Rank 2 free subgroups in autoequivalence groups of Calabi–Yau categories

Kohei Kikuta

§1 Intro.

§2 Spherical twists and an inequality about iterations

§3 Proof of main results

§1. Intro.

Homological mirror symmetry

X : Calabi-Yau variety / $\mathbb{C}$

$(\check{X}, \omega)$: symplectic manifold

$$D^b \mathrm{Fuk}(\check{X}, \omega) \cong D^b(X)$$

$\cup\!\!\!\!\!\!\;\;$ L : Lag. submfd. of $\check{X}$	$\mathcal{E}$: coh. sh. on X
$\#(L_1 \cap L_2)$	$\sum_{i \in \mathbb{Z}} \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(\mathcal{E}_1, \mathcal{E}_2[i])$
Symp. mapping class gp. of $(\check{X}, \omega)$ $\pi_0 \mathrm{Symp}(\check{X}, \omega) \curvearrowright D^b \mathrm{Fuk}(\check{X}, \omega)$	Autoequivalence gp. of $D^b(X)$ $\mathrm{Aut}(D^b(X)) \curvearrowright D^b(X)$

Real dimension 2 vs Complex dimension 2

conn. ori. closed surf. Σ_g of genus $g \geqq 2$	cpx. alg. K3 surf. X
$MCG(\Sigma_g) = \pi_0 \, \mathrm{Symp}(\Sigma_g, \omega_0)$	$\mathrm{Aut}(D^b(X))$
simple closed curve $\gamma \in D^b \mathrm{Fuk}(\Sigma_g, \omega_0)$	smooth rational curve C (or $\mathcal{O}_C(i) \in D^b(X)$) More generally, *spherical object* $E \in D^b(X)$

Dehn twist $T_\gamma \in MCG(\Sigma_g)$	spherical twists $T_{\mathcal{O}_C(i)} \in \mathrm{Aut}(X)$
$i(\gamma_1, \gamma_2) = \sum_i \dim \mathrm{Hom}(\gamma_1, \gamma_2[i])$	$\sum_i \dim \mathrm{Hom}(E_1, E_2[i])$
topological entropy of $\phi \in MCG(\Sigma_g)$	categorical entropy of $\Phi \in \mathrm{Aut}(D^b(X))$

$\rightsquigarrow$ Study $\mathrm{Aut}(D^b(K3))$

as an analogue of $MCG(\Sigma_g)$

MCG

Today : Rank 2 free subgroup of $MCG(\Sigma_g)$

generated by two Dehn twists

$F_2 := \mathbb{Z} * \mathbb{Z}$: the group generated by 2 elements

with no relation

<u>Fact</u> $\gamma_1, \gamma_2 \subset \Sigma_g$: distinct s.c.c.

(1) If $i(\gamma_1, \gamma_2) = 0$

then $\langle T_{\gamma_1}^{k_1}, T_{\gamma_2}^{k_2} \rangle \cong \mathbb{Z} \times \mathbb{Z}$ for all $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$.

(2) If $i(\gamma_1, \gamma_2) = 1$

then $\langle T_{\gamma_1}^{k_1}, T_{\gamma_2}^{k_2} \rangle \cong F_2$.

except for $(|k_1|, |k_2|) = (|k_1|, 0), (0, |k_2|),$

$$\underbrace{(1,1), (2,1), (3,1), (1,2), (1,3)}_{\text{relation アリ}}.$$

(3) If $i(\gamma_1, \gamma_2) \geqq 2$

then $\langle T_{\gamma_1}^{k_1}, T_{\gamma_2}^{k_2} \rangle \cong F_2$ for all $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$.

Setting

k : a field

$\mathcal{T}$: triangulated category of fin. type $/ k$

$\exists$ exact equiv. $\mathcal{T} \simeq D(R)^c$.

- R : smooth proper dga $/k$
- $D(R)$: derived cat. of dg R-mod.
- $D(R)^c$: fall subcat. of compact obj. in $D(R)$

(R is called a "Morita enhacement" of $\mathcal{T}$)

Let $\tilde{\mathcal{T}} := D(R)$

$\rightsquigarrow$ Since $\left\{ \begin{array}{l} \exists \text{ dg-enhancement } D_{dg}(R) \text{ of } D(R) \\ \qquad\qquad \simeq \qquad D_{dg}^{perf}(R) \text{ of } D(R)^c, \end{array} \right.$

we have $\left\{ \begin{array}{l} \tilde{\mathcal{T}} \cong H^0(D_{dg}(R)) \\ \mathcal{T} \cong \tilde{\mathcal{T}}^c \cong H^0(D_{dg}^{perf}(R)). \end{array} \right.$

bounded cpx. of fin. gen. mod. unbounded cpx. of mod.

$$\begin{array}{ccc} \mathcal{T} & \subset & \tilde{\mathcal{T}} \\ \parallel & & \parallel \\ D(R)^c & \subset & D(R) \end{array}$$

- $M, N \in \tilde{\mathcal{T}}$

$$\hom^i(M, N) := \dim_k \operatorname{Hom}_{\tilde{\mathcal{T}}}(M, N[i])$$

$$\mathrm{i}(M, N) := \sum_{i \in \mathbb{Z}} \hom^i(M, N)$$

- $d > 1$: integer

$E \in \mathcal{T}$: d-spherical obj. i.e. $\begin{cases} \cdot \ \hom^i(E, E) = \begin{cases} 1 & i = 0, d \\ 0 & \text{o/w} \end{cases} \\ \cdot \ \text{"CY property"} \end{cases}$

$\rightsquigarrow$ sph. twist $T_E \in \operatorname{Aut}(\tilde{\mathcal{T}})$ preserves $\mathcal{T}$ $(= \tilde{\mathcal{T}}^c)$.

E.g. $\quad X$: cpx. alg. K3 surf.

$$\mathcal{T} = D^b(X) := D^b(\mathrm{Coh}(X))$$
$$\cap$$
$$\tilde{\mathcal{T}} = D(\mathrm{QCoh}(X))$$

$C_1, C_2 \subset X$: distinct (-2) curves on X

Then $\mathcal{O}_{C_1}, \mathcal{O}_{C_2} \in D^b(X)$ are spherical, and

$$i(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) = \hom^1_{D^b(X)}(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) = C_1 . C_2 .$$

i.e. concentrate in deg 1

$i \neq j \Rightarrow E_i \not\cong E_j[m], \ \forall m \in \mathbb{Z}$

<u>Thm (K)</u> $E_1, E_2 \in \mathcal{T}$: distinct d-sph. obj.

If $i(E_1, E_2) = 0$

then $\langle T_{E_1}^{k_1}, T_{E_2}^{k_2} \rangle \cong \mathbb{Z} \times \mathbb{Z}$

for all $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$.

Thm (k) $E_1, E_2 \in \mathcal{T}$: distinct d-sph. obj.

Assume a Conjecture.

(1) If $i(E_1, E_2) = 1$

then $\langle T_{E_1}^{k_1}, T_{E_2}^{k_2} \rangle \cong F_2$.

except for $(|k_1|, |k_2|) = (|k_1|, 0), (0, |k_2|),$

$\underbrace{(1,1), (2,1), (3,1), (1,2), (1,3).}_{\text{relation アリ}}$

(2) If $i(E_1, E_2) = \hom_{\mathcal{T}}^{j}(E_1, E_2) \geq 2$ for some $j \in \mathbb{Z}$,

then $\langle T_{E_1}^{k_1}, T_{E_2}^{k_2} \rangle \cong F_2$

for all $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$.

Related works

Thm (Jongmyeong Kim)

$E_1, \cdots, E_m \in \mathcal{T}$: mutually distinct d-sph. obj.

s.t. the dga $\mathrm{End}_{D^{perf}_{dg}(R)}(\oplus E_i)$ is formal, $\forall \alpha$

(1) If $i(E_i, E_j) = 0$ for $i \neq j$,

then $\langle T_{E_1}, \cdots, T_{E_m} \rangle \cong \mathbb{Z}^{\oplus m}$.

(2) If $i(E_i, E_j) \geq 2$ for $i \neq j$,

then $\langle T_{E_1}, \cdots, T_{E_m} \rangle \cong F_m$.