§ Introduction

§ Matrix factorization

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(§ Proof)

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§1 Introduction

<u>Def</u> $\mathcal{T}$: triangulated category (tri cat).

$T \in \mathcal{T}$: <u>tilting</u> $\overset{\text{def}}{\iff}$ (1) $\mathrm{Hom}(T, T[i]) = 0$ $(\forall i \neq 0)$

(2) $\mathcal{T}$ = the smallest thick subcategory containing T.

<u>Thm</u> (Keller, Bondal-Orlov)

$\mathcal{T}$: idempotent complete algebraic tri cat. $T \in \mathcal{T}$: tilting obj.

Then, $\mathcal{T} \cong K^b(\mathrm{proj}\, \mathrm{End}(T))$. Moreover, if $\mathcal{T}$ has a strong generator, $\mathcal{T} \cong D^b(\mathrm{mod}\, \mathrm{End}(T))$.

$\boxed{X \in \mathcal{T} \text{ is \underline{strong generator}} \iff \exists n > 0 \text{ s.t. } \mathcal{T} = \langle T \rangle_n}$

<u>Question</u> G : $\overbrace{\text{abelian}}^{\text{f.g.}}$ grp of rank 1

R : G-graded comm Gorenstein ring.

When does $\underline{\mathrm{CM}}^G R$ admit a tilting object?

@ low dimensional case

$R = \bigoplus_{i \geq 0} R_i$: $\mathbb{Z}$-graded Gorenstein ring s.t. R_0 is a field.

<u>Thm</u> (Yamaura '13)

If $\dim R = 0$, $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ has a tilting obj.

<u>Thm</u> (Buchweitz - Iyama - Yamaura)

If $\dim R = 1$ and R is reduced, then $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ has a tilting obj.

@ hypersurfaces defined by invertible poly.

<u>Def</u> $f \in \mathbb{C}[x_1, \dots, x_n]$: quasi-homogeneous poly.

f : <u>invertible polynomial</u>

$\overset{\text{def}}{\Longleftrightarrow}$ (1) We can write

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} \left(\prod_{j=1}^{n} x_j^{E_{ij}} \right)$$

and $E_f := (E_{ij}) \in GL_n(\mathbb{Q})$.

(2) $\tilde{f} := \sum_{i=1}^{n} \left(\prod_{j=1}^{n} x_j^{E_{ji}} \right)$ is also quasi-homog.

$\tilde{f}$ is called the <u>Berglund-Hübsch transpose</u> of f.

(3) $1 \leqslant \mu(f) = \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n] / \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle < \infty$

$1 \leqslant \mu(\hat{f}) < \infty$.

<u>Thm</u> (Kreuzer-Skarke, '92)

An inv poly is the Thom-Sebastiani sums of several inv poly of the following two types

(chain type) $x_1^{a_1} + x_1 x_2^{a_2} + x_2 x_3^{a_3} + \cdots + x_{n-1} x_n^{a_n}$ ($a_i \geqslant 1$, $a_1 > 1$, $a_n > 1$)

(loop type) $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_n^{a_n} x_1$ ($a_i \geqslant 1$, $n \geqslant 2$)

(<u>Thom-Sebastiani sum</u> of $f \in \mathbb{C}[x_1, \dots, x_n]$ and $g \in \mathbb{C}[y_1, \dots, y_m]$ is $f \boxplus g := f \otimes 1 + 1 \otimes g \in \mathbb{C}[x_1, \dots, x_n] \otimes_{\mathbb{C}} \mathbb{C}[y_1, \dots, y_m]$
$= \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$)

<u>e.g.</u> (1) (Fermat poly) $x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$

(2) ADE poly

<u>Def</u> $f \in \mathbb{C}[x_1, \dots, x_n]$: inv poly, $E_f = (E_{ij})$

$$L_f := \left(\left(\bigoplus_{i=1}^{n} \mathbb{Z} \vec{x_i} \right) \oplus \mathbb{Z} \vec{f} \right) \Big/ \left\langle \vec{f} - \sum_{j=1}^{n} E_{ij} \vec{x_j} \,\middle|\, i \right\rangle$$

is an abelian grp of rk = 1. (called <u>maximal grading</u> of f)

Conj A (Takahashi, Lekili-Ueda)

$\underline{CM}^{Lf}(\mathbb{C}[x_1, \ldots, x_n]/f)$ admits a tilting obj.

Rmk (1) Conj A is reduced to the case when f is of chain type or loop type.

(2) Conj A is expected from view point from mirror symmetry

$$(\mathbb{C}^n, f) \xleftrightarrow{\text{LG mirror}} (\mathbb{C}^n, \widetilde{f})$$

(3) Conj A is proved for $\begin{cases} \text{special } n=3 \text{ cases by Kajiura-Saito-Takahashi ('07 and '09)} \\ \text{general } n \leq 3 \text{ cases by Kravets ('19).} \end{cases}$

Thm (H-Ouchi)

Conj A is true when f is of chain type.

§2 Matrix factorizations.

G : abelian grp

$R := \bigoplus_{g \in G} R_g$: G-graded comm ring.

$f \in R_d$: homogeneous element of $\deg = d \in G$.

<u>Def</u> (1) A (G-graded) <u>matrix factorization</u> (m.f.) of f is a sequence

$$F = \left(F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_1(d) \right)$$

s.t. F_i : G-graded free module of finite rank.

φ_i : homomorphism preserving degrees.

s.t. $\begin{cases} \varphi_0 \circ \varphi_1 = f \cdot id_{F_1} \\ \varphi_1(d) \circ \varphi_0 = f \cdot id_{F_0} \end{cases}$

where $F_i(d) := \bigoplus_{g \in G} F_i(d)_g$ with $F_i(d)_g := (F_i)_{g+d}$.

(2) $HMF^G_R(f)$ is defined

Obj := $\{$ m.f. of f $\}$

$$Hom(E,F) := \left\{ (\alpha_1, \alpha_0) \;\middle|\; \begin{tikzcd} E_1 \arrow[r, "\varphi_1^E"] \arrow[d, "\alpha_1"] \arrow[dr, phantom, "\circlearrowleft"] & E_0 \arrow[r, "\varphi_0^E"] \arrow[d, "\alpha_0"] \arrow[dr, phantom, "\circlearrowleft"] & E_1(d) \arrow[d, "\alpha_1(d)"] \\ F_1 \arrow[r, "\varphi_1^F"] & F_0 \arrow[r, "\varphi_0^F"] & F_1(d) \end{tikzcd} \right\} \Big/ \sim$$

homotopy equiv

Prop $HMF_R^G(f)$ is a tri cat.

e.g. $F[1] := \left(F_1 \xrightarrow{\varphi_0} F_1(d) \xrightarrow{\varphi_1(d)} F_0(d) \right)$.

Rmk If f is non-zero-divisor, then φ_i are injective

$rk(F_1) = rk(F_0)$

$$\rightsquigarrow \begin{cases} \cdot\ 0 \to F_1 \xrightarrow{\varphi_1} F_0 \to \mathrm{Cok}(\varphi_1) \to 0 & : \text{exact} \\ \cdot\ f \cdot \mathrm{Cok}(\varphi_1) = 0 \end{cases}$$

$$\rightsquigarrow \begin{cases} \cdot\ \mathrm{proj\,dim}_R\, \mathrm{Cok}(\varphi_1) = 1 \\ \cdot\ \mathrm{Cok}(\varphi_1) \in \mathrm{mod}^G R/f \end{cases}$$

$\rightsquigarrow$ If R is regular and $f \neq 0$, $\mathrm{Cok}(\varphi_1) \in \mathrm{CM}^G R/f$.

Thm (Eisenbud) R : regular, $f \neq 0$.

$$\begin{array}{rcl} \mathrm{Cok} : HMF_R^G(f) & \longrightarrow & \underline{\mathrm{CM}}^G(R/f) \quad \text{is an equiv.} \\ \rotatebox{90}{$\in$}\ \ \ & & \\ F & \longmapsto & \mathrm{Cok}(\varphi_1^F). \end{array}$$

§3 Main result.

Def $\mathcal{T}$: tri cat. $\mathcal{T}_1, \dots, \mathcal{T}_r \subseteq \mathcal{T}$: seq of full admissible tri sub cat.

(1) $\mathcal{T}_1, \dots, \mathcal{T}_r$ is semi-orthogonal decomposition (s.o.d) of $\mathcal{T}$

$\overset{\text{def}}{\iff}$ (a) for all $i > j$, $\mathrm{Hom}_{\mathcal{T}}(X_i, X_j) = 0$ for $\forall X_i \in \mathcal{T}_i$, $\forall X_j \in \mathcal{T}_j$.

(b) $\mathcal{T}$ = the smallest full tri sub cat containing all $\mathcal{T}_i$.

In this case, we write $\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_r \rangle$.

(2) $\mathcal{T}_1, \dots, \mathcal{T}_r$ is orthogonal decomposition of $\mathcal{T}$

$\overset{\text{def}}{\iff}$ (a)' for $\forall i \neq j$, $\mathrm{Hom}(\mathcal{T}_i, \mathcal{T}_j) = 0$

(b) in (1)

In this case, $\mathcal{T} = \bigoplus_{i=1}^{r} \mathcal{T}_i$.

Def $\mathcal{T}$: tri cat. $E \in \mathcal{T}$.

(1) E is exceptional $\overset{\text{def}}{\iff}$ $\mathrm{Hom}(E, E[i]) = \begin{cases} k & (i=0) \\ 0 & (i \neq 0) \end{cases}$

(2) $\mathcal{E}: E_1, \dots, E_r$: seq of excep obj

$\mathcal{E}$ is <u>full exceptional collection</u> (f.e.c.)

$\overset{\text{def}}{\iff} \mathcal{T} = \langle \langle E_1 \rangle, \dots, \langle E_r \rangle \rangle.$

(3) $\mathcal{E}: E_1, \dots, E_r$: f.e.c.

$\mathcal{E}$ is <u>strong</u> $\iff \mathrm{Hom}(E_i, E_j[l]) = 0$ if $l \neq 0$

<u>Rmk</u> $\mathcal{E}: E_1, \dots, E_r$: f.s.e.c $\Rightarrow E := \bigoplus_{i=1}^{r} E_i$ is tilting.

◎ conjectures from mirror symmetry

<u>Conj B</u> (Homological LG mirror symmetry, Takahashi).

$f \in S^n := \mathbb{C}[x_1, \dots, x_n]$: inv poly.

$\exists$ finite acyclic quive Q & $\exists I$: admissible relations in Q

s.t. $\exists$ equiv of tri cat's

$$\mathrm{HMF}^{L_f}_{S^n}(f) \cong D^b(\mathrm{mod}\, kQ/I) \cong D^b\mathrm{Fuk}^{\rightarrow}(\tilde{f})$$

Fukaya-Seidel cat.

Rmk $D^b \mathrm{Fuk}^{\to}(\tilde{f})$ has f.e.c. of length $\mu(\tilde{f})$.

Conj C $\mathrm{HMF}^{L_f}_{S^n}(f)$ admits a f.s.e.c of length $\mu(\tilde{f})$.

In particular, it has tilting obj

⊚ explicit construction of f.s.e.c. for chain polynomials.

$a_1, \dots, a_n \in \mathbb{Z}_{\geq 1}$ $(a_1 > 1,\ a_n > 1)$

$S^n := \mathbb{C}[x_1, \dots, x_n]$

$f_n := x_1^{a_1} + x_1 x_2^{a_2} + x_2 x_3^{a_3} + \cdots + x_{n-1} x_n^{a_n}$

$L_n := L_{f_n}$

$\mathcal{C}_n := \mathrm{HMF}^{L_n}_{S^n}(f^n)$ $(n \geq 3)$

• For $F \in \mathcal{C}_{n-1}$, $\left(\widetilde{(-)} := (-) \otimes_{S^{n-1}} S^n \right)$

$$\Psi F := \left(\widetilde{F_1} \oplus \widetilde{F_0}(-\vec{x}_n) \xrightarrow{\begin{pmatrix} \tilde{\varphi}_1 & x_n \\ -x_{n-1} x_n^{a_n - 1} & \tilde{\varphi}_0 \end{pmatrix}} \widetilde{F_0} \oplus \widetilde{F_1}(\vec{f}_n - \vec{x}_n) \xrightarrow{\begin{pmatrix} \tilde{\varphi}_0 & -x_n \\ x_{n-1} x_n^{a_n - 1} & \tilde{\varphi}_1 \end{pmatrix}} (-)(\vec{f}_n) \right)$$

$\in \mathcal{C}_n$

$\left(f_n = f_{n-1} + x_n \left(x_{n-1} x_n^{a_n - 1} \right) \right)$

· For $E \in \mathcal{C}_{n-2}$,

$$\Phi E := \left(\hat{E}_1 \oplus \hat{E}_0(-\vec{x}_{n-1}) \xrightarrow{\begin{pmatrix}\hat{\varphi}_1 & x_{n-1} \\ -g & \hat{\varphi}_0\end{pmatrix}} (\ —\ ) \xrightarrow{\begin{pmatrix}\hat{\varphi}_0 & -x_{n-1} \\ g & \hat{\varphi}_1\end{pmatrix}} (\ —\ ) \right)$$

$\in \mathcal{C}_n$

$$\left(g := x_{n-2} x_{n-1}^{a_{n-1}-1} + x_n^{a_n} \rightsquigarrow f_n = f_{n-2} + x_{n-1} g \right)$$

$$\Psi_i : \mathcal{C}_{n-1} \to \mathcal{C}_n \ ; \ F \mapsto \Psi F(i \vec{x}_n)$$

$$\Phi_j : \mathcal{C}_{n-2} \to \mathcal{C}_n \ ; \ E \mapsto \Phi E(j \vec{x}_{n-1} + (-a_n+1) \vec{x}_n)$$

<u>Thm</u> (H-Ouchi)

Ψ_i and Φ_j are fully faithful, and $\exists$ s.o.d.

$$\mathcal{C}_n = \left\langle \mathrm{Im}\,\Psi_0, \dots, \mathrm{Im}\,\Psi_{(-a_n+2)}, \bigoplus_{j=0}^{-a_{n-1}+1} \mathrm{Im}\,\Phi_j \right\rangle.$$

($a_n = 1$ case also holds).

Cor (1) Conj C is true if f is of chain type.

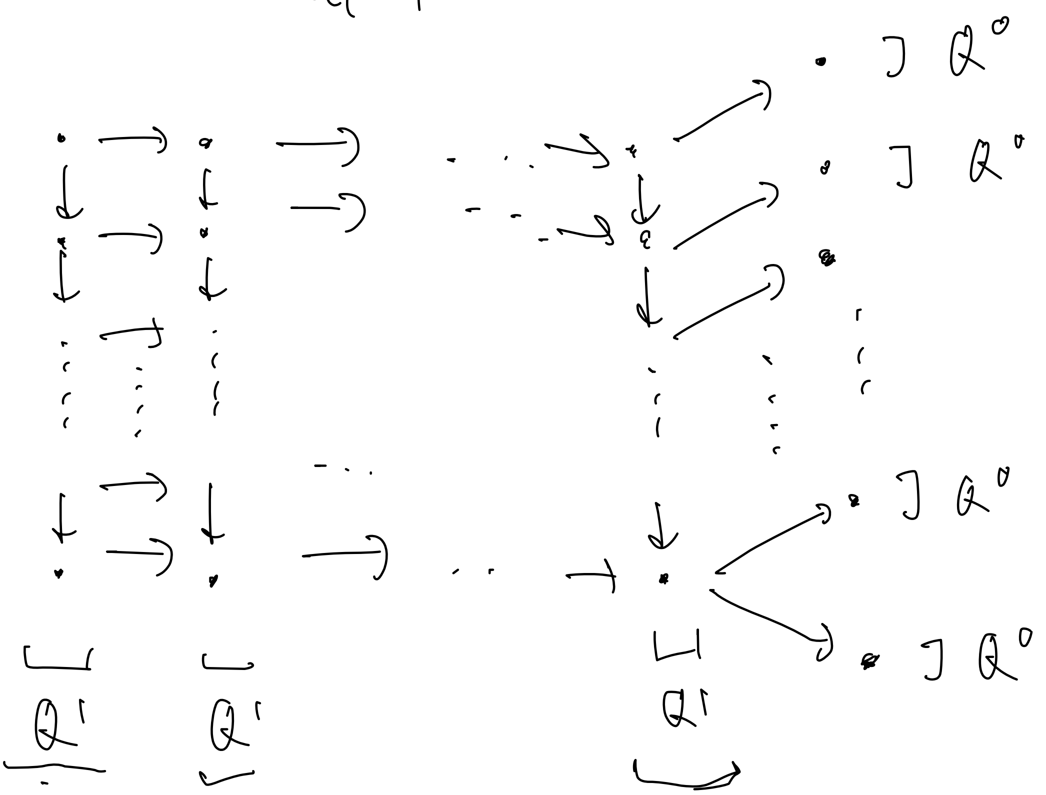
(2) $\exists$ explicit description of (Q^n, I^n) s.t. $\mathcal{E}_n \cong D^b(\mathrm{mod}\, kQ^n/I^n)$

$$Q^n : \quad \underbrace{Q^{n-1} \to Q^{n-1} \to \cdots \to Q^{n-1}}_{a_n - 1} \quad \left.\begin{array}{l} \to Q^{n-2} \\ \to Q^{n-2} \\ \quad \vdots \\ \to Q^{n-2} \end{array}\right\} a_{(n-1)}$$

$Q^0 : \quad \bullet$ $\quad \left(S^0 = \mathbb{C},\ f_0 = 0,\ L_0 = \mathbb{Z}\vec{f}_0 \cong \mathbb{Z} \to \mathcal{E}_0 \cong D^b(\mathrm{mod}\, \mathbb{C}). \right)$

$$Q^1 : \quad \underbrace{\bullet \to \bullet \to \cdots \to \bullet}_{a_1 - 1}$$

Q^2 :



Proof of s.o.d

$G := \mathrm{Spec}\, \mathbb{C}[L_n]$

$\mathbb{G}_m \times G \curvearrowright Q := \mathbb{A}^n_x \times \mathbb{A}^1_u$: certain action

$W := f_{n-1} + x_{n-1} x_n^{a_n} u \ : Q \to \mathbb{C}$.

$\lambda : \mathbb{G}_m \to \mathbb{G}_m \times G \ ; \ t \mapsto (t, 1)$

$Q \setminus Q_+ := \{ x \in Q \mid \lim_{a \to 0} \lambda(a).x \in Z \}$

$\lambda \rightsquigarrow \quad Q_+ = \mathbb{A}^n_x \times (\mathbb{A}^1_u \setminus \{0\}) \subseteq Q$

$Q_- = \mathbb{A}^{n-1}_x \times (\mathbb{A}^1_{x_n} \setminus \{0\}) \times \mathbb{A}^1_u \subseteq Q$

Z : fixed locus of λ-action on Q.

By VGIT by [Ballard-Favero-Katzarkov, Halpern-Leistner]

$\Longrightarrow$ $\exists\, \Phi : \mathrm{DMF}_G(Q_-, W) \hookrightarrow \mathrm{DMF}_G(Q_+, W)$: f.f.

$\exists\, \Psi_j : \mathrm{DMF}_{G/\mathrm{Im}\lambda}(Z, W) \hookrightarrow \mathrm{DMF}_G(Q_+, W)$: f.f.

$\exists$ s.o.d $\mathrm{DMF}_G(Q_+, W) = \langle \mathrm{Im}\,\Psi_0, \ldots, \mathrm{Im}\,\Psi_{-(a_n+2)}, \mathrm{Im}\,\Phi \rangle$

Then we show $\mathrm{DMF}(Q_+, W) \simeq \mathcal{C}_n$

$\mathrm{DMF}(Q_-, W) \simeq \bigoplus \mathcal{C}_{n-1}$]

$\mathrm{DMF}(Z, W) \simeq \mathcal{C}_{n-1}$) $\square$

Knörrer periodicity.