(-2) blow-up formula

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§1 Introduction

Setting

$Q = \mathbb{C}^2$: affine plane, $\quad \Gamma \subset \mathrm{SL}(Q)$: finite subgroup,

$$\Gamma \curvearrowright \mathbb{P}^2 = \mathbb{P}(\mathbb{C} \oplus Q) \ni [z_0, z_1, z_2]$$

$$\begin{tikzcd}
\ell_\infty \arrow[r, hook] & X = [\mathbb{P}^2/\Gamma] \arrow[d, "f"] & \text{: orbifold} \\
O \arrow[r, hook] & \mathbb{P}^2/\Gamma = \{\Gamma\text{-orbits in } \mathbb{P}^2\} & \text{: singularity}
\end{tikzcd}$$

where

$$\ell_\infty = \{z_0 = 0\} = [\mathbb{P}(Q)/\Gamma]$$

$$O = \{[1,0,0]\}$$

Diagram

Y : orbiofld with $Z \subset Y$ closed sub-stack

$$X \setminus f^{-1}(O) \cong Y \setminus Z$$

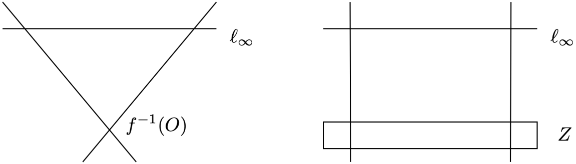


Figure: X and Y

Remark X and Y have common infinity line ℓ_∞

Example

Example 0 $Y = X$

Example 1 $\Gamma = \{\mathrm{id}_Q\}$, $X = \mathbb{P}^2$, $Y = \hat{\mathbb{P}}^2$ blow-up at O

Example 2 $\Gamma = \{\pm\,\mathrm{id}_Q\}$, $X = [\mathbb{P}^2/\Gamma]$, $Y = |\mathcal{O}_{\mathbb{P}^1}(-2)| \sqcup \ell_\infty$

$$\begin{tikzcd} X \setminus \ell_\infty \arrow[dr, "f"'] & & |\mathcal{O}_{\mathbb{P}^1}(-2)| \arrow[dl, "g"] \\ & Q/\Gamma & \end{tikzcd}$$

In the following, Y is one of these examples

Results (Theorem 1, 2)

Example 2 $\quad X = [\mathbb{P}^2/\{\pm \mathrm{id}_Q\}], \quad Y = |\mathcal{O}_{\mathbb{P}^1}(-2)| \sqcup \ell_\infty$

Compare integrations over $M_X(c)$ and $M_Y(c)$

$\rightsquigarrow$ (-2) blow-up formula

Motivation 1 : Nakajima-Yoshioka blow-up formula ($\Gamma = 1$)

Compare integrations over $M_{\mathbb{P}^2}(c)$ and $M_{\hat{\mathbb{P}}^2}(c)$

Motivation 2 : Painlevé τ function

Framed sheaf

W : fixed Γ-representation

($W = \mathbb{C}^r$ when $\Gamma = \{\mathrm{id}_Q\}$, $\quad W = W_0 \oplus W_1$ when $\{\pm\mathrm{id}_Q\}$)

Definition Framed sheaf on Y is a pair (E, Φ) such that

E : torsion free sheaf on Y

$\Phi\colon E|_{\ell_\infty} \cong \mathcal{O}_{\mathbb{P}^1} \otimes W$ on $\ell_\infty = [\mathbb{P}^1/\Gamma]$

Remark $\mathrm{Coh}(\ell_\infty) \cong \mathrm{Coh}_\Gamma(\mathbb{P}^1)$

We put $M_Y(c) := \{(E, \Phi) \mid \widetilde{\mathrm{ch}}(E) = c\}$ for $c \in A^*(IY)$

Fact (Nakajima-Yoshioka, Nakajima)

$M_Y(c)$ is smooth but non-compact.

Torus action

$\mathrm{GL}(Q)$-action on Y and $\mathrm{GL}(W)$-action on W

$\rightsquigarrow \mathrm{GL}(Q) \times \mathrm{GL}(W) \curvearrowright M_Y(c) \ni (E, \Phi)$

$$T^2 = \left\{ \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \in \mathrm{GL}(Q) \right\}, T^r = \left\{ \begin{bmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & e_r \end{bmatrix} \in \mathrm{GL}(W) \right\},$$

$\rightsquigarrow \mathbb{T} = T^2 \times T^r \curvearrowright M_Y(c)$ for $T = \mathbb{C}^*$ and $r = \dim W$

$t_1, t_2, e_1, \dots, e_r$: weight spaces for $\mathbb{T}$-action

$\varepsilon_1 = c_1(t_1), \varepsilon_2 = c_1(t_2), a_1 = c_1(e_1), \dots, a_r = c_1(e_r) \in A^*_{\mathbb{T}}(\mathrm{pt})$

Integrations

Fact (Nakajima-Yoshioka, Nakajima)

The fixed points set $M_Y(c)^{\mathbb{T}}$ is finite

For $\psi \in A^*_{\mathbb{T}}(M_Y(c))$

$$\int_{M_Y(c)} \psi := \sum_{p \in M_Y(c)^{\mathbb{T}}} \frac{\psi|_p}{e(T_p M_Y(c))} \in \mathbb{Q}(\varepsilon_1, \varepsilon_2, a_1, \ldots, a_r)$$

where $\psi|_p$ and the equivariant Euler class $e(T_p M_Y(c))$ belong to

$$A^*_{\mathbb{T}}(\mathrm{pt}) = \mathbb{Z}[\varepsilon_1, \varepsilon_2, a_1, \ldots a_r]$$

We put $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ and $\boldsymbol{a} = (a_1, \ldots, a_r)$

Nekrasov function ($Y = X = \mathbb{P}^2$, $\Gamma = \{\mathrm{id}_Q\}$)

$M(r, n) := M_{\mathbb{P}^2}(c)$ for $c = (r, 0, n) \in A^*(\mathbb{P}^2)$

$$Z(\varepsilon, \boldsymbol{a}, q) = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} 1$$

Combinatorial description

$$Z(\varepsilon, \boldsymbol{a}, q) = \sum_{n=0}^{\infty} \sum_{|\vec{Y}|=n} \frac{1}{G_{\vec{Y}}} q^n$$

$\vec{Y} = (Y_1, \ldots, Y_r)$: tuple of Young diagrams

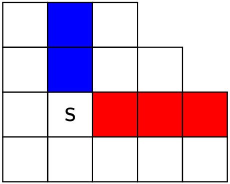
$|\vec{Y}| = |Y_1| + \cdots + |Y_r|$: sum of numbers of boxes in $Y_1, \ldots, Y_r$

Combinatorial description

$$Z(\varepsilon, \boldsymbol{a}, q) = \sum_{n=0}^{\infty} \sum_{|\vec{Y}|=n} \frac{1}{G_{\vec{Y}}} q^n$$

$$G_{\vec{Y}} = \prod_{\alpha,\beta=1}^{r} \left(\prod_{s \in Y_\alpha} \left(a_\beta - a_\alpha - L_{Y_\beta}(s)\varepsilon_1 + (A_{Y_\alpha}(s) + 1)\varepsilon_2\right) \right.$$

$$\left. \prod_{t \in Y_\beta} \left(a_\beta - a_\alpha + (L_{Y_\alpha}(t) + 1)\varepsilon_1 - A_{Y_\beta}(t)\varepsilon_2\right) \right).$$

$Y =$ 

Arms : $A_Y(s) = 2$

Legs : $L_Y(s) = 3$

Motivation 1 : Nakajima-Yoshioka blow-up formula

$C := \pi^{-1}(O) \subset Y = \hat{\mathbb{P}}^2 \xrightarrow{\pi} \mathbb{P}^2$: blow-up at $O = [1, 0, 0]$

Fix r and $c_1 = 0$ (for simplicity)

Put $c = (r, 0, c_2) \in A^*(\hat{\mathbb{P}}^2)$ moving c_2

$$\hat{Z}(\varepsilon, \boldsymbol{a}, q, t) := \sum_c q^{c_2} \int_{M_{\hat{\mathbb{P}}^2}} \mu(C)^d \frac{t^d}{d!} \in \mathbb{Q}(\varepsilon, \boldsymbol{a})[[q, t]]$$

where $\mu(C)$: Poincare dual of $p_*(c_2(\mathcal{E}) \cap [C \times M_{\hat{\mathbb{P}}^2}(c)]) \in A_*^{\mathbb{T}}(M_{\hat{\mathbb{P}}^2}(c))$ and

$\mathcal{E}$: universal sheaf on $\hat{\mathbb{P}}^2 \times M_{\hat{\mathbb{P}}^2}(c)$

$p \colon \hat{\mathbb{P}}^2 \times M_{\hat{\mathbb{P}}^2}(c) \to M_{\hat{\mathbb{P}}^2}(c)$: projection

Motivation 1 : Nakajima-Yoshioka blow-up formula

Theorem (Nakajima-Yoshioka)

$$\hat{Z}(\varepsilon, \boldsymbol{a}, q, t) = Z(\varepsilon, \boldsymbol{a}, q) + O(t^{2r})$$

equivalently

$$\int_{M_{\hat{\mathbb{P}}^2}(c)} \mu(C)^d = \begin{cases} 0 & 0 < d < 2r \\ \int_{M_{\mathbb{P}^2}(p_* c)} 1 & d = 0 \end{cases}$$

$\rightsquigarrow \lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 \log Z(\boldsymbol{e}, \boldsymbol{a}, q)$ coincides with the Seiberg-Witten prepotential

(Nekrasov conjecture

also proved by Nekrasov-Okounkov, Braverman-Etingov independently)

Motivation 2 : Painlevé τ function ($r = 2$)

Theorem (Bershtein-Shchechkin, Iorgov-Lisovyy-Teschner)

(Conjecture by Gamayun-Iorgov-Lisovyy)

$\tau(t) = \sum_{n \in \mathbb{Z}} s^n C(\sigma + n) Z(\sqrt{-1}, \sqrt{-1}, \sigma + n, -\sigma - n, t)$ satisfies

$$D_{III}(\tau, \tau) = 0$$

for $D_{III} = \frac{1}{2} D^4 - t \frac{d}{dt} D^2 + \frac{1}{2} D^2 + 2tD^0$

Here the Hirota differential D^k is defined by

$$f(e^{\alpha t}) g(e^{\alpha t}) = \sum_{k=0}^{\infty} D^k(f, g) \frac{\alpha^k}{k!}$$

§2 Main Results

$$X = [\mathbb{P}^2/\{\pm\,\mathrm{id}_Q\}], \quad Y = |\mathcal{O}_{\mathbb{P}^1}(-2)| \sqcup \ell_\infty$$

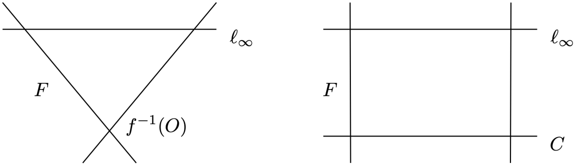


Figure: X and Y

Tautological bundle

$\mathcal{E}$: universal sheaf on $X \times M_X(c)$, or $Y \times M_Y(c)$

$$\begin{aligned} \mathcal{V}_0 &= \mathbb{R}^1 p_* \mathcal{E}(-\ell_\infty), \quad \mathcal{W}_0 = \mathcal{O}_M \otimes W_0 \\ \mathcal{V}_1 &= \mathbb{R}^1 p_* \mathcal{E}(-F), \quad \mathcal{W}_1 = \mathcal{O}_M \otimes W_1 \end{aligned}$$

where $M = M_X(c)$, or $M_Y(c)$

$p \colon X \times M_X(c) \to M_X(c)$, or $Y \times M_Y(c) \to M_Y(c)$: projection

$F = \{z_1 = 0\}$ in X, or its proper transform in Y

Another torus (Matter bundle)

$$(e^{m_1}, \dots, e^{m_{2r}}) \in T^{2r}, \quad \tilde{\mathbb{T}} = T^2 \times T^r \times T^{2r}$$

$$\boldsymbol{m} = (m_1, \dots, m_{2r}) = (c_1(e^{m_1}), \dots, c_1(e^{m_{2r}})) \in A^*_{\tilde{\mathbb{T}}}(\mathrm{pt})$$

$$c = (r, k[C], -n[P], (w_0 - w_1)\ell^1_\infty) \in A^*(IY)$$

(This c can be viewed in $A^*(IX)$ via the Mckay derived equivalence)

$$Z_X(\varepsilon, \boldsymbol{a}, \boldsymbol{m}, q) = \sum_n q^n \int_{M_X(c)} e\left(\bigoplus_{f=1}^{2r} \mathcal{V}_0 \otimes \frac{e^{m_f}}{\sqrt{t_1 t_2}}\right)$$

$$Z_Y(\varepsilon, \boldsymbol{a}, \boldsymbol{m}, q) = \sum_n q^n \int_{M_Y(c)} e\left(\bigoplus_{f=1}^{2r} \mathcal{V}_0 \otimes \frac{e^{m_f}}{\sqrt{t_1 t_2}}\right)$$

Conjecture by Ito-Maruyoshi-Okuda

Theorem 1

$$Z_Y^k(-\varepsilon, \boldsymbol{a}, \boldsymbol{m}, q) = \begin{cases} (1-(-1)^r q)^{u_r} Z_X^k(\varepsilon, \boldsymbol{a}, \boldsymbol{m}, q) & \text{for } k \geq 0, \\ Z_X^k(-\varepsilon, \boldsymbol{a}, \boldsymbol{m}, q) & \text{for } k \leq 0, \end{cases}$$

where

$$u_r = \frac{(\varepsilon_1 + \varepsilon_2)(2\sum_{\alpha=1}^r a_\alpha + \sum_{f=1}^{2r} m_f)}{2\varepsilon_1\varepsilon_2}$$

Remark When $k=0$, $Z_X^k(-\varepsilon, \boldsymbol{a}, \boldsymbol{m}, q) = (1-(-1)^r q)^{u_r} Z_X^k(\varepsilon, \boldsymbol{a}, \boldsymbol{m}, q)$

Remark When $\Gamma = \{\mathrm{id}_Q\}$, we have similar formula

(-2) blow-up formula

Theorem 2

$$\int_{M_Y(c_+)} (\mathrm{ch}_2(\mathcal{E})/[C])^d \;=\; \int_{M_X(c_\pm)} (\psi_\pm)^d$$

Here

$$\begin{cases} d \le 2-4k & \psi_+ = 2c_1(\mathcal{V}_0) - 2c_1(\mathcal{V}_1) + c_1(\mathcal{W}_1) + \varepsilon_+)(2k + w_1/2) \\ d \le 2r+2-4k & \psi_- = 2k\varepsilon_+ - \psi_+ \end{cases}$$

$$c_\pm = (w_0, w_1, \pm k[C], -nP) \in (\mathbb{Z}_{\ge 0})^2 \oplus A^1(Y) \oplus A^2(Y)$$

$\varepsilon_+ = \varepsilon_1 + \varepsilon_2$, and $\mathrm{ch}_2(\mathcal{E})/C$: slant product

§3 Outline of proof

(Simple Example for Mochizuki method)

Example 1 : Projective space

$W = \mathbb{C}^r$: vector space

Compute the Euler number of $\mathbb{P}(W) = \mathbb{P}^{r-1}$

by Mochizuki method.

Master space

We put $\mathcal{M} = \mathbb{P}(W \oplus \mathbb{C})$, and consider $\mathbb{C}^*_\hbar$-action defined by

$$[w_1, \ldots, w_r, x] \mapsto [w_1, \ldots, w_r, e^\hbar x].$$

The fixed points set $\mathcal{M}^{\mathbb{C}^*_\hbar}$ is decomposed as follows:

$$\mathcal{M}^{\mathbb{C}^*_\hbar} = \mathcal{M}_+ \sqcup \mathcal{M}_{exc},$$

where $\mathcal{M}_+ = \{x = 0\} = \mathbb{P}(W)$ and $\mathcal{M}_{exc} = \{[0, \ldots, 0, 1]\} = \mathrm{pt}$.

We put

$$\iota \colon \mathcal{M}^{\mathbb{C}^*_\hbar} = \mathcal{M}_+ \sqcup \mathcal{M}_{exc} \to \mathcal{M}$$

Equivariant Chow ring

For a proper variety X with $\mathbb{C}^*_\hbar$-action, we put

$A^\bullet_{\mathbb{C}^*_\hbar}(X)$: equivariant Chow ring

We have

$$A^\bullet_{\mathbb{C}^*_\hbar}(\mathrm{pt}) \cong \mathbb{Z}[\hbar]$$

where $\hbar = c_1(e^\hbar)$ for the weight space $e^\hbar$

※ $e^\hbar$ can be regarded as $\mathbb{C}^*_\hbar$-equivariant vector bundle over pt.

Localization formula

For the fixed points set $X^{\mathbb{C}^*_\hbar}$, we have

$$\iota_* \colon A^\bullet_{\mathbb{C}^*_\hbar}(X^{\mathbb{C}^*_\hbar}) \otimes \mathbb{Q}[\hbar, \hbar^{-1}]] \cong A^\bullet_{\mathbb{C}^*_\hbar}(X) \otimes \mathbb{Q}[\hbar, \hbar^{-1}]]$$

<u>Fact</u> When X is smooth, we have the following:

(1) $X^{\mathbb{C}^*_\hbar} = \bigsqcup_{\mathfrak{J}} X_{\mathfrak{J}}$ for smooth $X_{\mathfrak{J}}$

(2)

$$(\iota_*)^{-1}[X] = \sum_{\mathfrak{J}} \frac{[X_{\mathfrak{J}}]}{\mathrm{Eu}(N_{\mathfrak{J}})},$$

$X_{\mathfrak{J}}$

where $\mathrm{Eu}(N_{\mathfrak{J}})$ is the Euler class of the normal bundle $N_{\mathfrak{J}}$ of $X^{\mathbb{C}^*_\hbar}$ in X

Integral by localization (X: smooth)

For $\Pi \colon X \to \mathrm{pt}$ and $\Pi_{\mathfrak{J}} \colon X_{\mathfrak{J}} \to \mathrm{pt}$, we have the commutative diagram:

$$\begin{tikzcd}
A^{\bullet}_{\mathbb{C}^*_\hbar}(X) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]] \arrow[r, "(\iota_*)^{-1}", "\cong"'] \arrow[d, "\Pi_*(\cdot)\cap[X]"'] & A^{\bullet}_{\mathbb{C}^*_\hbar}(X^{\mathbb{C}^*_\hbar}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]] \arrow[d, "\sum_{\mathfrak{J}} \Pi_{\mathfrak{J}*}(\cdot)\cap[X_{\mathfrak{J}}]"] \\
A_{\bullet}^{\mathbb{C}^*_\hbar}(\mathrm{pt}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]] \arrow[r, equal] & A_{\bullet}^{\mathbb{C}^*_\hbar}(\mathrm{pt}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Q}[\hbar, \hbar^{-1}]]
\end{tikzcd}$$

$$\therefore \int_X c = \sum_{\mathfrak{J}} \int_{X_{\mathfrak{J}}} \frac{c|_{X_{\mathfrak{J}}}}{\mathrm{Eu}(N_{\mathfrak{J}})},$$

where $\begin{cases} \int_X c = \Pi_* c \cap [X] & \text{for } c \in A^{\bullet}_{\mathbb{C}^*_\hbar}(X) \\ \int_{X_{\mathfrak{J}}} c_{\mathfrak{J}} = \Pi_{\mathfrak{J}*} c_{\mathfrak{J}} \cap [X_{\mathfrak{J}}] & \text{for } c_{\mathfrak{J}} \in A^{\bullet}_{\mathbb{C}^*_\hbar}(X_{\mathfrak{J}}) \end{cases}$

Tautological bundle over $\mathcal{M} = \mathbb{P}(W \oplus \mathbb{C})$

If we put $\begin{cases} V = \mathbb{C} \\ \mathrm{Hom}^{\mathsf{inj}}(V, W) = \mathrm{Hom}(V, W) \setminus \{O\} \end{cases}$

$$\mathbb{P}(W) = \mathbb{P}(\mathrm{Hom}(V, W)) = \left[\mathrm{Hom}^{\mathsf{inj}}(V, W)/\, \mathsf{GL}(V)\right]$$

$$\mathcal{V} = \left[\left\{\mathrm{Hom}^{\mathsf{inj}}(V, W) \times V\right\} / \,\mathsf{GL}(V)\right] \cong \mathcal{O}(-1)$$

We have the Euler sequence

$$0 \to \mathcal{O} \to W \otimes \mathcal{V}^\vee \to T\mathbb{P}(W) \to 0$$

※ $\mathcal{V}$ is also defined on $\mathcal{M} = \mathbb{P}(\mathrm{Hom}(V, W) \oplus \det V^\vee)$, and $W \otimes \mathcal{V}^\vee - \mathcal{O}_{\mathcal{M}}$ in $K(\mathcal{M})$ restricts to $T\mathcal{M}_+$ on $\mathcal{M}_+ = \mathbb{P}(W)$.

Euler class for virtual vector bundle

$e^m \in \mathbb{C}^*_m$: equivariant parameter to define the Euler class

$\alpha = [E] - [F] \in K_{\mathbb{C}^*_\hbar}(\mathcal{M})$ with $\mathbb{C}^*_\hbar$-equivariant vector bundles E, F on $\mathcal{M}$

$$\mathrm{Eu}^m(\alpha) = \frac{c_{\mathrm{rk}\, E}(E \otimes e^m)}{c_{\mathrm{rk}\, F}(F \otimes e^m)} \in A^\bullet_{\mathbb{C}^*_m \times \mathbb{C}^*_\hbar}(\mathcal{M}) \otimes \mathbb{Q}(m, \hbar).$$

We put

$$\psi(m) = \mathrm{Eu}^m(W \otimes \mathcal{V}^\vee - \mathcal{O}_\mathcal{M}) \in A^*_{\mathbb{C}^*_m \times \mathbb{C}^*_\hbar}(\mathcal{M}) \otimes \mathbb{Q}(m)[\hbar].$$

Integral

N_+, N_{exc} : normal bundles of $\mathcal{M}_+, \mathcal{M}_{exc}$ in $\mathcal{M}$ respectively.

ℏ eち

$$\frac{1}{Eu(N_+)} = \frac{1}{\hbar + c_1(\mathcal{V}^\vee)} = \frac{1}{\hbar} \cdot \frac{1}{1 + c_1(\mathcal{V}^\vee)/\hbar}$$

$$= \frac{1}{\hbar}\left(1 - \frac{c_1(\mathcal{V}^\vee)}{\hbar} + \cdots\right) \in A^*_{\mathbb{C}^*_m \times \mathbb{C}^*_\hbar}(\mathcal{M}) \otimes \mathbb{Q}(m)[\hbar, \hbar^{-1}]]$$

Localization formula

$$\Longrightarrow \int_{\mathcal{M}} \psi(m) = \int_{\mathcal{M}_+} \frac{\psi(m)|_{\mathcal{M}_+}}{\mathrm{Eu}(N_+)} + \int_{\mathcal{M}_{exc}} \frac{\psi(m)|_{\mathcal{M}_{exc}}}{\mathrm{Eu}(N_{exc})}.$$

(LHS) in $\mathbb{C}(m)[\hbar]$ vs (RHS) in $\mathbb{C}(m)[\hbar, \hbar^{-1}]]$

$$\Longrightarrow \int_{\mathbb{P}(W)} \mathrm{Eu}(T\mathbb{P}(W)) = -\operatorname*{Res}_{\hbar=\infty} \int_{\mathcal{M}_{exc}} \frac{\psi(m)|_{\mathcal{M}_{exc}}}{\mathrm{Eu}(N_{exc})}.$$

Here $\mathrm{Res}_{\hbar=\infty}$ is taking the coefficient in $\hbar^{-1}$.

Residue

$$\int_{\mathbb{P}(W)} \mathrm{Eu}(T\mathbb{P}(W)) = -\operatorname*{Res}_{\hbar=\infty} \int_{\mathcal{M}_{exc}} \frac{\psi(m)|_{\mathcal{M}_{exc}}}{\mathrm{Eu}(N_{exc})}. \tag{1}$$

$$\begin{cases} \psi(m) = \mathrm{Eu}^m(W \otimes e^{-\hbar} - \mathcal{O}) \\ N_{exc} = W \otimes e^{-\hbar} \end{cases} \Longrightarrow \text{(RHS) of (1) is equal to}$$

$$\begin{aligned} \chi(\mathbb{P}(W)) &= -\operatorname*{Res}_{\hbar=\infty} \frac{(-\hbar+m)^r}{m} \cdot \frac{1}{(-\hbar)^r} \\ &= -\operatorname*{Res}_{\hbar=\infty} \frac{1}{m} \cdot \frac{(\hbar-m)^r}{\hbar^r} = -\frac{1}{m} \cdot r(-m) = r \end{aligned}$$

$$※ \operatorname*{Res}_{\hbar=\infty} \prod_{\alpha=1}^{r} \frac{\hbar + a_\alpha}{\hbar + b_\alpha} = \sum_{\alpha=1}^{r} (a_\alpha - b_\alpha)$$

Example 2 : Grassmann manifold

$$\begin{cases} W = \mathbb{C}^r \\ V = \mathbb{C}^n \end{cases} \quad (\ n \leq r\ )$$

Compute the Euler number of the Grassmann manifold

$$G(r,n) = G(W,V) = \{w \in \mathrm{Hom}_{\mathbb{C}}(V,W) \mid w \text{ is injective}\}/\ \mathrm{GL}(V)$$

by Mochizuki method.

Enhanced master space

$$\begin{aligned} \mathbb{M} &= \mathrm{Hom}_{\mathbb{C}}(V, W) \\ \widetilde{\mathbb{M}} &= \mathbb{M} \times Fl(V) \\ \hat{\mathbb{M}} &= \widetilde{\mathbb{M}} \times \det V^{\vee}, \end{aligned}$$

where $Fl(V)$ is the full flag manifold of V.

$G(r,n) = G(W,V)$

We put $\mathcal{M} = \hat{\mathbb{M}}^{ss} / \mathrm{GL}(V)$, and consider $\mathbb{C}^*_\hbar$-action on $\mathcal{M}$ such that

$$\mathcal{M}^{\mathbb{C}^*_\hbar} = \mathcal{M}_+ \sqcup \mathcal{M}_{exc}$$

$$\begin{cases} \mathcal{M}_+ \cong FL(\mathcal{V}) & \text{over } G(r,n) = G(W,V) \\ \mathcal{M}_{exc} \cong FL(\mathcal{V}_\flat) & \text{over } G(r,n-1) = G(W,V_\flat) \end{cases}$$

where $\mathcal{V}$, $\mathcal{V}_\flat$ are universal bundles over $G(r,n), G(r,n-1)$

$$\begin{aligned} \Longrightarrow \int_{G(r,n)} \mathrm{Eu}(TG(r,n)) &= \frac{r-n+1}{n} \cdot \int_{G(r,n-1)} \mathrm{Eu}(TG(r,n-1)) \\ &\vdots \\ &= \frac{r-n+1}{n} \cdot \frac{r-n+2}{n-1} \cdots \frac{r}{1} = \binom{r}{n}. \end{aligned}$$

GL(W)-action

$W = \mathbb{C}\boldsymbol{e}_1 \oplus \cdots \oplus \mathbb{C}\boldsymbol{e}_r, \quad V = \mathbb{C}^n \ (\ n \leq r \ )$

$$G(r,n) = \{w \in \mathrm{Hom}_{\mathbb{C}}(V,W) \mid w \text{ is injective}\}/\,\mathrm{GL}(V) \curvearrowleft \mathrm{GL}(W)$$

In particular, the diagonal torus $\mathbb{T} = (\mathbb{C}^*)^r \subset \mathrm{GL}(W)$ acts on $G(r,n)$

$$I = \{1 \leq i_1 < \cdots < i_n \leq r\} \in G(r,n)^{\mathbb{T}}$$

For $\psi \in A^{\bullet}_{\mathbb{T}}(G(r,n))$, we have

$$\int_{G(r,n)} \psi = \sum_{I \in G(r,n)^{\mathbb{T}}} \frac{\psi|_I}{\mathrm{Eu}(T_I G(r,n))} \in \mathbb{Q}(a_1,\ldots,a_r)$$

where $a_1 = c_1(e_1), \ldots, a_r = c_1(e_r)$ for diag $(e_1,\ldots,e_r) \in \mathbb{T}$.

Schur polynomial

For partition $\lambda = (\lambda_1, \lambda_2, \dots,)$ of length m

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > \lambda_{m+1} = 0 \cdots$$

we put

$$S_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\det(x_i^{n-j})_{1 \leq i,j \leq n}}$$

When $m \leq n$, we define $S_\lambda(\mathcal{V}) \in A^\bullet_{\mathbb{T}}(G(r,n))$ by

$$S_\lambda(\mathcal{V}) = S_\lambda(\beta_1, \dots, \beta_n)$$

where $\beta_1, \dots, \beta_n$ are Chern roots of the universal sub-bundle $\mathcal{V}$ over $G(r,n)$

Example : pt $= G(r, r)$

For partition $\lambda = (\lambda_1, \lambda_2, \ldots,)$ of length $m \leq r$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > \lambda_{m+1} = 0 \cdots$$

we have

$$\int_{G(r,r)} S_\lambda(\mathcal{V}) = S_\lambda(a_1, \ldots, a_r)$$

Jacobi-Trudi formula

$$\int_{G(r,n)} S_\lambda(\mathcal{V}) = \operatorname*{Res}_{\hbar_1\cdots\hbar_n=\infty} \frac{(-1)^{nr+n(n+1)/2}}{n!} \frac{\det(\hbar_i^{\lambda_j+n-j})\det(\hbar_i^{n-j})}{\prod_{i=1}^n \prod_{\alpha=1}^r (\hbar_i - a_\alpha)}$$

$\displaystyle\prod_{\alpha=1}^r (\hbar_i - a_\alpha)^{-1} = \sum_{\ell=0}^\infty h_\ell(a_1,\dots,a_r)\hbar_i^{-r-\ell}$: generating functions of complete homogeneous symmetric polynomials

$$\rightsquigarrow \int_{G(r,n)} S_\lambda(\mathcal{V}) = (-1)^{n(r+1)} \det_{1\le i,j\le n}(h_{\lambda_j-j+i+n-r}) \qquad (2)$$

In particular when $n=r$, we have

$$S_\lambda(a_1,\dots,a_r) = \det_{1\le i,j\le r}(h_{\lambda_j-j+i}). \qquad (3)$$

(-2)-curve

$\Gamma = \{\pm \mathrm{id}_Q\} \subset \mathrm{SL}(Q), \quad Q = \mathbb{C}^2$

$$
\begin{aligned}
\mathbb{M} &= \mathrm{Hom}_\Gamma(Q^\vee \otimes V, V) \oplus \mathrm{Hom}_\Gamma(\wedge^2 Q^\vee \otimes V, W) \\
&\oplus \mathrm{Hom}_\Gamma(W, V) \xrightarrow{\mu} \mathrm{Hom}_\Gamma(\wedge^2 Q^\vee \otimes V, V) \\
\widetilde{\mathbb{M}} &= \mu^{-1}(\mathbf{0}) \times Fl(V_i) \\
\hat{\mathbb{M}} &= \widetilde{\mathbb{M}} \times \mathbb{P}(L_- \oplus L_+),
\end{aligned}
$$

$\zeta^+ = (\zeta_0^+, \zeta_1^+)$.

$\zeta^- = (\zeta_0^-, \zeta_1^-)$.

where $W = W_0 \oplus W_1$, $V = V_0 \oplus V_1$ are Γ-representations, and

$L_+ = \det V_0^{\otimes \zeta_0^+} \otimes \det V_1^{\otimes \zeta_1^+}$, $L_- = \det V_0^{\otimes \zeta_0^-} \otimes \det V_1^{\otimes \zeta_1^-}$, and

$Fl(V_i)$ is the full flag manifold of V_i for $i = 0, 1$.

Outline of proof

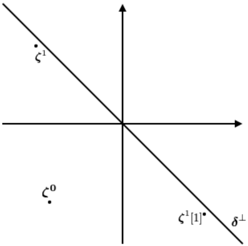


Figure: ζ^0 and ζ^1

Isomorphism from moduli of ζ-stable ADHM data $M^\zeta(\boldsymbol{w}, \boldsymbol{v})$ (Nakajima)

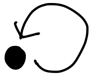
$$M^\zeta(\boldsymbol{w}, \boldsymbol{v}) \cong \begin{cases} M_X(c) \text{ for } \zeta = \zeta^0 \\ M_Y(c) \text{ for } \zeta = \zeta^1 \end{cases}$$

Here, $\boldsymbol{w} = (\dim W_0, \dim W_1)$, $\boldsymbol{v} = (\mathrm{rank}\mathcal{V}_0, \mathrm{rank}\mathcal{V}_1)$

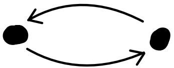
Summary

Grassmmanian $\leftrightarrow$ one point

Framed moduli on $\mathbb{P}^2$ $\leftrightarrow$ Jordan quiver



Framed moduli on (-2) curve $\leftrightarrow$ $A_1^{(1)}$



ADE singularity

$\Gamma \subset \mathrm{SL}(Q)$ corresponding to a Dynkin diagram, $\quad Q = \mathbb{C}^2$

Q/Γ : *ADE* isolated singularity

$$\begin{aligned} \mathbb{M} &= \mathrm{Hom}_\Gamma(Q^\vee \otimes V, V) \oplus \mathrm{Hom}_\Gamma(\wedge^2 Q^\vee \otimes V, W) \\ &\oplus \mathrm{Hom}_\Gamma(W, V) \xrightarrow{\mu} \mathrm{Hom}_\Gamma(\wedge^2 Q^\vee \otimes V, V) \end{aligned}$$

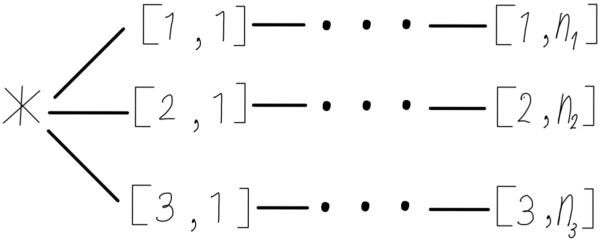
where W, V are Γ-representations,

Introducing $\hat{\mathbb{M}}$ and $\widetilde{\mathbb{M}}$ suitably

$\rightsquigarrow$

Wall-crossing for framed moduli on $[\mathbb{P}^2/\Gamma]$

Star-shaped graph $\mathcal{G} = (I, E)$ of ADE type



$$I = \{*\} \cup \{[i, m_i] \mid i = 1, 2, 3, m_i = 1, \ldots, n_i\}$$

$$* \in J \subset I$$

S_J : contraction of (-2)-curves in $I \setminus J$

X_J : (stacky resolution of S_J) $\sqcup \ell_\infty$

Here ℓ_∞ is the infinity line in $X = [\mathbb{P}^2/\Gamma]$

Weighted projective line associated to $\mathcal{G} = (I, E)$

$$\pi\colon \mathcal{C} = \mathbb{P}^1\left[\frac{1}{n_1+1}(0), \frac{1}{n_2+1}(1), \frac{1}{n_3+1}(\infty)\right] \to \mathbb{P}^1$$

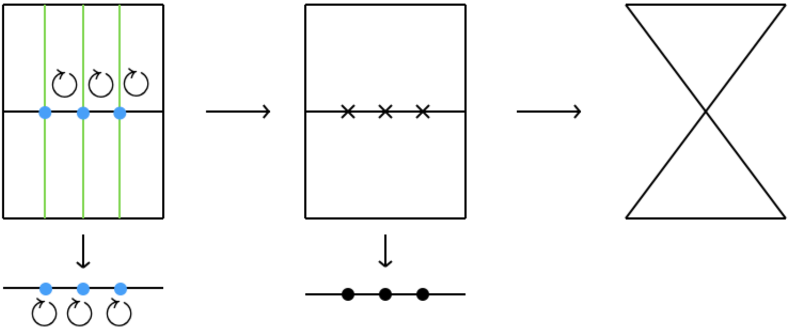
$$\omega_{\mathcal{C}} = \pi^*\omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathcal{C}}\left(-\tfrac{n_1}{n_1+1}(0) - \tfrac{n_2}{n_2+1}(1) - \tfrac{n_3}{n_3+1}(\infty)\right)$$

$$\begin{aligned} X_J \setminus \ell_\infty = |\omega_{\mathcal{C}}| &= \mathit{Spec} \bigoplus_{d=0}^{\infty} \omega_{\mathcal{C}}^{-d} \\ S_J &= \mathit{Spec} \bigoplus_{d=0}^{\infty} \pi_*\omega_{\mathcal{C}}^{-d} \end{aligned}$$

for $J = \{*\}$

Resolution X_J for $J = \{*\}$

$$\begin{tikzcd} X_J \setminus \ell_\infty \arrow[r] \arrow[d] & S_J \arrow[r] \arrow[d] & \operatorname{Spec} \bigoplus_{d=0}^{\infty} \Gamma(\mathcal{C}, \omega_{\mathcal{C}}^{-d}) \\ \mathcal{C} \arrow[r, "\pi"] & \mathbb{P}^1 & \end{tikzcd}$$



Higher dimensions

M. Herschend, O. Iyama, H. Minamoto, S. Oppermann,

Representation theory of Geigle-Lenzing complete intersections,

arXiv:1409.0668

M. Tomari and K. Watanabe,

Cyclic covers of normal graded rings,

Kodai Math. J. 24 (2001), 436-457

Variation

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \subset \mathrm{GL}(Q) \qquad Q = \mathbb{C}^2$$

$$\begin{aligned} \mathbb{M} &= \mathrm{Hom}_\Gamma(Q^\vee \otimes V, V) \oplus \mathrm{Hom}_\Gamma(\wedge^2 Q^\vee \otimes V, W) \\ &\oplus \mathrm{Hom}_\Gamma(W, V) \xrightarrow{\mu} \mathrm{Hom}_\Gamma(\wedge^2 Q^\vee \otimes V, V) \\ \widetilde{\mathbb{M}} &= \mu^{-1}(\mathbf{0}) \times Fl(V_i) \\ \hat{\mathbb{M}} &= \widetilde{\mathbb{M}} \times \mathbb{P}(L_- \oplus L_+) \end{aligned}$$

$\rightsquigarrow$ Wall-crossing for Handsaw quiver variety

Vortex partition functions (joint with Yutaka Yoshida)

Future work

(1) ADE singularity (affine quiver variety)

(2) K-theoretic version

(3) (-2) blow-up formula for Matter theory

- Handsaw quiver variety
- Flag manifold of type $ABCDEFG$
- Finite quiver variety