorem *(S.) It holds that* $\operatorname{seq-depth}(A) \leq \dim(\mathrm{H}^0(A))$*, with equality if and only if A is local-Cohen-Macaulay. In particular, if* $\dim(\mathrm{H}^0(A)) = 0$ *then A is local-Cohen-Macaulay.*

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It turns out that even if A is local-Cohen-Macaulay, it is possible that $A_{\bar{\mathfrak{p}}}$ is not local-Cohen-Macaulay for some $\bar{\mathfrak{p}} \in \operatorname{Spec}(\mathrm{H}^0(A))$.

Example Let A be the localization of $\Bbbk[x,y,z]/(y^2z,xyz)$ at the origin. Let $M = A/zA$. Then M is a Cohen-Macaulay module over the local ring A. It is possible to show that the trivial extension $B = A \ltimes M[3]$ is a noetherian local DG-ring which is local-Cohen-Macaulay. However, for $\mathfrak{p} = (x,y)$, the localization $B_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ is a ring which is not Cohen-Macaulay.

Definition A commutative noetherian DG-ring A with bounded cohomology is called a Cohen-Macaulay DG-ring if for all $\bar{\mathfrak{p}} \in \operatorname{Spec}(\mathrm{H}^0(A))$, the local DG-ring $A_{\bar{\mathfrak{p}}}$ is local-Cohen-Macaulay.

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- The failure of local-Cohen-Macaulay DG-rings to be Cohen-Macaulay only happens when $\mathrm{amp}(A_{\bar{\mathfrak{p}}}) < \mathrm{amp}(A)$.
- It follows that if $\mathrm{Supp}(\mathrm{H}^{\inf(A)}(A)) = \mathrm{Spec}(\mathrm{H}^0(A))$ and A is local-Cohen-Macaulay then A is Cohen-Macaulay.
- If $\mathrm{Supp}(\mathrm{H}^{\inf(A)}(A)) = \mathrm{Spec}(\mathrm{H}^0(A))$ we say that A has constant amplitude.
- If A is local-Cohen-Macaulay and $\mathrm{Spec}(\mathrm{H}^0(A))$ is irreducible, then A is Cohen-Macaulay.
- In particular, if A is local-Cohen-Macaulay and $\mathrm{H}^0(A)$ is an integral domain, then A is Cohen-Macaulay.

We now seek to return the following basic result:

Proposition *Let A be a Cohen-Macaulay ring, and let $a_1, \dots, a_n$ be an A-regular sequence. Then the quotient ring $A/(a_1, \dots, a_n)$ is also Cohen-Macaulay.*

We will explain that this result holds without any assumption on the sequence if one replaces quotient by derived quotient.

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Theorem *Let A be a Cohen-Macaulay ring, and let $a_1, \dots, a_n \in A$ be any sequence of elements in A. Then the Koszul complex $K(A; a_1, \dots, a_n)$ is a Cohen-Macaulay DG-ring.*

More generally:

Theorem *Let A be a Cohen-Macaulay DG-ring with constant amplitude, and let $\bar{a}_1, \dots, \bar{a}_n \in \mathrm{H}^0(A)$ be any sequence of elements in $\mathrm{H}^0(A)$. Then the Koszul complex $K(A; \bar{a}_1, \dots, \bar{a}_n)$ is a Cohen-Macaulay DG-ring.*

This result is false if A does not have constant amplitude.

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Proof.

- Let A be a Cohen-Macaulay ring, and let $a_1, \dots, a_n \in A$.
- Since Koszul complexes commute with localization and completion, and the Cohen-Macaulay property is preserved by completion, we may assume without loss of generality that $(A, \mathfrak{m})$ is a local ring which is $\mathfrak{m}$-adically complete.
- This implies that there is a finitely generated A-module R which is a dualizing complex over A.
- Let $K = K(A; a_1, \dots, a_n)$, and let $D = \mathrm{R}\operatorname{Hom}_A(K, R)$. Then D is a dualizing DG-module over K, so it is enough to show that $\operatorname{amp}(D) = \operatorname{amp}(K)$.

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Proof (Cont.)

- Let $I = (a_1, \dots, a_n)$. By the depth-sensitivity property of the Koszul complex, it is known that

$$\operatorname{amp}(K) = n - \operatorname{depth}(I, A)$$

- The fact that A is a Cohen-Macaulay ring implies that we can calculate I-depth via the formula:

$$\operatorname{depth}(I, A) = \dim(A) - \dim(A/I).$$

- We deduce that

$$\operatorname{amp}\left(K(A; a_1, \dots, a_n)\right) = n - \dim(A) + \dim(A/I).$$

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Proof (Cont.)

- To compute the amplitude of $D = \mathrm{R}\operatorname{Hom}_A(K,R)$, since K is a perfect complex over A, we know that

$$\mathrm{R}\operatorname{Hom}_A(K,R) \cong \mathrm{R}\operatorname{Hom}_A(K,A) \otimes^{\mathrm{L}}_A R.$$

- The self-duality property of the Koszul complex says that $\mathrm{R}\operatorname{Hom}_A(K,A) \cong K[-n]$.
- It follows that $\operatorname{amp}(D) = \operatorname{amp}(K \otimes^{\mathrm{L}}_A R)$.
- Let us shift R so that it sits in cohomological degree $-\dim(A)$. A dualizing complex with infimum being $-\dim(A)$ is called a normalized dualizing complex.
- We then have that $\sup(K \otimes^{\mathrm{L}}_A R) = \sup(R) = -\dim(A)$.

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Proof (Cont.)

- By the depth-sensitivity property of the Koszul complex,

$$\inf(K \otimes^{\mathrm{L}}_A R) = \operatorname{depth}(I,R) - n.$$

- By basic properties of depth:

$$\operatorname{depth}(I,R) =$$
$$\inf\left\{i \mid \operatorname{Ext}^i_A(A/I,R) \neq 0\big)\right\} =$$
$$\inf\left(\mathrm{R}\operatorname{Hom}_A(A/I,R)\right)$$

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Proof (Cont.)

- By properties of dualizing complexes $\mathrm{R}\operatorname{Hom}_A(A/I, R)$ is a normalized dualizing complex over A/I, so its infimum is $-\dim(A/I)$.
- We see that

$$\operatorname{amp}(D) = -\dim(A) - (-\dim(A/I) - n) =$$
$$n - \dim(A) + \dim(A/I) = \operatorname{amp}(K).$$

- This shows that K is local-Cohen-Macaulay.
- Because everything here commutes with localization, this also shows that K is Cohen-Macaulay.

□

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Corollary *Let $f : X \to Y$ be a morphism of noetherian schemes, such that X is Cohen-Macaulay and Y is regular. Then the homotopy fiber of f at every point is a Cohen-Macaulay DG-ring.*

Proof.

This is a local statement. It is equivalent to: given a local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ between noetherian local rings, such that R is regular and S is Cohen-Macaulay, it holds that $R/\mathfrak{m} \otimes^{\mathrm{L}}_R S$ is a Cohen-Macaulay DG-ring. Since R is regular, the ideal $\mathfrak{m}$ is generated by a regular sequence $\mathfrak{m} = (a_1, \dots, a_n)$. This means that $K(R; a_1, \dots, a_n) \cong R/\mathfrak{m}$. Hence:

$$R/\mathfrak{m} \otimes^{\mathrm{L}}_R S \cong K(R; a_1, \dots, a_n) \otimes^{\mathrm{L}}_R S \cong K(S; \varphi(a_1), \dots, \varphi(a_n)).$$

Since S is Cohen-Macaulay, it follows from the theorem that $K(S; \varphi(a_1), \dots, \varphi(a_n))$ is also Cohen-Macaulay. □

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Corollary *Let* $(R, \mathfrak{m})$ *be a regular local ring, let* $(S, \bar{\mathfrak{n}})$ *be a Cohen-Macaulay local DG-ring with constant amplitude and* S^0 *noetherian. Let* $\varphi : R \to S$ *be a local homomorphism. Then* $\operatorname{flat}\dim_R(S)$ *is equal to the number*

$$\dim(R) - \dim(\mathrm{H}^0(S)) + \dim(\mathrm{H}^0(S)/\mathfrak{m}\mathrm{H}^0(S)) + \operatorname{amp}(S).$$

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Proof.

It holds that

$$\operatorname{flat\,dim}_R(S) = \operatorname{amp}(R/\mathfrak{m} \otimes^{\mathrm{L}}_R S)$$

We have seen above that if $(a_1, \dots, a_n)$ is a regular sequence that generates $\mathfrak{m}$ then

$$R/\mathfrak{m} \otimes^{\mathrm{L}}_R S = K(S; \bar{\varphi}(a_1), \dots, \bar{\varphi}(a_n))$$

where $\bar{\varphi} = \mathrm{H}^0(\varphi) : R \to \mathrm{H}^0(S)$.
The fact that S is a Cohen-Macaulay local DG-ring with constant amplitude implies that

$$\operatorname{amp}(K(S; \bar{\varphi}(a_1), \dots, \bar{\varphi}(a_n)))$$
$$= n - \dim(\mathrm{H}^0(S)) + \dim(\mathrm{H}^0(S)/\mathfrak{m}\mathrm{H}^0(S)) + \operatorname{amp}(S).$$

Finally, as R is a regular local ring, $n = \dim(R)$. □

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Corollary *Let $\mathbb{k}$ be a regular local ring, let A be a commutative noetherian local DG-ring with bounded cohomology and constant amplitude, and let $\mathbb{k} \to A$ be a local homomorphism such that the induced map $\mathbb{k} \to \mathrm{H}^0(A)$ is finite and injective. Then A is a Cohen-Macaulay DG-ring if and only if*

$$\operatorname{flat dim}_{\mathbb{k}}(A) = \operatorname{amp}(A).$$

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Proof.

If $\operatorname{flat\,dim}_{\Bbbk}(A) = \operatorname{amp}(A)$, then $R = \mathrm{R}\operatorname{Hom}_{\Bbbk}(A, \Bbbk)$ is a dualizing DG-module over A which satisfies that $\operatorname{amp}(R) = \operatorname{amp}(A)$, so A is Cohen-Macaulay.
Conversely, if A is Cohen-Macaulay, we know by the above corollary $\operatorname{flat\,dim}_{\Bbbk}(A)$ is equal to

$$\dim(\Bbbk) - \dim(\mathrm{H}^0(A)) + \dim(\mathrm{H}^0(A)/\mathfrak{m}\mathrm{H}^0(A)) + \operatorname{amp}(A).$$

As the map $\Bbbk \to \mathrm{H}^0(A)$ is finite and injective, it follows that $\dim(\Bbbk) = \dim(\mathrm{H}^0(A))$ and $\dim(\mathrm{H}^0(A)/\mathfrak{m}\mathrm{H}^0(A)) = 0$, which shows that $\operatorname{flat\,dim}_{\Bbbk}(A) = \operatorname{amp}(A)$. □

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Thank you for your attention.

ご清聴ありがとうございました。