§0. Introduction

Keywords

- (local) cohomological dimension of a specialization closed subset of Spec R
  ⋯ geometry
- n-wide subcategories of Mod R / n-uniform subcategories of D(Mod R)
  ⋯ category theory, representation theory
- n-coherent subsets of Spec R
  ⋯ ideal theory.

Aim

Study the relation among them using "small support".

Def

For $M \in \operatorname{Mod} R$ (D(Mod R)), the small support of M is

$$\operatorname{supp}(M) := \{ \mathfrak{p} \in \operatorname{Spec} R \mid \underbrace{\operatorname{Tor}_i^R(k(\mathfrak{p}), M) \neq 0 \quad (\exists i \geq 0)}_{(\Leftrightarrow\ k(\mathfrak{p}) \otimes^{L}_{R} M \neq 0)} \} \qquad k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$$

$$\underset{\text{Chen-Iyengar.}}{=} \bigcup_{i \geq 0} \underset{\{\mathfrak{p} \mid E(R/\mathfrak{p}) <_{\oplus} E^i(M)\}}{\operatorname{Ass}(E^i(M))}$$

Note: $\operatorname{Ass} M \subset \operatorname{supp} M \subset \operatorname{Supp} M := \{ \mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \neq 0 \}$

For $W \subset \operatorname{Spec} R$, set

$$\operatorname{supp}_M^{-1}(W) := \{ M \in \operatorname{Mod} R \mid \operatorname{supp}(M) \subset W \}$$

$$\operatorname{supp}_D^{-1}(W) := \{ X \in D(\operatorname{Mod} R) \mid \operatorname{supp}(X) \subset W \}$$

For $W \subset \operatorname{Spec} R$,

$\exists\ \gamma_W$: local cohomology functor

$$\begin{tikzcd} \operatorname{supp}_D^{-1}(W) \arrow[r, hook] & D(\operatorname{Mod} R) \arrow[l, bend right, "\top"'] \\ \operatorname{supp}_M^{-1}(W) \arrow[r, hook] \arrow[u, phantom, "\cup" rotate=90] & \operatorname{Mod} R \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

If W is specialization closed,

$$H^i(\gamma_W(M)) \cong H^i_W(M) := R^i\Gamma_W(M)$$

$$\left(\quad \Gamma_W(M) = \{ x \in M \mid \operatorname{Supp}(Rx) \subset W \} \quad \right)$$

Def

For a specialization closed subset $W \subset \operatorname{Spec} R$,

$$\operatorname{cd}(W) := \inf \{ i \geq 0 \mid H^{>i}_W(M) = 0 \text{ for } {}^\forall M \in \operatorname{Mod} R \}$$

: the (local) cohomological dimension of W.

(if W is closed,

$\operatorname{cd}(W) = $ (the cohomological dimension of the scheme $\operatorname{Spec} R \setminus W$) $+1$)

◎ Known results.

<u>Thm</u> (Gabriel 1962, Neeman 1992)

$\exists$ bij

$$\left\{\begin{array}{l}\oplus\text{-closed Serre}\\ \text{subcat. of } \mathrm{Mod}R\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{specialization closed}\\ \text{subsets of } \mathrm{Spec}R\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{smashing subcat}\\ \text{of } D(\mathrm{Mod}R)\end{array}\right\}$$

$$\mathcal{X} \longmapsto \mathrm{supp}(\mathcal{X}) := \bigcup_{M\in\mathcal{X}} \mathrm{supp}(M) \longleftarrow\!\shortmid\ \mathcal{X}$$

$$\mathrm{supp}_M^{-1}(W) \longleftarrow\!\shortmid\ W \longmapsto \mathrm{supp}_D^{-1}(W)$$

Here,

- $\mathcal{X} \subset \mathrm{Mod}R$: <u>Serre</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{X}$ is closed under submodules, quotient modules, extensions.

- $\mathcal{X} \subset D(\mathrm{Mod}R)$: <u>smashing</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{X}$ is $\oplus$-closed thick & $\mathcal{X} \hookrightarrow D(\mathrm{Mod}R)$ has a right adj. which preserves direct sums.

- $W \subset \mathrm{Spec}R$: <u>specialization closed</u> (sp. cl.) $\overset{\text{def}}{\Longleftrightarrow}$ $\forall\, p \subset q \subset \mathrm{Spec}R$, $p \in W \Rightarrow q \in W$.

<u>Easy Fact</u>

For a sp. cl. subset $W \subset \mathrm{Spec}R$,

$$\mathrm{cd}(W) \leq 0 \Leftrightarrow W^c \text{ is sp. cl.}$$

Thm (Krause 2008)

$\exists$ bij.

$$\left\{\begin{array}{l}\oplus\text{-closed wide}\\ \text{subcat. of } \mathrm{Mod}R\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{coherent subsets}\\ \text{of } \mathrm{Spec}R\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}(\oplus, H)\text{-closed thick}\\ \text{subcat. of } D(\mathrm{Mod}R)\end{array}\right\}$$

Here,

- $\mathcal{X} \subset \mathrm{Mod}R$: <u>wide</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{X}$ is closed under kernels, cokernels, extensions.
- $\mathcal{X} \subset D(\mathrm{Mod}R)$: <u>H-closed</u> $\overset{\text{def}}{\Longleftrightarrow}$ " $X \in \mathcal{X} \Rightarrow H^i(X) \in \mathcal{X}$ ($\forall i$)"
- $W \subset \mathrm{Spec}R$: <u>coherent</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\forall\, I^0 \xrightarrow{f} I^1$ with I^i : inj, $\mathrm{Ass}\, I^i \subset W$,

  then

  $\exists\, I^0 \to I^1 \to I^2$: ex. seq.

  with I^2 : inj., $\mathrm{Ass}\, I^2 \subset W$

Thm (Angeleri Hügel, Marks, Šťovíček, Takahashi, Vitória 2018)

For a sp. cl. subset $W \subset \mathrm{Spec}R$,

$$\mathrm{cd}(W) \leq 1 \Longleftrightarrow W^c : \text{coherent}$$

We want to consider higher cohomological dim $\mathrm{cd}(W) \leq n$.

$\rightsquigarrow$ n-wide subcat, n-uniform subcat, , n-coherent subset

Plan

§0 Introduction

§1 Subcategories of $\mathrm{Mod}R$ / $D(\mathrm{Mod}R)$

§2. Subsets of $\mathrm{Spec}R$

§3. Main Result

§1. Subcategories of $\mathrm{Mod}R$ / $D(\mathrm{Mod}R)$

◎ Subcategories of $\mathrm{Mod}R$

Def

A subcat. $\mathcal{X} \subset \mathrm{Mod}\,R$ is

- <u>closed under n-kernels</u> if
  "$0 \to M \to X^0 \to X^1 \to \cdots \to X^n$, $X^i \in \mathcal{X}$"
  $\Longrightarrow M \in \mathcal{X}$
- <u>Closed under n-cokernels</u> if
  "$X_n \to \cdots \to X_1 \to X_0 \to M \to 0$, $X_i \in \mathcal{X}$"
  $\Longrightarrow M \in \mathcal{X}$
- <u>n-wide</u> if $\mathcal{X}$ is closed under n-ker, n-cok, ext.

Observation

(1) closed under 0-kernel $\Longleftrightarrow$ closed under submod.
 — " — 1 — " — $\qquad$ — " — kernels
 closed under 0-cok $\Longleftrightarrow$ closed under quot. mod
 — " — 1 — " — $\qquad$ — " — cokernels

In particular,

0-wide $=$ Serre

1-wide $=$ wide

(2) 0-wide $\Longrightarrow$ 1-wide $\Longrightarrow \cdots \Longrightarrow$ ∞-wide
 $\;\;\|\qquad\qquad\|$
 Serre $\qquad$ wide

Prop 1.1 $(n \in \mathbb{N})$

Let $F : \mathrm{Mod}\,R \longrightarrow \mathrm{Mod}\,R$ be a left exact functor with $R^{>n}F = 0$.
Then

$$\{ M \in \mathrm{Mod}\,R \mid R^{\geq 0}F(M) = 0 \}$$

is n-wide.

Similar result holds for the left derived functor of a right ex. funct.

$H_W^{>n}(-) = 0$

$\Updownarrow$

Ex 1.2

(1) Let $W \subset \mathrm{Spec}\,R$: sp. cl. with $\mathrm{cd}(W) \leq n$.
Then

$$\{ M \in \mathrm{Mod}\,R \mid H_W^{\geq 0}(M) = 0 \} = \mathrm{supp}^{-1}(W^c)$$

is n-wide:

$$\mathrm{cd}(W) \leq n \Longrightarrow \mathrm{supp}^{-1}(W^c) : n\text{-wide}.$$

(2) Let $T \in \mathrm{Mod}\,R$, $\mathrm{id}\,T \leq n$.
Then

$$\{ M \in \mathrm{Mod}\,R \mid \mathrm{Ext}_R^{\geq 0}(M, T) = 0 \}$$

is n-wide

(3) Let $T \in \mathrm{Mod}\,R$, $\mathrm{pd}\,T \leq n$.
Then

$$\{ M \in \mathrm{Mod}\,R \mid \mathrm{Ext}_R^{\geq 0}(T, M) = 0 \}$$

$$\{ M \in \mathrm{Mod}\,R \mid \mathrm{Tor}^R_{\geq 0}(T, M) = 0 \}$$

are n-wide

◎ Subcategories of $D(\mathrm{Mod} R)$

Def

A thick subcat. $\mathcal{X} \subset \mathcal{D}(R)$ is <u>n-uniform</u>

$\overset{\text{def}}{\iff}$ Let $X \in \mathcal{X}$, $i \in \mathbb{Z}$ s.t.

- $H^j(X) = 0$ for $i \neq j \in (i-n, i+n)$
- $X^j \in \mathcal{X}$ for $j \in [i-n, i+n]$

Then

$$Z^i(X), \ X^i/B^i(X) \in \mathcal{X}.$$

Observation

(1) If $n \geq 1$ and $\mathcal{X}$ is $\oplus$-closed, then

$\mathcal{X}$ is n-uniform

$\iff$ Let $X \in \mathcal{X}$, $i \in \mathbb{Z}$ s.t.

- $H^j(X) = 0$ for $i \neq j \in (i-n, i+n)$

Then $H^i(X) \in \mathcal{X}$

This uses the results by Neeman, Nakamura-Yoshino :

- (Neeman) $\mathcal{X} = \mathrm{supp}_D^{-1}(W)$ for $\exists W \subset \mathrm{Spec} R$
- (Nakamura-Yoshino) $\mathrm{supp}(X) \subset W \Rightarrow X \cong {}^{\exists}Y$ in $D(\mathrm{Mod} R)$ s.t. $\mathrm{supp}\, Y^i \subset W$ $(\forall i)$

(2) $\oplus$-closed 0-uniform = smashing

-"— 1-uniform = H-closed

-"— ∞-uniform = $\forall$ $\oplus$-closed thick subcat.

(3) 0-uniform $\Rightarrow$ 1-uniform $\Rightarrow$ 2-uniform $\Rightarrow \cdots \Rightarrow$ ∞-uniform.

Prop 1.3 $(n \in \mathbb{N})$

Let $F : \mathrm{Mod}\, R \longrightarrow \mathrm{Mod}\, R$ be a left exact functor with $R^{>n}F = 0$.
Then

$$\{ X \in D(\mathrm{Mod}\, R) \mid RF(X) = 0 \}$$

is n-uniform

Ex 1.4

(1) Let $W \subset \mathrm{Spec}\, R$: sp. cl. with $\mathrm{cd}(W) \leq n$
Then

$$\{ X \in D(\mathrm{Mod}\, R) \mid R\Gamma_W(X) \cong 0 \} = \mathrm{supp}_D^{-1}(W^c)$$

is n-uniform.

$\therefore$ $\mathrm{cd}(W) \leq n \Longrightarrow \mathrm{supp}_D^{-1}(W^c)$: n-uniform.

(2) Let $T \in \mathrm{Mod}\, R$ s.t. $\mathrm{id}\, T \leq n$.
Then

$$\{ X \mid R\mathrm{Hom}(X, T) \cong 0 \}$$

is n-uniform

(3) Let $T \in \mathrm{Mod}\, R$ s.t. $\mathrm{pd}\, T \leq n$.
Then

$$\{ X \mid R\mathrm{Hom}_R(T, X) \cong 0 \}$$

$$\{ X \mid T \overset{L}{\otimes}_R X \cong 0 \}$$

are n-uniform.

§2 Subsets of Spec R

Def

A subset $W \subset \operatorname{Spec} R$ is <u>n-coherent</u>

$\overset{\text{def}}{\iff}$ For an ex. seq

$$I_n \to \cdots \to I_1 \xrightarrow{d_1} I_0 \to C \to 0 \quad \text{with } I_i \in \operatorname{Inj} R, \ \operatorname{Ass} I_i \subset W$$

$\operatorname{Ass}(C) \subset W$.

Observation

(1) 0-coherent = sp. cl.

1-coherent = coherent

∞-coherent = $\forall$ subsets

(2) 0-coh $\Rightarrow$ 1-coh $\Rightarrow \cdots \Rightarrow$ ∞-coh.

◎ How to find n-coherent subsets

Prop 2.1

TFAE for $n \geq 0$

(1) $\forall$ subset of Spec R is n-coh

(2) $\forall$ sp. cl. subset W of Spec R, $\operatorname{cd}(W) \leq n$

(3) $\dim R \leq n$

Rem

This result uses a big Cohen-Macaulay module M,

whose existence is { one of "the homological conjectures"
proved by André (2016).

To prove (1) $\Rightarrow$ (3), use the min. inj. resol. of M.

Prop 2.2

$W \subset \operatorname{Spec} R$

$\forall \mathfrak{p} \in W,\ \operatorname{ht} \mathfrak{p} \leq n \Rightarrow W^c$: n-coh.

Ex 2.3

R : 1-dim ring, $W \subset \operatorname{Spec} R$

- W : 0-coh $\overset{\text{Obs (1)}}{\Longleftrightarrow}$ W : sp. cl.
- $\forall$ subsets are 1-coh. (by Prop 2.1)

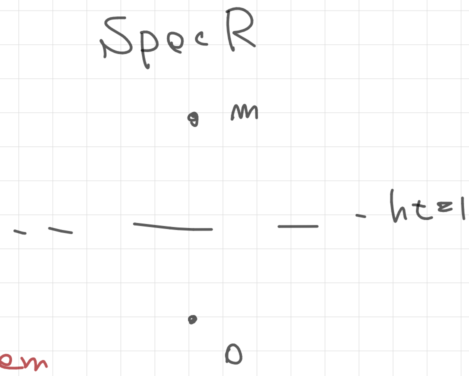
Ex 2.4

$R = k[\![x, y]\!]$, W ($\neq \emptyset$) : generalization closed (i.e. W^c : sp. cl)

- W : 0-coh $\Leftrightarrow$ W : sp. cl
- W : 1-coh $\Leftrightarrow$ $\mathfrak{m} \in W \Leftrightarrow W = \operatorname{Spec} R$

  ($\Leftarrow$) : Prop 2.2

  ($\Rightarrow$) : Hartshorne - Lichtenbaum vanishing theorem



- $\forall$ (gen. cl.) subsets are 2-closed

  by Prop 2.1

§3. Main Result.

Summary

	0	1	...	∞
n-wide	Serre	wide	...	
⊕-closed n-uniform	smashing	H-closed	...	$\forall$ ⊕-closed thick subcat
n-coh.	sp.cl.	coherent	...	$\forall$ subset

Main Thm

For $n \in \mathbb{N} \cup \{\infty\}$, $\exists$ bij.

$$\underset{\mathcal{W}_n}{\left\{ \begin{array}{l} (\oplus, E)\text{-closed } n\text{-wide} \\ \text{subcat. of } \operatorname{Mod} R \end{array} \right\}} \rightleftarrows \overset{\mathcal{C}_n}{\left\{ \begin{array}{l} n\text{-coherent subsets} \\ \text{of } \operatorname{Spec} R \end{array} \right\}} \rightleftarrows \overset{\mathcal{U}_n}{\left\{ \begin{array}{l} \oplus\text{-closed} \\ n\text{-uniform subcat.} \\ \text{of } D(\operatorname{Mod} R) \end{array} \right\}}$$

$$\mathcal{X} \longmapsto \operatorname{supp}(\mathcal{X}) \longleftarrow\!\shortmid \mathcal{X}$$

$$\operatorname{supp}_M^{-1}(W) \longleftarrow\!\shortmid W \longmapsto \operatorname{supp}_D^{-1}(W)$$

Here, $\mathcal{X} \subset \operatorname{Mod} R$: $\underline{E\text{-closed}} \overset{\text{def}}{\iff} {}^\forall M \in \mathcal{X},\ E^i(M) \in \mathcal{X}$ $({}^\forall i)$

$$\begin{array}{ccccccc} \mathcal{W}_0 & \subset & \mathcal{W}_1 & \subset & \cdots & \subset & \mathcal{W}_\infty \\ \downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\ \mathcal{C}_0 & \subset & \mathcal{C}_1 & \subset & \cdots & \subset & \mathcal{C}_\infty \\ \downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\ \mathcal{U}_0 & \subset & \mathcal{U}_1 & \subset & \cdots & \subset & \mathcal{U}_\infty \end{array}$$

Fact

$\mathcal{X} \subset \operatorname{Mod} R$: ⊕-closed (1-) wide $\Longrightarrow$ $\mathcal{X}$: E-closed.

Cor 3.1

(1) $n = 0$

Gabriel (1962) Neeman (1992)

$$\left\{\begin{array}{c}\oplus\text{-closed Serre subcat.}\\ \text{of } \mathrm{Mod}\, R\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{sp. cl. subsets}\\ \text{of } \mathrm{Spec}\, R\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{smashing subcat.}\\ \text{of } D(\mathrm{Mod}\, R)\end{array}\right\}$$

(2) $n = 1$

Krause (2008)

$$\left\{\begin{array}{c}\oplus\text{-closed wide subcat.}\\ \text{of } \mathrm{Mod}\, R\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{coherent subsets}\\ \text{of } \mathrm{Spec}\, R\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}(\oplus, H)\text{-closed thick}\\ \text{subcat of } D(\mathrm{Mod}\, R)\end{array}\right\}$$

(3) $n = \infty$

Takahashi (2009) Neeman (1992)

$$\left\{\begin{array}{c}(\oplus, E)\text{-closed } \infty\text{-wide subcat.}\\ \text{of } \mathrm{Mod}\, R\end{array}\right\} \rightleftarrows 2^{\mathrm{Spec}\, R} \rightleftarrows \left\{\begin{array}{c}\oplus\text{-closed thick subcat.}\\ \text{of } D(\mathrm{Mod}\, R)\end{array}\right\}$$

Thm 3.2 (M-Nam-Takahashi-Tri-Yen)

For a sp. cl. subset $W \subset \mathrm{Spec}\, R$,

$$\mathrm{cd}(W) \leq n \Longrightarrow \mathrm{supp}_M^{-1}(W^c) : n\text{-wide}$$

and "$\Longleftarrow$" holds if $n \leq 1$ or $n \geq \dim R - 1$.

Cor 3.3

For a sp. cl. subset $W \subset \mathrm{Spec}\, R$,

$$\mathrm{cd}(W) \leq n \Longrightarrow W^c : n\text{-coherent}$$

and "$\Longleftarrow$" holds if $n \leq 1$ or $n \geq \dim R - 1$

new

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Rem

For any $2 \leq n \leq \dim R - 2$, $\exists$ counter example to "$\Longleftarrow$".