有限群のブロック上のτ-傾理論

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τ-tilting theory.

In this section,

- $k = \overline{k}$
- Λ : fin. dim. symmetric k-alg.
- τ : Auslander-Reiten translation
  ($\tau \simeq \Omega^2$ ∵ Λ : symmetric alg.)
- for $M \in \Lambda\text{-mod}$,
  $|M| :=$ the number of iso. classes of indec. direct summand.

  In particular, $|\Lambda| := |{}_\Lambda \Lambda|$

<u>Def</u> (Adachi-Iyama-Reiten)

- $M \in \Lambda\text{-mod}$ is τ-tilting module

$\Leftrightarrow$
- $\mathrm{Hom}_\Lambda(M, \tau M) = 0$
- $|M| = |\Lambda|$

- M is support τ-tilting module

$\Leftrightarrow {}^\exists e$: idemp. of Λ s.t. M is τ-tilting $\Lambda/\Lambda e \Lambda$-module

☆ $s\tau\text{-tilt}\,\Lambda$:= the set of iso. classes of support τ-tilting Λ-modules

<u>Def.</u>

- $S = S_1 \oplus \cdots \oplus S_n \in \Lambda\text{-mod}$ が semibrick

$$:\Longleftrightarrow \quad \mathrm{Hom}_\Lambda(S_i, S_j) \simeq \begin{cases} k & (i = j) \\ 0 & (i \neq j) \end{cases}$$

- semibrick S が left finite

$:\Longleftrightarrow$ $\mathsf{T}(S)$ is functorially finite

the smallest torsion class containing S

☆ $\mathrm{sbrick}\,\Lambda$:= the set of iso. classes of semibricks over Λ

$f_L\text{-sbrick}\,\Lambda$:= 〃 left finite semibricks over Λ.

* $2\text{-silt}\,\Lambda$:= the set of iso. classes of 2 term silting complexes over Λ
$2\text{-tilt}\,\Lambda$ tilting complex

($2\text{-silt}\,\Lambda = 2\text{-tilt}\,\Lambda$ ☺ Λ : sym. alg.)

* $2\text{-smc}\,\Lambda$:= the set of iso. classes of 2 term simple-minded collections over Λ.

Thm (Adachi-Iyama-Reiten 2014, Koenig-Yang 2014, Asai 2018)

$$\begin{tikzcd}
s\tau\text{-tilt}\,\Lambda \arrow[r, leftrightarrow, "1:1", "①"'] \arrow[rrr, leftrightarrow, bend right, "④", "1:1"'] & 2\text{-silt}\,\Lambda \arrow[r, leftrightarrow, "1:1", "②"'] & 2\text{-smc}\,\Lambda \arrow[r, leftrightarrow, "③"'] & f_L\text{-sbrick}\,\Lambda
\end{tikzcd}$$

① (AIR)

$s\tau\text{-tilt}\,\Lambda \ni M \mapsto (P_1 \oplus P \xrightarrow{(f,0)} P_0) \in 2\text{-silt}\,\Lambda$

where $P_1 \xrightarrow{f} P_0 \to M$ is a min. proj. pres.

and P is a proj mod. s.t. $|P| + |M| = |\Lambda|$

and $\mathrm{Hom}_\Lambda(P, M) = 0$

② (KY)

omit

③ (Asa) by Brüstle-Yang 2013

$2\text{-smc}\,\Lambda \ni \underbrace{(\oplus X_i) \oplus (\oplus Y_i)[1]} \mapsto \oplus X_i \in f_L\text{-sbrick}\,\Lambda$

④ (Asa)

$s\tau\text{-tilt}\,\Lambda \ni M \mapsto M \Big/ \sum\limits_{f \in \mathrm{rad}\,\mathrm{End}_\Lambda(M)} \mathrm{Im} f \in f_L\text{-sbrick}\,\Lambda$

Modular representation

- $k = \bar{k}$, char $k = p > 0$
- G : finite group
- $\tilde{G}$: finite group s.t. $G \trianglelefteq \tilde{G}$
- $kG = \{ \sum_{g \in G} a_g g \mid a_g \in k \}$

Def

kG has a unique decomposition into a direct product of indecomp alg s

$$kG = B_1 \times \cdots \times B_\ell \quad \cdots (*)$$

We call each B_i a block of kG.

Example.

ch $k = 2$, $\quad kS_3 = \underbrace{k[\bullet \circlearrowright \alpha] / \langle \alpha^2 \rangle}_{} \times \underbrace{M_2(k)}_{}$

blocks of kS_3

Rem

The decomposition (*) gives us a direct decomposition

$$kG\text{-mod} \simeq B_1\text{-mod} \times \cdots \times B_\ell\text{-mod}.$$

★ The study of kG-mod

$\Leftrightarrow$ each study of B_i-mod.

Def.

- $\tilde{B}$: a block of $k\tilde{G}$
- B : a block of kG

$\tilde{B}$ covers B ($\tilde{B}$がBを被覆する.)

$:\Leftrightarrow 1_{\tilde{B}} \cdot 1_B \neq 0$

Rem.

In this case, there are some similarities between B-mod and $\tilde{B}$-mod.

easier to consider

Example.

$\operatorname{ch} k = 5$, $\quad G := A_5 \trianglelefteq S_5 =: \tilde{G}$

$$kS_5 \cong \underbrace{\circ \overset{1_1}{\text{———}} \circ \overset{3_1}{\text{———}} \circ \overset{3_2}{\text{———}} \circ \overset{1_2}{\text{———}} \circ} \times \underbrace{M_5(k)} \times \underbrace{M_5(k)}$$

$\downarrow$ cover $\qquad\qquad\qquad\qquad$ $\swarrow \ \swarrow$ cover

$$kA_5 \cong \overbrace{\circ \overset{1_1}{\text{———}} \circ \overset{3_1}{\text{———}} \bullet_{2}} \times \overbrace{M_5(k)}$$

Fact

$\operatorname{ch} k = p > 0$, $\quad G \trianglelefteq \tilde{G}$, $\quad |\tilde{G} : G| = p^n$

For any block B of kG,

there exists a unique block $\tilde{B}$ of $k\tilde{G}$

covering the block B.

Thm (Koshio-K, 2021)

$k = \bar{k}$, $\mathrm{ch}\, k = p$, $G \trianglelefteq \tilde{G}$, $|\tilde{G} : G| = p^n$.

B : a block of kG

$\tilde{B}$: the unique block of $k\tilde{G}$ covering B

Assume that

(★) $\begin{cases} \bullet\ B : \tau\text{-tilting finite } (\# \mathrm{s\tau\text{-}tilt}\, B < \infty) \\ \bullet\ {}^\forall U \in B\text{-mod}, {}^\forall \tilde{g} \in \tilde{G} ; \tilde{g}U \simeq U \text{ as } kG\text{-mods.} \end{cases}$

a kG-module s.t. ${}^\forall g \in G$ acts on U as follows:

$$g \cdot (\tilde{g}u) := \tilde{g}(\underline{\tilde{g}^{-1} g \tilde{g}} \cdot u)$$

$$\in \tilde{g}^{-1} G \tilde{g} = G$$

Then $\mathrm{Ind}_G^{\tilde{G}} := k\tilde{G} \otimes_{kG} - : B\text{-mod} \to \tilde{B}\text{-mod}$

induces poset isomorphisms

$$\mathrm{s\tau\text{-}tilt}\, B \simeq \mathrm{s\tau\text{-}tilt}\, \tilde{B}$$

and

$$2\text{-tilt}\, B \simeq 2\text{-tilt}\, \tilde{B}$$

Remark

If G has a cyclic Sylow p-subgroup, then the assumption ($\star$) is satisfied automatically.

Question

Can we generalize our thm? That is,

- $G \trianglelefteq \tilde{G}$
- B : a block of kG
- $\tilde{B}$: a block of $k\tilde{G}$ covering B

(1) Are there similarities between $s\tau$-tilt B and $s\tau$-tilt $\tilde{B}$?

(1)′ 〃 2-tilt B and 2-tilt $\tilde{B}$?

(2) Can we obtain sbrick $\tilde{B}$ from sbrick B?

(2)′ 〃 2smc $\tilde{B}$ from 2smc B?

Main Theorem 9

From now on we assume the following conditions hold:

$$
(*)\begin{cases}
\bullet\ {}^\forall \tilde{g} \in \tilde{G},\ {}^\forall S:\text{brick in } B; \\
\qquad \tilde{g}S \simeq S \text{ as } kG\text{-modules} \\
\bullet\ H^2(\tilde{G}/G, k^\times) = \{1\} \\
\bullet\ k[\tilde{G}/G]: \text{basic algebra}
\end{cases}
$$

Example.
If G has a cyclic Sylow p-subgp and $\tilde{G}/G$ is a p-group or a cyclic group, then $(*)$ holds automatically.

Thm (Koshio-K)

- The following maps are well-def and injective.

(1) $\mathrm{s\tau\text{-}tilt}\, B \ni M \longmapsto \tilde{B}\,\mathrm{Ind}_G^{\tilde{G}} M \in \mathrm{s\tau\text{-}tilt}\, \tilde{B}$

(1)′ $2\text{-}\mathrm{tilt}\, B \ni T \longmapsto \tilde{B}\,\mathrm{Ind}_G^{\tilde{G}} T \in 2\text{-}\mathrm{tilt}\, \tilde{B}$

Moreover these maps preserve partial orders.

- For any brick S in B-mod, there exists $e := \left| k[\tilde{G}/G] \right|$ bricks $\tilde{S}^{(1)}, \cdots, \tilde{S}^{(e)}$ in $k\tilde{G}$-mod s.t. $\mathrm{Res}_G^{\tilde{G}} \tilde{S}^{(i)} \simeq S$.

Moreover the following maps are well-def and injective.

(2) $\mathrm{sbrick}\, B \ni \bigoplus_{k=1}^{n} S_k \mapsto \bigoplus_{k=1}^{n} \bigoplus_{i=1}^{e} \tilde{B}\tilde{S}_k^{(i)} \in \mathrm{sbrick}\, \tilde{B}$

(2)′ $2\text{-}\mathrm{smc}\, B \ni \bigoplus_{k=1}^{n} S_k \oplus \bigoplus_{l=1}^{m} S_l[1]$

$\mapsto \bigoplus_{k=1}^{n} \bigoplus_{i=1}^{e} \tilde{B}\tilde{S}_k^{(i)} \oplus \bigoplus_{l=1}^{m} \bigoplus_{j=1}^{e} \tilde{B}\tilde{S}_l^{(j)}[1]$

<u>Thm</u> (Koshio - K)

We have the following commutative diagram:

$$
\begin{tikzcd}
 & \text{2-smc}\,B \arrow[rr, "(2)'"] \arrow[dd, "(\mathrm{Asai})"] & & \text{2-smc}\,\tilde{B} \arrow[dd, "(\mathrm{Asai})"] \\
\text{2-tilt}\,B \arrow[ur, "(\mathrm{KY})"] \arrow[rr, "(1)'"] & & \text{2-tilt}\,\tilde{B} \arrow[ur, "(\mathrm{KY})"] & \\
 & f_L\text{-sbrick}\,B \arrow[rr, "(2)"] & & f_L\text{-sbrick}\,\tilde{B} \\
s\tau\text{-tilt}\,B \arrow[uu, "(\mathrm{AIR})"] \arrow[ur, "(\mathrm{Asai})"] \arrow[rr, "(1)"'] & & s\tau\text{-tilt}\,\tilde{B} \arrow[uu, "(\mathrm{AIR})"] \arrow[ur, "(\mathrm{Asai})"] &
\end{tikzcd}
$$

Example.

$$p=3,\quad G := S_3 \simeq \overset{\langle a\rangle}{\overset{\shortparallel}{C_3}} \rtimes \overset{\langle t\rangle}{\overset{\shortparallel}{C_2}} \qquad t: a \mapsto a^{-1}$$

$$\widetilde{G} := (\underset{\langle a\rangle}{\underset{\cup}{C_3}} \times \underset{\langle b\rangle}{\underset{\shortparallel}{C_3}}) \rtimes \underset{\langle t\rangle}{\underset{\shortparallel}{C_2}} \qquad t: \begin{array}{l} a \mapsto a^{-1} \\ b \mapsto b \end{array}$$

$$\widetilde{\widetilde{G}} := (\overset{\shortparallel}{C_3} \times \overset{\shortparallel}{C_3}) \rtimes (\overset{\shortparallel}{C_2} \times \overset{\langle u\rangle}{\overset{\shortparallel}{C_2}}) \qquad u: \begin{array}{l} a \mapsto a^{-1} \\ b \mapsto b^{-1} \end{array}$$

Then $G \trianglelefteq \widetilde{G} \trianglelefteq \widetilde{\widetilde{G}}$.

$$kG = P(S_1) \oplus P(S_2) = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \\ 2 \end{array} = \circ \overset{1}{\text{———}} \circ \overset{2}{\text{———}} \circ$$

$$k\widetilde{G} = P(T_1) \oplus P(T_2) = \begin{array}{ccccc} & & 1 & & \\ & 2 & & 1 & \\ 1 & & 2 & & 1 \\ & 1 & & 2 & \\ & & 1 & & \end{array} \oplus \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 2 & & 1 & & 2 \\ & 2 & & 1 & \\ & & 2 & & \end{array}$$

$$(\operatorname{Res}^{\widetilde{G}}_{G} T_i = S_i)$$

$$k\widetilde{\widetilde{G}} = \bigoplus_{i=1}^{4} P(U_i) = \bigoplus_{i=1}^{4} \begin{array}{ccccc} & & i & & \\ & i+1 & & i+2 & \\ i & & i+3 & & i \\ & i+2 & & i+1 & \\ & & i & & \end{array} \qquad (i \in \mathbb{Z}/4\mathbb{Z})$$

$$\left(\operatorname{Res}^{\widetilde{\widetilde{G}}}_{\widetilde{G}} U_1 = \operatorname{Res}^{\widetilde{\widetilde{G}}}_{\widetilde{G}} U_3 = T_1,\ \operatorname{Res} U_2 = \operatorname{Res} U_4 = T_2\right)$$

For $M \in s\tau\text{-tilt}\, kG$,

$$\mathrm{Ind}_G^{\hat{G}} M = k\tilde{G} \otimes_{kG} M \in s\tau\text{-tilt}\, k\tilde{G}.$$

Moreover $\mathrm{Ind}_{\tilde{G}}^{\hat{\tilde{G}}} \mathrm{Ind}_G^{\tilde{G}} M = \mathrm{Ind}_G^{\hat{\tilde{G}}} M \in s\tau\text{-tilt}\, k\tilde{\tilde{G}}$

If $M = 1 \oplus P(S_1) \in s\tau\text{-tilt}\, kG$, then

$$\Longleftarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \in f_L\text{-sbrick}\, kG$$

$$\mathrm{Ind}_G^{\hat{G}} M = \begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix} \oplus P(T_1),$$

$$\Longleftarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \in f_L\text{-sbrick}\, kG$$

$$\mathrm{Ind}_G^{\hat{\hat{G}}} M = \begin{smallmatrix} 1 \\ 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \\ 3 \end{smallmatrix} \oplus P(U_1) \oplus P(U_3).$$

$$\Longleftarrow \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \in \text{sbrick}\, kG$$

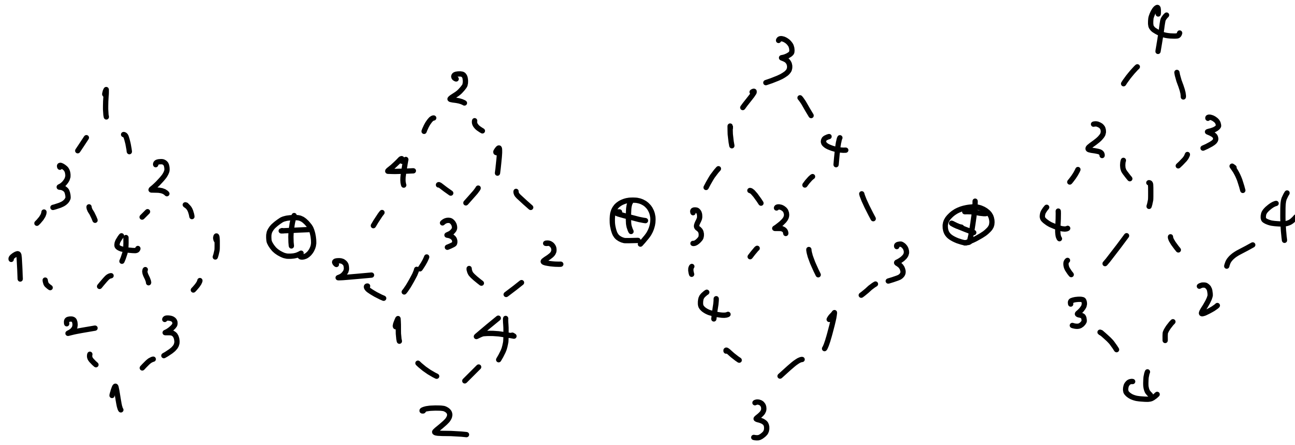
Remark

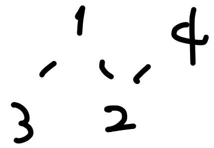
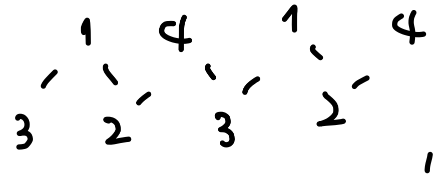
$k\widetilde{G}$ is a τ-tilting finite.

($\because$ $s\tau\text{-tilt}\, kG \simeq s\tau\text{-tilt}\, k\widetilde{G}$)

But $k\widetilde{\widetilde{G}}$ is not a τ-tilting infinite alg.

$\because$ For

$k\widetilde{\widetilde{G}}$ = 

$S^{(1)} :=$  , $S^{(2)} :=$  , ...

are bricks, so $k\widetilde{\widetilde{G}}$ has infinitely many bricks.