Sign-stable mutation loops & pseudo-Anosov mapping classes

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[IK'20-1] arXiv: 2010.05214 [IK'20-2] arXiv: 2011.14320

§1. Introduction

Integrable systems

e.g. - realizations of q-Painlevé eq's

(Cremona grp.) $\hookrightarrow \Gamma_{\mathfrak{s}(q\text{-Painlevé})}$

[Bershtein-Gavrylenko-Marshakov'17]

[Mizuno'20]

[Fock-Goncharov'09]

Cluster ensembles

$(\mathcal{A}_{\mathfrak{s}}, \mathcal{X}_{\mathfrak{s}})$

$\circlearrowleft$

$\Gamma_{\mathfrak{s}} = \{$mutation loops$\}$

Representation theory

e.g. - cluster str's on $G^{u,v}$

- realizations of Weyl groups

$W(\mathfrak{g}) \hookrightarrow \Gamma_{\mathfrak{s}_m(\mathfrak{g})}$

[Inoue-I.-Oya'21]

Teichmüller theory

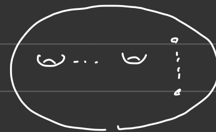
$\circlearrowleft$

$MC(\Sigma) \hookrightarrow \Gamma_{\mathfrak{s}_\Sigma}$

mapping class group

Goal: Generalize the classification / dynamical study of mapping classes to those of mutation loops $\in \Gamma_{\mathfrak{s}}$.

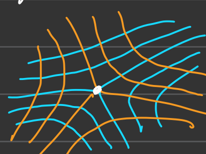
Nielsen–Thurston classification

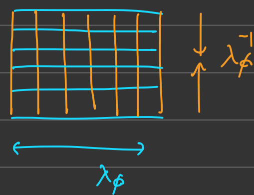
$\Sigma =$ 

$MC(\Sigma) \ni {}^{\forall}\phi$ is either :

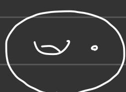
- periodic (fin. order)
- reducible ($\exists$ an inv. multicurve)

- pseudo-Anosov ($\exists$ a pair of inv. measured foliations)
  (pA)



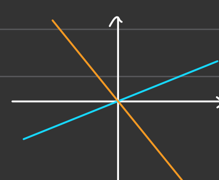


Topological entropy $= \log \lambda_\phi > 0$
("rich" dynamical systems).

Example $\Sigma =$ 

$$MC(\Sigma) \cong PSL_2(\mathbb{Z}) \quad \left(\curvearrowright (H_1(\Sigma;\mathbb{Z}) \setminus \{0\}) / \{\pm 1\} \right)$$
$$\phi \in PSL_2(\mathbb{Z})$$

- periodic $\Leftrightarrow$ $|\mathrm{tr}\,\phi| < 2$
- reducible $\Leftrightarrow$ $|\mathrm{tr}\,\phi| = 2$, $\phi \overset{\mathbb{Z}}{\sim} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, ${}^{\exists!}$eigendirection $\longleftrightarrow$ ${}^{\exists!}$inv. curve in Σ
- pA $\Leftrightarrow$ $|\mathrm{tr}\,\phi| > 2$, $\phi \overset{\mathbb{R}}{\sim} \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ for some $\lambda > 1$,
  ${}^{\exists}2$ eigendirections $\longleftrightarrow$ ${}^{\exists}2$ inv. fol. (w/ slope $\in \mathbb{R} \setminus \mathbb{Q}$)



Theorem (I.–Kano)

For a mapping class $\phi \in MC(\Sigma) \hookrightarrow \Gamma_{\mathfrak{s}_\Sigma}$,

1) ϕ: "generic" pA $\iff$ $\phi \in \Gamma_{\mathfrak{s}_\Sigma}$ is uniformly sign-stable

[IK'19~]

2) In this case, the stable presentation matrix of $\phi \in \Gamma_{\mathfrak{s}_\Sigma}$ satisfies a Perron–Frobenius property, and cluster stretch factor $= \cancel{\log}\, \lambda_\phi$.

Remark

- A general pA can be characterized by a weaker version of unif. SS.
- For a marked surface, pA $\iff$ weak SS + $\mathcal{C}(R_g)$-hereditariness

[IK'20-2]

§2. Mutation loops

Fix : - a fin. set $I = \{1, \cdots, N\}$

- $\mathcal{F}_A \cong \mathcal{F}_X \cong \mathbb{Q}(u_1, \cdots, u_N)$

A <u>seed</u> in $(\mathcal{F}_A, \mathcal{F}_X)$ is a tuple $(B, \mathbb{A}, \mathbb{X})$,

where · $B = (b_{ij})_{i,j \in I}$ is an integral skew-sym. matrix

(exchange matrix)

· $\mathbb{A} = (A_i)_{i \in I} \subset \mathcal{F}_A$, $\mathbb{X} = (X_i)_{i \in I} \subset \mathcal{F}_X$: alg. indep. elem's

cluster variables

For $k \in I$, the mutation $\mu_k : (B, \mathbb{A}, \mathbb{X}) \longrightarrow (B', \mathbb{A}', \mathbb{X}')$

affect

is defined by an explicit rule

$$b'_{ij} = \cdots$$

$$A'_i = \cdots$$

$$X'_i = \begin{cases} X_k^{-1} & (i = k) \\ X_i (1 + X_k^{-\mathrm{sgn}\, b_{ik}})^{-b_{ik}} & (i \neq k) \end{cases}$$

$$\begin{tikzcd}
\cdots \arrow[dr, leftrightarrow] & & & \cdots \arrow[dl, leftrightarrow] \\
& (B, \mathbb{A}, \mathbb{X}) \arrow[r, leftrightarrow, "\mu_k"] \arrow[dl, leftrightarrow] & (B', \mathbb{A}', \mathbb{X}') \arrow[dr, leftrightarrow] & \\
\cdots & & & \cdots
\end{tikzcd}$$

Permutations $\sigma : (B, \mathbb{A}, \mathbb{X}) \longrightarrow (B', \mathbb{A}', \mathbb{X}')$ defined by

$$b'_{ij} := b_{\sigma^{-1}(i),\sigma^{-1}(j)}, \quad A'_i := A_{\sigma^{-1}(i)}, \quad X'_i := X_{\sigma^{-1}(i)}.$$

Two seeds in $(\mathcal{F}_A, \mathcal{F}_X)$ are <u>mutation-equiv.</u>

if they are connected by a seq. of mutations & permutations.

$\rightsquigarrow$ equiv. class $\mathcal{S}$ is called a <u>mutation class.</u>

<u>Exchange graph</u>

$\mathrm{Exch}_{\mathcal{S}}$: a connected graph w/

<u>vertices</u> : seeds in $\mathcal{S}$ $\quad (v \in V(\mathrm{Exch}_{\mathcal{S}}) \longleftrightarrow \mathcal{S}^{(v)} = (B^{(v)}, \mathbb{A}^{(v)}, \mathbb{X}^{(v)}))$

<u>edges</u> : $\quad v \overset{k}{\text{---}} v' \iff \mathcal{S}^{(v')} = \mu_k \mathcal{S}^{(v)} \quad (k \in I)$

$$\begin{array}{c} v' \\ \big|\,\sigma \\ v \end{array} \iff \mathcal{S}^{(v')} = \sigma\, \mathcal{S}^{(v)} \quad (\sigma = (i\,j) \in \mathfrak{S}_I)$$

An edge path in $\mathrm{Exch}_{\mathcal{S}}$ is usually called a <u>mutation seq.</u>

$\exists$ a proj. $B^\bullet : V(\mathbb{E}\mathrm{xch}_s) \longrightarrow \mathrm{Mat}$

$\qquad\qquad\qquad\qquad v \longmapsto B^{(v)}$

<u>Def</u> A <u>mutation loop</u> is a graph autom ϕ of $\mathbb{E}\mathrm{xch}_s$ which preserves each fiber of $B^\bullet$.

$\Gamma_s := \{\text{mutation loop}\} < \mathrm{Aut}(\mathbb{E}\mathrm{xch}_s)$: <u>cluster modular grp.</u>

- $\phi \in \Gamma_s$, $v_0 \in \mathbb{E}\mathrm{xch}_s$

an edge path $\gamma : v_0 \longrightarrow \phi^{-1}(v_0)$ in $\mathbb{E}\mathrm{xch}_s$ is called a <u>representation path</u> of ϕ.

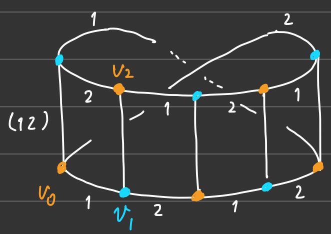
$\rightsquigarrow \exists \mu_\gamma : s^{(v_0)} \longrightarrow s^{(\phi^{-1}(v_0))}$ s.t. $B^{(v_0)} = B^{(\phi^{-1}(v_0))}$

<u>Rem</u> Alternatively, $\Gamma_s = \{\text{edge paths in } \mathbb{E}\mathrm{xch}_s\} / \sim$ [IK'21-1]

Example (Type A_2)

$$\mathfrak{s}(v_0) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, (A_1, A_2), (X_1, X_2) \right) \quad \text{in } \mathcal{F}_A = \mathbb{Q}(A_1, A_2),\ \mathcal{F}_X = \mathbb{Q}(X_1, X_2)$$

$$\mathrm{Exch}_{A_2} =$$



$$\xrightarrow{B^\bullet} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

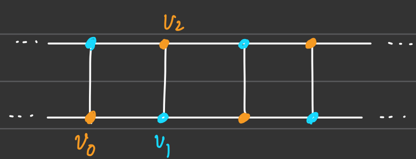
$\Gamma_{\mathfrak{s}} = \langle \phi \rangle \cong \mathbb{Z}/5$, $\gamma : v_0 \xrightarrow{1} v_1 \xrightarrow{(12)} v_2$ is a rep. path of ϕ.

Example (Kronecker quivers)

$$\mathfrak{s}(v_0) = \left(\begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix}, (A_1, A_2), (X_1, X_2) \right) \quad \text{in } \mathcal{F}_A = \mathbb{Q}(A_1, A_2),\ \mathcal{F}_X = \mathbb{Q}(X_1, X_2)$$

$(\ell \in \mathbb{Z}_{\geq 2})$

$$\mathrm{Exch}_{K_\ell} =$$



$$\xrightarrow{B^\bullet} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

$\Gamma_{\mathfrak{s}} = \langle \phi \rangle \cong \mathbb{Z}$, $\gamma : v_0 \xrightarrow{1} v_1 \xrightarrow{(12)} v_2$ is a rep. path of ϕ.

<u>Sign stability</u> $\quad \delta \rightsquigarrow (\mathcal{A}_\delta, \mathcal{X}_\delta)$: cluster varieties [FG'09]

$$\underline{\text{e.g.}} \quad \mathcal{X}_\delta = \bigcup_{v \in \mathbb{E}\mathrm{xch}_\delta} \underbrace{\mathcal{X}_{(v)}}_{\cong (\mathbb{C}^\times)^I} \qquad \mathcal{X}_{(v)} \overset{\mu_k^x}{\dashleftarrow\dashrightarrow} \mathcal{X}_{(v')}$$

$$\circlearrowright \Gamma_\delta \qquad\qquad \mathcal{X}_{(v')} \overset{\sigma^x}{\updownarrow} \mathcal{X}_{(v)}$$

Tropical cluster variety

- $v \in \mathbb{E}\mathrm{xch}_\delta \rightsquigarrow \mathcal{X}^{trop}_{(v)} := \mathbb{R}^I$ w/ linear coord's $(x_i^{(v)})_{i \in I}$
- $v \overset{k}{\text{———}} v' \rightsquigarrow \mu_k^{trop} : \mathcal{X}^{trop}_{(v)} \xrightarrow{\sim} \mathcal{X}^{trop}_{(v')}$ <u>PL map</u>

$$(\mu_k^{trop})^* x_i^{(v')} = \begin{cases} -x_k^{(v)} \\ x_i^{(v)} - b_{ik} \min\{0, -(\mathrm{sgn}\, b_{ik}) x_k^{(v)}\} \end{cases}$$

- $\begin{matrix} v' \\ \big| \sigma \\ v \end{matrix} \rightsquigarrow \sigma^{trop}$: ——— permutation

$$\Rightarrow \mathcal{X}^{trop}_\delta = \bigcup_{v \in \mathbb{E}\mathrm{xch}_\delta} \mathcal{X}^{trop}_{(v)} \ : \text{PL mfd.}$$

$$\circlearrowright \Gamma_\delta$$

<u>Explicitly</u>: $\phi \in \Gamma_\delta$, $\gamma : v_0 \longrightarrow \phi^{-1}(v_0)$ rep. path

$$\rightsquigarrow \phi_{(v_0)} : \mathcal{X}^{trop}_{(v_0)} \xrightarrow{\mu_\gamma^{trop}} \mathcal{X}^{trop}_{(\phi^{-1}(v_0))} \cong \mathcal{X}^{trop}_{(v_0)}$$

$$x_i \longleftrightarrow x_i, \ \forall i \in I$$

Note: $\mu_k^{trop}: \mathcal{X}_{(v)}^{trop} \longrightarrow \mathcal{X}_{(v')}^{trop}$ is linear on $\mathcal{H}_{k,\varepsilon}^{(v)} := \{\varepsilon x_k^{(v)} \geq 0\}$.

$\gamma: v_0 \xrightarrow{k_0} v_1 \xrightarrow{k_1} v_2 \cdots \cdots \xrightarrow{k_{m-1}} v_m = \phi^{-1}(v_0)$: a rep. path of ϕ

$\{k_{i(0)}, \cdots, k_{i(h-1)}\} \subset \{k_0, \cdots, k_m\}$: horizontal indices

Define $\mathcal{C}_\gamma^\varepsilon := \bigcap_{\nu=0}^{h-1} \mathcal{H}_{k_{i(\nu)}, \varepsilon_\nu}^{(v_{i(\nu)})} \subset \mathcal{X}_{(v_0)}^{trop}$ for $\varepsilon = (\varepsilon_0, \cdots, \varepsilon_{h-1}) \in \{+,-\}^h$

Then $\phi_{(v_0)}$ is linear on $\mathcal{C}_\gamma^\varepsilon$, and $\mathcal{X}_{(v_0)}^{trop} = \bigcup_\varepsilon \mathcal{C}_\gamma^\varepsilon$

Def γ is $\underbrace{\text{sign-stable}}^{\text{weakly}}$ on an $\mathbb{R}_{>0}$-inv. set $\Omega \subset \mathcal{X}_{(v_0)}^{trop}$

$:\Longleftrightarrow \exists\, \underbrace{\varepsilon_\gamma^{stab}}_{\text{stable sign}} \in \{+,-\}^h$ s.t. $\forall w \in \Omega \setminus \{0\}$ $\exists n_0 \geq 0$:

$\forall n \geq n_0 \quad \phi_{(v_0)}^n(w) \in (\mathrm{int})\,\mathcal{C}_\gamma^{\varepsilon_\gamma^{stab}}$

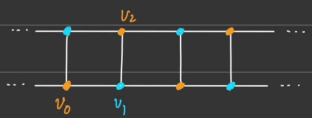
Rem • γ: sign-stab $\underset{\not\Leftarrow}{\Longrightarrow}$ $\{\phi_{(v_0)}^n\}$ is stably a linear dynamical system

$\phi_{(v_0)}|_{\mathcal{C}^{stab}} =: E_{\phi,\Omega}^{(v_0)}$ (stable presentation mat.)

• γ determines a factorization $E_{\phi,\Omega}^{(v_0)} = \underbrace{J_{k_0} \cdots J_{k_m}}_{\substack{\text{pres. mat. of } \mu_k|_{\mathcal{H}_{k,\varepsilon}} \text{ (hori.)} \\ \text{or } \sigma \text{ (vert.)}}}$

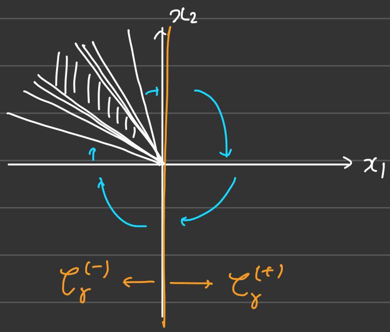
Example (Kronecker quivers)

$$\left[\begin{array}{l} s^{(v_0)} = \left(\begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix}, (A_1, A_2), (X_1, X_2) \right) \quad (\ell \in \mathbb{Z}_{\geq 2}) \quad \text{in } \mathcal{F}_A = \mathbb{Q}(A_1, A_2),\ \mathcal{F}_X = \mathbb{Q}(X_1, X_2) \\ \mathbb{E}\mathrm{xch}_{K_\ell} = \cdots \xrightarrow{B^\circ} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \\ \Gamma_s = \langle \phi \rangle \cong \mathbb{Z}, \end{array}\right]$$



γ: $v_0 \xrightarrow{\ 1\ } v_1 \xrightarrow{(12)} v_2$ is a rep. path of $\phi \in \Gamma_{K_\ell}$

$\mathcal{X}^{\mathrm{trop}}_{(v_0)}$:



$$\phi_{(v_0)} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_2 + \ell \min\{0, x_1\} \\ -x_1 \end{pmatrix}$$

- γ is sign-stab. on $\Omega = \mathcal{X}^{\mathrm{trop}}_{(v_0)}$
- $E^{(v_0)}_{\phi, \Omega} = \begin{pmatrix} \ell & 1 \\ -1 & 0 \end{pmatrix}$
- eigenvalues : $\lambda^2 - \ell\lambda + 1 = 0$

$$\lambda = \frac{\ell \pm \sqrt{\ell^2 - 4}}{2}$$

<u>Theorem</u> (Perron–Frobenius property [IK'21])

γ : sign-stable on a "tame" set Ω

$\Rightarrow$ spectral radius of $E^{(v_0)}_{\phi,\Omega}$ is an eigenvalue $\lambda_{\phi,\Omega} \geq 1$.

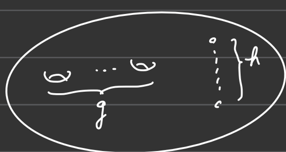
Under a mild assumption, the eigenvector $\in \mathcal{C}^{\mathrm{stab}}_{\gamma} \setminus \{0\}$.

- When $\Omega \supset \mathcal{C}^{\pm}_{(v_0)} := \{ \pm x^{(v_0)}_k \geq 0 \ , \forall k \in I \}$,

  we call $\lambda_\phi := \lambda_{\phi,\Omega}$ the <u>cluster stretch factor</u> of ϕ.

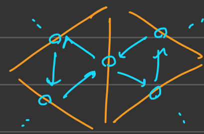
<u>Remark</u> The algebraic entropies of $\phi \circlearrowright \mathcal{A}_{\mathfrak{s}}, \mathcal{X}_{\mathfrak{s}}$ are estimated by $\log \lambda_\phi$ [IK'21]

§3. Mapping classes on a punctured surface

$\Sigma = \Sigma^h_g =$  , $\chi(\Sigma) = 2-2g-h < 0$, $h > 0$

$MC(\Sigma) := \mathrm{Homeo}^+(\Sigma, \{\text{puncture}\}) / \text{isotopy}$

exchange matrices

Δ : an ideal triangulation of Σ $\rightsquigarrow$ $Q_\Delta :=$ 

Then $\{Q_\Delta\}_\Delta$ are mutation-equiv.

$\rightsquigarrow$ $\exists$ a canonical class $s = s_\Sigma$

Rem $\exists$ constructions of seeds in $\mathcal{F}_A = K(\mathcal{A}_{SL_2,\Sigma})$, $\mathcal{F}_X = K(\mathcal{X}_{PGL_2,\Sigma})$

[FG'06]

Since $MC(\Sigma) \overset{\text{free}}{\curvearrowright}$ $\{$ideal triangulations$\}$, $MC(\Sigma) \hookrightarrow \Gamma_{s_\Sigma}$.

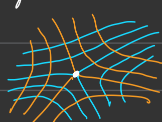
(II) Geometric models of $\mathcal{X}^{trop}_{s_\Sigma}$: measured foliations / laminations

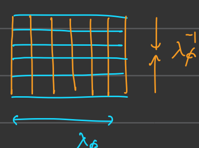
Nielsen–Thurston classification

$MC(\Sigma) \ni {}^\forall \phi$ is either :

- periodic (fin. order)
- reducible ($\exists$ an inv. multicurve)

- pseudo-Anosov ($\exists$ a pair of inv. measured foliations)
  (pA)





<u>Def</u> A rational (unbounded) lamination consists of:

- a collection $L = \{(\gamma_j, \underset{\in \mathbb{Q}_{>0}}{w_j})\}$ of mutually disjoint weighted curves

- a tuple $\sigma_L = (\sigma_p) \in \{+, 0, -\}^{\mathbb{P}}$ of signs

  s.t. $\exists \gamma_j$ incident to $p \iff \sigma_p \in \{+, -\}$

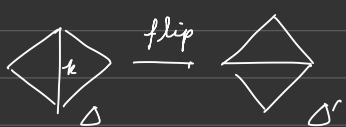
$\mathcal{L}^x(\Sigma, \mathbb{Q}) := \{\text{rational lamination}\}/\text{isotopy}$

<u>Shear coordinates</u> $x_\Delta = (x_e^\Delta) : \mathcal{L}^x(\Sigma, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^{e(\Delta)}$ defined by:

① make a spiralling diagram:

② $x_e^\Delta :=$ weighted sum of the contributions

<u>Lemma</u>

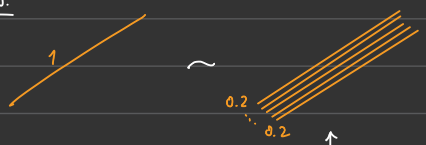


$$x_{\Delta'} \circ x_\Delta^{-1} = \mu_k^{trop} : \mathbb{Q}^{e(\Delta)} \longrightarrow \mathbb{Q}^{e(\Delta')}$$

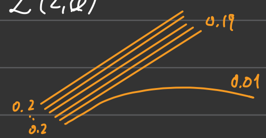
$\mathcal{L}^x(\Sigma, \mathbb{R}) := \overline{\mathcal{L}^x(\Sigma, \mathbb{Q})}$ how it looks like?

$\xrightarrow[\sim]{PL} \mathcal{X}_{s_\Sigma}^{trop}$

obs.



close in $Z^x(\Sigma,\mathbb{Q})$ $\updownarrow$

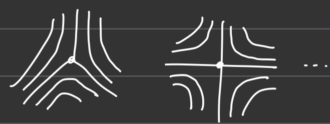


$\cdots\longrightarrow$ $\left\{ \begin{array}{l} \text{lamination} \\ \text{foliation} \end{array} \right.$

weights (numbers) Σ $\cdots\longrightarrow$ "measures" $\int$

Def A measured foliation consists of:

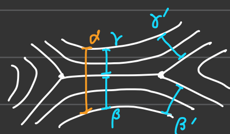
- $\mathcal{F}$: a foliation on Σ w/ admissible sing.



- μ : a transverse measure on $\mathcal{F}$

$\alpha \longmapsto \mu(\alpha) = $ "$\displaystyle\int_{\mathcal{F}^\perp} \alpha$" $\in \mathbb{R}_{>0}$

transverse arc to $\mathcal{F}$



$\mu(\alpha) = \mu(\beta) + \mu(\gamma)$
$= \mu(\beta') + \mu(\gamma')$

$(\mathcal{F},\mu)$ is considered modulo isotopy &

Whitehead equivalence :

canonical model

(w/ no saddle connections)

<u>Rem</u> $\mathcal{MF}(\Sigma) \cong \{ \sum_{\partial e \ni p} x_e^\Delta = 0 \mid p \in \mathbb{P} \} < \mathcal{X}^{trop}_{\mathfrak{S}_\Sigma}$

stretch factor

$\phi \in MC(\Sigma)$ is pA $:\Longleftrightarrow$ $\exists (\mathcal{F}_\pm, \mu_\pm) \in \mathcal{MF}(\Sigma)$, $\exists \Lambda_\phi > 1$

s.t. $\phi(\mathcal{F}_\pm, \mu_\pm) = (\mathcal{F}_\pm, \Lambda_\phi^{\pm 1} \cdot \mu_\pm)$

Moreover if the canonical model of $(\mathcal{F}_\pm, \mu_\pm)$ only have 3-plonged singularities, then ϕ is called a <u>generic pA</u>.

<u>Important lemma</u>:

If the can. model of $(\mathcal{F}, \mu)$ only have 3-plonged sing's, then $x_e^\Delta(\mathcal{F}, \mu) \neq 0$, $\forall e \in \forall \Delta$.

In particular, for any mutation seq. γ, $(\mathcal{F}, \mu) \in \text{int}\, \mathcal{C}_\gamma^\varepsilon$ for some ε.

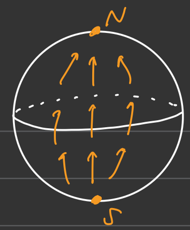
<u>Theorem</u>

$\phi \in MC(\Sigma)$ is generic pA $\Leftrightarrow$ any rep. path of ϕ is sign-stable on

$$\Omega_{\mathbb{Q}} := \mathbb{R}_{>0} \cdot \underbrace{\mathcal{X}_{\mathfrak{S}_\Sigma}(\mathbb{Q}^{trop})}_{= \mathcal{L}^x(\Sigma, \mathbb{Q})}$$

<u>i.e</u> ϕ is uniformly sign stable.

Moreover, $\Lambda_\phi = \lambda_\phi$.



Sketch of proof:

- A pA mapping class induces the North-South dynamics on $\mathbb{S}\mathcal{X}^{trop}_{\mathfrak{S}_\Sigma}$:

$\mathbb{S}V := (V \setminus \{0\})/\mathbb{R}_{>0}$

$$\forall [w] \in \mathbb{S}\mathcal{X}^{trop}_{\mathfrak{S}_\Sigma} \setminus \{[\mathcal{F}_\mp, \mu_\mp]\}, \qquad \lim_{n\to\infty} \phi^{\pm n}([w]) = [\mathcal{F}_\pm, \mu_\pm]$$

- Assume ϕ is generic pA. Then for any rep. path γ,

$$\exists\, \varepsilon_\gamma \in \{+,-\}^{h(\gamma)} \quad \text{s.t.} \quad (\mathcal{F}_+, \mu_+) \in \operatorname{int} \mathcal{C}_\gamma^{\varepsilon_\gamma}.$$

The NS dynamics implies

$\mathcal{L}^x(\Sigma, \mathbb{Q})$

$$\forall w \in \mathcal{X}^{trop}_{\mathfrak{S}_\Sigma} \setminus \mathbb{R}_{>0} \cdot (\mathcal{F}_-, \mu_-), \qquad \phi^n(w) \in \operatorname{int} \mathcal{C}_\gamma^{\varepsilon_\gamma} \quad \text{for } n \gg 1.$$

Thus "generic pA $\Rightarrow$ unif. SS".

The converse follows from the NT classification:

$$\phi : \begin{cases} \text{periodic} \\ \text{reducible} \\ \text{non-generic pA} \end{cases} \Rightarrow \exists \text{ non-sign-stable rep. path.}$$

- $T_w \mathcal{X}^{trop}_{\mathfrak{S}_\Sigma} \cong T_w \mathcal{ML}(\Sigma) \oplus \mathbb{R}^h$, $\quad E^{(v_0)}_{\phi,\Omega} = (d\phi)_{(\mathcal{F}_+, \mu_+)} = \left(\begin{array}{c|c} T_\phi & * \\ \hline 0 & \underbrace{\sigma}_{\text{permutation of punctures}} \end{array}\right)$

- It is classically known that $\lambda_\phi = \rho(T_\phi) = \rho(E^{(v_0)}_{\phi,\Omega})$ $\blacksquare$