# 周期三角圏上の傾理論

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## §0. Intro

In rep. thy. of f.d. alg, tilting theory gives rise to many tri. eq
However, it doesn't work in periodic triangulated categories.

Today's aim : introduce "periodic" tilting theory.

| Tilting theory | usual | periodic |
|---|---|---|
| Tri. cat. $(\mathcal{T}, \Sigma)$ | $\Sigma^m \not\simeq \mathrm{Id}_{\mathcal{T}}$ for $0 \neq {}^{\forall} m \in \mathbb{Z}$ | $\Sigma^m \simeq \mathrm{Id}_{\mathcal{T}}$ for $0 \neq {}^{\exists} m \in \mathbb{Z}$ (periodic tri. cat.) |
| Tilting obj. $T \in \mathcal{T}$ | $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ for ${}^{\forall} i \neq 0$ | $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ for ${}^{\forall} i \notin m\mathbb{Z}$ |
| Tilting theorem | $\mathcal{T} \xrightarrow[\text{tri.}]{\simeq} D^b(\mathrm{mod}\,\Lambda)$ for ${}^{\exists}\Lambda$: alg. | $\mathcal{T} \xrightarrow[\text{tri.}]{\simeq} D_m(\mathrm{mod}\,\Lambda)$ for ${}^{\exists}\Lambda$: alg. periodic derived cat. |

**Setting**

- $k$ : a perfect field
- All categories and functors are $k$-linear.
- All subcategories are full subcat. and closed under isom.
- For a f.d. $k$-alg $\Lambda$,
  - $\mathrm{Mod}\,\Lambda$ : the cat. of right $\Lambda$-modules.
  - $\mathrm{mod}\,\Lambda$ : 〃 f.g. 〃 .
  - $\mathrm{proj}\,\Lambda$ : 〃 f.g. proj. 〃 .

## §1. Backgrounds

① Tilting theory = A way to relate a tri. cat. with the derived cat. of a f.d. k-alg.

Let $\mathcal{T}$ : a tri. cat. with suspension functor $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$.
For $\mathcal{I} \subset \mathcal{T}$ : a class of objects,
- $\mathrm{thick}_{\mathcal{T}}(\mathcal{I}) \subset \mathcal{T}$ : the smallest thick subcat. containing $\mathcal{I}$
  ↳ closed under { · cone, · shift, · summand

__Def__
$T \in \mathcal{T}$ : a tilting object
:$\Leftrightarrow$ { (i) $T \in \mathcal{T}$ : a thick generator i.e. $\mathrm{thick}_{\mathcal{T}}(T) = \mathcal{T}$
(ii) $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ for $i \neq 0$.

__Thm__ (Keller 94')
Let $\mathcal{T}$ : an (idem. comp. algebraic) tri. cat.
- $\mathcal{T}$ has a tilt. obj. $T$

$\Rightarrow$ $\mathcal{T} \xrightarrow[\text{tri.}]{\sim} K^b(\mathrm{proj}\,\Lambda)$, where
- $\Lambda := \mathrm{End}_{\mathcal{T}}(T)$
- $K^b(\mathrm{proj}\,\Lambda)$ : the perfect derived cat.

__Rmk__
(1) $\Lambda$ : f.d. k-alg. $\Rightarrow \Lambda \in K^b(\mathrm{proj}\,\Lambda)$ : a tilt. obj.
(In general, ring)
(2) $\mathrm{gl.dim}\,\Lambda < \infty \Rightarrow K^b(\mathrm{proj}\,\Lambda) \xrightarrow[\text{tri.}]{\sim} D^b(\mathrm{mod}\,\Lambda)$ : the bounded derived cat.

## ② Periodic tri. cat.

Let $m \in \mathbb{Z}_{\geq 1}$.

**Def**

A tri. cat. $(\mathcal{T}, \Sigma)$ is $m$-periodic
$\overset{!}{\iff}$ $\Sigma^m \simeq \mathrm{Id}_{\mathcal{T}}$ as additive functors.

**Ex**

(1) $\Lambda$: a self-inj. f.d. $k$-alg. of fin. rep. type.
$\Longrightarrow$ $\underline{\mathrm{mod}}\Lambda$ is a periodic tri. cat. (Dugas 10')
( For $\mathcal{F}$: a Frob. ~~ex~~ exact cat,
$\underline{\mathcal{F}}$: the stable cat. of $\mathcal{F}$ )

(2) $R$: a hypersurface singularity $(= k[[x_0, \cdots, x_d]]/(f))$
$\Longrightarrow$ $\underline{\mathrm{CM}}(R)$ is a 2-periodic tri. cat, (Eisenbud 80')
where $\mathrm{CM}(R)$: the cat. of maximal Cohen-Macaulay $R$-modules

**⚠**

An $m$-periodic tri. cat. $\mathcal{T}$ has no tilt. obj.
Indeed, for $0 \neq T \in \mathcal{T}$,
$$0 \neq \mathrm{Hom}_{\mathcal{T}}(T, T) = \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^m T).$$
In particular, $\mathcal{T}$ is not equivalent to $K^b(\mathrm{proj}\,\Lambda)$
$(\simeq D^b(\mathrm{mod}\,\Lambda))$.

## §2. Periodic derived categories

Let
- $m \in \mathbb{Z}_{\geq 1}$
- $\mathbb{Z}_m$: the cyclic grp. of order $m$.
- $\mathcal{C}$ : an additive cat.

**Def**

$C_m(\mathcal{C})$ : the cat. of $m$-periodic complexes.

$:=$
- obj: $M = (M^i, d_M^i)_{i \in \mathbb{Z}_m}$, where $d_M^i : M^i \to M^{i+1}$ in $\mathcal{C}$ s.t. $d_M^{i+1} d_M^i = 0$ for $\forall i \in \mathbb{Z}_m$.

  e.g.)
  - 2-periodic complex

  $$\begin{tikzcd} M^0 \arrow[r, shift left, "d_M^0"] & M^1 \arrow[l, shift left, "d_M^1"] \end{tikzcd}$$

  - 3-periodic complex ....

  $$\begin{tikzcd} & M^2 \arrow[dl, "d_M^2"'] & \\ M^0 \arrow[rr, "d_M^0"'] & & M^1 \arrow[ul, "d_M^1"'] \end{tikzcd}$$

- morph: $f : M \to N$ in $C_m(\mathcal{C})$

  $\parallel$

  $\{ f^i : M^i \to N^i \}_{i \in \mathbb{Z}_m}$

  s.t.
  $$\begin{tikzcd} M^i \arrow[r, "d_M^i"] \arrow[d, "f^i"'] & M^{i+1} \arrow[d, "f^{i+1}"] \\ N^i \arrow[r, "d_N^i"] & N^{i+1} \end{tikzcd}$$

**Prop**

(1) $C_m(\mathcal{E})$ : a Frobenius exact cat.
(2) $K_m(\mathcal{E}) := \underline{C_m(\mathcal{E})}$ : a tri. cat.

Let $\mathcal{A}$ : an abelian cat.
Then we can define $\begin{cases} \cdot \text{ cohomology} \\ \cdot \text{ quasi-isomorphism (qis)} \end{cases}$ for m-periodic complexes

**Def**

$D_m(\mathcal{A}) := K_m(\mathcal{A})[\mathrm{qis}^{-1}]$ : the localization of $K_m(\mathcal{A})$ w.r.t. qis is called the <u>m-periodic derived cat.</u> of $\mathcal{A}$

**⚠**

$m$ : even $\Longrightarrow$ $D_m(\mathcal{A})$ is an m-periodic tri. cat.
$m$ : odd $\Longrightarrow$ $D_m(\mathcal{A})$ is not necessarily an m-periodic tri. cat.

$$\left( \underline{\text{e.g.}} \quad M \circlearrowleft d \xmapsto{\Sigma} M \circlearrowleft -d \right)$$

**Rmk**

Let $\Lambda$ : a f.d. k-alg. of fin. gl. dim.
Then $D_m(\mathrm{mod}\Lambda)$ : a triangulated hull of $D^b(\mathrm{mod}\Lambda)/\Sigma^m$
(= the smallest tri. cat. containing $D^b(\mathrm{mod}\Lambda)/\Sigma^m$) (Zhao 14')
In particular,
$\Lambda = kQ$ : a path alg. $\Longrightarrow$ $D_m(\mathrm{mod}\,kQ) \simeq D^b(\mathrm{mod}\,kQ)/\Sigma^m$

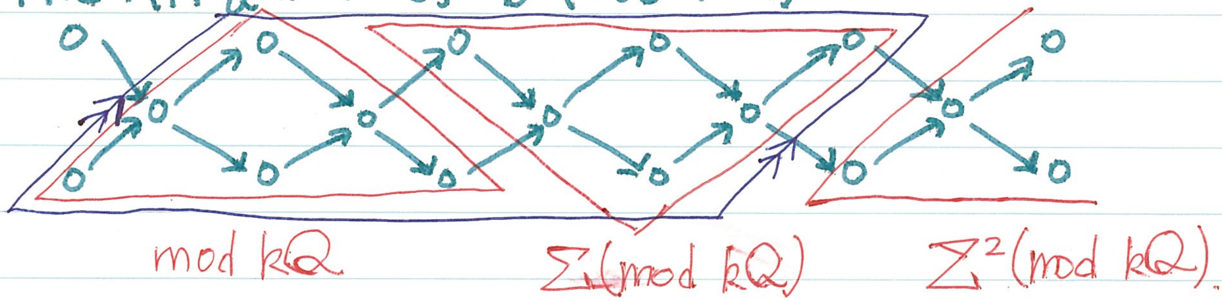
- $D_2(\mathrm{mod}\,kQ) = D^b(\mathrm{mod}\,kQ)/\Sigma^2$ is called the root cat. of $Q$ (Happel 87')

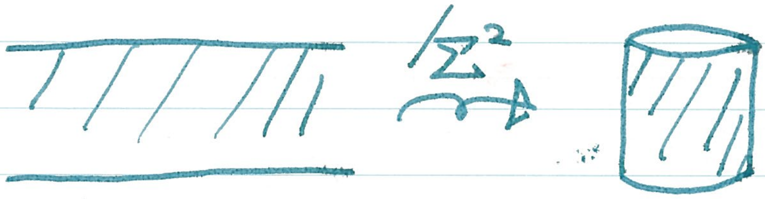
It naturally arises in a categorification of quantum groups. (Bridgeland 13').

## Ex

$Q = 1 \longleftarrow 2 \longleftarrow 3$

- The AR quiver of $D^b(\mathrm{mod}\, kQ)$




$D^b(\mathrm{mod}\, kQ)$ $\qquad$ $D_2(\mathrm{mod}\, kQ) = D^b(\mathrm{mod}\, kQ)/\Sigma^2$

## §3. Periodic tilting theorem

Let $m \in \mathbb{Z}_{\geq 1}$, $\mathcal{T}$: an $m$-periodic tri. cat.

**Def** (S)

$T \in \mathcal{T}$: an <u>$m$-periodic tilting object</u>

$:\Leftrightarrow$ (i) $T$ is a thick gen. of $\mathcal{T}$.

(ii) $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ for $i \notin m\mathbb{Z}$.

**Thm** (S)

Assume $\mathcal{T}$: idem. comp. and algebraic.

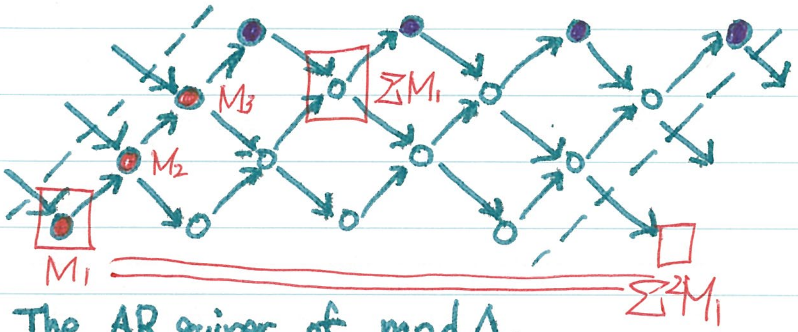
Let $T \in \mathcal{T}$: an $m$-periodic tilt. obj, $\Lambda := \mathrm{End}_{\mathcal{T}}(T)$

- $\mathrm{gl.dim}(\Lambda) \leq m \Longrightarrow \mathcal{T} \xrightarrow[\text{tri.}]{\simeq} D_m(\mathrm{mod}\Lambda)$

__Ex__

Let $Q := \begin{array}{ccc} 4 & \longrightarrow & 3 \\ \uparrow & & \downarrow \\ 1 & \longleftarrow & 2 \end{array}$ , $\Lambda := kQ/\langle \text{path of length } 4 \rangle$

Then $\Lambda$ : a self-inj. f.d. $k$-alg.




$\bullet$ : proj. = inj.

The AR quiver of $\operatorname{mod}\Lambda$.

- $\underline{\operatorname{mod}}\Lambda$ : a 2-periodic tri. cat.
- $T := M_1 \oplus M_2 \oplus M_3$ : a 2-periodic tilt. obj.
- $\operatorname{End}_{\underline{\operatorname{mod}}\Lambda}(T) \simeq kA_3$ , $A_3 = 1 \leftarrow 2 \leftarrow 3$.

Since $\operatorname{gl.dim}(kA_3) = 1 \leq 2 = (\text{the period of } \underline{\operatorname{mod}}\Lambda)$,
$\underline{\operatorname{mod}}\Lambda \xrightarrow{\simeq} D_2(\operatorname{mod} kA_3)$ by periodic tilt. thm.

# §4. The proof

Notation:

Let $A$: a differential graded (DG) algebra.

( = a complex $(A, d_A)$ with ass. mult. $A \otimes_k A \longrightarrow A$
satisfying the graded Leibniz rule: $d_A(a \cdot b) = d_A(a) \cdot b + (-1)^{\deg(a)} a \cdot d_A(b)$ )

- $D(A)$: the derived cat. of $A$.
  $\cup$
  $\mathrm{perf}(A) := \mathrm{thick}_{D(A)}(A)$: the perfect derived cat. of $A$.
- A f.d. $k$-alg $\Lambda$ can be considered as a DG alg. concentrated in degree $0$.
  Then
  $D(\Lambda)$: the usual derived cat. of $\Lambda$.
  $\cup$
  $\mathrm{perf}(\Lambda) = K^b(\mathrm{proj}\,\Lambda)$

**Key theorem** (Keller 94')

Let $\mathcal{T}$: an idem. comp. algebraic tri. cat.
- $T \in \mathcal{T}$: a thick generator.

$\Longrightarrow \exists A$: a DG alg s.t.

(1) $\mathcal{T} \xrightarrow[\text{tri.}]{\simeq} \mathrm{perf}(A) \subset D(A)$

(2) $H^*(A) = \bigoplus_{i \in \mathbb{Z}} H^i(A) \simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T)$ as graded $k$-alg's.

Let $\mathcal{T}$ : $\overbrace{\text{an } m\text{-periodic}}^{\text{idem. comp, algebraic}}$ tri. cat.

having an $m$-periodic tilt. obj. $T$.

$\mapsto$ (i) a thick. gen.

(ii) $\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ for $i \notin m\mathbb{Z}$.

Assume $\Lambda := \mathrm{End}_{\mathcal{T}}(T)$ : a f.d. $k$-alg. of fin. gl. dim.

By Keller's thm, $\exists A$ : a DG alg s.t. $\mathcal{T} \xrightarrow{\simeq} \mathrm{perf}(A) \subset D(A)$

and $H^*(A) \simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = \bigoplus_{i \in m\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i T) = \bigoplus_{i \in m\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(T,$

$\simeq \Lambda[t, t^{-1}]$ : the Laurent poly. ring over $\Lambda$,

where $\deg(t) = m$.

Note that

$$\begin{array}{ccc} D(\Lambda[t,t^{-1}]) & \simeq & D_m(\mathrm{Mod}\,\Lambda) \\ \cup & & \cup \\ \mathrm{perf}(\Lambda[t,t^{-1}]) & \simeq & D_m(\mathrm{mod}\,\Lambda) \end{array}$$

It is enough to show that:

$$\begin{array}{ccccccc} & & D(A) & \overset{\simeq}{\underset{\exists}{\dashrightarrow}} & D(\Lambda[t,t^{-1}]) & \simeq & D_m(\mathrm{Mod}\,\Lambda) \\ & & \cup & & \cup & & \cup \\ \mathcal{T} & \xrightarrow{\simeq} & \mathrm{perf}(A) & \overset{\simeq}{\dashrightarrow} & \mathrm{perf}(\Lambda[t,t^{-1}]) & \simeq & D_m(\mathrm{mod}\,\Lambda) \end{array}$$

In general, for two DG alg's $A$ and $B$,

- $A \underset{\text{q.is}}{\simeq} B$ as DG alg's $\Longrightarrow D(A) \underset{\text{tri.}}{\simeq} D(B)$.

(= quasi-isom).

It suffices to show that:

$$A \underset{\text{qis}}{\simeq} \Lambda[t, t^{-1}] \quad \text{as DG alg's}$$
$$\qquad\qquad\quad \| $$
$$\qquad\qquad\quad H^*(A).$$

Def

(1) A DG alg $A$ is formal $:\Longleftrightarrow$ $A \underset{\text{qis}}{\simeq} H^*(A)$ as DG alg's.

(2) A graded alg $B$ is intrinsically formal

$:\Longleftrightarrow$ ${}^\forall$DG alg $A$ s.t. $H^*(A) \simeq B$ as graded alg's,

$A \underset{\text{qis}}{\simeq} B$ as DG alg's.

$\exists$ a sufficient condition of being intrinsically formal.

Key Lem (Kadeishvili 88')

Let $B$: a graded alg.

- $HH^{p,2-p}(B) = 0 \ {}^\forall p \geq 3 \Longrightarrow B$: intrinsically formal.

Hochschild cohomology

Claim (S)

Let $\Lambda[t, t^{-1}]$: the Laurent poly ring with $\deg(t) = m$.

- $\mathrm{gl.dim}(\Lambda) \leq m \Longrightarrow HH^{p,2-p}(\Lambda[t,t^{-1}]) = 0$ for ${}^\forall p \geq 3$

In particular, $\Lambda[t,t^{-1}]$: intrinsically formal

//.

Rmk (S)

$\exists$DG alg $A$ s.t. $\begin{cases} \cdot\ H^*(A) \simeq \Lambda[t,t^{-1}] \\ \cdot\ A \underset{\text{qis}}{\not\simeq} \Lambda[t,t^{-1}] \text{ as DG alg's} \end{cases}$, where $\mathrm{gl.dim}(\Lambda) = \infty$

- $A$ is constructed by the DG cat. of matrix factorizations of $y^2 - x^{2n}$.