Yoneda algebras of quasi-hereditary algebras, and simple-minded systems of triangulated categories

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Declaration

I declare that I have composed this thesis myself, that it has not been accepted in any previous application for a degree. I also declare that this dissertation is the result of my own work and contains nothing which is the outcome of my work done in collaboration with others, except as specified below.

Chapter 6 and Section 8.1 are modified from a joint work with Steffen Koenig and Yuming Liu. Proposition 8.2.13 is a result obtained in a discussion with Takuma Aihara and Takahide Adachi.

All quotations have been distinguished by quotation marks and the sources of information specifically acknowledged.

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Abstract

This thesis is divided into two parts. The first part studies homological algebra of quasi-hereditary algebras, with the underlying theme being to understand properties of the Yoneda algebra of standard modules. We will first show how homological properties of a quasi-hereditary algebra are carried over to its tensor products and wreath products. We then determine the ext-groups between indecomposable standard modules of a Cubist algebra of Chuang and Turner. We will also determine generators, hence the quiver, of the Yoneda algebra of standard modules for the rhombal algebras of Peach. We also obtain a higher multiplication vanishing condition for certain rhombal algebras.

The second part of this thesis studies the notion of simple-minded systems, introduced by Koenig and Liu. Such systems were designed to generate the stable module categories of artinian algebras by extension, in the same way as the sets of simple modules. We classify simple-minded systems for representation-finite self-injective algebras, and establish connections of them to various notions in combinatorics and related derived categories. We also look at the notion of simple-minded systems defined on triangulated categories, and obtain some classification results using a connection between the simple-minded systems of a triangulated category and of its orbit category.

Introduction

In the modern representation theory of algebras, homological algebra can very often be used to reveal the beauty hidden in the structure of the representations (modules). While this thesis touches on a variety of classes of algebras, our main interests are the quasi-hereditary algebras and the self-injective algebras. We explore various homological aspects for these two classes of algebras in this thesis.

For a non-semi-simple self-injective algebra, the fact that it has infinite global dimension already indicates difficulties in understanding its homological structure. Even for algebras with much more exploitable structure, such as blocks of group algebras of finite groups, there are well-known long-standing conjectures about their homological structure. One prominent example is Broué's abelian defect conjecture, which asserts that a block of group algebra with abelian defect is derived equivalent to its local block. This is one of the holy grails in group representation theory these days; and yet, we still lack a grand unified theory which can solve the problem efficiently.

If we look to an even smaller class of algebras, namely the group algebras of the symmetric groups, then we have a much richer source of combinatorics which can be used to study representations. Moreover, the long established Schur-Weyl duality hints that we can relate these algebras to Lie theory; the result of this investigation is the Schur algebra. The Schur algebra allows us to study representations of symmetric group, and representations of algebraic groups simultaneously. Unlike self-injective algebras, the Schur algebras have finite global dimension. This means that their homological structure should be relatively easy. In particular, the higher extension groups of modules can only go up to a certain degree. In fact, Schur algebras belong to an extremely nice (at least in terms of homological behaviour) class of algebras of finite global dimension - the quasi-hereditary algebras. Another important class of examples of quasi-hereditary algebras comes from Lie theory, namely the BGG category $\mathcal{O}$ of a finite dimensional semi-simple complex Lie algebra.

In the first part of this thesis, we study the homological structure of a quasi-hereditary algebra.

For a quasi-hereditary algebra, there is a special family of modules, called the standard modules, which plays the role of Specht modules in the representation theory of symmetric groups. This family of modules is indexed in the same way as the simple modules and projective modules; it also fuses properties of simple modules and projective modules - the endomorphism ring of a standard module is one dimensional, and quotienting out the algebra with some heredity ideal sends the standard module to a projective module. Out of the several families of structural modules, the Yoneda algebra of (the direct sum of) standard modules is the least understood. The motivation to the first part of the thesis is to investigate this Yoneda algebra for quasi-hereditary algebras which are related to the Schur algebras.

In a recent study [Mad1, Mad2], based on investigation by Drozd-Marzorchuk [DM], Madsen showed that the Yoneda algebra of standard modules for certain quasi-hereditary algebras admits a duality theory similar to Koszul algebras. Such algebras are said to satisfy the condition (H). A class of examples of such algebras are the blocks of Schur algebras which are of finite type. Some of these algebras appear as quasi-hereditary covers of the weight 1 blocks of the group algebras of symmetric groups. Homological properties of Schur algebras often boil down to the so-called Rouquier blocks (or RoCK blocks), as each block of a Schur algebra is derived equivalent to some Rouquier block. Each Rouquier block is Morita equivalent to the wreath product of a block of finite type. Taking this point of view, we study homological properties of the wreath products of a quasi-hereditary algebra. After going through preliminary material in Chapter 1, we then show in Chapter 2 that if a quasi-hereditary algebra satisfies the condition (H), then so do its wreath products (Proposition 2.3.6). Along the way, we give proofs of various other folklore which are needed to build our result. These lemmas are well-known to experts, but most of them are not written in the literature.

In Chapter 3, we focus on a class of infinite dimensional algebras which are simultaneously symmetric and quasi-hereditary - the Cubist algebras of Chuang and Turner [CT3]. This class of algebras provides an abundance of examples for which the condition (H) is not satisfied. On the other hand, the nature of the Cubist algebras allows one to obtain homological structure using almost pure combinatorics of tilings. This gives us an appropriate starting point to look at the Yoneda algebras of standard modules in the more general case. Our main result in the chapter is the complete description of the Ext-groups between standard modules (Theorem 3.3.1), obtained by exploiting the combinatorics of cubical tilings in Euclidean space. Moreover, we will see that the Ext-group non-vanishing condition closely resembles that of a quasi-hereditary algebra satisfying condition (H) - an interesting phenomenon which has not been observed before (Proposition 3.3.2).

We then further investigate into the structure of the Yoneda algebras of standard modules for rhombal algebras in Chapter 4. The rhombal algebras form a subclass of the Cubist algebras, and are known to be closely related to the weight 2 blocks of symmetric groups (and associated blocks of Schur algebras). Our main result is the calculation of the quiver of the Yoneda algebra (Theorem 4.1.1), i.e. a set of generators for the Yoneda algebra.

It is often natural to look at the so-called A_∞-structure of Yoneda algebras. Yoneda algebras are built from Ext-groups, which only encode homological information "up to homotopy" from the viewpoint of topology. A_∞-algebras (vector spaces with A_∞-structure) are designed to keep track of the information hidden from taking homotopies (and its higher analogue) via "higher multiplication". The Yoneda algebras come with a natural A_∞-structure. It is well-known that if there is some A_∞-structure on the Yoneda algebra with vanishing higher multiplication, then one could establish a derived equivalence between the original algebra and the Yoneda algebra. In the last part of Chapter 4, we determine the higher multiplication vanishing condition for the Yoneda algebra of standard modules for some rhombal algebras (Theorem 4.5.5).

The second part of this thesis starts from Chapter 5. The motivation for this part comes from the stable module category of a self-injective algebra. Stable module category - category of modules with morphisms quotiented by those that factor through projective modules, comes with a different type of homological algebra compared to, for instance, category of complexes. Interestingly, for a self-injective algebra, its stable module category and derived category are both triangulated. In fact, the triangulated structure of the stable module category can be obtained by localising the derived category at a perfect subcategory. However, the central tool for studying equivalence between triangulated categories - tilting theory - is of no use in the stable module category, as there is no non-zero tilting object in the stable module category. The lack of "projective-minded" generator (progenerator and tilting objects) has become the main obstacle in studying the homological algebra of stable module categories and stable equivalences.

Conceptually, the dual notion of projective objects are simple objects. This inspired researchers to seek for "simple-minded" generators instead of "projective-minded" ones for stable module categories. However, it is still unclear how we can organise information from the set of simple modules to obtain an analogous theory of tilting. For example, an unproven conjecture by Auslander and Reiten predicts that two stably equivalent algebras have the same number of non-projective simple modules. If we replace "stably" by "derived", and "non-projective simple modules" by "indecomposable summands of a tilting object (up to isomorphism)", then the statement becomes one of the rather apparent properties of tilting objects.

Recently, Koenig and Liu introduced the simple-minded system as an attempt to find a suitable notion of generating set for the stable module category. Roughly speaking, it is a system of modules which generates the stable module category by extensions, and satisfies the "stable Schur lemma" (Hom-orthgonality condition). A striking feature enjoyed by such a system is that it is stably invariant, i.e. a simple-minded system is mapped to a simple-minded system under a stable equivalence. This is a property for which other attempts of finding a simple-minded generator for the stable module category have failed to prove. This opens up a new way to attack the Auslander-Reiten conjecture which has the following advantage: we only need to study simple-minded systems of one algebra instead of all the algebras in the same stable equivalence class.

Another topic we are interested in is the use of mutation technique in triangulated categories and simple-minded systems. Mutation technique for representation theory dates back to the study of quiver representations by Bernstein-Gelfand-Ponomarev, which was then subsequently developed into a central theme of representation theory of algebras - tilting theory. Since the introduction of the cluster algebras of Fomin-Zelevinsky and the cluster categories of Buan-Marsh-Reiten-Reineke-Todorov, mutation theories was revitalised and popularised in the last decade. As its name suggest, mutation simply means that by changing a local structure of a mathematical object, the resultant becomes a different mathematical object with the same intrinsic properties as the original. For instance, the Okuyama-Rickard tilting complex can be seen as a mutation of the canonical tilting complex (the algebra itself) by replacing a projective summand with a two-term complex. We will use the mutation theories developed around triangulated categories to investigate the relations between simple-minded systems and other objects important to triangulated categories.

We provide a more comprehensive guide to our investigations around simple-minded theory and various mutation theories in Chapter 5. This includes all the definitions of the objects and theories we are interested in, and addresses the main results we obtain.

In Chapter 6, we present some of the results in a joint work with Steffen Koenig and Yuming Liu - a study of simple-minded system theory for representation-finite self-injective (RFS) algebras. The first main result in that chapter is the identification of simple-minded systems with combinatorial objects called combinatorial configurations (Theorem 6.1.1). For the so-called standard RFS algebras, this identification is in some sense a simple translation of Riedtmann's definition, after applying a result in Koenig-Liu's article. The not-so-trivial result is that simple-minded systems of non-standard RFS algebras are also classified by configurations. As an application of this identification and mutation theory of simple-minded systems, we found a connection between simple-minded theory of stable module categories and projective-minded (tilting) theory of derived categories. That is, every simple-minded system is in fact given by images of simple modules under possibly the nicest kind of stable equivalence - liftable stable equivalence of Morita type (stable equivalence which is induced by a two-sided-complex-tensoring functors on the derived category) (Theorem 6.1.3). We also give some consequences of this result in Section 6.3. One notable consequence is the connection between the tilting theory and simple-minded theory of the derived category, and the simple-minded theory of the stable module category (Theorem 6.3.4) for an RFS algebra.

We then divert slightly to simple-minded systems of other types of triangulated categories in Chapter 7. Our main results are the classifications of simple-minded systems of several families of triangulated categories. Namely, certain triangulated orbit categories (Theorem 7.1.4), the derived categories of representation-finite hereditary algebras (Theorem 7.2.1), and finite 1-Calabi-Yau triangulated categories (Theorem 7.3.1). In particular, we can easily write down the simple-minded systems of a stable category of maximal Cohen-Macaulay modules of a Kleinian singularity - one of the central objects in non-commutative geometry and invariant theory (Corollary 7.3.2).

In the last chapter, we look closely into the connection between the mutation theory of tilting complexes and the mutation theory of simple-minded systems of RFS algebras. In Section 8.1, we show the so-called tilting-connectedness property for RFS algebras (Theorem 8.1.3). This is done by generalising the proof of Aihara [Aih1] for the representation-finite symmetric algebras. This investigation grew out of discussions with Steffen Koenig and Yuming Liu, and will also be included in our collaborative article. In Section 8.2, we show that a natural mutation-respecting map from the set of two-term tilting complexes to the set of simple-minded systems is always surjective for a self-injective Nakayama algebra (Theorem 8.2.1). This result is far from obvious, and our proof exploits various connections of combinatorial objects developed around Nakayama algebras.

Part I

Homological algebras of certain quasi-hereditary algebras

Chapter 1

Preliminaries

Throughout Chapter 1 to 4, $\mathbf{k}$ is an algebraically closed field of arbitrary characteristic unless otherwise specified. Any algebras are assumed to be an $\mathbf{k}$-algebra. By A-modules we mean finitely generated *left* A-modules, whose category is denoted A-mod. The category of finitely generated right A-module is denoted by mod-A. We denote an isoclass of simple A-module by $L(i)$, indecomposable projectives by $P(i)$, and indecomposable injectives by $Q(i)$.

The main theme of these chapters is the homological behaviour of *quasi-hereditary algebras*.

Definition 1.0.1 ([CPS1]). *An algebra A is* quasi-hereditary*, if the isoclasses of simple A-modules are indexed by an interval-finite poset $(I, \leq)$, and there exists a collection of modules (*standard modules*) $\{\Delta(i)|i \in I\}$ with the following properties:*

(a) $\Delta(i) \twoheadrightarrow L(i)$ with kernel filtered by $L(j)$ such that $j < i$;

(b) $P(i) \twoheadrightarrow \Delta(i)$ with kernel filtered by $\Delta(j)$ such that $j > i$.

*Equivalently, if there exists a collection of modules (*costandard modules*) $\{\nabla(i)|i \in I\}$ with the following properties:*

(a') $L(i) \hookrightarrow \nabla(i)$ with cokernel filtered by $L(j)$ such that $j < i$;

(b') $\nabla(i) \hookrightarrow Q(i)$ with cokernel filtered by $\nabla(j)$ such that $j > i$.

The poset I is called weight poset.

Remark 1.1. Using the fact that A is quasi-hereditary if and only if its opposite ring A^{op} is also quasi-hereditary, one can deduce that the costandard (left) A-modules are simply the k-linear dual of the standard (right) A^{op}-module.

Denote $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$)the full subcategory of A-module filtered by standard (resp. costandard) modules. It is well-known [Rin] that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ is additively closed with a generator $T \in A$-mod whose isoclasses of indecomposable summands are also parameterised by I. T is called the (characteristic) *tilting module*

I is now the indexing set for the isoclasses of six families of modules, namely the standard modules $\{\Delta(i)\}$, costandard modules $\{\nabla(i)\}$, tilting modules $\{T(i)\}$, projectives $\{P(i)\}$, injectives $\{Q(i)\}$ and simples $\{L(i)\}$. We will call these six families of modules the *structural families* or *structural modules* of A. We use the symbol X, for $X \in \{P, Q, L, T, \Delta, \nabla\}$, to denote the direct sum of representatives of all the isomorphism classes of the corresponding structural family, i.e. $X = \bigoplus_{i \in I} X(i)$.

We assume all algebras are *positively graded*, i.e. such an algebra A can be decomposed as F-vector space into $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_n$ with $A_m A_n \subset A_{m+n}$. For a graded A-module $M = \bigoplus_{n \in \mathbb{Z}} M_n$, we let $M\langle k \rangle$ denote the grading shift such that $(M\langle k \rangle)_n = M_{k+n}$. A homomorphism of graded modules $M \to N$ is *homogeneous of degree* j if $M_n \to N_{n+j}$ for all $n \geq 0$. An A-module M is *locally finite dimensional A-module* if each graded piece of M is finite dimensional. The category of locally finite dimensional graded (left) modules is denoted A-gr, note that $\hom_A(M, N) := \mathrm{Hom}_{A\text{-gr}}(M, N)$ consists of maps from M to N which are homogeneous of degree 0. In particular, $\hom_A(M, N\langle j \rangle)$ consists of maps from M to N which are homogeneous of degree j.

A positively graded algebra A is called *quadratic* if $A = T_{A_0}(V)/R$ where $T_{A_0}(V)$ is the tensor algebra of an A_0-A_0-bimodule V over A_0, with V being in degree 1, and the relation ideal R is generated by elements of degree 2 (hence elements of $V^{\otimes 2}$). The quadratic dual of A, denoted $A^!$, is given by $T_{A_0}(V^*)/R^\perp$ where V^* is the right A_0-module formed by the homomorphisms of left A_0-modules $\mathrm{Hom}_{A_0}(V, A_0)$, and $R^\perp$ is the space orthogonal to R with respect to the natural pairing of V and V^* induced on the corresponding tensor algebras. Details of this construction can be found in [BGS]. If furthermore A is generated in degree 1, then A is called *Koszul*. We call that specific grading the *Koszul grading* on A.

When A is graded with $A_0 \cong A/\operatorname{rad} A$, the structural modules of A have a canonical graded lifts as follows. As any simple A-module $L(i)$ can be identified with summands of A_0, the canonical graded lift of $L(i)$ is concentrated in degree 0, i.e. $L(i) = L(i)_0$. The standard graded lifts of the structural A-module are chosen such that all the maps above live in A-gr. Also recall the natural morphisms on the structural A-modules:

$$\begin{tikzcd} P(i) \arrow[r, two heads] & \Delta(i) \arrow[r, two heads] \arrow[dr, hook] & L(i) \arrow[r, hook] & \nabla(i) \arrow[r, hook] & Q(i) \\ & & T(i) \arrow[ur, two heads] & & \end{tikzcd} \tag{1.0.1}$$

If $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded A-module, the graded multiplicity of $L(i)$ in M_n is denoted by

$$[M : L(i)]_q := \sum_{n\in\mathbb{Z}} q^n \dim \hom(P(i), M\langle n\rangle).$$

Following the notion in the classical Koszul theory, a complex of structural modules $\mathcal{X}^\bullet$ given by

$$\cdots \to \mathcal{X}^{n-1} \xrightarrow{d^{n-1}} \mathcal{X}^n \xrightarrow{d^n} \mathcal{X}^{n+1} \to \cdots$$

is said to be *linear* if all indecomposable summands of $\mathcal{X}^n$ are isomorphic to $X(i)\langle n\rangle$ for some $i \in I$. Let $C^\bullet = \cdots \to C^n \to C^{n+1} \to \cdots$ be a complex of (graded) A-modules. Then the i-th homological shift is denoted $C^\bullet[i]$, which is the complex with $(C[i])^n = C^{i+n}$.

For $X \in \{P, Q, L, T, \Delta, \nabla\}$, we denote by A^X the opposite ring of the *Yoneda algebra* of a structural family of A-modules, i.e. $A^X := \mathrm{Ext}^\bullet_A(X, X)^{\mathrm{op}}$. We simply call A^X as the *Ext-algebra* of structural family X. Understanding the structure of the Ext-algebra A^X for a given algebra A is then a natural and interesting question to ask. In the case of $X = P$ (resp. $X = Q$), one gets the basic algebra associated to A (resp. A^{op}). If $X = T$, then one gets the *Ringel dual* A' of A. This is a quasi-hereditary algebra with respect to $(I, \leq^{\mathrm{op}})$. In these cases, we get a derived equivalence between A and A^X given by the tilting complex $\mathcal{X}$, which is the projective resolution of X. However, for $X \in \{L, \Delta, \nabla\}$, the properties of A^X are generally much more obscure. One then has to restrict to subclasses of quasi-hereditary algebras which exhibit nice homological properties. A quasi-hereditary algebra A is *standard Koszul* if there is a grading on A, so that for each $i \in I$, a minimal graded projective resolution $\widetilde{\Delta}(i)^\bullet$ of the standard modules $\Delta(i)$ and a minimal graded injective coresolution $\widetilde{\nabla}(i)^\bullet$ of $\nabla(i)$ are both linear. This notion was first established in [ÁDL], where they have proved that A is then a Koszul algebra (for the same grading) and the Koszul dual $A^! \cong A^L$ of A is quasi-hereditary with respect to $(I, \leq^{\mathrm{op}})$ (also see recap in Chapter 3). By [BGS], one then gets a derived equivalence between A and $A^!$; also see [Mad1] on unifying such derived equivalences with the one arising from Ringel duality.

Also recall from [Irv1] that A is a *BGG algebra* (or quasi-hereditary with *duality*) if it is quasi-hereditary and there is a duality functor on the category of finitely generated modules A-mod, i.e. a contravariant exact functor δ on A-mod such that $\delta^2 \cong \mathrm{id}_{A\text{-mod}}$, and for all $i \in I$, we have $\delta L(i) \cong L(i)$. When A is a BGG algebra, $\delta P(i) \cong Q(i)$ and $\delta\Delta(i) \cong \nabla(i)$, see [Irv1]. In particular, linearity on the resolutions of standard modules will suffice to show standard Koszulity for BGG algebras.

Following [Maz], A is said to be *balanced* if A is standard Koszul, and for each $i \in I$, a minimal graded tilting coresolution $\mathcal{T}_\Delta(i)^\bullet$ of the standard module $\Delta(i)$ and a minimal graded tilting resolultion $\mathcal{T}_{\nabla}(i)^{\bullet}$ of the costandard module $\nabla(i)$ are both linear. In this case, the Ringel dual of A is also Koszul, and $(A^T)^L \cong (A^L)^T$, see [Maz].

Much less is known about A^{Δ} and A^{∇} in general. This forms the motivation of Chapter 2 to 4.

Chapter 2

Some homological properties of tensor and wreath products of quasi-hereditary algebras

2.1 Introduction

We retain all notations from the previous chapter. The first property one can say about A^{Δ} is that it is quasi-hereditary with respect to both $(I, \leq)$ and $(I, \leq^{\mathrm{op}})$ since it is directed algebras. It is then desirable to ask for a derived equivalence between A and A^{Δ} as the homological algebra for a directed algebra is usually relatively easier to understand. Madsen approached this problem using generalised Koszul duality [Mad2], and we will come to this soon. Another natural problem is under what conditions A^{Δ} will be Koszul.

Drozd-Mazorchuk showed that if a graded quasi-hereditary algebra A is equipped with a function $h : I \to \{0, 1, \ldots, n\}$, where n is a natural number, and satisfies the following four conditions,

(I) $\mathcal{T}_{\Delta}(i)^k \in \operatorname{add}\left(\oplus_{j:h(j)=h(i)-k} T(j)\langle k\rangle\right)$ for all $k \geq 0$;

(II) $\mathcal{T}_{\nabla}(i)^k \in \operatorname{add}\left(\oplus_{j:h(j)=h(i)+k} T(j)\langle k\rangle\right)$ for all $k \leq 0$;

(III) $\widetilde{\Delta}(i)^k \in \operatorname{add}\left(\oplus_{j:h(j)=h(i)-k} P(j)\langle k\rangle\right)$ for all $k \leq 0$;

(IV) $\widetilde{\nabla}(i)^k \in \operatorname{add}\left(\oplus_{j:h(j)=h(i)+k} Q(j)\langle k\rangle\right)$ for all $k \geq 0$.

then A^{Δ} is Koszul. More explicitly,

Theorem 2.1.1 (Drozd-Mazorchuk [DM])**.** *Let A be a quasi-hereditary algebra equipped with a function satisfying conditions (I)-(IV). Then A^Δ is Koszul, with Koszul dual the Ext-algebra of costandard A-modules, i.e. $(A^\Delta)^! \cong (A^\nabla)^{\mathrm{op}}$.*

In fact, the original theorem contains more information, but we will omit this for now, since the results of Madsen [Mad1, Mad2] also recover the same information. Madsen's works use the theory of T-Koszulity, which was first introduced in [GRS], and combining with inspiration from [DM], to unify the theory of A^Δ (or A^∇) with that of A^X for $X \in \{P, Q, T, L\}$ when A satisfies the following condition:

Definition 2.1.2. *Let $(A, (I, \leq))$ be a standard Koszul BGG algebra. Then we say A* satisfies condition (H) *when A is equipped with a function $h : I \to \{0, 1, \dots, n\}$ with the following property. If the k-th radical layer of $\Delta(x)$ contains $L(y)$, or equivalently, the polynomial $[\Delta(x) : L(y)]_q$ has non-zero coefficient of q^k, then $h(y) = h(x) - k$.*

It was originally proved in [DM] that if the algebra associated to a block of category $\mathcal{O}$ of a complex semisimple Lie algebra is multiplicity-free, then the block satisfies condition (H), which gives a function satisfying conditions (I)-(IV). In [Mad2], Madsen relaxes this to standard Koszul BGG algebras (rather than just specific blocks of category $\mathcal{O}$); one can then reproduce most of the results in [DM] through the use of T-Koszulity. Moreover, one can now get a derived equivalence of graded A and A^Δ-modules, with a grading different from the Koszul grading and homological grading, termed as Δ-grading by Madsen. This grading has actually been seen "in disguise" in [DM], and also has appeared in other investigations of A^Δ such as [MT].

Theorem 2.1.3 (Madsen, [Mad1], [Mad2])**.** *Let A be a standard Koszul BGG algebra satisfying (H). Then*

1. *A satisfies conditions (I)-(IV), hence A is balanced and A^Δ is Koszul.*
2. *There is a Δ-grading on A, i.e. A is positively graded with $A_0 = \Delta$ and $A_i \in \mathrm{add}(\Delta)\langle i \rangle$ for all $i > 0$. Note the shift $\langle i \rangle$ here is on the Koszul grading.*
3. *Taking $T = A_0$ in the Δ-grading, then T satisfies the axioms of T-Koszulity. In particular*
   (a) *$\Delta^* := \mathrm{Hom}_{\mathbf{k}}(\Delta, \mathbf{k})$ is an A^Δ-module, and $A \cong (A^\Delta)^{\Delta^*}$*
   (b) *There is a graded derived equivalence:*

$$D^b(A\text{-gr}) \xrightarrow[\sim]{R\mathrm{Hom}_A(\Delta, -)} D^b(A^\Delta\text{-gr})$$

which sends costandard A-modules to simple A^Δ-modules.

(3)(b) above now gives a rigorous meaning to the idea of Drozd-Mazorchuk (originated from Ovsienko) that costandard A-modules can be "aligned" in such a way that they are simple A^{Δ}-modules, inducing Koszulity of A^{Δ}.

We have introduced different subclasses of quasi-hereditary algebras which have nice homological properties, and these nice properties give information on how the Ext-algebras of the structural families behave. In this chapter, we show that these nice properties can be carried over to tensor products of such algebras, as well as wreath products of such algebras with the symmetric group.

Throughout this chapter, any tensor product of vector spaces $\otimes$ without a subscript is the tensor product over the underlying field $\mathbf{k}$. Given an algebra A and $w \in \mathbb{Z}_{>0}$, there is a natural action of the symmetric group $\mathfrak{S}_w$ on the tensor product $A^{\otimes w}$ by permuting components. For simplicity, we assume at the moment that A is an ungraded algebra. The wreath product of A with the symmetric group $\mathfrak{S}_w$ is the vector space $A^{[w]} := A^{\otimes w} \otimes \mathbf{k}\mathfrak{S}_w$, with multiplication

$$(a_1 \otimes \cdots \otimes a_w \otimes \sigma)(b_1 \otimes \cdots \otimes \tau) = a_1 b_{\sigma^{-1}(1)} \otimes \cdots \otimes a_w b_{\sigma^{-1}(w)} \otimes \sigma\tau \tag{2.1.1}$$

for $\sigma, \tau \in \mathfrak{S}_n$. We will simply call such an algebra the *wreath product algebra* or *wreath product of A*.

In the representation theory of symmetric groups and their quasi-hereditary covers (Schur algebras) over prime characteristic, the "complexity" of blocks are measured by weights $w \in \mathbb{Z}_{>0}$ (not to be confused with the notion of weights in highest weight theory). The weight zero blocks are the semisimple blocks and the weight one blocks are Morita equivalent to the Brauer tree algebras, and their quasi-hereditary covers. These algebras have been thoroughly studied throughout the literature. For each given weight w, with $w > \operatorname{char}\mathbf{k}$, there is a special kind of block, called the Rouquier block or RoCK block, which is the simplest block to understand in terms of its homological behaviour. The reason for this is because the RoCK block of weight w is Morita equivalent to the wreath product of the weight one block with $\mathfrak{S}_w$; similar situations also occur in other areas of "type A representation theory", see for example [CT1]. This particular example is our motivation to show that wreath product algebras inherit nice homological properties of the original algebra. Since we will need results from [CT2] in our exposition, we will impose the extra condition that $w!$ *is invertible in the field* $\mathbf{k}$ when we study wreath product algebras in section 2.3.

The rest of this chapter consists of two sections. The first surveys some results on tensor products of quasi-hereditary algebras with nice homological properties (BGG and/or standard

Koszul and/or balanced and/or condition (H)). We will also show that taking the Ext-algebra of a structural families over the tensor product of algebras is the same as taking the tensor products of the Ext-algebras of the structural families. Most of these results are folklore, but as we have yet to find good enough references for them, we will include a simple proof for each of them. In the second section, we show the analogous results for wreath product algebras, using roughly the same ideas from the proofs in Section 2.2.

2.2 Tensor product algebras

Let A_1, A_2 be quasi-hereditary algebras and $(I_1, \leq_1), (I_2, \leq_2)$ be the respective weight posets. The tensor product algebra $A := A_1 \otimes A_2$ is then quasi-hereditary with respect to $(I := I_1 \times I_2, \leq)$ where the partial order $\leq$ is defined by: $(x_1, x_2) \leq (y_1, y_2)$ if $x_k \leq y_k$ for $k = 1, 2$. This comes from the fact that each structural A-module is the tensor product of structural modules of A_1 and A_2, namely $X(x_1, x_2) = X_{A_1}(x_1) \otimes X_{A_2}(x_2)$ for $X \in \{P, Q, L, \Delta, \nabla, T\}$ and all $(x_1, x_2) \in I$. For simplicity, we denote structural A_i-module $X_{A_i}(x)$ by $X_i(x)$ for $i = 1, 2$.

When A_1, A_2 are graded algebras, we note that the multiplication of elements respects the Koszul sign convention, that is $(a \otimes a')(b \otimes b') = (-1)^{|a'||b|}(ab \otimes a'b')$ for homogeneous element a, a', b, b' with $|x|$ being the degree of x. The action of $A_1 \otimes A_2$ on the module $M_1 \otimes M_2$ with M_i being A_i-module will also inherit a sign as follows. Let a_i be homogeneous elements of A_i and m_i be homogeneous elements of M_i of degree $|m_i|$. The action of A is given by $(a_1 \otimes a_2)(m_1 \otimes m_2) = (-1)^{|a_2||m_1|} a_1 m_1 \otimes a_2 m_2$. The Koszul sign convention for a tensor product of maps of graded modules is the same as the convention for tensoring maps of complexes. More explicitly, let $f_1 : C_1 \to D_1$ and $f_2 : C_2 \to D_2$ be maps of complexes (respectively modules). The notation $f_1 \otimes f_2$ is understood as the map which sends $c_1 \otimes c_2 \mapsto (-1)^{|f_2||c_1|} f_1(c_1) \otimes f_2(c_2)$ where $|x|$ denotes the (homological or $\mathbb{Z}$-grading) degree of a homogeneous element or function x.

Proposition 2.2.1. *Tensoring and taking the Ext-algebra of structural modules are commuting operations on algebras. i.e.* $(A_1 \otimes A_2)^X \cong A_1^{X_1} \otimes A_2^{X_2}$

Proof. Since X_A is isomorphic to $X_1 \otimes X_2$ for $X \in \{P, Q, L, \Delta, \nabla, T\}$. It then follows from a well-known folklore that the Ext-algebra of the tensor product of modules is isomorphic (as an algebra) to the tensor product of Ext-algebras. The closest reference we can find is the generalisation of this result in [BO, Theorem 3.7]. □

We first show that BGG duality can be induced naturally:

Lemma 2.2.2. *If A_1, A_2 are BGG algebras, then so is $A_1 \otimes A_2$.*

Proof. From [CPS2, Prop 2.1], a duality functor (not necessarily fixing simple modules) corresponds to an anti-automorphism ι_i of A_i such that ι_i^2 is an inner automorphism α_i of A_i. We see that $\iota := \iota_1 \otimes \iota_2$ is an anti-automorphism on $A_1 \otimes A_2$ such that $\iota^2 = \alpha_1 \otimes \alpha_2$ which is also an inner automorphism of A. Consequently, ι induces a duality functor δ, which maps $M \in A$-mod to the vector space $M^* = \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$, with the A-action given by $a \cdot f(m) = f(\iota(a)m)$. In particular, for finite dimensional modules $M_i \in A_i$-mod for $i = 1, 2$, since $(M_1 \otimes M_2)^* \cong M_1^* \otimes M_2^*$, we have $\delta(M_1 \otimes M_2) \cong \delta_1 M_1 \otimes \delta_2 M_2$, where δ_i is the BGG duality functor on A_i-mod. Since $\delta_i L_i(x) \cong L_i(x)$ for all simple A_i-module $L_i(x)$, it follows that the duality on $A_1 \otimes A_2$ fixes simple $A_1 \otimes A_2$-modules. □

Remark 2.1. When A_1, A_2 are positively $\mathbb{Z}$-graded algebras, we note that the duality functor δ_i sends a simple module concentrated in degree n to a simple module concentrated in degree $-n$, and the associated anti-automorphisms preserves gradings [Irv2, section 3]. Moreover, under Koszul sign convention, since ι_1 and ι_2 are degree 0 maps, ι is defined in the same way as in the non-graded setting, i.e. $\iota(a_1 \otimes a_2) = \iota_1 \otimes \iota_2(a_1 \otimes a_2) = \iota_1(a_1) \otimes \iota_2(a_2)$.

Lemma 2.2.3. *If A_1, A_2 are standard Koszul (resp. balanced) algebras, then so is $A_1 \otimes A_2$.*

Proof. We use the fact that the total complex of the tensor product of (graded) projective resolutions is a projective resolution of the corresponding tensor product of modules, see for example [BO, Lemma 3.6]. This tensor product of resolutions also preserves minimality and linearity. Applying the dual argument on the injective coresolution, we have the claim for standard Koszulity.

For (graded) tilting (co)resolutions, one just does the same trick. Using the fact that $T_1(x_1) \otimes T_2(x_2) = T(x_1, x_2)$ is a tilting $A_1 \otimes A_2$-module, the tensor product of the tilting (co)resolutions will then be the tilting (co)resolution of the tensor product of the modules, which also preserves minimality and linearity. Hence the claim for balancedness. □

Proposition 2.2.4. *Let A_1, A_2 be standard Koszul BGG algebra satisfying condition (H). Then so is the tensor product algebra $A_1 \otimes A_2$.*

Proof. The grading of $A = A_1 \otimes A_2$ comes naturally from the usual grading on tensor products, and Koszulity for this grading follows from Lemma 2.2.3 above. BGG duality follows from Lemma 2.2.2. Given functions $h_j : I_j \to \{0, \ldots, n_i\}$ of A_j's so that A_j's satisfy condition (H), we define the required function $h : I \to \{0, 1, \ldots, n = n_1 + n_2\}$ as follows

$$\begin{array}{rcl} h : I & \to & \{0, 1, \ldots, n\} \\ (x_1, x_2) & \mapsto & h_1(x_1) + h_2(x_2) \end{array}$$

Examining the (Koszul) graded structure of $\Delta(x)$ closely, its l-th graded piece is given by:

$$(\Delta(x))_l = \bigoplus_{l_1+l_2=l} (\Delta_1(x_1))_{l_1} \otimes (\Delta_2(x_2))_{l_2}$$

Therefore, when $\dim \hom(P(y), \Delta(x)\langle l \rangle) \neq 0$, there are some l_1, l_2 such that $\dim \hom(P(y_j), \Delta_j(x_j)\langle l_j \rangle) \neq 0$ for $j = 1, 2$. Since the A_j's satisfy condition (H) with respect to the h_j's, we have

$$\begin{array}{rrcll} & h_j(x_j) - h_j(y_j) & = & l_j & \text{for } j = 1, 2 \\ \Rightarrow & (h_1(x_1) + h_2(x_2)) - (h_1(y_1) + h_2(y_2)) & = & l_1 + l_2 \;=\; l & \\ \Rightarrow & h(x) - h(y) & = & l & \end{array}$$

and the condition (H) is satisfied. □

By induction, one shows that the above results extend to all finite tensor products of quasi-hereditary algebras.

2.3 Wreath product algebras

We first remark that if A is a graded algebra, then we have an induced grading on $A^{\otimes w}$ given by the usual $\mathbb{Z}$-grading on the tensor product algebras. This further induces a grading on $A^{[w]}$ by putting the "tensor component" $\mathbf{k}\mathfrak{S}_w$ in degree zero. The induced sign convention on multiplication is described as follows. The sign induced by the action of transposition s_i swapping i and $i+1$ on $b_1 \otimes \cdots b_w \in A^{\otimes w}$ (with b_i homogeneous for all i) is given by

$$s_i \cdot (b_1 \otimes \cdots b_w) = (-1)^{|b_i||b_{i+1}|} b_1 \otimes \cdots \otimes b_{i+1} \otimes b_i \otimes \cdots \otimes b_w.$$

This generates the sign convention for $\mathfrak{S}_w$-action on $A^{\otimes w}$. The sign convention for multiplications in $A^{[w]}$ is to put a $(-1)^d$ in the right-hand side of (2.1.1), where d is determined by the action of $\sigma \in \mathfrak{S}_w$ on $b_1 \otimes \cdots b_w$, and the sign convention of the multiplication $(a_1 \otimes \cdots a_w)(b_{\sigma^{-1}(1)} \otimes \cdots \otimes b_{\sigma^{-1}(w)})$ in $A^{\otimes w}$.

Given any (graded) A-module M, one can take the wreath product of M with symmetric group, i.e. $M^{[w]} := M^{\otimes w} \otimes \mathbf{k}\mathfrak{S}_w$, to get an $A^{[w]}$-module, with the $A^{[w]}$-action induced from $A^{\otimes w}$-action. Note that if M is graded, then so is $M^{[w]}$, where the induced grading is given by the same rule as the induced grading when wreathing A. Wreath product preserves many nice properties of an algebra, its modules, and its complexes (see later paragraphs). We start by collecting some results from [CT2] and [CLS, Section 2] that will be useful to us. We remind the reader again that throughout this section, w is a positive integer with $w!$ invertible in the underlying field $\mathbf{k}$. For convenience, we simply say "wreathing an object" instead of "taking the wreath product of an object with the symmetric group".

Let $\{X(i)|i \in I\}$ be a (structural) family of A-modules, and the cardinality of I be $n \in \mathbb{N}$. Then there is a family of $A^{[w]}$-modules, $\{X(\boldsymbol{\lambda})|\boldsymbol{\lambda} \in \Lambda_w^I\}$, which are indexed by the set of I-tuples of partitions such that the sum of the size of the entries is w, i.e.

$$\Lambda_w^I := \left\{ \boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)}) \;\middle|\; \lambda^{(i)} \vdash \omega_i \text{ and } \sum_{i\in I} \omega_i = w \right\}$$

Lemma 2.3.1. *For $X \in \{P, Q, L, \Delta, \nabla, T\}$, the family $\{X(\boldsymbol{\lambda})|\boldsymbol{\lambda} \in \Lambda_w^I\}$ is the structural family of $A^{[w]}$-modules.*

Proof. For each $\boldsymbol{\lambda} \in \Lambda_w^I$, the $A^{[w]}$-module $X(\boldsymbol{\lambda})$ occur as indecomposable summands of wreathing the direct sum of structural modules [CT2, Lemma 3.8]:

$$\left(\bigoplus_{i\in I} X(i) \right)^{[w]} \cong \bigoplus_{\boldsymbol{\lambda}\in\Lambda_w^I} X(\boldsymbol{\lambda})^{\oplus m(\boldsymbol{\lambda})}$$

for some $m(\boldsymbol{\lambda}) \in \mathbb{N}$ which depend only on $\boldsymbol{\lambda}$. For $X \in \{P, Q, L, \Delta, \nabla\}$, [CT2, Lemma 3.8, 3.9, Section 6] already showed that the new family $\{X(\boldsymbol{\lambda})\}$ indeed is the corresponding structural family. Also from [CT2, Section 4], one can see that the induced family $\{T(\boldsymbol{\lambda})|\boldsymbol{\lambda} \in \Lambda_w^I\}$ from the family of tilting A-modules is filtered by $\{\Delta(\boldsymbol{\lambda})\}$ as well as $\{\nabla(\boldsymbol{\lambda})\}$. Hence $\{T(\boldsymbol{\lambda})\}$ is indeed the family of tilting $A^{[w]}$-modules. □

This construction of a new family of objects can also be applied to maps. Given two families $\{X(i)\}$, $\{Y(i)\}$ of A-modules indexed by $i \in I$, and a family of maps $\{f_i : X(i) \to Y(i)|i \in I\}$, then there is a family of maps of $A^{[w]}$-modules $\{f_{\boldsymbol{\lambda}} : X(\boldsymbol{\lambda}) \to Y(\boldsymbol{\lambda})|\boldsymbol{\lambda} \in \Lambda_w^I\}$, see [CT2, 3.9(2)]. In particular, if $\{f_i\}$ is a family of one of the structural maps appearing in (1.0.1), then $\{f_{\boldsymbol{\lambda}}\}$ is the corresponding family of structural maps.

From now on, we assume all the modules are graded. As mentioned in the first paragraph of this section, the wreathing construction applies to bounded complexes of (graded) A-modules as follows. Let $C^\bullet$ be a bounded complex of finite dimensional A-modules with differential d, then there is a differential on the $A^{\otimes w}$-complex $C^{\otimes w \bullet}$, which can be written as

$$\sum_{a+b=w-1} 1^{\otimes a} \otimes d \otimes 1^{\otimes b}.$$

We remind the reader again that our notation of tensoring maps here has implicitly used the Koszul sign convention, see discussion in the beginning of the previous section.

The differential on the $A^{[w]}$-complex $C^{[w]\bullet}$ is given by tensoring the above differential with $1_{\mathbf{k}\mathfrak{S}_w}$. One can also wreath a chain map of A-complexes, which will consequently make $(-)^{[w]}$ a functor that preserves homotopy [CLS, Lemma 2.4]. In another words, one can regard $(-)^{[w]}$ as a functor from the bounded homotopy category $K^b(A\text{-gr})$ to the bounded homotopy category $K^b(A^{[w]}\text{-gr})$. Note that this functor is polynomial, non-linear, non-additive, and non-triangulated (or non-exact on the full subcategory of modules), see [CLS, Section 2]. On the other hand, the wreathing functor preserves monomorphisms and epimorphisms on modules. We also have a stronger result:

Lemma 2.3.2. *Let $C^\bullet$ be a bounded complex of finite dimensional A-modules. Then the homology of wreathing $C^\bullet$ is the isomorphic to wreathing the homology of $C^\bullet$. i.e. $H(C^{[w]\bullet}) \cong H(C^\bullet)^{[w]}$. In particular, $f^{[w]}$ is a quasi-isomorphism in $K^b(A^{[w]}\text{-gr})$ if f is a quasi-isomorphism in $K^b(A\text{-gr})$.*

Proof. Since we are working over a field $\mathbf{k}$ and $C^{[w]\bullet}$ is a bounded complex of finite dimensional A-modules, $C^{[w]\bullet}$ is only a tensor of complexes of finite dimensional vector spaces $(C^\bullet)^{\otimes w} \otimes \mathbf{k}\mathfrak{S}_w$. Applying Kúunneth theorem, the homology of this complex is isomorphic to $H^\bullet(C^\bullet)^{\otimes w} \otimes H^\bullet(\mathbf{k}\mathfrak{S}_w)$, which is isomorphic to $H^\bullet(C^\bullet) \otimes \mathbf{k}\mathfrak{S}_w \cong H^\bullet(C^\bullet)^{[w]}$ as $\mathbf{k}\mathfrak{S}_w$ is a just stalk complex.

If $f^\bullet : C^\bullet \to D^\bullet$ is a quasi-isomorphism, then we have vector space isomorphism

$$H^\bullet(f^\bullet) = f^\bullet|_{H^\bullet(C^\bullet)} : H^\bullet(C^\bullet) \xrightarrow{\sim} H^\bullet(D^\bullet).$$

Therefore, the wreath $f^{[w]\bullet}$ of $f^\bullet$ induces:

$$H^\bullet(f^{[w]\bullet}) = f^{[w]\bullet}|_{H^\bullet(C^{[w]\bullet})} = f^{[w]\bullet}|_{H^\bullet(C^\bullet)^{[w]}} = (f^\bullet|_{H^\bullet(C^\bullet)})^{[w]},$$

which is an isomorphism from $H^\bullet(C^\bullet)^{[w]} = H^\bullet(C^{[w]\bullet})$ to $H^\bullet(D^\bullet)^{[w]} = H^\bullet(D^{[w]\bullet})$. $\square$

Remark 2.2. The lemma can therefore be stated using derived category $D^b(A\text{-gr})$ instead of homotopy category, as homologies and quasi-isomorphisms are preserved under wreath product.

The following three propositions are folklore that have not been shown in the literature explicitly to the best of our knowledge.

Proposition 2.3.3. *Wreathing and taking Ext-algebra of structural modules are commuting operations on algebras. i.e.* $\mathrm{Ext}^\bullet_{A^{[w]}}(X^{[w]}, X^{[w]})^{\mathrm{op}} \cong (A^X)^{[w]}$.

Proof. Let $\widetilde{X}$ be the projective resolution of X. We claim that we have the following algebra isomorphisms:

$$\begin{aligned}
\mathrm{Ext}^\bullet_{A^{[w]}}(X^{[w]}, X^{[w]}) &= H^\bullet(\mathrm{End}_{A^{[w]}}(\widetilde{X}^{[w]})) \\
&\cong H^\bullet(\mathrm{End}_A(\widetilde{X})^{[w]}) \\
&\cong H^\bullet(\mathrm{End}_A(\widetilde{X}))^{[w]} \\
&= (\mathrm{Ext}^\bullet_A(X, X))^{[w]}
\end{aligned}$$

Note that the endomorphism rings above are taken over the dg algebra A (dga concentrated in degree zero). The third (algebra) isomorphism is justified by Lemma 2.3.2 above.

To justify the second isomorphism, we note that for any dg A-modules M, N, we have

$$\begin{aligned}
& \mathrm{Hom}_{A^{[w]}}(M^{\otimes w} \otimes k\mathfrak{S}_w, N^{\otimes w} \otimes \mathbf{k}\mathfrak{S}_w) \\
\cong\ & \mathrm{Hom}_{A^{[w]}}(A^{[w]} \otimes_{A^{\otimes w}} M^{\otimes w}, A^{[w]} \otimes_{A^{\otimes w}} N^{\otimes w}) \\
\cong\ & \mathrm{Hom}_{A^{\otimes w}}(M^{\otimes w}, \mathrm{Hom}_{A^{[w]}}(A^{[w]}, A^{[w]} \otimes_{A^{\otimes w}} N^{\otimes w})) \\
\cong\ & \mathrm{Hom}_{A^{\otimes w}}(M^{\otimes w}, \bigoplus_{\sigma \in \mathfrak{S}_w} \sigma \otimes N^{\otimes w}) \\
\cong\ & \bigoplus_{\sigma \in \mathfrak{S}_w} \sigma \otimes \mathrm{Hom}_{A^{\otimes w}}(M^{\otimes w}, N^{\otimes w}) \\
\cong\ & A^{[w]} \otimes_{A^{\otimes w}} \mathrm{Hom}_A(M, N)^{\otimes w} \\
\cong\ & \mathrm{Hom}_A(M, N)^{[w]}
\end{aligned}$$

Therefore, we have vector space isomorphism $\mathrm{End}_{A^{[w]}}(\widetilde{\Delta}^{[w]}) \cong \mathrm{End}_A(\widetilde{\Delta})^{[w]}$, and this becomes an algebra isomorphism because $f^{[w]}g^{[w]} = (fg)^{[w]}$ by construction. Moreover, we regard $\mathrm{End}_A(\widetilde{X})$ as an algebra graded by the (homological) degree of the maps of complexes. The Koszul sign convention then ensures the isomorphism is a graded algebra isomorphism. □

Lemma 2.3.4. *Let A be a BGG algebra. Then its wreath product $A^{[w]}$ is also a BGG algebra.*

Proof. The fact that $A^{[w]}$ is quasi-hereditary is the result of [CT2, Section 6]. For proving BGG duality on $A^{[w]}$, similarly to Lemma 2.2.2, suppose ι is the anti-automorphism of A

corresponding to the BGG duality on A, and ι_w the natural involutary anti-automorphism ι_w on the group algebra structure of $\mathbf{k}\mathfrak{S}_w$, which sends $\sigma \in \mathfrak{S}_w$ to σ^{-1}. These anti-automorphisms are compatible with the action of permuting the components of a tensor space, so they combine to give the anti-automorphism $\iota^{[w]}$ on $A^{[w]}$, which is given by

$$\iota^{[w]}(a_1 \otimes \cdots \otimes a_w \otimes \sigma) = \iota(a_{\iota_w(\sigma)(1)}) \otimes \cdots \otimes \iota(a_{\iota_w(\sigma)(n)}) \otimes \iota_w(\sigma).$$

Indeed,

$$\begin{aligned}
& \iota^{[w]}((a_1 \otimes \cdots \otimes a_w \otimes \sigma)(b_1 \otimes \cdots \otimes b_w \otimes \tau)) \\
= \ & \iota^{[w]}(a_1 b_{\sigma^{-1}(1)} \otimes \cdots \otimes a_w b_{\sigma^{-1}(w)} \otimes \sigma\tau) \\
= \ & \iota(a_{\tau\sigma(1)} b_{\tau(1)}) \otimes \cdots \otimes \iota(a_{\tau\sigma(w)} b_{\tau(w)}) \otimes \tau^{-1}\sigma^{-1} \\
= \ & \iota(b_{\tau(1)})\iota(a_{\tau\sigma(1)}) \otimes \cdots \otimes \iota(b_{\tau(w)})\iota(a_{\tau\sigma(w)}) \otimes \tau^{-1}\sigma^{-1} \\
= \ & (\iota(b_{\tau(1)}) \otimes \cdots \otimes \iota(b_{\tau(w)}) \otimes \tau^{-1})(\iota(a_{\sigma(1)}) \otimes \cdots \otimes \iota(a_{\sigma(1)}) \otimes \sigma^{-1}) \\
= \ & \iota^{[w]}(b_1 \otimes \cdots \otimes \tau)\iota^{[w]}(a_1 \otimes \cdots \otimes \sigma)
\end{aligned}$$

Note that as $\iota^2 = \alpha$ is an inner automorphism of A, so $(\iota^{[w]})^2 = \alpha^{\otimes w} \otimes 1_{\mathfrak{S}_w}$ is also an inner automorphism of $A^{[w]}$. Let the duality functor of A be δ and the corresponding functor on $A^{[w]}$ be $\delta^{[w]}$. Now the duality induced by $\iota^{[w]}$ maps any simple $A^{[w]}$-modules to its linear dual (cf. Lemma 2.2.3), then the fact that $\delta^{[w]}$ fixes simples follows from Remark (3) of [CT2, Coro 3.9]. □

Remark 2.3. Note in the graded setting, ι is a degree 0 map. $\mathbf{k}\mathfrak{S}_w$ being placed in degree 0 means that ι_w is also a degree 0 map. Using Koszul sign convention, we see that $\iota^{[w]}$ is defined with the same formula when A is graded.

Remark 2.4. Recall that the cell datum of a cellular algebra consists of an involutary anti-automorphism. As cellular algebras have such close resemblance of BGG algebras, one may expect the same result to hold for cellular algebras as well. A special case is already known, see [GG].

Proposition 2.3.5. *If A is a standard Koszul (resp. balanced) algebra, then so is $A^{[w]}$.*

Proof. By Lemma 2.3.2, the wreath of the minimal projective resolution $\widetilde{X}$ of the direct sum X of structural modules is quasi-isomorphic to $X^{[w]}$. $\widetilde{X}^{[w]}$ is then a projective resolution of $X^{[w]}$. Note that $\widetilde{X}^{\otimes w}$ is the minimal $A^{\otimes w}$-projective resolution of $X^{\otimes w}$, which is also linear. Wreathing $\widetilde{X}$ is just tensoring $\widetilde{X}^{\otimes w}$ with the $\mathbf{k}$-vector space $\mathbf{k}\mathfrak{S}_w$ concentrated in degree 0 at every component, so $\widetilde{X}^{\bullet[w]}$ is the minimal projective resolution of $X^{[w]}$ which is also linear.

This proves standard Koszulity of $A^{[w]}$.

If A is balanced, then A and its Ringel dual A^T are Koszul. We already have $A^{[w]}$ Koszul. By (a Morita equivalent version of) Proposition 2.3.3, the Ringel dual of $A^{[w]}$ is isomorphic to the basic algebra of $(A^T)^{[w]}$. Since A^T is Koszul, $(A^T)^{[w]}$ is also Koszul, and hence the Ringel dual of $A^{[w]}$ is Koszul. This is sufficient for A to be balanced. $\square$

We finish by showing that wreathing also inherits condition (H).

Proposition 2.3.6. *A be a standard Koszul BGG algebra satisfying condition (H). Then $A^{[w]}$ is also a standard Koszul BGG algebra satisfying condition (H).*

Proof. By Lemma 2.3.4 and Proposition 2.3.5, it is sufficient to show that condition (H) is satisfied for $A^{[w]}$. Let h_A be the function so that A satisfies condition (H). We prove that condition (H) holds in $A^{[w]}$ using the following function:

$$\begin{array}{rcl} h : \Lambda_w^I & \to & \{0, 1, \dots, n\} \\ (x_1, \dots, x_w) & \mapsto & \left(\sum_{k=1}^{w} h_A(x_k)|\lambda^{(k)}| \right) - (w-1) \end{array}$$

Suppose $\dim \hom(P(\boldsymbol{\mu}), \Delta(\boldsymbol{\lambda})\langle l \rangle) \neq 0$. From [CT2, Prop 4.4], there exist $(\rho^{(i,s)})_{(i,s)\in K} \in \Lambda_w^K$ such that

$$\sum_{s=0}^{m_i} |\rho^{(i,s)}| = |\lambda^{(i)}| \quad \forall i \in I, \tag{2.3.1}$$

$$\sum_{(j,s)\in K_i} |\rho^{(j,s)}| = |\mu^{(i)}| \quad \forall i \in I, \tag{2.3.2}$$

$$\sum_{(i,s)\in K} l_{i,s}|\rho^{(i,s)}| = l \tag{2.3.3}$$

where

$$\begin{array}{rcl} K & = & \{(i,s) \in I \times \mathbb{Z} \mid \ 0 \leq s \leq m_i\}, \\ K_i & = & \{(j,s) \in K \mid \ \Delta(j,s)/\Delta(j,s+1) \cong L(i)\}, \\ \text{and for all } i \in I, \ \Delta(i) & = & \Delta(i,0) \supset \Delta(i,1) \supset \cdots \supset \Delta(i, m_i+1) = 0 \end{array}$$

is a refinement of the radical filtration of $\Delta(i)$ such that each subquotient is a single simple A-module; and the $l_{i,s}$-th radical layer of $\Delta(i)$ contains $\Delta(i,s)/\Delta(i,s+1)$.

We can see that $K = \coprod_{i\in I} K_i$. Indeed, by definition K is the union of K_i, and every (j,s) lies in K_i for precisely one i as $\Delta(j,s)/\Delta(j,s+1)$ is a single simple module by definition. To prove that condition (H) holds, it suffices to show that $h(\boldsymbol{\lambda}) - h(\boldsymbol{\mu}) = l$. Expanding the left hand side using the definition and conditions (2.3.1) and (2.3.2), we get

$$\begin{aligned} h(\boldsymbol{\lambda}) - h(\boldsymbol{\mu}) &= \sum_{i\in I} h_A(i)|\lambda^{(i)}| - \sum_{i\in I} h_A(i)|\mu^{(i)}| \\ &= \sum_{i\in I} h_A(i)\left(\sum_{s=0}^{m_i}|\rho^{(i,s)}|\right) - \sum_{i\in I} h_A(i)\left(\sum_{(j,s)\in K_i}|\rho^{(j,s)}|\right). \end{aligned}$$

Since $(j,s) \in K_i$ implies $\Delta(j,s)/\Delta(j,s+1) \cong L(i)$ and $L(i)$ is in the $l_{j,s}$-th radical layer of $\Delta(j)$, i.e. $\dim \hom(P(i), \Delta(j)\langle l_{j,s}\rangle) \neq 0$, by condition (H) on A, we get $h_A(i) = h_A(j) - l_{j,s}$. We substitute this back into our expansion and get:

$$\begin{aligned} h(\boldsymbol{\lambda}) - h(\boldsymbol{\mu}) &= \sum_{i\in I}\sum_{s=0}^{m_i} h_A(i)|\rho^{(i,s)}| - \sum_{i\in I}\sum_{(j,s)\in K_i}(h_A(j) - l_{j,s})|\rho^{(j,s)}| \\ &= \sum_{(i,s)\in K} h_A(i)|\rho^{(i,s)}| - \sum_{(i,s)\in K}\Big(h_A(i)|\rho^{(i,s)}| - l_{i,s}|\rho^{(i,s)}|\Big) \\ &= \sum_{(i,s)\in K} l_{i,s}|\rho^{(i,s)}| \\ &= l\ , \end{aligned}$$

where the second equality here uses $K = \coprod_{i\in I} K_i$ and the third one uses condition (2.3.3). This completes the proof. $\square$

Remark 2.5. Note that the ending term $-(w-1)$ in the definition of h is not necessary to prove that condition (H) holds; but it is convenient in practice to include such a "shift", because if $\min_{i\in I}\{h_A(i)\} = 0$, then $\min_{\boldsymbol{\lambda}\in\Lambda^I_w} h(\boldsymbol{\lambda}) = 0$.

Chapter 3

The ext-groups of standard modules for the Cubist algebras

3.1 Notations and facts

A Koszul algebra A is positively graded, and we will call such grading on A as r-grading.. We call the homological grading (in the Yoneda algebras/Ext-algebras) the h-grading, that is the h-degree i-th component of A^X is $\mathrm{Ext}^i_A(X,X)$. We assume an algebra A is concentrated in h-degree 0 with zero differential, unless otherwise specified.

Although Ext-algebras of a positively graded algebra are naturally bigraded, we will use only the r-grading for their categories of graded modules. Recall that the (graded) ext-groups are defined by $\mathrm{ext}^i_A(M,N) := H^i(\mathrm{hom}_A(P_M,N))$ where P_M is a minimal graded projective resolution of M. Since elements of $\mathrm{ext}^i_A(M,N\langle j\rangle)$ are map of complexes (up to homotopy), this space comes naturally with an h-degree and an r-degree. More explicitly, an element $\alpha \in \mathrm{ext}^i_A(M,N\langle j\rangle)$ is induced by a map between graded modules in $\mathrm{hom}_A(P_M^{-i},N\langle j\rangle)$. We therefore may use $\deg(\alpha) = (\deg_h(\alpha),\deg_r(\alpha))$ for $\alpha \in \mathrm{ext}^i_A(M,N\langle j\rangle)$. When P_M is a linear complex (i.e. P_M^{-i} is concentrated in r-degree i), the r-degree of α is then $i+j$. Since any graded modules M,N can be regarded as an ordinary A-module by forgetting the grading, we have

$$\mathrm{Ext}^i_A(M,N) \quad = \quad \bigoplus_{j\in\mathbb{Z}} \mathrm{ext}^i_A(M,N\langle j\rangle). \tag{3.1.1}$$

This generalises the relation between Hom-groups and (graded) hom-groups.

We record some facts about Koszul algebras, from [BGS, Kel, MOS]. We also refer the reader to the Appendix of [MT] for a concise exposition to understand the difference between the approaches of [BGS,Kel], as well as the technicalities and terminologies around Koszul algebras. First recall that for $M \in A$-gr, we denote M^* to be the graded (right A-module) $\mathbf{k}$-linear dual $\hom_{\mathbf{k}}(M, \mathbf{k})$ with i-th component being $\hom_{\mathbf{k}}(M_{-i}, \mathbf{k})$.

Theorem 3.1.1 ([BGS, Kel, MOS]). *Let A be positively graded quadratic algebra. Denote by $K = K_A = A \otimes A^{!*}$ the* Koszul complex *of A. If A Koszul, then*

(1) The canonical map $K_A \twoheadrightarrow A_0$ is a quasi-isomorphism, i.e. it induces $H^\bullet(K_A) \cong H^\bullet(A_0)$.

(2) The grading of A is compatible with radical and socle filtration of A. In particular, $L = A_0$.

(3) The Yoneda algebra $E(A) = \oplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_A(A_0, A_0) = (A^L)^{\mathrm{op}}$ is also Koszul and $A^! \cong A^L$ canonically.

(4) $(A^!)^! \cong A$ and $E(E(A)) = A$

(5) The Koszul grading coincides with homological grading of $E(A)$ (or $A^!$) in the following way: $\mathrm{Ext}^i_A(L, L) = \mathrm{ext}^i_A(L, L\langle i\rangle)$

(6) Regard A as a positively (r-)graded quadratic algebra, then there is a triangulated equivalence between certain full triangulated subcategory of $D(A\text{-gr})$ and $D(A^!\text{-gr})$:

$$D^{\downarrow}(A\text{-gr}) \underset{G}{\overset{F:=\mathrm{Hom}_A(K,-)}{\rightleftarrows}} D^{\uparrow}(A^!\text{-gr})$$

such that $F(A_0p) = A^!p$ (simples to projectives) and $F(A^*p) = A^!_0p$ (injectives to simples) for $p \in A_0$, and $F(M\langle j\rangle) = (FM)\langle -j\rangle[j]$. The pair of functors (F, G) is an adjoint pair, with $G \cong \mathrm{Hom}_{A_0}(A, -)$. Note that in [MOS], these categories are notated as $D^{\uparrow}(A\text{-gr})$ and $D^{\downarrow}(A^!\text{-gr})$. This equivalence induces the derived equivalence on the bounded derived categories:

$$D^b(A\text{-gr}) \underset{G}{\overset{F}{\rightleftarrows}} D^b(A^!\text{-gr})$$

(7) The derived equivalence in (6) restricts to equivalence of (abelian) categories:

$$\mathcal{LC}(A\text{-inj}) \xrightarrow{\sim} A^!\text{-gr} \tag{3.1.2}$$

$$\textit{injective coresolution of } L(x) \mapsto P^!(x) \tag{3.1.3}$$

$$(0 \to Q(x) \to 0) \mapsto L^!(x) \tag{3.1.4}$$

where

(a) $\mathcal{LC}(A\text{-inj})$ is the category of linear complexes such that each term is injective A-module.

(b) $L(x)$ (resp. $L^!(x)$) is simple A (resp. $A^!$) module.

(c) $Q(x)$ (resp. $P^!(x)$) is indecomposable injective A (resp. projective $A^!$) module.

Remark 3.1. Our Koszul grading shift is in reverse direction to the one chosen in [BGS], but aligned with the choice in [MOS] and many other more recent literature on graded representation theory.

For quasi-hereditary A with duality, the criteria of A being standard Koszul can be reduced to $\Delta(x)$ admits linear projective resolution $\widetilde{\Delta}(x)$ for all $x \in \mathcal{X}$. We restate some results of [ÁDL] here in the way that is compatible with [BGS].

Theorem 3.1.2 ([ÁDL]). *If A is standard Koszul, then A is Koszul. Furthermore, under the derived equivalence in 3.1.1(6), $\nabla(x) \in D^{\downarrow}(A\text{-gr})$ gets sent to the standard $A^!$-module $\Delta^!(x)$.*

By the above theorem, and the implicit result of [MOS], we have the following:

Proposition 3.1.3. *If A standard Koszul, under the equivalence of categories in (7) of Theorem 3.1.1, we have the correspondence:*

$$\nabla(x) \quad \leftrightarrow \quad \widetilde{\Delta}^!(x) \overset{q.i.}{\simeq} \Delta^!(x) \tag{3.1.5}$$

Moreover, $L(z)$ is a composition factor of $\operatorname{soc}^i(\nabla(x))/\operatorname{soc}^{i-1}(\nabla(x))$ if and only if $P^!(z)$ appears in the $-i$-th term of $\widetilde{\Delta}^!(x)$.

Let $(\mathcal{X}, \preceq)$ be the weight poset of a quasi-hereditary algebra A. For $x, y \in \mathcal{X}$, let

$$\mu x \quad = \quad \{z \in \mathcal{X} \mid P(z) \in \operatorname{add}(\widetilde{\Delta}^i(x)) \text{ for some } i\}, \tag{3.1.6}$$

$$\text{and } \lambda y \quad = \quad \{z \in \mathcal{X} \mid [\Delta(y) : L(z)] \neq 0\}. \tag{3.1.7}$$

Note that $z \in \mu x$ implies $x \leq z$ and $z \in \lambda y$ implies $z \leq y$. In particular, if $x \leq y$, then $\mu y \subset \mu x$ and $\lambda x \subset \lambda y$. If A is BGG standard Koszul, then combining with the proposition, we see that $\mu(x)$ in $A^!$ is the $\lambda(x)$ in A, and vice versa as $\Delta(x) \mapsto \nabla(x)$ under the duality functor defining BGG algebra (see [Xi]).

Proposition 3.1.4. *Suppose U is a BGG standard Koszul algebra, and V its Koszul dual, then there is a vector space isomorphism:*

$$\operatorname{ext}^i_U(\Delta_U(x), \Delta_U(y)\langle j\rangle) \cong \operatorname{ext}^{i+j}_V(\Delta_V(y), \Delta_V(x)\langle -j\rangle).$$

In particular, there is a vector space isomorphism between U^Δ and V^Δ.

Proof. Applying the contravariant BGG duality on U:

$$\begin{aligned}\mathrm{ext}^i_U(\nabla(y), \nabla(x)\langle j\rangle) &\cong \mathrm{ext}^i_U(\Delta(x)\langle -j\rangle, \Delta(y)) \\ &\cong \mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle j\rangle) \qquad (3.1.8)\end{aligned}$$

On the other hand, by Koszul duality (Theorem 3.1.1, Proposition 3.1.3), we have:

$$\begin{aligned}\mathrm{ext}^i_U(\nabla(y), \nabla(x)\langle j\rangle) &= \mathrm{Hom}_{D^\downarrow(U\text{-gr})}(\nabla(y), \nabla(x)\langle j\rangle[i]) \\ &\cong \mathrm{Hom}_{D^\uparrow(V\text{-gr})}(\Delta(y), \Delta(x)\langle -j\rangle[i+j]) \\ &\cong \mathrm{ext}^{i+j}_V(\Delta(y), \Delta(x)\langle -j\rangle) \qquad (3.1.9)\end{aligned}$$

□

3.2 Cubist algebras

We work with Cubist algebras $U_\mathcal{X}$ as introduced in [CT3]; these are the Cubist algebras with parameter $w = r - 1$ in [Tur]. We go through the construction of these infinite dimensional algebras in this section.

3.2.1 Cubist combinatorics

Given $x, y \in \mathbb{R}^r$, we write $x \leq y$ if $y - x \in \mathbb{R}^r_{\geq 0}$. This defines a partial order on $\mathbb{R}^r$. We denote by $\epsilon_1, \ldots, \epsilon_r$ the standard basis of $\mathbb{R}^r$. An *orientation* is a choice of linear ordering on the standard basis of $\mathbb{R}^r$, labelled $\{\epsilon_1, \ldots, \epsilon_r\}$. For $x \in \mathbb{R}^r$ and $\zeta \in \mathbb{R}$, let $x[\zeta] = x + \zeta(\epsilon_1 + \cdots + \epsilon_r) \in \mathbb{R}^r$.

A subset $\mathcal{X} \subset \mathbb{Z}^r$ is *Cubist*, if $\mathcal{X} = \mathcal{X}^- \backslash \mathcal{X}^-[-1]$, where $\mathcal{X}^-$ is a nonempty proper ideal of $\mathbb{Z}^r$ (with respect to the partial order $\leq$). Equivalently [CT3, Lem 4], $\mathcal{X} = \mathcal{X}^+ \backslash \mathcal{X}^+[1]$, where $\mathcal{X}^+$ is a nonempty proper coideal of $\mathbb{Z}^r$.

Let $x \in \mathcal{X}$ with $\mathcal{X}$ Cubist. The *distance* of $x, y \in \mathcal{X}$, denoted $d(x, y)$, is shortest path length from x to y in $\mathcal{X}$. This coincides with the sense of distance on $\mathbb{Z}^r$, i.e. $y = x + (a_1, \ldots, a_r)$ with $a_j \in \mathbb{Z}$, then $d(x, y) = \sum_j |a_j|$. A k-dimensional *cube*, or a k-*cube*, F of $\mathcal{X}$ is a set of size 2^k such that, for any $x, y \in F$, we have $d(x, y) \leq k$ and $y = x + (a_1, \ldots, a_r)$ with $a_j \in \{-1, 0, 1\}$. The $(r-1)$-cubes are of particular importance, and we call them *facets*. For any fixed orientation of $\mathbb{R}^r$, there are following maps defined on $\mathcal{X}$ [CT3, Sec 2.3]:

(i) $\lambda x := x + F_{i_x}$ for some i_x, where $F_{i_x} = \{\sum_{j<i_x} a_j\epsilon_j - \sum_{j>i_x} a_j\epsilon_j | a_j = 0, 1\}$. λx is a facet in $\mathcal{X}$ and defines a bijection between $\mathcal{X}$ and the set of facets of $\mathcal{X}$. We define furthermore $\lambda_j x$ to be the set $\{z \in \lambda x | d(z, x) = j\}$. We will reserve the notation i_x for this purpose from now on.

(ii) $\mu x = x + C_{i_x}$ with $C_{i_x} = \mathbb{Z}_{\leq 0}^{i_x - 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}^{r - i_x} \cap \mathcal{X}$; we also define $\mu_i x = \{z \in \mu x | d(x, z) = i\}$.

The set λx can be thought of as a cube emanating from x, and μx can be seen as a convex polyhedral cone in $\mathbb{Z}^r$ emanating from x, in all directions opposite to λx. We denote by x^{op} the *opposite vertex* of x in λx, i.e. $x^{\mathrm{op}} = x + \sum_{j < i_x} \epsilon_j - \sum_{j > i_x} \epsilon_j$.

3.2.2 Algebraic setup

We first define a graded associative $\mathbf{k}$-algebra U_r by quiver and relations. The (Ext-)quiver $Q = Q(U_r)$ of U_r has vertices (which we identify with the primitive idempotents)

$$\{e_x \mid x \in \mathbb{Z}^r\},$$

and arrows

$$\{a_{x,i}, b_{x,i} \mid x \in \mathbb{Z}^r, 1 \leq i \leq r\}.$$

The arrow $a_{x,i}$ is directed from e_x to $e_{x+\epsilon_i}$, and $b_{x,i}$ is directed from e_x to $e_{x-\epsilon_i}$. U_r is defined to be the path category kQ of Q, modulo *square relations*,

$$\begin{aligned} a_{x,i} a_{x+\epsilon_i,i} &= 0, \\ b_{x,i} b_{x-\epsilon_i,i} &= 0, \end{aligned} \tag{U0}$$

for $x \in \mathbb{Z}^r,\ 1 \leq i \leq r$, as well as *supercommutation relations*,

$$\begin{aligned} a_{x,i} a_{x+\epsilon_i,j} + a_{x,j} a_{x+\epsilon_j,i} &= 0, \\ b_{x,i} b_{x-\epsilon_i,j} + b_{x,j} b_{x-\epsilon_j,i} &= 0, \\ a_{x,i} b_{x+\epsilon_i,j} + b_{x,j} a_{x-\epsilon_j,i} &= 0, \end{aligned} \tag{U1}$$

for $x \in \mathbb{Z}^r,\ 1 \leq i, j \leq r,\ i \neq j$, and *Heisenberg relations*,

$$b_{x,i} a_{x-\epsilon_i,i} + a_{x,i} b_{x+\epsilon_i,i} = b_{x,i+1} a_{x-\epsilon_{i+1},i+1} + a_{x,i+1} b_{x+\epsilon_{i+1},i+1}, \tag{U2}$$

for $x \in \mathbb{Z}^r,\ 1 \leq i < r$.

Let $\mathcal{X}$ be a Cubist subset of $\mathbb{Z}^r$. The *Cubist algebra* associated to $\mathcal{X}$ is

$$U_{\mathcal{X}} := U_r / \sum_{x \in \mathbb{Z}^r \setminus \mathcal{X}} U_r e_x U_r.$$

When $r = 1$, $U_{\mathcal{X}} \cong \mathbf{k}$. When $r = 2$, $U_{\mathcal{X}}$ is isomorphic to the Brauer tree algebra of an infinite line (a direct limit of Brauer tree algebras of a finite line), which is also isomorphic to U_1. When $r = 3$, $U_{\mathcal{X}}$ is isomorphic to *rhombal algebras* introduced by Peach [Pea] (after using another choice of signs in the relations). We will review the combinatorics in detail for the rhombal algebras in the introduction of the next chapter.

Theorem 3.2.1 (Section 5,6 in [CT3]). *Let $\mathcal{X}$ be a Cubist set with $x, y \in \mathcal{X}$, and $U = U_{\mathcal{X}}$. Then $U_{\mathcal{X}}$ is super-symmetric, standard Koszul, with BGG duality.*

(i) The standard U-module $\Delta(y) = \Delta_U(y)$ has Loewy structure described by λy with the formula $[\Delta(y) : L(z)]_q = \begin{cases} q^{d(z,y)} & \text{if } z \in \lambda y, \\ 0 & \text{otherwise.} \end{cases}$

(ii) The minimal projective resolution of $\Delta(x)$ (retaining the notation $\widetilde{\Delta}(x)^\bullet$) is linear and can be completely described by μx as follows. The i-th term of $\widetilde{\Delta}(x)^\bullet$ is given by $\widetilde{\Delta}(x)^{-i} := \bigoplus_{z \in \mu_i x} P(z)\langle -i \rangle$, i.e. $\widetilde{\Delta}(x)$ is given by

$$\cdots \xrightarrow{d^2} \bigoplus_{z \in \mu_2 x} P(z)\langle -2 \rangle \xrightarrow{d^1} \bigoplus_{y \in \mu_1 x} P(y)\langle -1 \rangle \xrightarrow{d^0} P(x) \to 0$$

with $P(x)$ in h-degree 0. Moreover, the differential d^i is given by multiplying by a sum $\sum \alpha_{u,v}$ of arrows, where the sum is over all arrows $u \xrightarrow{\alpha_{u,v}} v$ in the quiver of U with $u \in \mu_{i+1} x$, $v \in \mu_i x$ whenever such arrow exists.

(iii) U has a duality induced by the anti-automorphism of the underlying quiver, which swaps the pair of arrows $x \xrightarrow{a_{x,i}} x + \epsilon_i$ and $x \xleftarrow{b_{x+\epsilon_i,i}} x + \epsilon_i$.

The partial order on $\mathcal{X}$ defining quasi-heredity is *not* the restriction of the partial order on $\mathbb{Z}$ described earlier. Instead, this partial order $\succeq$ is generated by the relations $x \succeq y$ for $y \in \lambda x$ [CT3, Prop 30]. In particular, an orientation, which uniquely determines i_x, defines λx, and in turn, determines the quasi-hereditary structure of $U_{\mathcal{X}}$. Consequently, the choice of quasi-hereditary structure is not unique. We note that, when an orientation is specified, then for any $x \in \mathcal{X}$ we always have $x \succ x + \epsilon_1$ and $x \succ x - \epsilon_r$ on the weight poset. Moreover, $C_1 = \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0}^{r-1} \cap \mathcal{X}$ and $C_n = \mathbb{Z}_{\leq 0}^{r-1} \times \mathbb{Z}_{\geq 0} \cap \mathcal{X}$. This has significant effect when we calculate the Ext-algebra in the case of rhombal algebras (i.e. when $r = 3$).

3.3 Main result

The main result of this chapter is a necessary and sufficient condition for $\mathrm{Ext}^{\bullet}_U(\Delta(x), \Delta(y)) \neq 0$, where U is a Cubist algebra:

Theorem 3.3.1 ((Non-)vanishing condition of Ext-groups)**.** *Let U be a Cubist algebra. Then we have the following vanishing condition for the Ext-groups of standard modules:* $\mathrm{Ext}^{\bullet}_U(\Delta(x), \Delta(y))$ *is non-zero, if and only if, $\lambda y \cap \mu x \neq \emptyset$ and for all $z \in \lambda y \cap \mu x$, $d(x, z) + d(z, y) = d(x, y)$. In this case, we have:*

$$\mathrm{Ext}^{\bullet}_U(\Delta(x), \Delta(y)) = \bigoplus_{i=i_0}^{i_0+s} \mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle d(x, y) - 2i\rangle)$$

where $s = \dim \lambda x \cap \mu y$ and $i_0 = \min\{d(x, z)|z \in \lambda y \cap \mu x\}$. Moreover, the basis of each ext-group $\mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle d(x, y) - 2i\rangle)$ can be chosen such that it is indexed by elements $z \in \lambda y \cap \mu x$ with $d(x, z) = i$.

The value $d(x, y)$ for U will be explained in the next section. Now we would like to remark on a phenomenon that appears in several other examples of standard Koszul algebra with duality.

Proposition 3.3.2. *Suppose U is one of the following classes of algebras:*

(1) Cubist algebra.

(2) A BGG standard Koszul algebra A which satisfies the condition (H) (Def 2.1.2). In particular, the principal blocks of category $\mathcal{O}$ of a complex semi-simple finite dimensional Lie algebra which are multiplicity-free, and the weight 1 blocks of Schur algebras $S(n, n)$.

Then there is a function $d : I \times I \to \mathbb{N}_0$ such that the following implication holds:

$$\mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle j\rangle) \neq 0 \;\Rightarrow\; 2i + j = d(x, y). \tag{3.3.1}$$

Proof. (1) By Theorem 3.3.1, we have $\mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle j\rangle) \neq 0$ if and only if $j = d(x, y) - 2i$.

(2) Define $d(x, y) := h(y) - h(x)$. $\mathrm{ext}^i(\Delta(x), \Delta(y)\langle j\rangle) \neq 0$ means that $\hom_A(\widetilde{\Delta}^{-i}(x), \Delta(y)\langle j\rangle) \neq 0$. Condition (H) implies that $\widetilde{\Delta}^{-i}(x) \in \mathrm{add}(\oplus_{z:h(z)=h(x)+i} P(z)\langle -i\rangle)$. So there is some weight z such that $h(z) = h(x) + i$ and $\hom(P(z)\langle -i\rangle, \Delta(y)\langle j\rangle) \neq 0$. In particular, the q^{i+j} monomial in $[\Delta(y) : L(z)]_q$ has non-zero coefficient. Condition (H) then implies that $h(y) - h(z) = i + j$. Combining the two formulae, we have

$$\begin{aligned} d(x, y) = h(y) - h(x) &= h(y) - h(z) + h(z) - h(x) \\ &= i + j + i = 2i + j \end{aligned}$$

Hence the claim. $\square$

Remark 3.2. Note that most of the Cubist algebras do not satisfy (the infinite version of) condition (H).

From a Lie theoretic perspective, it is an interesting and important to know of an quasi-hereditary algebra A, whether a function d exists such that implication (3.3.1) holds. More generally, one could ask if it holds for the following classes of algebras:

(i) BGG standard Koszul,

(ii) BGG balanced,

(iii) BGG balanced and Ringel self-dual.

Clearly, the implication holds for (i) implies that it holds for (ii); if the implication holds for (ii), then it also holds for (iii). Algebras satisfying condition (H) lie in (ii) in general, but most examples coming from Lie theory are in (iii). Cubist algebras lie in family (iii).

3.4 More Cubist combinatorics

First note that for any $x, y \in \mathcal{X}$, we can see from Theorem 3.2.1 that if $\lambda y \cap \mu x = \emptyset$, then $\mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle j \rangle) = 0$ for all i, j. Therefore, it is natural to just look at the case when $\lambda y \cap \mu x \neq \emptyset$.

Proposition 3.4.1. *For any $x, y \in \mathcal{X}$ with $\lambda y \cap \mu x \neq \emptyset$, then $\lambda y \cap \mu x$ is an s-cube $C_{x,y}$ for some $s \leq r-1$ such that $s = 0$ if an only if $y = x$. If $x \neq y$, then for all $z \in C_{x,y}$, there is some $k \in \{1, \ldots, r\}, \sigma \in \{\pm 1\}$ such that $z' = y + \sigma\epsilon_k \in \lambda y \cap \mu x$ and $d(x, z') + d(z', y) = d(x, z) + d(z, y)$.*

Proof. The first statement is proved in [CT3] Proposition 33 and Corollary 34. We proceed by induction on r. The case $r = 1$ is trivial. The case $r = 2$ gives $C_{x,y} = \lambda y$ is a 2-vertex set, so the statement is clear. Assume now $r > 2$. Assume without loss of generality that x is zero.

Case 1. $i_x = r = i_y$. One can observe that $y \in \mu x$. Note that the claim is true for $z = y$ by taking any $k < r$ (and $\sigma = +1$) so that $y + \epsilon_k \in \mu x$. Now take the maximal subset S of $\{1, \ldots, r-1\}$ so that for all $i \in S$, $y + \epsilon_i \in \mu x$. Then $C_{x,y} = \{y + \sum_{i \in S} a_i\epsilon_i | a_i = 0, 1\}$. For each $z \in C_{x,y}$ not equal to y, there is an $k \in S$ so that the k-th coordinate of z is greater than the k-th coordiante of y. The claim now follows by taking $\sigma = -1$.

Case 2. $i_x = r$ and $i_y < r$. Observe that the r-th coordinate of y must be in $\mathbb{Z}_{>0}$. If the r-th coordinate of $z \in C_{x,y}$ is the same as that of y, then we take (k, σ) as $(r, -1)$, otherwise as

$(r, +1)$. It follows that $z' \in C_{x,y}$, $d(x, z') = d(x, z) - 1$ (resp. $d(x, z) + 1$), and $d(z', y) = d(z, y) + 1$ (resp. $d(z, y) - 1$). This proves the claim.

Case 3. $i_x < r$ and $i_y = r$. $z \in C_{x,y}$ implies that the j-th coordinate z_j of z is in $\mathbb{Z}_{\leq 0}$ for all $j < i_x$. Suppose that $z_j < 0$ for some $j < i_x$, then $z_j \pm 1 \leq 0$, so there is $z' = z + \sigma\epsilon_j \in C_{x,y}$, where σ is uniquely determined. Similar to case 2, one can observe that there are two cases for $d(x, z')$ and $d(z', y)$, but both of them agrees with the equality in the claim.

On the other hand, suppose z_j is zero for all $j = 1, \ldots, i_x - 1$. If the i_x-th coordinate of z is the same as that of y, then we take $(k, \sigma) = (i_x, 1)$, otherwise take $(i_x, -1)$. Again, we can deduce the claim with similar situations as those in case 2.

Case 4. $i_x < r, i_y < r$. If the r-th coordinate of y is greater than 0, then we take $k = r$, and observe similar (sub)case-splitting (depending on σ, which is uniquely determined by r-th coordinate of z relative to that of y) as in case 2 and 3. Similar calculation shows the validity of the claim.

If the r-th coordinate of y is zero, then we consider the Cubist subset $\mathcal{X}_0$ of $\mathcal{X}$ given by vertices with r-th coordinate zero, and the claim follows from the induction hypothesis. $\square$

From now on, we will fix any $x, y \in \mathcal{X}$ with $x \neq y$ such that $\lambda y \cap \mu x \neq \emptyset$, and adopt the following notations:

(i) $C_{x,y} = \lambda y \cap \mu x$.

(ii) $s := \dim C_{x,y}$.

(iii) $i_0 := \min\{i \geq 0 | \lambda y \cap \mu_i x \neq \emptyset\}$.

(iv) z_0 be the unique vertex in $C_{x,y}$ such that $d(x, z_0) = i_0$.

(v) $B_{x,y} := \{z \in \mathbb{Z}^r | d(x, z) + d(z, y) = d(x, y)\}$.

Note that $B_{x,y}$ defines a cuboid (*box*) in $\mathbb{Z}^r$ with x, y being opposite corners. Now the vanishing condition in the statement of Theorem 3.3.1 is equivalent to saying $C_{x,y} \cap B_{x,y} = C_{x,y} \neq \emptyset$.

Lemma 3.4.2. *$C_{x,y} \cap B_{x,y}$ is non-empty.*

Proof. We do this by induction on r. For $r = 1$, the statement is trivial. Now assume this is true for all cubist sets of $\mathbb{Z}^{r-1}$, and let $\mathcal{X}$ a Cubist set of $\mathbb{Z}^r$. We will find a vertex in $z \in C_{x,y} \cap B_{x,y}$. Assume without loss of generality that $x = 0$. Note that, as $z \in \lambda y$, $z = (z_1, \ldots, z_r) = (y_1 + \delta_1, \ldots, y_r + \delta_r)$ with suitable $\delta_j \in \{0, \pm 1\}$ for all $j = 1, \ldots, r$. Now we have:

$$
\begin{aligned}
z \in B_{x,y} &\Leftrightarrow d(x,z) + d(z,y) = d(x,y) \\
&\Leftrightarrow \sum_{j=1}^{r} |z_j| + \sum_{j=1}^{r} |\delta_j| = \sum_{j=1}^{r} |y_j| \\
&\Leftrightarrow |y_j + \delta_j| + |\delta_j| = |y_j| \text{ for all } j = 1, \dots, r \\
&\Leftrightarrow \begin{cases} \delta_j \in \{0,1\} & \text{if } y_j < 0; \\ \delta_j = 0 & \text{if } y_j = 0; \\ \delta_j \in \{0,-1\} & \text{if } y_j > 0. \end{cases}
\end{aligned} \tag{3.4.1}
$$

Case 1. $i_x = r = i_y$. Then we must have $y \in \mu x$, and obviously $y \in B_{x,y}$, so take $z = y$.

Case 2. $i_x = r$ and $i = i_y < r$. Let us first look at what the condition $C_{x,y} \neq \emptyset$ tells us about the j-th coordinate of y. If $z \in \mu x = C_r$, we have $z_i \leq 0$ for all $i < r$ and $z_r \geq 0$. On the other hand, since $z \in \lambda y$, we have $z_j = y_j + \delta_j \leq 0$ with $\delta_j \in \{0,1\}$ for all $j \in \{1, \dots, i-1\}$. Combining the two conditions we have $y_j \leq 0$ for all $j \in \{1, \dots, i-1\}$. Since $\delta_i = 0$, we obtain $y_i \leq 0$. For $j > i$ and $j \neq r$, we have $z_j = y_j + \delta_j \leq 0$ for some $\delta_j \in \{0,-1\}$, which means that $y_j \leq 1$ always. If $j = r$, then we have $y_r > 0$ by similar argument. Now define $z = y + (\delta_1, \dots, \delta_j)$ as follows:

$$
\delta_j = \begin{cases} 1 & \text{if } j < i, y_j < 0 \\ 0 & \text{if } j < i, y_j = 0, \text{ or } j = i \\ -1 & \text{if } j > i, y_j > 0 \\ 0 & \text{if } j > i, y_j \leq 0 \end{cases}
$$

Now each coordinate of z satisfies (3.4.1) and guarantees that $z \in C_{x,y}$, and so $z \in C_{x,y} \cap B_{x,y}$. Note that, our construction of z here guarantees that $z = z_0$.

Case 3. $i = i_x < r$ and $i_y = r$. Similarly to case 2, for $j < i$, we get $y_j \leq 0$. For $j = i$, we only have $y_i \in \mathbb{Z}$. For $j > i$ (including $j = r$), $z \in \mu x$ implies $z_j \geq 0$, and so $y_j \geq -1$ (for $j = r, y_j \geq 0$). Now we define $z = y + (\delta_1, \dots, \delta_r)$ as follows:

$$
\delta_j = \begin{cases} 0 & \text{if } y_j \geq 0 \\ 1 & \text{if } y_j < 0 \end{cases}
$$

This construction of z again makes $z \in C_{x,y} \cap B_{x,y}$. We also have $z = z_0$ in this case.

Case 4. $i_x < r, i_y < r$. $C_{x,y} \neq \emptyset$ implies $y_r \geq 0$. If $y_r > 0$, then y and $y - \epsilon_r$ are both in

$C_{x,y}$. Obviously $y \in B_{x,y}$. If $y_r = 0$, then by ignoring the r-th coordinate of y and x, we get two vertices which lie in a Cubist set $\mathcal{X}_0$ of $\mathbb{Z}^{r-1}$ (cf. proof of [CT3] Prop 30, Prop 33). So induction hypothesis finishes the proof. $\square$

Lemma 3.4.3. *Suppose $z, z' \in \mu x \cap \lambda y$, such that $d(z,z') = 1$ and $d(x,z') = d(x,z)+1$, then we have the following two cases*

(i) $d(z',y) = d(z,y) - 1$, hence $d(x,z') + d(z',y) = d(x,z) + d(z,y)$.

(ii) $d(z',y) = d(z,y) + 1$, hence $d(x,z') + d(z',y) = d(x,z) + d(z,y) + 2$.

Proof. Follows easily from the fact that $d(z,z') = 1$ then $d(z',y) = d(z,y) \pm 1$. $\square$

In Lemma 3.4.2, we have already shown $z_0 \in B_{x,y}$ in two out of four possible cases. This is actually always true:

Lemma 3.4.4. *$z_0 \in B_{x,y}$.*

Proof. Suppose to the contrary that $z_0 \notin B_{x,y}$. Note that Prop 3.4.1 implies that $s \geq 1$ for $x \neq y$ and $C_{x,y} \neq \emptyset$, so $\mu_{i_0+1}x \cap \lambda y \neq \emptyset$. Now for all $z \in \mu_{i_0+1}x \cap \lambda y$, we have $d(z_0, z) = 1$ and $d(x,z) = d(x,z_0) + 1$ The condition for Lemma 3.4.3 is now satisfied, and we have either $d(z,y) = d(z_0,y) - 1$ or $d(z,y) = d(z_0,y) + 1$. In the former case, we get $d(x,z) + d(z,y) = d(x,z_0) + d(z_0,y)$, which is strictly greater than $d(x,y)$ by the assumption, so $z \notin B_{x,y}$. In the other case, we get $d(x,z) + d(z,y) = d(x,z_0) + d(z_0,y) + 2 \gneqq d(x,y)$. Repeating this procedure, it follows that all vertices in $C_{x,y}$ are not in $B_{x,y}$, which contradicts Lemma 3.4.2. $\square$

Lemma 3.4.5. *$C_{x,y} \cap B_{x,y}$ is a cube of dimension $t \geq 1$.*

Proof. Since $C_{x,y}$ is a cube and $B_{x,y}$ is a cuboid, their intersection is a t-cube for some $t \leq s$. $t \geq 1$ follows from Proposition 3.4.1. $\square$

Let S denote the maximal subset of $\{1, \ldots, r\}$ satisfying the property: for all $k \in S$, $z_0 + \sigma_k \epsilon_k \in C_{x,y}$ for some $\sigma_k \in \{\pm 1\}$. Note that σ_k are determined by the fact that $C_{x,y}$ is a cube.

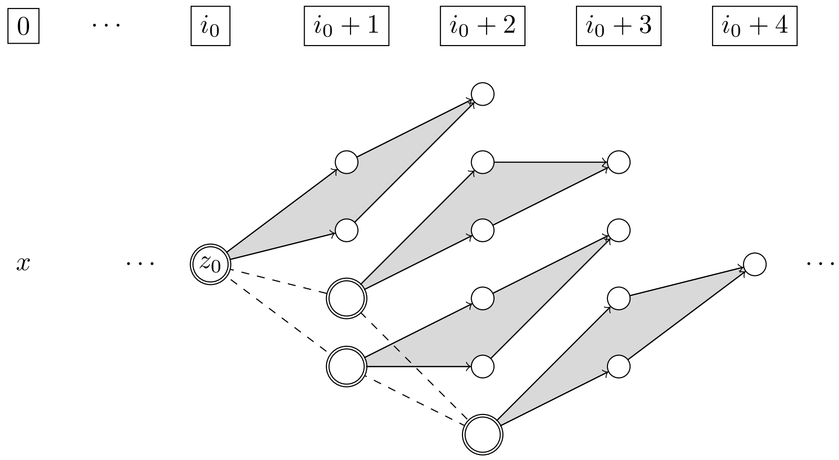
Similarly, define a subset T which is maximal in S satisfying the property: for all $k \in T$, $z_0 + \sigma_k \epsilon_k \in C_{x,y} \cap B_{x,y}$. It follows that the dimension of the cube $C_{x,y} \cap B_{x,y}$ is the size of the set T.

Using Lemma 3.4.3, we observe that $S \setminus T$ give rise to a subcube $D^{z_0}_{x,y}$ in $C_{x,y}$ of dimension $s - t$, containing vertex z_0, such that all other vertices in this cube are not in $B_{x,y}$. In fact, by the same argument, for every vertex $z \in C_{x,y} \cap B_{x,y}$, one has $z + \sigma_k \epsilon_k \notin B_{x,y}$ for all $k \in S \setminus T$.

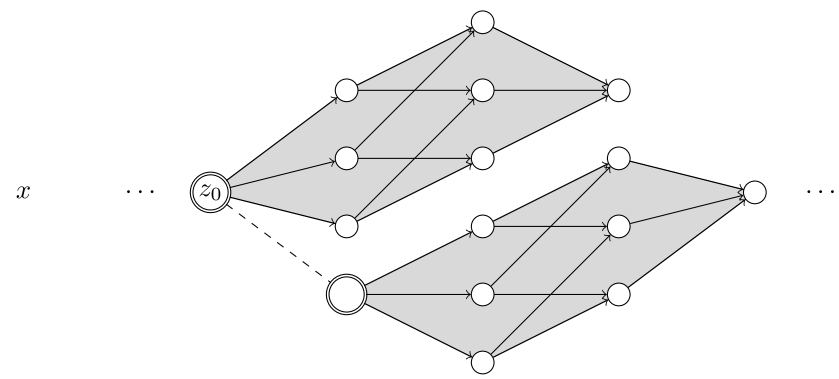
Hence, z induces a subcube $D^z_{x,y}$ of dimension $s-t$. By counting the number of vertices, we get the decomposition:

$$C_{x,y} = (C_{x,y} \cap B_{x,y}) \sqcup \Big(\bigsqcup_{z \in C^T_{x,y}} D^z_{x,y} \setminus \{z\} \Big). \tag{3.4.2}$$

Example 3.3. We briefly give some possible scenarios in Figure 3.1. In these examples, we have $s = \dim C_{x,y} = 4$. We circled the vertices of $C_{x,y}$. The double-circled nodes are vertices in $C_{x,y} \cap B_{x,y}$, and the single-circled ones are those in $C_{x,y} \setminus B_{x,y}$. We arrange the vertices in the order of their distances from x, which are labelled in the framed box in the top row. We will explain the arrows between the vertices later in the proof of Theorem 3.5.3.



Case: $t = \dim C_{x,y} \cap \Omega(x,y) = 2$



Case: $t = \dim C_{x,y} \cap \Omega(x,y) = 1$

Figure 3.1: Visualising $C_{x,y}$ and $C_{x,y} \cap B_{x,y}$

3.5 Proof of Theorem 3.3.1

Strategy of Proof: First we use the combinatorics to deduce existence of maps from $\widetilde{\Delta}(x)$ to $\Delta(y)$, and existence of non-zero differential (Lemma 3.5.1). Then we prove in Theorem 3.5.2 that if $C_{x,y} \cap B_{x,y} = C_{x,y}$, we can obtain non-zero ext-groups, and we describe them explicitly. This contributes to half of the Theorem 3.3.1. Finally, in theorem 3.5.3 we show that if $C_{x,y} \cap B_{x,y} \neq C_{x,y}$, then all ext-groups $\mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle j\rangle)$ vanish. This shows the remaining statement of Theorem 3.3.1. $\square$

Consider the Cubist algebra U as a dg algebra concentrated in (h-)degree 0, with zero differential (of degree +1). The differential grading will appear in superscripts whenever needed; the Koszul (r-)grading, will appear in subscripts. Hence $U = \bigoplus_{h,r\geq 0} U^h_r = \bigoplus_{r\geq 0} U^0_r$. The projective resolution $\widetilde{\Delta}^\bullet(x)$ (with differential denoted $d_x = d^\bullet_x$) of $\Delta(x)$ is naturally a dg U-module. When we ignore the Koszul grading, the complex $\mathrm{Hom}^\bullet_U(\widetilde{\Delta}(x), \Delta(y))$ is viewed as a dg $\mathbf{k}$-module with the i-th component being the $\mathbf{k}$-space $\mathrm{Hom}_U(\widetilde{\Delta}^i(x), \Delta(y))$, and differential $\widetilde{d}(f) := f \circ d_x$. In particular, the Ext-groups are given by

$$\mathrm{Ext}^\bullet_U(\Delta(x), \Delta(y)) = \bigoplus_{i\geq 0} \mathrm{Ext}^i_U(\Delta(x), \Delta(y)) = \bigoplus_{i\geq 0} H^i\left(\mathrm{Hom}^\bullet_U(\widetilde{\Delta}(x), \Delta(y))\,;\, \widetilde{d}\right)$$

Taking the Koszul grading into consideration, we have dg $\mathbf{k}$-module $\mathrm{hom}^\bullet_U(\widetilde{\Delta}(x), \Delta(y)\langle j\rangle)$, where the i-th dg component is the $\mathbf{k}$-space $\mathrm{hom}_U(\widetilde{\Delta}^i(x), \Delta(y)\langle j\rangle)$, with the same differential $\widetilde{d}(f) := f \circ d_x$ as before. The graded ext-group $\mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle j\rangle)$ is obtained by taking homology. Note that the dg $\mathbf{k}$-modules defined above are different from the "internal Hom-space" $\mathcal{H}\mathrm{om}^\bullet_U(\widetilde{\Delta}(x), \widetilde{\Delta}(y))$, which is a dg $\mathbf{k}$-module with differential given by $\partial_{x,y}(f) = d_y \circ f + f \circ d_x$.

Lemma 3.5.1. *(i)* $\mathrm{hom}(\widetilde{\Delta}^{-i}(x), \Delta(y)\langle j\rangle) \neq 0$ *if and only if there exists some z in $\lambda_{i+j}y \cap \mu_i x$. In this case, we can choose a basis of* $\mathrm{hom}(\widetilde{\Delta}^{-i}(x), \Delta(y)\langle j\rangle)$ *indexed by elements in $\lambda_{i+j}y \cap \mu_i x$.*

(ii) Suppose there exists $z \in \lambda_{i+j}y \cap \mu_i x$, and let $(\alpha : P(z)\langle -i\rangle \to \Delta(y)\langle j\rangle) \in \mathrm{hom}(\widetilde{\Delta}^i(x), \Delta(y)\langle j\rangle)$ be the corresponding map, then $\widetilde{d}(\alpha)$ is non-zero if and only if there exists some $v \in \lambda_{i+j+1}y \cap \mu_{i+1}x$ with $d(v,z) = 1$.

Proof. (i): There is a non-zero map from $\widetilde{\Delta}^{-i}(x)$ to $P(y)\langle j\rangle$ if and only if there is a direct summand $P(z)\langle -i\rangle$ of $\widetilde{\Delta}^{-i}(x)$ such that $L(z)$ is a composition factor in the $(i+j)$-th radical of $\Delta(y)$. This is equivalent to having a $z \in \lambda y \cap \mu_i x$ with $d(z,y) = i+j$, i.e. there exists $z \in \lambda_{i+j}y \cap \mu_i x$. In particular, since the coefficient of $[\Delta(y) : L(z)]_q$ is always 0 or 1,

$\dim \hom_U(\widetilde{\Delta}^{-i}(x), \Delta(y)\langle j\rangle) = \#\lambda_{i+j}y \cap \mu_i x$, and so we can choose a basis using elements of $\lambda_{i+j}y \cap \mu_i x$ which represent the corresponding multiplication of elements of U.

(ii): $\widetilde{d}(\alpha) \neq 0$ if and only if there is a direct summand $P(v)\langle -(i+1)\rangle$ of $\widetilde{\Delta}^{-(i+1)}(x)$ such that d_x^{i+1} maps $P(v)\langle -(i+1)\rangle$ non-trivially to $P(z)\langle -i\rangle$, and $\hom_U(P(v)\langle i+1\rangle, \Delta(y)\langle j\rangle) \neq 0$. This is equivalent to having $v \in \lambda_{i+j+1}y \cap \mu_{i+1}x$ with a r-degree 1 map from $P(v)$ to $P(z)$, i.e. $d(z, v) = 1$. □

Combining with case (1) of Lemma 3.4.3, we have part of the result for Theorem 3.3.1. Recall that i_0 is the minimal positive integer i such that $\lambda y \cap \mu_i x \neq \emptyset$, and s is the dimension of $C_{x,y}$.

Theorem 3.5.2. *If $C_{x,y} \cap B_{x,y} = C_{x,y} \neq \emptyset$, then the induced differential $\tilde{d} \equiv 0$. In particular, for each $i \in \{i_0, \dots, i_0 + s\}$, there is a vector space isomorphism*

$$\mathrm{ext}_U^i(\Delta(x), \Delta(y)\langle d(x,y) - 2i\rangle) \cong \bigoplus_z \mathbf{k} \cdot z,$$

where the summation is over all elements of the non-zero set $\lambda_{d(x,y)-i}y \cap \mu_i x$. For any $i \notin \{i_0, \dots, i_0 + s\}$, the ext-groups $\mathrm{ext}_U^i(\Delta(x), \Delta(y)\langle j\rangle)$ vanish for all j.

Proof. By Lemma 3.4.3, the condition $C_{x,y} \cap B_{x,y} = C_{x,y}$ implies that for all $z, z' \in C_{x,y}$ with $d(z, z') = 1$ and $d(x, z') = d(x, z) + 1$, we have $d(z', y) = d(z, y) - 1$. By Lemma 3.5.1 (2), this implies the induced differential $\widetilde{d_x}$ on $\mathrm{Hom}(\widetilde{\Delta}(x), \Delta(y))$ is zero everywhere. So the ext-groups are just $\hom_U(\widetilde{\Delta}^{-i}(x), \Delta(y))$. The rest follows from the description in Lemma 3.5.1 (1). □

Recall from Lemma 3.4.5 that t is the dimension of the subcube $C_{x,y} \cap B_{x,y}$ in the s-cube $C_{x,y}$, and $t \geq 1$. Also recall that the vector space $\mathbf{k}[X]/(X^2) = (\mathbf{k} \xrightarrow{X} \mathbf{k} \cdot X)$ is a dg $\mathbf{k}$-module concentrated in degree 0 and 1, with differential given by multiplying X. This is a dg $\mathbf{k}$-module with zero homology. In particular $(\mathbf{k}[X]/(X^2))^{\otimes n}$ also has zero homology for all $n \geq 1$.

Theorem 3.5.3. *If $C_{x,y} \cap B_{x,y} \subsetneqq C_{x,y}$, then we have the following dg $\mathbf{k}$-module isomorphisms*

$$\mathrm{Hom}_U^\bullet(\widetilde{\Delta}(x), \Delta(y)) \cong \bigoplus_{k=0}^{t} (\mathbf{k}[X]/(X^2))^{\otimes(s-t)})^{\oplus\binom{t}{k}}[-(i_0 + k)], \tag{3.5.1}$$

In particular, the ext-groups vanish:

$$\mathrm{Ext}_U^\bullet(\Delta(x), \Delta(y)) = \bigoplus_{i,j} \mathrm{ext}_U^i(\Delta(x), \Delta(y)\langle j\rangle) = 0.$$

Proof. As shown in the proof of Lemma 3.4.5 (cf. Figure 3.1), for each $z \in C_{x,y} \cap B_{x,y}$, we have an $(s-t)$-cube $D_{x,y}^z$ which satisfies the following conditions:

(i) $z \in D^z_{x,y}$ and it is the unique element in $D^z_{x,y}$ such that $d(x,z) < d(x,z')$ for all $z' \in D^z_{x,y}$,

(ii) for all $z' \in D^z_{x,y}$ with $d(z,z') = 1$, we have $d(z',y) = d(z,y) + 1$.

Using Lemma 3.5.1 (1), we identify the basis of $\mathrm{Hom}_U(\widetilde{\Delta}(x), \Delta(y))$ with $C_{x,y}$. That is, for each $z \in C_{x,y}$, we have the basis by α_z which lives in $\mathrm{Hom}(\widetilde{\Delta}^{-d(x,z)}(x), \Delta(y))$. On the other hand, Lemma 3.5.1 (2) says that each vertex $z' \neq z$ in $D^z_{x,y}$, we get $\alpha_{z'} = \widetilde{d}(\alpha_z)$. For simplicity, we work without Koszul grading. Now we have a dg-module isomorphism

$$(\mathrm{Hom}(P(z), \Delta(y)) \xrightarrow{\delta} \mathrm{Hom}(P(z'), \Delta(y))) = (\mathbf{k} \cdot \alpha_z \xrightarrow{\delta} \mathbf{k} \cdot \alpha_{z'}) \cong \mathbf{k}[X]/(X^2)$$

where δ is the restriction of $\widetilde{d}$ on $\mathrm{Hom}(P(z), \Delta(y))$. Because of decomposition (3.4.2), we can identify $D^z_{x,y}$ with (the basis of) a subspace of $\mathrm{Hom}(\widetilde{\Delta}(x), \Delta(y))$, thus obtaining an isomorphism of dg modules $D^z_{x,y} \cong (\mathbf{k}[X]/(X^2))^{\otimes(s-t)}$. These dg modules can be visualised as the each darkened cubes $D^z_{x,y}$ in Figure 3.1, with the arrows representing the non-zero differential. Now using a standard combinatorial argument on choosing vertices z of a t-cube with $d(z_0, z) = k$ for each fixed $k \in \{0, \ldots, t\}$, and the decomposition (3.4.2), we obtain the isomorphism 3.5.1 of dg modules. $\square$

Chapter 4

Ext-algebra of standard modules for the rhombal algebras

The main aim of this chapter is to get a glimpse of the structure of the Ext-algebra U^Δ for a *rhombal algebra*, i.e. Cubist algebra with Cubist set $\mathcal{X} \subset \mathbb{Z}^3$, using the the basis obtained from the previous chapter. Note that such a Cubist set can now be projected onto $\mathbb{R}^2$ to form a rhombic tiling of the plane. In particular, each facet λx is precisely a rhombus in the tiling, so we sometimes call a facet λx a *rhombus* (cf. [CT3, Figure 3]). Many of our combinatorial observations in this chapter can be drawn on paper easily. Pictures presented in [Pea, Tur] and the introduction of [CT3] are particularly useful to understand the combinatorics.

After going through some conventions, we state our main result on the description of the quiver $Q(E_\Delta)$ of E_Δ in section 4.1. We then prove some useful lemmas in section 4.2, which will help us to determine the generators for $E_\Delta := \mathrm{Ext}^\bullet_U(\Delta, \Delta)$, i.e. the arrows on the quiver $Q(E_\Delta)$, in section 4.3. We also determine the complete structure of U^Δ for some special types of Cubist sets in section 4.4. At the end of the chapter, we investigate the A_∞-structure of U^Δ.

4.1 Conventions and statement of main result

Since $\widetilde{\Delta}$ should be regarded as a left U- right $\mathrm{End}(\widetilde{\Delta})$-bimodule, we compose maps of complexes from left to right. This means the multiplicative structure of the Yoneda algebra

$E_\Delta = \mathrm{Ext}^\bullet_U(\Delta, \Delta)$ is given by:

$$\begin{array}{rcl} \mathrm{Ext}^i_U(\Delta(x), \Delta(y)) \otimes \mathrm{Ext}^j_U(\Delta(y), \Delta(z)) & \to & \mathrm{Ext}^{i+j}_U(\Delta(x), \Delta(z)) \\ [\alpha] \otimes [\beta] & \mapsto & [\alpha \circ \beta] \end{array}$$

where $[\alpha]$ denotes the homotopy class of the map $\alpha : \widetilde{\Delta}(x) \to \widetilde{\Delta}(y)$, etc. Since path multiplication of the path algebras also go from left to right, the arrows of the quiver $Q(E_\Delta)$ of E_Δ are identified with maps of complexes. Equivalently, $Q(E_\Delta)$ is the quiver so that the right module structure of the path algebra $kQ(E_\Delta)/I$ and E_Δ coincide, with indecomposable projective right modules given by

$$e_x kQ(E_\Delta)/I \cong e_x E_\Delta = \mathrm{Ext}^\bullet_U(\Delta(x), \Delta).$$

We choose to work with these conventions throughout this chapter, and so all relations presented are actually relations of the Yoneda algebra. Since U^Δ is just the opposite ring of the Yoneda algebra, its quiver $Q(U^\Delta)$ is just the opposite quiver of $Q(E_\Delta)$, and relations can be obtained by reading the relations for the Yoneda algebra in reverse direction (i.e. from right to left).

Theorem 4.1.1 (Quiver of the Yoneda algebra)**.** *Let U be a rhombal algebra. There is a combinatorial construction for the quiver $Q(E_\Delta)$ of the Yoneda algebra $\mathrm{Ext}_U(\Delta, \Delta)$ from $\mathcal{X}$ as follows:*

(i) The set of vertices of $Q(E_\Delta)$ is identified with the set of facets $\{\lambda x | x \in \mathcal{X}\}$.

(ii) For each $x \prec y$ with λx and λy sharing an edge, assign a pair of arrows from λx to λy.

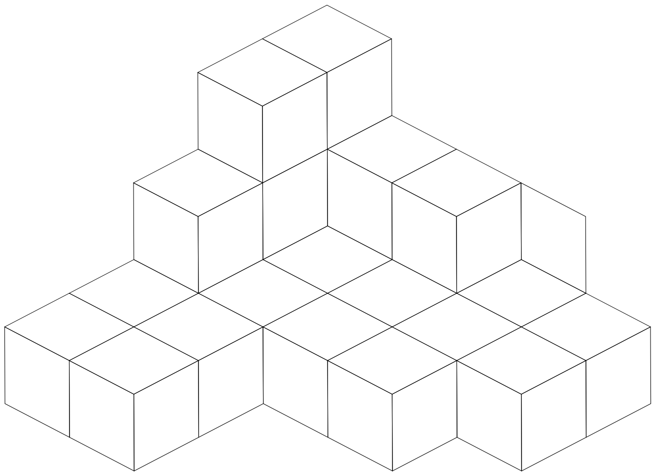
(iii) For each corner configuration (4.3.4), remove the pair of arrow from $x + F_3$ to $y + F_1$ (or $x + F_1$ to $y + F_3$) as shown in (4.3.4).

(iv) For each $x \in \mathcal{X}$ with $i_x = 1$. If there is some $y = x - k\epsilon_1 + \epsilon_3 \in \mathcal{X}$ (with $k > 0$), then add a pair of arrows from λx to λy, as in (4.3.3).

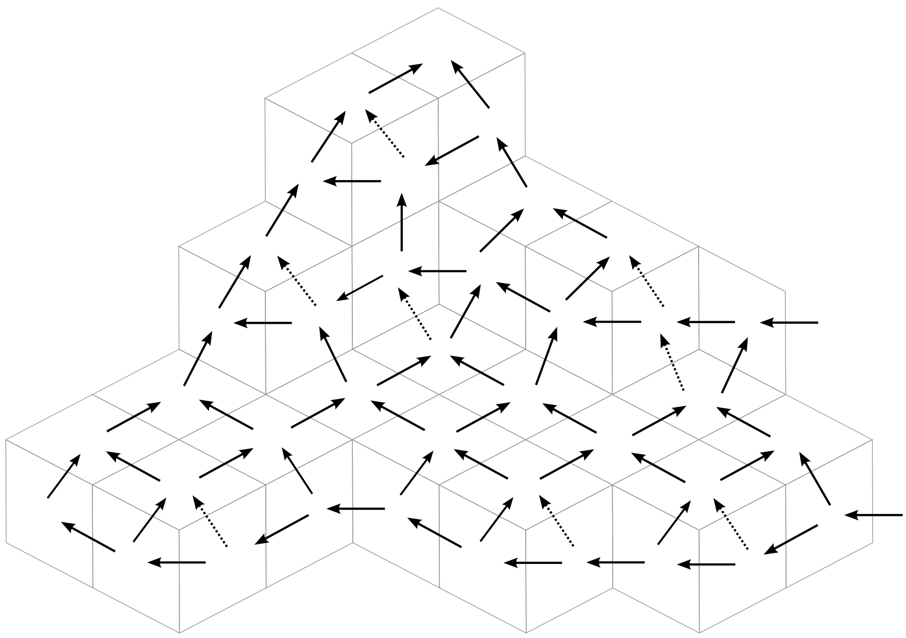
(v) For each $x \in \mathcal{X}$ with $i_x = 3$. If there is some $y = x - \epsilon_1 + k\epsilon_3 \in \mathcal{X}$ (with $k > 0$), then add a pair of arrows from λx to λy, as in (4.3.3).

(vi) For each $x \in \mathcal{X}$ with $i_x = 2$. If there is some $y = x - k\epsilon_1 + k'\epsilon_3 \in \mathcal{X}$, such that $i_z \neq 2$ for all $z \in B_{x,y} \cap \mathcal{X} \setminus \{x, y\}$, then add four arrows from λx to λy.

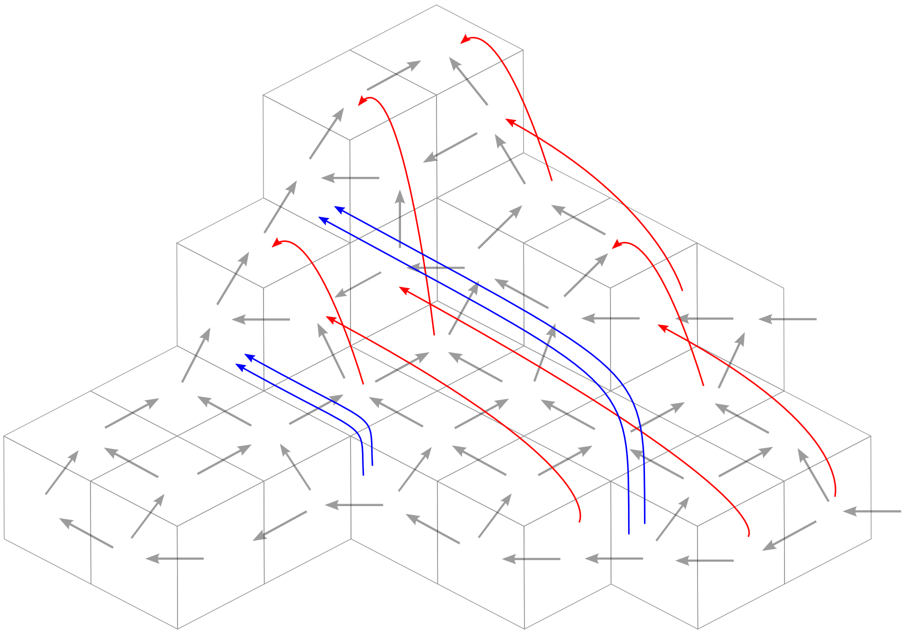
Example 4.1. We choose an orientation (i.e. quasi-hereditary structure of $U = U_\mathcal{X}$) as in Figure 4.1. Note $\mathcal{X}$ is an infinite set. We can only look at a "local portion" of the algebra. Here is a an example of a (local portion of) rhombic tiling:



To construct the quiver of U^{Δ}, first take the Poincaré dual of the tiling, with arrows pointing in the correct direction (this is determined by the orientation, i.e. the quasi-hereditary structure).



Note first that all the arrows we draw here represent a *pair* of arrows, in order to make presentation concise. Now we need to delete some pairs of arrows - one pair from each triangle of pairs from corner configuration. These are drawn as dotted arrows in the above diagram. Finally we need to put the "jumps" (i.e. arrows of the form (iii), (iv), (v) in the statement of the theorem):



This completes the description of quiver of U^Δ.

By Theorem 3.3.1, if $\mathrm{Ext}_U(\Delta(x), \Delta(y)) \neq 0$, then its basis is indexed by $z \in \lambda y \cap \mu x$. We label the corresponding map $(m,n)_z \in \mathrm{ext}_U^m(\Delta(x), \Delta(y)\langle n-m\rangle)$. In particular, $(m,n) = (d(x,z), d(z,y))$. We will omit z in the labelling if there is no ambiguity.

As in the previous chapter, we always assume $x \neq y$ and $\lambda y \cap \mu x \neq \emptyset$. For $x = y$, $\mathrm{Ext}_U(\Delta(x), \Delta(y)) = \mathbf{k} \cdot [\mathrm{id}_x]$, where id_x is the identity map of $\Delta(x)$. This corresponds to a primitive idempotent e_x in E_Δ for each $x \in \mathcal{X}$.

We will also use $E_{x,y}$ to denote $\mathrm{Ext}_U(\Delta(x), \Delta(y))$ for typographical convenience. For two basis elements $\alpha \in E_{x,y}$ and $\beta \in E_{y,z}$, of degree (h_1, r_1) and (h_2, r_2) respectively, the product is an element of degree $(h_1 + h_2, r_1 + r_2)$ in $E_{x,y}$, if it is non-zero.

To visualise the combinatorics in the proofs, we fix the orientation shown in Figure 4.1. Each choice of orientation of the Cubist set (i.e. choice of ordering ϵ_i's) corresponds to a choice of division in the projection of coordinate axes. This choice will be labelled by a circle-headed arrow stemming from each vertex in $\mathcal{X}$. The rhombus λx corresponding to a vertex x is then the one containing the circle-headed arrow, as shown in Figure 4.1.

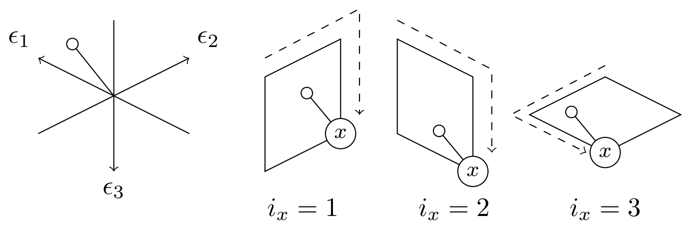


Figure 4.1: A choice of orientation and corresponding visualisation of λx.

Let $\lambda y = \{y, u, v, y^{\mathrm{op}}\}$. If $(d(x,y)-2,2)_{y^{\mathrm{op}}} \in E_{x,y}$, then it is induced by $P(y^{\mathrm{op}}) \to \Delta(y)$. Because of the supercommutation relation, there is a choice for this map, given by multiplication of the path $\rho = (y^{\mathrm{op}} \to u \to y)$ or $(y^{\mathrm{op}} \to v \to y)$. From now on, we will fix a choice of $(d(x,y)-2,2)_{y^{\mathrm{op}}}$ for all y as follows:

i_y	choice of ρ
1	$y^{\mathrm{op}} \to y - \epsilon_3 \to y$
2	$y^{\mathrm{op}} \to y - \epsilon_3 \to y$
3	$y^{\mathrm{op}} \to y + \epsilon_1 \to y$

The path ρ is visualised as the dotted arrow in Figure 4.1.

When presenting our calculations of multiplying (composing) the maps of complexes, we adopt the following notation. We will write z instead of $P(z)$ for the projective indecomposable module appearing in the complexes, and use comma instead of $\oplus$ to save spaces. A map between components of complexes are drawn vertically and labelled by $x_1 + \cdots + x_k \to y$ to represent the map $P(x_1) \oplus \cdots \oplus P(x_k) \to P(y)$ by mapping each $P(x_i)$ onto the same radical layer of $P(y)$. Equivalently, this means that the map is given by multiplication of $\sum_i (x_i \to y)$ for some (choice of) path from x_i to y. If $d(x_i, y) = 1$, there is no ambiguity. The choice of maps for $d(x_i, y) = 2$ is shown in Figure 4.1. We will not use any paths of length greater than 2. We omit the labelling if the map is clear (e.g. from a single projective indecomposable to another). Moreover, we will not list all the (possible) projective indecomposable appearing at a given component of a complex. We only list those modules that are important to the calculations.

4.2 Some Lemmas

Definition 4.2.1. *We say that two rhombi λx and λy in the Cubist set $\mathcal{X}$ are* on the same strip *of direction ϵ_a, if there is a sequence of vertices $x_0 = x, x_1, \ldots, x_n = y$ such that for each $i \in \{1, \ldots, n\}$, $\lambda x_{i-1} \cap \lambda x_i = \{u, u + \sigma_a \epsilon_a\}$ for some u (depending on i).*

Remark 4.2. See Figure 3 in [Tur].

Lemma 4.2.2. *$\#\lambda y \cap \mu x = 2$ if and only if $\dim E_{x,y} = 2$.*

Proof. Let $\{a, b, c\} = \{1, 2, 3\}$. If λx and $\lambda y = \{y, u, v, y^{\mathrm{op}}\}$ are on the same strip of direction a, then there is $\alpha_b, \alpha_c \in \mathbb{Z}$ such that $x + \alpha_b \epsilon_b + \alpha_c \epsilon_c \in \lambda y$. In particular, (as $x \neq y$) $\#\lambda y \cap \mu x = 2$. It is also not difficult to see that $\lambda y \cap \mu x \cap B_{x,y} = \lambda y \cap \mu x$ in this case, and so $\dim E_{x,y} = 2$. Conversely, if $\dim E_{x,y} = 2$, then basis of $E_{x,y}$ bijects with $C_{x,y} \cap B_{x,y} = C_{x,y}$ and $\#\lambda y \cap \mu x = 2$. ∎

So the following are all possibilities of $\lambda y \cap \mu x$ when its size is 2, along with the corresponding basis of $E_{x,y}$:

(i) $\{y, w\}$ with $E_{x,y}$ spanned by $\{(d,0)_y, (d-1,1)_w\}$, where w is either u or v.

(ii) $\{w, y^{\mathrm{op}}\}$ with $E_{x,y}$ spanned by $\{(d-1,1)_w, (d-2,2)_{y^{\mathrm{op}}}\}$, where w is either u or v.

As remarked in a previous chapter and [CT3, Def 11], the slogan to think about μx is "a convex polyhedral cone in $\mathbb{Z}^3$ emanating from x in all directions opposite to λx". This cone is bounded by the two strips (in directions not equal to i_x) containing λx, and so $\#\lambda y \cap \mu x = 2$ forces λy to be on one of these two strips (otherwise, $\#\lambda y \cap \mu x = 4$). Visualisation of this can be found in [CT3, Fig 3] or [Tur, Fig 7].

We also note that if $\lambda y = \{y, u, v, y^{\mathrm{op}}\}$ and $\dim E_{x,y} = 4$, then $E_{x,y}$ is spanned by $\{(d,0)_y, (d-1,1)_u, (d-1,1)v, (d-2,2)_{y^{\mathrm{op}}}\}$.

Lemma 4.2.3. *If $i_y = 2$, then $E_{x,y} \neq 0$ and $y \in \lambda y \cap \mu x$.*

Proof. Let $\lambda y = \{y, y+\epsilon_1, y-\epsilon_3, y+\epsilon_1-\epsilon_3\}$. Assume without loss of generality, $x = (0,0,0)$.

Case 1 $\lambda y = \lambda y \cap \mu x$: For any $i_x \in \{1,2,3\}$, and $z = (z_1, z_2, z_3) \in C_{i_x}$, we have $z_1 \leq 0$. Since $y = (y_1, y_2, y_3), y+\epsilon_1 \in C_{i_x}$, we have $y_1 \leq -1$, and so $d(x, y+\epsilon_1) = d(x,y) - 1$. On the other hand, $y, y-\epsilon_3 \in C_{i_x}$, so $y_3 \geq 1$, and we get $d(x, y-\epsilon_3) = d(x,y) - 1$. This implies that the cube $\lambda y \cap \mu x \cap B_{x,y}$ contains $y, y+\epsilon_1, y-\epsilon_3$, which means $\lambda y \cap \mu x \cap B_{x,y} = \lambda y$. Hence, $\dim E_{x,y} = 4$.

Case 2 If $\#\lambda y \cap \mu x = 2$, it follows from Lemma 4.2.2 that $\dim E_{x,y} = 2$. In this case, λy and λx are on the same strip. For $i_x = 1$, λy must be on a strip of direction ϵ_3, and so $\lambda y \cap \mu x = \{y, y+\epsilon_1\}$. For $i_x = 2$, if λy is on a strip of direction ϵ_1 (resp. ϵ_3), then $\lambda y \cap \mu x = \{y, y-\epsilon_3\}$ (resp. $\{y, y+\epsilon_1\}$). For $i_x = 3$, λy must be on a strip of direction ϵ_1, and $\lambda y \cap \mu x = \{y, y-\epsilon_3\}$. □

Lemma 4.2.4. *(1) Suppose $E_{x,y} \neq 0$ and $(d = d(x,y), 0)_y \in E_{x,y}$, then the map $(d,0)_y$ is given by the identity map of projective modules in the components of $\widetilde{\Delta}(x)^\bullet$ and $\widetilde{\Delta}(y)^\bullet$.*

(2) Let $E_{x,y}, E_{y,w}$ be non-zero ext-groups, and $(d = d(x,y), 0)_u \in E_{x,y}$ and $(m,n)_z \in E_{y,w}$, then $(d,0)_u \cdot (m,n)_z$ is equal to $(d+m,n)_z$ if $(d+m,n)_z$ exists (in $E_{x,w}$); or zero otherwise.

Proof. (1): Let $(d,0)_y = (f^i)_{i\leq 0}$ with $f^i : \widetilde{\Delta}(x)^i \to \widetilde{\Delta}(y)^{i-d}$. We know that $(d,0)_y$ represents the module map $P(y) \to \Delta(y)$. So f^{-d} is the identity map on $P(y)$. This lifts to identity maps on the rest of the components.

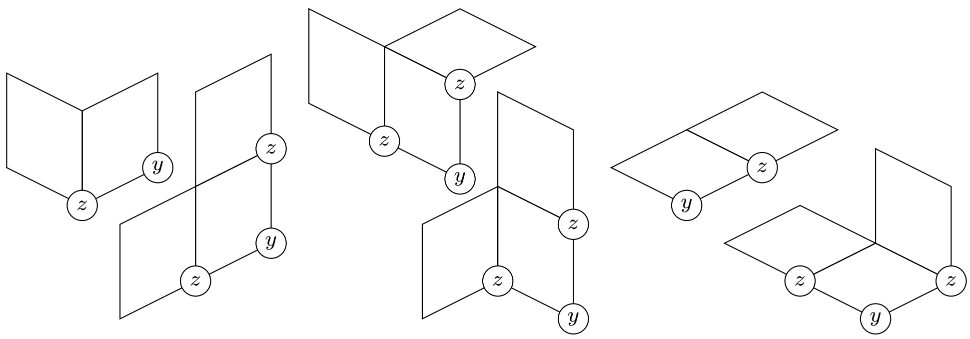
(2): By (1), $(d,0)_y$ is given by the identity maps of its components, so precomposing with $(d,0)_y$ only increases h-degree but does not change the mapping. □

4.2.1 Reduction Lemmas

The exposition is orientation-independent, as one can just rotate and/or reflect the projection of the Cubist set on $\mathbb{R}^2$ until the orientation matches up with Figure 4.1 and work from there. For convenience, we say a basis element α can be *factorised* if $\alpha = \sum_i c_i \beta_i \gamma_i$ for some basis elements β_i, γ_i and non-zero coefficient $c_i \in \mathbf{k}^\times$.

Lemma 4.2.5. *Suppose $z \prec y$ are vertices in μx with $d(z, y) = 1$, λy shares an edge with λz, and both $E_{x,z}, E_{x,y}$ non-zero. Then any element of $E_{x,y}$ factor through elements of $E_{x,z}$.*

Proof. The assumption implies $\lambda y \cap \lambda z = \{z, y^{\mathrm{op}}\}$. In particular, $E_{z,y}$ is spanned by $\{(1,0)_y, (0,1)_z\}$. Possible configurations of λy and λz are listed in Figure 4.2.



$i_y = 1$ $\quad$ $i_y = 2$ $\quad$ $i_y = 3$

Figure 4.2: Possible configurations for Lemma 4.2.5

Case 1 $\lambda y \cap \mu x = \lambda y = \{y, z, u, y^{\mathrm{op}}\}$: We have $y^{\mathrm{op}} \in \lambda z \cap \mu x$. Possible compositions of basis elements from $E_{x,z}$ and $E_{z,y}$ are as follows:

$$
\begin{tikzcd}
\widetilde{\Delta}(x): \arrow[d, "(d-2,1)_{y^{\mathrm{op}}}"] & x & \cdots \arrow[l] & y^{\mathrm{op}} \arrow[l] \arrow[d] & z, u \arrow[l] \arrow[d, "\substack{z-u\to y \\ \text{or} \\ -u\to y}"] & \cdots \arrow[l] \\
\widetilde{\Delta}(z): \arrow[d] & & & z \arrow[d] & y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\widetilde{\Delta}(y): & & & y & y &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\Delta}(x): \arrow[d, "(d-1,0)_z"] & x & \cdots \arrow[l] & z \arrow[l] \arrow[d] & y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\widetilde{\Delta}(z): \arrow[d] & & & z \arrow[d] & y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\widetilde{\Delta}(y): & & & y & y &
\end{tikzcd}
$$

The choice of the map $\widetilde{\Delta}(x)^{-(d-1)} \to \widetilde{\Delta}(z)^{-1}$ is as follows:

condition	map
$i_y = i_z$	$-u \to y$
$i_y \neq i_z$	$z - u \to y$

Set $d = d(x, y)$, we obtain the following relations:

$$\begin{aligned}
(d-1,0)_z \cdot (1,0)_y &= (d,0)_y \\
(d-1,0)_z \cdot (0,1)_z &= (d-1,1)_z \\
(d-2,1)_{y^{\mathrm{op}}} \cdot (1,0)_y &= (1-\delta_{i_y,i_z})(d-1,1)_z - (d-1,1)_u \\
(d-2,1)_{y^{\mathrm{op}}} \cdot (0,1)_z &= \pm(d-2,2)_{y^{\mathrm{op}}}.
\end{aligned} \tag{4.2.1}$$

In the last relation, the sign on the right-hand-side is determined by i_y and coordinate of z:

$i_y = 1$		$i_y = 2$		$i_y = 3$	
z	sign	z	sign	z	sign
$y - \epsilon_2$	$-$	$y + \epsilon_1$	$-$	$y + \epsilon_1$	$+$
$y - \epsilon_3$	$+$	$y - \epsilon_3$	$+$	$y - \epsilon_2$	$-$

Case 2 $\#\lambda y \cap \mu x = 2$: Assumption requires $z, y \in \mu x$, so $\lambda y \cap \mu x = \{y, z\}$. Since $y^{\mathrm{op}} \notin C_{x,y}$, it is also not in $C_{x,z}$. So $\lambda x, \lambda y, \lambda z$ are all on the same strip of direction $a \in \{1, 2, 3\}$, where $z \pm \epsilon_a = y^{\mathrm{op}}$, and $\lambda z \cap \mu x = \{z, v\}$ for some $v \neq y^{\mathrm{op}}$. Setting $d = d(x, y)$, we will show that there are relations:

$$\begin{aligned}
(d-2,1)_v \cdot (0,1)_z &= 0 \\
(d-1,0)_z \cdot (1,0)_y &= (d,0)_y \\
-\delta_{i_z,i_y}(d-2,1)_v \cdot (1,0)_y = (d-1,0)_z \cdot (0,1)_z &= (d-1,1)_z.
\end{aligned} \tag{4.2.2}$$

The first relation follows from the fact that there is no map of r-degree 2 from $P(v)$ to $\Delta(y)$. Calculation for the second relation and the last equality is similar to the right-hand-side calculation in Case 1, but with u deleted from $\widetilde{\Delta}(x)$. The left-most multiplication in the last relation can be displayed as:

$$\boxed{\begin{tikzcd}
\widetilde{\Delta}(x): \arrow[d, "(d-2,1)_v"] & x & \cdots \arrow[l] & v \arrow[l] \arrow[d] & z, w \arrow[l] \arrow[d, "\substack{\pm z\to y \\ \text{or} \\ z-w\to y}"] & \cdots \arrow[l] \\
\widetilde{\Delta}(z): \arrow[d] & & & z \arrow[d] & y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\widetilde{\Delta}(y): & & & y & y &
\end{tikzcd}}$$

The choice of labelled map depends on the configurations:

condtion	map
$i_y = i_z$	$-z \to y$
$i_y \neq i_z, i_y \neq 2$	$z \to y$
$i_y = 2, i_z = 1, w = y - \epsilon_2 \in \mathcal{X}$	$z - w \to y$

In the third case, we claim that $f = (d-2,1)_v \cdot (1,0)_y$ is homotopy to $g = (d-1,1)_z$. If the claim is true, then we can merge it with the other cases to get $(d_2,1)v \cdot (1,0)_y = (-\delta_{i_y,i_z})(d-1,1)_z$.

$$\boxed{\begin{tikzcd} \widetilde{\Delta}(x): \arrow[d, "f"] & x & \cdots \arrow[l] & z, w, \cdots \arrow[l] \arrow[d, "(z-w)\to y"] & y, \cdots \arrow[l] \\ \widetilde{\Delta}(y): & & & y & \cdots \arrow[l] \end{tikzcd}} \quad \text{homotopic to} \quad \boxed{\begin{tikzcd} \widetilde{\Delta}(x): \arrow[d, "g"] & x & \cdots \arrow[l] & z, u, \cdots \arrow[l] \arrow[d, "z\to y"] & y, \cdots \arrow[l] \\ \widetilde{\Delta}(y): & & & y & \cdots \arrow[l] \end{tikzcd}}$$

Since $i_y = 2$, $w = y - \epsilon_2 \in \mu y$ and $d(y,u) = 1$. In particular, we have $P(w) \in \widetilde{\Delta}(y)^{-1}$. The (minus) identity map $-\mathrm{id}_{P(w)}$ induces a map h of complexes of degree $(i-1,0)$, such that $f - g = dh + hd$, and so f is homotopic to g. $\square$

Lemma 4.2.6. *Suppose λx and λz are of the same shape (i.e. $i_x = i_z$), with $d(x,z) = 1$, $x \prec z$ and $y \in \mu z$. If $\dim E_{x,y} = 2$, then $\dim E_{z,y} = 2$ and any element in $E_{x,y}$ factors through elements of $E_{x,z}$.*

Proof. Since $y \in \mu z$, we have $\#\lambda y \cap \mu z \geq 2$. Suppose $\lambda y \cap \mu z = \lambda y$, then any u in λy are also in $z + C_{i_z} = x + \sigma_i \epsilon_i + C_{i_z}$ for some $i \in \{1,2,3\}$ and $\sigma_i \in \{\pm 1\}$. But $i_x = i_z$ implies $z + C_{i_z} \subset x + C_{i_x}$, so $u \in x + C_{i_x} \cap \mathcal{X}$. This gives $\lambda y \cap \mu x = \lambda y$, contradicting the assumption of $\#\lambda y \cap \mu x = 2$. Therefore, $\#\lambda y \cap \mu x = 2$ means that λy and λx are on the same strip. By Lemma 4.2.2, we get $\dim E_{z,y} = 2$. In particular, we have λx, λy, and λz are all in the same strip. So $E_{x,z}, E_{z,y}$ and $E_{x,y}$ have the following basis respectively:

$$\{(1,0)_z, (0,1)_x\} \ , \ \{(d-1,0)_y, (d-2,1)_u\} \ , \ \{(d,0)_y, (d-1,1)_u\},$$

where $d = d(x,y)$. Now we can calculate the following factorisation of basis elements of $E_{x,y}$ into those of $E_{x,z}$ and $E_{z,y}$:

$$\begin{aligned} (d,0)_y &= (1,0)_z \cdot (d-1,0)_y \\ (d-1,1)_u &= (1,0)_z \cdot (d-2,1)_u. \end{aligned} \tag{4.2.3}$$

$\square$

4.3 Calculating the Ext-quiver

In this section, we will construct the quiver $Q(E_\Delta)$ of the Yoneda algebra $\mathrm{Ext}_U(\Delta, \Delta)$.

Recall from Section 3.2.2 that $x \succ x+(1,0,0)$ for all $x \in \mathcal{X}$. In particular, any $y = (y_1, y_2, y_3) \in \mu x$ with $x = (x_1, x_2, x_3)$ implies $y_1 \leq x_1$ and $y_3 \geq x_3$. Visualising this with our chosen orientation in previous section 4.2.1, we can think of the cubist set as stacking layers of rhombi of the form $z+F_3$. Hence, in each layer of such rhombi, all the vertices have the same coefficient of ϵ_3. Two layers are connected by a strip of direction 3, formed by rhombi $\lambda y = y + F_i$ with $i \neq 1$.

Since E_Δ "projectivises" standard modules, which "are" rhombi, we will draw the arrows of $Q(E_\Delta)$ from one rhombus to another, instead of the conventional approach which goes from vertex to vertex. We say that an arrow λx and λy *crosses layer* if $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ with $x_3 < y_3$. We say an arrow $\lambda x \to \lambda y$ is a *jump* if $\#\lambda x \cap \lambda y \leq 1$.

We identify the arrows in $Q(E_\Delta)$ with the a subset of basis elements $\{(m,n)_z \in E_{x,y} | x, y \in \mathcal{X}\}$. This subset consist of maps which can not be factorised. We start with the criteria for which a basis element cannot be factorised.

Lemma 4.3.1. *If a non-zero homogeneous element $\alpha \in E_{x,y}$ with $\deg \alpha = (h,r)$ can be factorised. Let $\alpha = \sum_a c_a \beta_a \gamma_a$ be the factorisation of α in terms of linear combinations of basis elements $\beta_a \in E_{x,z_a}$ and $\gamma_a \in E_{z_a,y}$. Then $z_a \in R_{x,y} := B_{x,y} \cap \mathcal{X} \setminus \{x,y\}$ for all a.*

Proof. Let $\deg \beta_a = (h_{a,1}, r_{a,1})$ and $\deg \gamma_a = (h_{a,2}, r_{a,2})$, then we have $h_{a,1} + h_{a,2} = h$ and $r_{a,1} + r_{a,2} = r$ for all a. Also by Theorem 3.3.1, $d(x, z_a) = h_{a,1} + r_{a,1}$, $d(z_a, y) = h_{a,2} + r_{a,2}$, and $d(x,y) = h + r$. These combine to give $d(x, z_a) + d(z_a, y) = d(x,y)$, so $z_a \in B_{x,y}$. $\square$

Lemma 4.3.2. *If $R_{x,y} = \emptyset$, or for all $z \in R_{x,y}$ we have*

$$B_{x,y} \cap C_{x,z} \neq C_{x,z}, \text{ or } B_{x,y} \cap C_{z,y} \neq C_{z,y},$$

then the basis elements of $E_{x,y}$ cannot be factorised.

Proof. Denote $d = d(x,y)$. Suppose on contrary that $0 \neq \alpha \in E_{x,y}$ can be factorised. By Lemma 4.3.1, we have some $z \in B_{x,y}$ with $E_{x,z} \neq 0$ and $E_{z,y} \neq 0$, hence $C_{x,z} \cap B_{x,z} = C_{x,z}$

and $C_{z,y} \cap B_{z,y} = C_{z,y}$ by Theorem 3.3.1. On the other hand,

$$\begin{aligned} z \in B_{x,y} &\Rightarrow B_{x,z} \text{ and } B_{z,y} \subset B_{x,y} \\ &\Rightarrow \begin{cases} B_{x,y} \cap C_{x,z} = B_{x,z} \cap C_{x,z} \\ B_{x,y} \cap C_{z,y} = B_{z,y} \cap C_{z,y} \end{cases} \\ &\Rightarrow \begin{cases} B_{x,y} \cap C_{x,z} = C_{x,z} \\ B_{x,y} \cap C_{z,y} = C_{z,y} \end{cases} \end{aligned}$$

which contradicts the assumption. □

In what follows, we will first calculate the condition for which *linking arrows* (going from one rhombus to a neighbouring rhombus sharing an edge) appear in $Q(E_\Delta)$. We then give a detailed analysis to find out the precise conditions for jumps to appear. In particular we show that all jumps cross layers, and only occur when the source and target are of the same shape. These combine to give full description of the Ext-quiver of U^Δ for any cubist set $\mathcal{X}$.

4.3.1 Strip configurations

Let $\lambda x = \lambda x^{(0)}, \lambda x^{(1)}, \ldots, \lambda x^{(n)} = \lambda y$ be the strip connecting λx and λy. If $d(x^{(i-1)}, x^{(i)}) = 1$ for all $i = 1, \ldots, n$, then clearly $x^{(i+1)} \in \mu x^{(i)}$ for all i. In particular, $x^{(i)} \in \mu x$ for all i. In this case, we can apply reduction lemmas 4.2.5 4.2.6 repeatedly to obtain the following "strip configuration relations".

By *strip configuration*, we mean a local collection of three consecutive rhombi, all of them being on the same strip. So there are 24 such configurations in total.

The first group of strip configurations are shown in Figure 4.3. For these configurations, the relations associated to the arrows are:

$$\begin{aligned} (0,1)_x \cdot (0,1)_z &= 0 \\ (-\delta_{i_z,i_y})(0,1)_x \cdot (1,0)_y = (1,0)_z \cdot (0,1)_z &= (1,1)_z \\ (1,0)_z \cdot (1,0)_y &= (2,0)_y \end{aligned} \tag{4.3.1}$$

These relations comes from proof of Lemma 4.2.5 (Case 2).

There are six other strip configurations left. Four of them, shown in 4.4 have relations similar

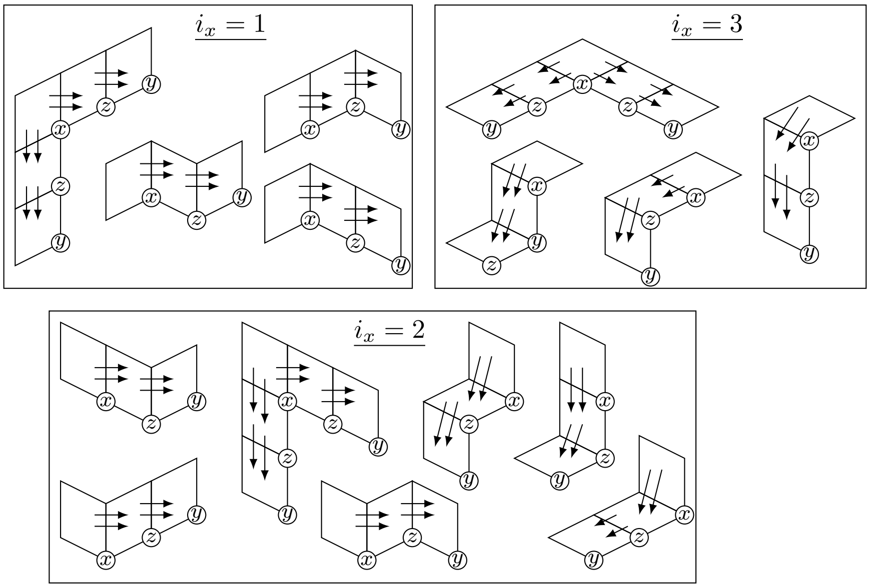


Figure 4.3: Strip configurations 1

to the first group, but the arrows are of different degrees.

$$\begin{aligned} (0, r_1)_x \cdot (0, r_2)_z &= 0 \\ (0, r_1)_x \cdot (1, r_2 - 1)_y = (1, r_1 - 1)_z \cdot (0, r_2)_z &= \pm(1, 2)_{y^{\text{op}}} \\ (1, r_1 - 1)_z \cdot (1, r_2 - 1)_z &= (2, 1)_y \end{aligned} \tag{4.3.2}$$

The choice of r_1, r_2 and the sign in the second relation is as follows.

Group 1 $r_1 = 2, r_1 = 1$, sign $= -$.

Group 2 $r_1 = 1, r_1 = 2$, sign $= +$.

The calculations are similar to the previous set of relations.

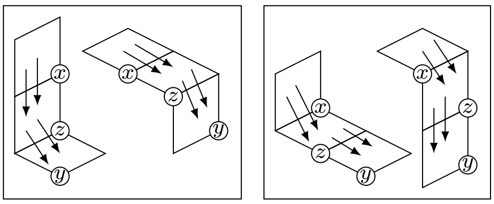


Figure 4.4: Strip configurations 2

The last two strip configurations are shown in Figure 4.5. Any composition of arrows shown in Figure 4.5 is zero. In particular, as $\dim E_{x,y} = \#\lambda y \cap \mu x = 2$, there is a pair of arrows, which are both jumps that cross layer. These two arrows correspond to $(1, 1)_u$ and $(2, 0)_y$.

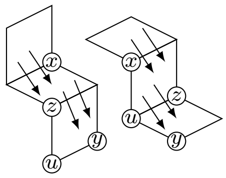
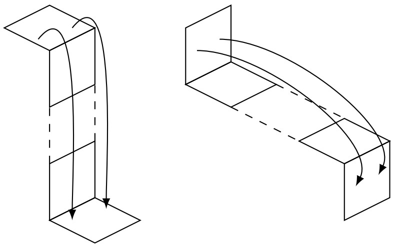


Figure 4.5: Strip configurations 3

In general, there are a pair jumps appearing on strips of the following form:

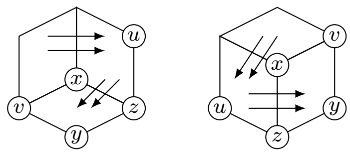


$i_x = 3$ $\qquad\qquad$ $i_x = 1$ (4.3.3)

The maps corresponding to the pair of jumps have degree $(n-1, 1)$ and $(n, 0)$, where $n = d(x, y)$. The existence of these jumps follows from Lemma 4.3.2.

4.3.2 Corner configuration

At first glance, one may think that all the arrows in the strip configurations appear as arrows in the quiver $Q(E_\Delta)$. There are some exceptional cases, which we call *corner configuration*:



(4.3.4)

We call the vertex x in these two case as *corner of a corner configuration*. The basis elements of $E_{x,y}, E_{x,z}$ and $E_{z,y}$ are respectively

$$\{(0,2)_x, (1,1)_z\}\ ,\ \{(0,1)_x, (1,0)_z\}\ ,\ \{(0,1)_z, (1,0)_y\}.$$

We obtain the following relations:

$$\begin{aligned} (1,0)_z \cdot (1,0)_y &= 0 \\ (1,0)_z \cdot (0,1)_z = (0,1)_x \cdot (1,0)_y &= (1,1)_z \\ (0,1)_x \cdot (0,1)_z &= -(0,2)_x \end{aligned} \tag{4.3.5}$$

$(1,0)_z \cdot (0,1)_z = (1,1)_z$ follows from Lemma 4.2.4. The other two non-zero compositions can be calculated from:

$$\begin{tikzcd}
\widetilde{\Delta}(x) \arrow[d, "(0,1)_x"] & x \arrow[d] & z \arrow[l] \arrow[d, "z\to y"] & \cdots \arrow[l] \\
\widetilde{\Delta}(z) \arrow[d] & z \arrow[d] & y, \cdots \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\widetilde{\Delta}(y) & y & y &
\end{tikzcd}$$

Note that the map $z \to y$ always exists: We look for maps from z to its neighbour z' so that $(z \to x \to z)$ equals to summation of some paths $z \to z' \to z$ of length 2 (up to sign). There are two candidates appearing in such summation: (i) $z \to u \to z$, and (ii) $z \to y \to z$. But $u \notin \mu z$ as $u \prec z$, so we cannot have $z \to u$ appearing in the degree 1 component of the map of complexes $(0,1)_x$. This forces $z \to y \to z$ to be our choice.

4.3.3 Jumps from $\lambda x = x + F_1$

We now investigate the configurations which induce jumps in the Ext-quiver of U^Δ.

Lemma 4.3.3. *If* $C_{x,y} \neq \emptyset$ *with* $i_x = 1$ *and* $i_y = 3$, *then either* $\#C_{x,y} = 2$ *or* $C_{x,y} \cap B_{x,y} \neq C_{x,y}$. *In particular,* $\dim E_{x,y} = 0$ *or* 2.

Proof. We only need to consider the case $\#C_{x,y} = 4$. Note $\lambda y = \{y, y+\epsilon_1, y+\epsilon_2, y+\epsilon_1+\epsilon_2\}$, and $\mu x = x + \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \cap \mathcal{X}$. So $d(x, y+\epsilon_1) = d(x,y) - 1$ and $d(x, y+\epsilon_2) = d(x,y)+1$, which implies $\#C_{x,y} \cap B_{x,y} = 2 \lneq \#C_{x,y}$. □

We visualise this lemma in Figure 4.6.

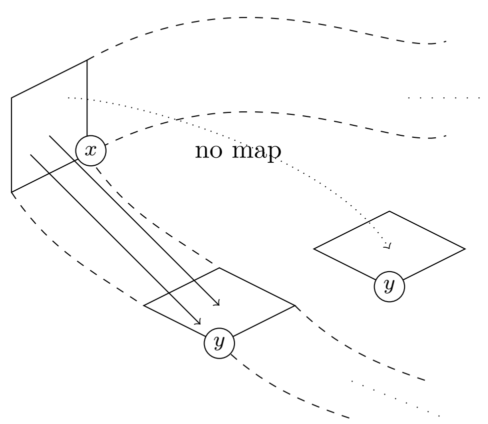


Figure 4.6: Visualising Lemma 4.3.3

Lemma 4.3.4. *If* $\dim E_{x,y} = 4$ *with* $i_x = 1$ *and* $i_y \neq 3$*, then there exists* $z \in \mathcal{X}$ *with* $d(z, y) = 1$ *such that* $E_{x,z} \neq 0$ *and elements of* $E_{x,y}$ *factor through elements of* $E_{x,z}$*.*

Proof. Assume without loss of generality $x = (0, 0, 0)$, and denote $y = (y_1, y_2, y_3)$.

Case 1 $i_y = 1$: We have $\lambda y = \{y, y-\epsilon_2, y-\epsilon_3, y-\epsilon_2-\epsilon_3\}$. Consider $z = y-\epsilon_2 = (y_1, y_2-1, y_3)$. Note that $y^{\text{op}} \in \lambda z$.

If $i_z = 1$, then $\lambda z = \{z = (y_1, y_2 - 1, y_3), u = (y_1, y_2 - 2, y_3), y^{\text{op}} = (y_1, y_2 - 1, y_3 - 1), (y_1, y_2 - 2, y_3 - 1)\}$. Since $z \in \mu x$, $y_2 \geq 1$. If $y_2 > 1$, we have $C_{x,z} = \lambda z$ and $d(x, u) = d(x, z) - 1$. This implies $C_{x,z} \cap B_{x,z} = \lambda z$, hence $\dim E_{x,z} = 4$. If $y_2 = 1$, $C_{x,z} = \{z, y^{\text{op}}\}$, and so $\dim E_{x,z} = 2$.

If $i_z = 2$, then by Lemma 4.2.3 we have $\dim E_{x,z} \neq 0$.

Now the condition for reduction Lemma 4.2.5 is satisfied, and so the statement follows.

Case 2 $i_y = 2$: We have $\lambda y = \{y, y+\epsilon_1, y-\epsilon_3, y+\epsilon_1-\epsilon_3\}$. Consider $z = y+\epsilon_1 = (y_1+1, y_2, y_3)$. Again, $y^{\text{op}} \in \lambda z$. (The proof is essentially the same as the previous case, but with different values.)

If $i_z = 1$, then $\lambda z = \{z, y^{\text{op}}, u = (y_1 + 1, y_2 - 1, y_3), (y_1 + 1, y_2 - 1, y_3 - 1)\}$. Since $z \in \mu x$, $y_2 \geq 0$. If $y_2 > 0$, we have $C_{x,z} = \lambda z$ and $d(x, u) = d(x, z) - 1$. This implies $C_{x,z} \cap B_{x,z} = \lambda z$, hence $\dim E_{x,z} = 4$. If $y_2 = 0$, $C_{x,z} = \{z, y^{\text{op}}\}$, and so $\dim E_{x,z} = 2$.

If $i_z = 2$, then by Lemma 4.2.3 we have $\dim E_{x,z} \neq 0$.

Now the condition for reduction Lemma 4.2.5 is satisfied, and so the statement follows. □

Proposition 4.3.5. *All the jumps starting from* λx *with* $i_x = 1$ *appear in the form of (4.3.3). In particular, the arrows in Ext-quiver of* U^Δ *starting from* x *with* $i_x = 1$ *consist of*

(i) pair of jumps to λy*, with* $y = x - m\epsilon_1 + \epsilon_3$ *for some* $m \in \mathbb{Z}_{\geq 1}$ *(see (4.3.3)),*

(ii) pair of arrows to λy*, where* λy *and* λx *share an edge, and* $x \succ y$*. Unless* x *is the corner of a corner configuration. In which case, there is no pair of arrow from* λx *to* $\lambda(x - \epsilon_1 - \epsilon_2)$*.*

Proof. Any arrow starting from λx not of the form (i) or (ii) will be a jump. So we are going to show that any elements in $E_{x,y} \neq 0$ factor through elements in $E_{x,z}$ and $E_{z,y}$, unless y satisfies the conditions given in (i) or (ii).

It follows from the previous Lemma 4.3.4 that $\dim E_{x,y} \neq 4$, i.e. jumps from λx can only appear on the strips containing λx.

Suppose λy is on the strip of direction ϵ_3. Then as we have argued in the strip configurations section, any basis elements of $E_{x,y}$ factors through basis elements of $E_{x,z}, E_{z,y}$ for some $z \prec y$, unless $d(x,y)=1$.

Suppose λy is on the strip of direction ϵ_2. If $x-\epsilon_1 \in \mathcal{X}$, then $i_{x-\epsilon_1}=1$ and we can replace x by $x-\epsilon_1$ by reduction lemma 4.2.6. So now we only need to consider x such that $z=x-\epsilon_1-\epsilon_2 \in \mathcal{X}$, in which case $i_z=3$ and $z^{\mathrm{op}}=x$. This is the same as saying the next rhombus on the strip from λx to λy is $z+F_3$.

Suppose furthermore that $i_y=3$ and $d=d(x,y)$. Write $\lambda y=\{y, u=y+\epsilon_1, v=y+\epsilon_2, y^{\mathrm{op}}\}$, and let $a=z+\epsilon_2$.

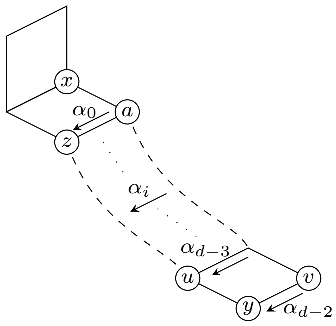


Figure 4.7: Calculation for Proposition 4.3.5

We write $(1,1)_b \in E_{x,z}$ as the sequence of maps $(\alpha_i : \widetilde{\Delta}^{-i+1}(x) \to \widetilde{\Delta}^{-i}(z))$. In Figure 4.7, we indicate some (constituents) of the α_i's. By supercommutation relation, if α_i corresponds to the multiplication of an arrow $\pm(c \to c-\epsilon_2)$, then $\alpha_{i\pm1}$ corresponds to multiplication of an arrow $-(d \to d-\epsilon_2)$. In particular, we have $\alpha_{d-3}=(-1)^{d-3}(y^{\mathrm{op}} \to u)$ and $\alpha_{d-2}=(-1)^{d-2}(v \to y)$. The relations are:

$$\begin{aligned}(1,1)_a \cdot (d-2,0)_y &= (-1)^d(d-1,1)_v \\ (1,1)_a \cdot (d-1,1)_u &= (-1)^{d-1}(d-1,2)_{y^{\mathrm{op}}}\end{aligned} \tag{4.3.6}$$

with the calculation being as follows:

$$\begin{tikzcd}
\widetilde{\Delta}(x): \arrow[d, "(1,1)_a"] & x & a \arrow[l] \arrow[d] & \cdots \arrow[l] & y^{\mathrm{op}} \arrow[l] \arrow[d, "(-1)^d(y^{\mathrm{op}}\to u)"'] & v \arrow[l] \arrow[d, "(-1)^{d+1}(v\to y)"'] & \cdots \arrow[l] \\
\widetilde{\Delta}(z): \arrow[d] & & z & \cdots \arrow[l] & u \arrow[l] \arrow[d] & y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\widetilde{\Delta}(y): & & & & y & y
\end{tikzcd}$$

This reduces the possibility of have arrows from λx to λy to the case when $y=x-\epsilon_1-\epsilon_2$. If x is the corner of a configuration, then we see from previous section that maps from $E_{x,y}$ can be factorised. Otherwise, condition for Lemma 4.3.2 is satisfied, and there will be a pair of arrows from λx to λy.

Finally, we have λy on the strip of direction ϵ_2 with $i_y=1$. If $z=y-\epsilon_3 \in \mathcal{X}$, then we can replace y by $y - \epsilon_3$ due to reduction lemma 4.2.5. So we are left with the case shown in Figure 4.8. with $E_{x,z} = \{(k,0)_z, (k-1,1_a)\}$, $E_{z,y} = \{(n,0)_y, (n-1,1)_z\}$. Now elements of $E_{x,y}$ can be factorised:

$$\begin{aligned} (n+k,0)_y &= (k,0)_z \cdot (n,0)_y \\ (n+k-1,1)_z &= (k,0)_z \cdot (n-1,1)_u \end{aligned} \tag{4.3.7}$$

using Lemma 4.2.4 (2). □

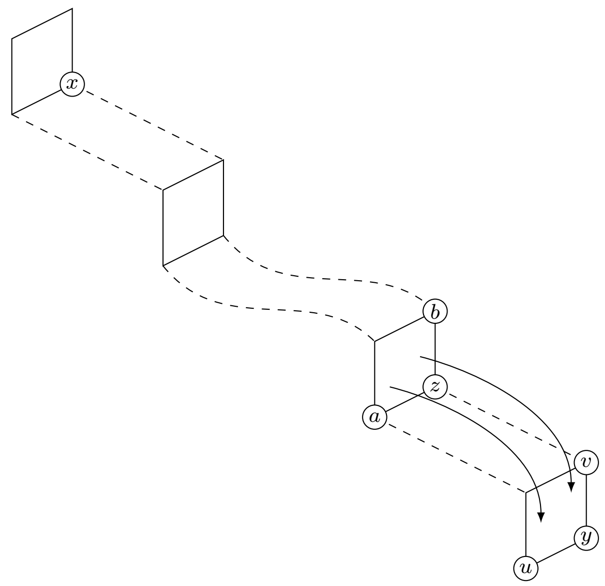


Figure 4.8: Reduction of jumps in Proposition 4.3.5

4.3.4 Jumps from $\lambda x = x + F_3$

Analogous statement to Lemma 4.3.3, Lemma 4.3.4 and Proposition 4.3.5 are also true for x with $i_x = 3$:

Lemma 4.3.6. *If* $C_{x,y} \neq \emptyset$ *with* $i_x = 3$ *and* $i_y = 1$*, then either* $\#C_{x,y} = 2$ *or* $C_{x,y} \cap B_{x,y} \neq C_{x,y}$*. In particular,* $\dim E_{x,y} = 0$ *or* 2*.*

Lemma 4.3.7. *If* $\dim E_{x,y} = 4$ *with* $i_x = 1$ *and* $i_y \neq 3$*, then there exists* $z \in \mathcal{X}$ *with* $d(z,y) = 1$ *such that* $E_{x,z} \neq 0$ *and elements of* $E_{x,y}$ *factors through elements of* $E_{x,z}$*.*

Proposition 4.3.8. *All the jumps starting from* λx *with* $i_x = 3$ *appear in the form of (4.3.3). In particular, the arrows in Ext-quiver of* U^{Δ} *starting from* x *with* $i_x = 3$ *consist of*

(i) pair of jumps to λy*, with* $y = x - \epsilon_1 + m\epsilon_3$ *for some* $m \in \mathbb{Z}_{\geq 1}$ *(see (4.3.3)),*

(ii) pair of arrows to λy*, where* λy *and* λx *share an edge, and* $x \succ y$*. Unless* x *is the corner of a corner configuration. In which case, there is no pair of arrow from* λx *to* $\lambda(x + \epsilon_2 + \epsilon_3)$*.*

The proofs of these statements are almost the same as those for $i_x = 1$ case, except that one needs to switch the role of ϵ_1 and ϵ_3, with some sign changes in the ϵ_i-coefficients (coordinates) appearing in the proofs.

4.3.5 Jumps from $\lambda x = x + F_2$

Lemma 4.3.9. *Let $x, y \in \mathcal{X}$ with $i_x = 2$ and $\#C_{x,y} = 2$, then $y \in \mu x$, and basis elements of $E_{x,y}$ factors through via strip relations.*

Proof. Observe that for any λy on the same strip as λx with $x \prec y$, $y \in \mu x$. The rest follows from reduction lemma 4.2.5. □

Lemma 4.3.10. *Suppose $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathcal{X}$ with $\lambda y \cap \mu x = \lambda y$ and $i_x = 2$, then we have*

(i) if $i_y = 1$, then $E_{x,y} \neq 0 \Leftrightarrow y_2 > x_2$;

(ii) if $i_y = 2$, then $E_{x,y} \neq 0$.

(iii) if $i_y = 3$, then $E_{x,y} \neq 0 \Leftrightarrow y_2 < x_2$.

Proof. Suppose $x = (0, 0, 0)$.

(1): If $y_2 \geq 0$, then $d(x, y + \epsilon_2) = d(x, y) + 1$, so $y + \epsilon_2 \notin B_{x,y}$. Then $E_{x,y} = 0$ follows Theorem 3.3.1. Otherwise, $d(x, y) - 1 = d(x, y + \epsilon_2)$. If $y_1 < 0$, then we also have $d(x, y + \epsilon_1) = d(x, y) - 1$, which gives $\dim E_{x,y} = 4$. If $y_1 = 0$, then λy and λx is on the same strip, hence $\dim E_{x,y} \neq 0$.

(2): Immediate from Lemma 4.2.3.

(3): Similarly as in (1) but switch the role of ϵ_1 and ϵ_3. □

Lemma 4.3.11. *If $E_{x,y}$ with $i_x = 2$ and $i_y \neq 1$, then there exists $z \in \mathcal{X}$ with $E_{x,z} \neq 0 \neq E_{z,y}$ such that basis elements of $E_{x,y}$ factor through elements of $E_{x,z}$ and elements of $E_{z,y}$. In particular, there is no jump starting from $\lambda x = x + F_2$ to $y + F_i$ with $i \neq 2$.*

Proof. By Lemma 4.3.9, we only need to consider the case when $\#\lambda y \cap \mu x = 4$.

Case 1 $i_y = 3$: By the previous lemma 4.3.10, we only need to consider y with $y_2 < x_2$. Let $\lambda y = \{y, u = y + \epsilon_1, v = y + \epsilon_2, y^{\mathrm{op}}\}$. Observe that $i_v = 2$ or 3. For $i_v = 2$, Lemma 4.3.10 says $E_{x,v} \neq 0$, and we can apply reduction lemma 4.2.5 to factorise elements of $E_{x,y}$. If $i_v = 3$ and $i_u = 3$, then $E_{x,u} \neq 0$ by Lemma 4.3.10, and we can apply reduction lemma 4.2.5 to factorise $E_{x,y}$. Otherwise, we get $i_v = 3$ and $i_{y^{\mathrm{op}}} = 1$. Observe that there exists a

$z = (z_1, z_2, z_3) = y + \epsilon_1 - k\epsilon_3$ for some unique $k \geq 0$ with $i_z = 3$. The situation is shown in Figure 4.9.

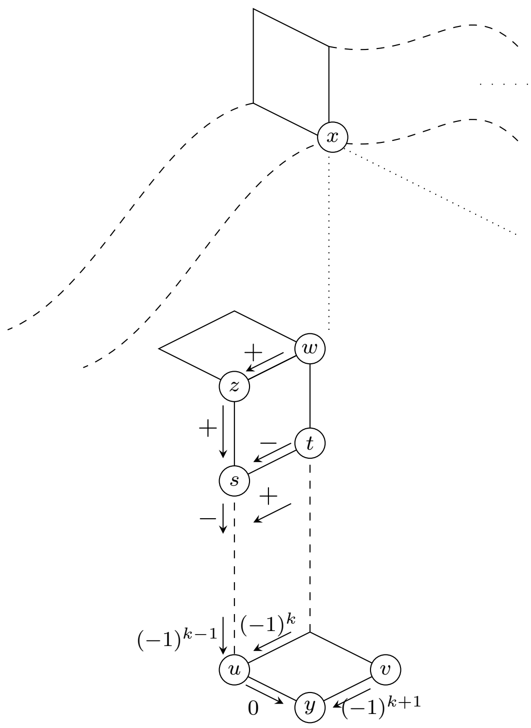


Figure 4.9: Calculation for Lemma 4.3.11

Now we know $E_{x,z}$ contains two elements $(n-1,1)_w, (n,0)_z$, and we obtain the following multiplications:

$$\begin{aligned} (n,0)_z \cdot (k+1,0)_y &= (n+k+1,0)_y \\ (n,0)_z \cdot (k,1)_u &= (n+k,1)_u \\ (n-1,1)_w \cdot (k+1,0)_y &= (-1)^{k+1}(n+k,1)_v \\ (n-1,1)_w \cdot (k,1)_u &= (-1)^k(n+k-1,2)_{y^{\mathrm{op}}} \end{aligned} \tag{4.3.8}$$

The calculation for the last two relations are shown in the diagram below. We now explain the reason for the constituents of the map $(n-1,1)_w$. In the first commutative square, to cancel out $z \to w \to z$, we need to use the Heisenberg relation and so we get degree 1 constituent $z \to s$. On the other hand, $t \to w \to z$ needs to be cancelled out by $-(t \to s \to z)$, so the degree 1 constituent also contains $-(t \to s)$. (See above picture) Repeating this process until reaching u. At u, the relation $(-1)^{k-1}(u \to u - \epsilon_3 \to u)$ will be cancelled out by $(-1)^k(u \to y^{\mathrm{op}} \to u)$, so there is no need to map u to y. Supercommutation in λy gives $(-1)^{k+1}(v \to y)$. (See Figure

4.9) The multiplication of the maps of complexes become:

$$\begin{tikzcd}
 & & \boxed{n-1} & \boxed{n} & & & \boxed{n+k-1} & \boxed{n+k} \\
\widetilde{\Delta}(x): \arrow[d, "{(n-1,1)_w}"] & \cdots & w \arrow[l] \arrow[d, "+"] & z,t \arrow[l] \arrow[d, "z-t"] & s \arrow[l] \arrow[d] \arrow[r, dotted, no head] & \cdots & y^{\mathrm{op}} \arrow[l] \arrow[d, "(-1)^k"'] & u,v \arrow[l] \arrow[d, "(-1)^{k+1} v\to y"'] \\
\widetilde{\Delta}(z): \arrow[d] & & z & s \arrow[l] & \cdots \arrow[l] \arrow[r, dotted, no head] & \cdots & u \arrow[l] \arrow[d] & y \arrow[l] \arrow[d] \\
\widetilde{\Delta}(y): & & & & & & y & y
\end{tikzcd}$$

Case 2 $i_y = 1$: The proof is almost the same as the previous one, but the role of ϵ_1 and ϵ_3 swapped, and some signs change. The involved vertices are $v = y - \epsilon_2$, $u = y - \epsilon_3$, $z = y + k_1\epsilon_1 - k_2\epsilon_3$ with $i_z = 1$, $w = z - \epsilon_2$. $\square$

Proposition 4.3.12. *Suppose $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathcal{X}$ and $i_x = i_y = 2$. Then there is a jump from λx to λy (equivalently, basis elements of $E_{x,y}$ cannot be factorised) if and only if for all all of the following conditions are satisfied:*

(1) $x_1 > y_1$, (2) $x_2 = y_2$, (3) $x_3 < y_3$, (4) $x \pm \epsilon_2 \in \mathcal{X}$, (5) $y^{\mathrm{op}} \pm \epsilon_2 \in \mathcal{X}$,
(6) there is no $z \in \mathcal{X}$ with $d(x,z) < d(x,y)$ such that z satisfies (1)-(5) above y replaced by z.

Proof. Now assume again $x = (0,0,0)$, and so have $\mu x = \mathbb{Z}_{\leq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0} \cap \mathcal{X}$. Note that for any λy on the same strip as λx with $x \prec y$, we have $y \in \mu x$. So for any y with $C_{x,y} \neq \emptyset$, we have $y_1 \leq 0$ and $y_3 \geq 0$.

By Lemma 4.3.9, we can assume λy is not on the same strip as λx. In particular, we can assume $\dim E_{x,y} = \#C_{x,y} = 4$. Let $\lambda y = \{y, u = y + \epsilon_1, v = y - \epsilon_3, y^{\mathrm{op}}\}$.

We first show that if each of the conditions (1)-(6) is violated, then basis elements of $E_{x,y}$ can be factorised.

(1) and (3): If $y_1 = 0$ (resp. $y_3 = 0$), then λy is on the same strip of direction ϵ_1 (resp. ϵ_3) as λx, and we are done.

(2): Suppose $y_2 > 0$, then $\lambda u = \{u, y^{\mathrm{op}}, w, u^{\mathrm{op}}\}$ with $i_u = 1$ or 2. We then have $w = u - \epsilon_2$ or $w = u + \epsilon_1$ respectively. In both situations, $d(x,w) = d(x,u) - 1$, and so $\lambda u \cap \mu x \cap B_{x,u} = \lambda u$. Now we can apply reduction lemma 4.2.5 to factorise basis elements of $E_{x,y}$. The case of $y_2 < 0$ can done similarly by replaces u by v.

(4): Suppose $(0,1,0) \notin \mathcal{X}$, then one can observe that $z = (-1,0,0) \in \mathcal{X}$ with $i_z = 2$. Since $y = (y_1, 0, y_3)$ with $y_1 < 0$, $y_3 > 0$, and $\mu z = \mathbb{Z}_{<0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0} \cap \mathcal{X}$, we deduce $y \in \mu z$. By Lemma

4.2.3, $E_{z,y} \neq 0$. If $\dim E_{z,y} = 4$, then

$$(1,0)_z \cdot (d-r-1,r)_w = (d-r,r)_w \quad \text{for all } w \in \lambda y. \tag{4.3.9}$$

If $\dim E_{z,y} = 2$, then we have $y = z + k\epsilon_3$ for some $k > 0$. By combinatorial calculation similar to the proof of relations (4.3.6) in Proposition 4.3.5, we have

$$\boxed{\begin{tikzcd}
\widetilde{\Delta}(x): \arrow[d, "(0,1)_x"] & x & z \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] & y^{\mathrm{op}} \arrow[l] \arrow[d, "(-1)^{d-1}"'] & u,v \arrow[l] \arrow[d, "(-1)^d u \to y"] & y \arrow[l] \\
\widetilde{\Delta}(z): \arrow[d] & z & \cdots \arrow[l] & \cdots \arrow[l] & v \arrow[l] \arrow[d] & y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\widetilde{\Delta}(y): & & & & y & y &
\end{tikzcd}}$$

Now we have the following relations:

$$\begin{aligned}
(1,0)_z \cdot (d-1,0)_y &= (d,0)_y \\
(1,0)_z \cdot (d-2,1)_v &= (d-1,1)_v \\
(0,1)_x \cdot (d-2,1)_y &= (-1)^d (d-1,1)_u \\
(0,1)_x \cdot (d-2,1)_v &= (-1)^{d-1} (d-2,2)_{y^{\mathrm{op}}}
\end{aligned} \tag{4.3.10}$$

where $d = d(x,y)$.

Suppose now $(0,-1,0) \notin \mathcal{X}$, then consider $z = (0,0,1)$ and everything follows similarly as above. Then we get the same calculations as (4.3.9) and (4.3.10) with u and v swapped. Therefore, violation of (4) implies reducibility of elements of $E_{x,y}$.

(5): Suppose now $y^{\mathrm{op}} + \epsilon_2$ (resp. $y^{\mathrm{op}} - \epsilon_2$) does not lie in $\mathcal{X}$, then one can observe that $z = (y_1, y_2, y_3 - 1)$ (resp. $z = (y_1 - 1, y_2, y_3)$) satisfies $i_z = 2$, so $E_{x,z} \neq 0$. We can then apply reduction lemma 4.2.5 to factorise $E_{x,y}$.

(6): If there is an $z \in \mathcal{X}$ with $d(x,z) < d(x,y)$ and z satisfies (1)-(5) with y replaced by z, then $z = (z_1, z_2, z_3)$ with $z_1 \geq y_1$, $z_2 = y_2 = 0$ and $z_3 \leq y_1$. We can assume the inequalities are strict, otherwise we are back in the situation where (5) is not true. Also, as $i_z = 2$, we have $E_{x,z}$ contain $(n,0)_z$ where $n = d(x,z)$. Suppose $E_{z,y} = \mathrm{span}\{(k,0)_y, (k-1,1)_u, (k-1,1)_v, (k-2,2)_{y^{\mathrm{op}}}\}$ where $k = d(z,y)$. By denoting a basis element in $E_{z,y}$ as $(k-i,i)_w$ with appropriate i and w, we can calculate the following factorisation:

$$(n+k-i,0)_w = (n,0)_z \cdot (k-i,i)_w \tag{4.3.11}$$

We have now shown that, if any of (1)-(6) is not satisfied, then basis elements of $E_{x,y}$ can be factorised.

To finish the proof, we need to show that when (1)-(6) are satisfied, then elements of $E_{x,y}$ cannot be factorised. (1)-(3) says that $B_{x,y}$ is a box in the ϵ_1-ϵ_3-plane. Combining with conditions (4)-(6), it implies that there is no $z \neq x, y$ in $B_{x,y} \cap \mathcal{X}$ have $\lambda z = z + F_2$. In another words, there is no $z \in B_{x,y} \setminus \{x, y\}$ which satisfies $\lambda z \subset B_{x,y} \cap \mathcal{X}$, so elements of $E_{x,y}$ cannot be factorised by Lemma 4.3.2. $\square$

Proof of Theorem 4.1.1. Combine Proposition 4.3.5, Proposition 4.3.8, Proposition 4.3.12. $\square$

We have now completed the calculation for the quiver of E_Δ.

4.4 Ext-algebra of standard modules for the hyperplane Cubist algebra

The *infinite Brauer line* algebra $Z_{\infty,\infty}$ is the Cubist algebra associated to *any* Cubist subset of $\mathbb{Z}^2$. Its quiver presentation is:

$$\begin{tikzcd} \cdots & \overset{1}{\bullet} \arrow[r, bend left, "a"] & \overset{2}{\bullet} \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] & \overset{3}{\bullet} \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] & \overset{4}{\bullet} \arrow[l, bend left, "b"] & \cdots \end{tikzcd}$$

with relations $ba + ab = a^2 = b^2 = 0$.

Denote by e_i the primitive idempotent of the vertex i in the quiver. For any finite interval $[a, b]$ of $\mathbb{Z}$, let $e_{a,b} = \sum_{i=a}^{b} e_i$, then $Z_{a,b} = e_{a,b} Z_{\infty,\infty} e_{a,b}$ is a quasi-hereditary algebra. There are many instances of (Morita equivalent version of) $Z_{a,b}$ in representation theory. For example, it is the quasi-hereditary cover of a (finite) Brauer tree algebra associated to a (Brauer) line with $b - a$ edges. We denote $Z_n := Z_{0,n-1} \cong Z_{a,b}$ where $b - a + 1 = n$.

The Ext-algebra Z_n^Δ has been calculated several times in the literature [Mad2, MT, Kla]. In particular, [MT] also presented the quiver presentation of $Z_{\infty,\infty}^\Delta$. We will just state their result here:

Proposition 4.4.1 (Proof of Prop 25 in [MT]). *The algebra Z_∞^Δ is given by the (infinite) quiver*

$$\begin{tikzcd} \cdots & \overset{1}{\bullet} & \overset{2}{\bullet} \arrow[l, bend right, "A"'] \arrow[l, bend left, "b"] & \overset{3}{\bullet} \arrow[l, bend right, "A"'] \arrow[l, bend left, "b"] & \overset{4}{\bullet} \arrow[l, bend right, "A"'] \arrow[l, bend left, "b"] & \cdots \end{tikzcd}$$

with the set of vertices being $\mathbb{Z}$, and relations generated by $bA + Ab = 0 = b^2$.

Remark 4.3. (1) The algebra $Z_{\infty,\infty}$ satisfies "an infinite version" of the condition (H), i.e. replacing the function $h : I \to \{1, \dots, n\}$ in the definition by $h : \mathcal{X} \to \mathbb{Z}$. In particular, $Z_{\infty,\infty}$ and $Z^{\Delta}_{\infty,\infty}$ are derived equivalent by Madsen's theorem.

(2) b corresponds to the homotopy class of the (h, r)-degree $(0, 1)$ map $P(x) \to \Delta(x + 1)$ in $\mathrm{Hom}_U(\widetilde{\Delta}(x), \Delta(x + 1))$. A corresponds to the homotopy class of the (h, r)-degree $(1, 0)$ map $P(x + 1) \to \Delta(x + 1)$ in $\mathrm{Hom}_U(\widetilde{\Delta}(x), \Delta(x + 1))$.

Fix some $i \in \{1, \dots, r\}$ and $c \in \mathbb{Z}$, let $\mathcal{X} = \{x \in \mathbb{Z}^r | x_i = c\}$. We call this type of Cubist sets the *hyperplane Cubist set*. The Cubist algebra associated to a hyperplane Cubist in $\mathbb{Z}^{r+1}$ is in fact isomorphic to the algebra U_r presented in the definition of Cubist algebras (see Section 3.2.2). In particular, the Cubist algebra associated to a hyperplane Cubist in $\mathbb{Z}^2$ (i.e. the infinite Brauer line algebra) is isomorphic to U_1. Since $U_r = U_1^{\otimes r}$, using an infinite analogue of the result in Chapter 2 (Proposition 2.2.1), we obtain

Proposition 4.4.2. *Let $\mathcal{X} \subset \mathbb{Z}^{r+1}$ be a hyperplane Cubist subset, then we have the following chain of (graded) algebra isomorphism:*

$$U^{\Delta}_{\mathcal{X}} \cong (U_1^{\otimes r})^{\Delta} \cong (Z^{\Delta}_{\infty,\infty})^{\otimes r}.$$

Remark 4.4. Proposition 2.2.1 was proved for finite dimensional algebra. But the result can be transferred to our setting. Alternatively, one can also use the direct limit of $\bigotimes_{i=1}^{r} Z_{n_i}$ over $\{(n_1, \dots, n_r) \in \mathbb{Z}^r\}$ to realise U_r.

Example 4.5. Recall that the quiver of U_2 is given by the vertex set $\mathbb{Z}^2$, and arrows $\{a_{x,i}, b_{x,i} | x \in \mathbb{Z}^2; i = 1, 2\}$. For simplicity, we let $a_{x,i} = a_i$, $b_{x,i} = b_i$ for all x. Then, the relations are

- (square relation) $b_i^2 = 0 = a_i^2$ for $i = 1, 2$;
- (supercommutation) $a_i a_j + a_j a_i = 0$, $b_i b_j + b_j b_i = 0$, $b_i a_j + a_j b_i = 0$ with $\{i, j\} = \{1, 2\}$;
- (Heisenberg relations) $b_1 a_1 + a_1 b_1 = b_2 a_2 + a_2 b_2$.

Then quiver of U_2^{Δ} has vertex set $\mathbb{Z}^2$ and arrows $\{A_{x,i}, b_{x,i} | x \in \mathbb{Z}^2; i = 1, 2\}$. Here $A_{x,i}$ is an arrow from x to $x - \epsilon_i$. We abbreviate the subscripts $x \in \mathbb{Z}^2$ again. Then the relations are given by

- $b_i^2 = 0$ for $i = 1, 2$;
- $A_i A_j + A_j A_i = 0$, $b_i b_j + b_j b_i = 0$, $b_i A_j + A_j b_i = 0$.

4.5 Investigation on an A_∞-model of the Ext-algebra

Let $\mathcal{E} = \bigoplus_{p\in\mathbb{Z}} \mathcal{E}^p$ be a $\mathbb{Z}$-graded vector space. $\mathcal{E}$ is called an A_∞*-algebra* if it is equipped with a family of graded $\mathbf{k}$-linear maps $m_n : \mathcal{E}^{\otimes n} \to \mathcal{E}$ for all $n \geq 1$ of degree $2-n$ satisfying the Stasheff identities $\mathsf{SI}(n)$ for all $n \geq 1$.

$$\mathsf{SI}(n): \quad \sum (-1)^{r+st} m_u(\mathrm{id}_{\mathcal{E}}^{\otimes r} \otimes m_s \otimes \mathrm{id}_{\mathcal{E}}^{\otimes t}) = 0$$

with the sum runs over all $r,t \geq 0$ and $s \geq 1$ such that $n = r+s+t$. An A_∞*-model* of the vector space $\mathcal{E}$ is a choice of $\{m_i\}_{i\geq 1}$ so that $(\mathcal{E};\{m_i\}_{i\geq 1})$ is an A_∞-algebra. An A_∞-algebra with *G-Adams grading* is a bigraded vector space $\mathcal{E} = \bigoplus_{p\in\mathbb{Z}, i\in G} A_i^p$ for some abelian group G, with degree $2-n$ multiplications $\{m_i : \mathcal{E}^{\otimes i} \to \mathcal{E}\}_{i\geq 1}$ satisfying Stasheff identities and preserving G-grading (i.e. each m_i is a G-degree 0 map).

The degree 1 map m_1 is also called the differential of A as it satisfies $m_1 m_1 = 0$ (from $\mathsf{SI}(1)$). By $\mathsf{SI}(2)$, the differential m_1 is a graded derivation with respect to m_2. By $\mathsf{SI}(3)$, if m_1 or m_3 vanishes, then $\mathcal{E}$ becomes an associative algebra. So m_2 plays the role of (classical) multiplication. We call all the m_n for $n \geq 3$ *higher multiplication*. In particular, an A_∞-algebra with all higher multiplications zero is just a dg algebra.

A morphism of A_∞-algebras $f : \mathcal{E} \to \mathcal{F}$ is a family of graded $\mathbf{k}$-linear maps $f_n : \mathcal{E}^{\otimes n} \to \mathcal{F}$ of degree $1-n$ satisfying Stasheff morphism identities (see for example [LPWZ, Sec 2]). In the case of a G-Adams graded A_∞-algebra, we also require all f_i's to preserve the G-Adams grading. f is called a *quasi-isomorphism* (or *quism*) if f_1 is quasi-isomorphism of the underlying complexes $(\mathcal{E};\{m_1,m_2\})$ and $(\mathcal{F};\{m_1',m_2'\})$.

A famous theorem of Kadeishvili shows that, if $\mathcal{E}$ is an A_∞-algebra, then its cohomology ring $H\mathcal{E} := \ker m_1 / \operatorname{Im} m_1$ inherits an A_∞-model.

Theorem 4.5.1 ([Kad])**.** *Let $\mathcal{E}$ be an A_∞-algebra and HA be the cohomology ring of $\mathcal{E}$. There is an A_∞-model on $H\mathcal{E}$ with $m_1 = 0$ and m_2 induced by the multiplication on $\mathcal{E}$, constructed from the A_∞-structure of A, such that there is a quasi-isomorphism of A_∞-algebras $H\mathcal{E} \to A$ lifting the identity of $H\mathcal{E}$. This A_∞-model on $H\mathcal{E}$ is unique up to quasi-isomorphism.*

In practice, Kadeishvili's theorem is very difficult to work with, making the A_∞-model of $H\mathcal{E}$ obscure. Merkulov [Mer], on the other hand, constructed an A_∞-model on $H\mathcal{E}$ which can be defined inductively.

We are interested in the following setting. Let $U = U_{\mathcal{X}}$ be a rhombal algebra and $\widetilde{\Delta}$ be the minimal projective resolution of the direct sum of standard U-modules. Then the endomorphism ring $\mathcal{E} = \mathcal{E}\mathrm{nd}_U(\widetilde{\Delta})$ is a natural dg algebra, hence an A_∞-algebra. Note that this (internal) endomorphism ring contains all maps $\widetilde{\Delta} \to \widetilde{\Delta}$, not necessarily chain maps. Since U is Koszul, $\mathcal{E}\mathrm{nd}_U(\widetilde{\Delta}) = \bigoplus_{i\in\mathbb{Z}} hom(\widetilde{\Delta}, \widetilde{\Delta}\langle i\rangle)$ becomes a $\mathbb{Z}$-Adams graded A_∞-algebra.

In the rest of this chapter, we follow the approach in [LPWZ], which applied Merkulov's construction [Mer] to Ext-algebras, like $U^\Delta = H\mathcal{E}$, in an attempt to find an accessible A_∞-structure of U^Δ. Similar investigation were also carried out in [Kla] for A^Δ where $A = K_1^n$ and $A = K_2^n$, two subfamilies of generalised Khovanov arc algebras.

We briefly review the material in [LPWZ, Kla] here. We suppress all superscripts whenever possible. Let $\mathcal{E}$ be an A_∞-algebra. Let B and $Z = B \oplus H$ be coboundaries and cocycles of $\mathcal{E}$ respectively. We have decomposition of $\mathcal{E}$ into $L \oplus H \oplus B$, where H can be identified as the homology $H\mathcal{E}$. Note that different choices of bases of H and L can induce different A_∞-models, but Kadeishvili's theorem says that all such A_∞-models are quasi-isomorphic.

In our setting $\mathcal{E} = \mathcal{E}\mathrm{nd}_U(\widetilde{\Delta})$, the space Z is precisely the subspace of chain maps. We choose the basis of H to be indexed by the multiplicative basis $\cup C_{x,y}$ (union over x, y satisfying Theorem 3.3.1) as in the previous chapters 3. We denote, as conventional, d for the differential m_1 on $\mathcal{E}$.

Let $\Pi : \mathcal{E} \to \mathcal{E}$ be the projection to $H\mathcal{E}$, and let $Q : \mathcal{E} \to \mathcal{E}$ be a homotopy from $\mathrm{id}_\mathcal{E}$ to Π, i.e. $\mathrm{id}_\mathcal{E} - \Pi = dQ + Qd$. The map Q is not unique, but there is a canonical choice for it:

$$\begin{aligned} Q^n : \mathcal{E}^n &\to \mathcal{E}^{n-1} \\ \alpha &\mapsto \begin{cases} (d|_{L^{n-1}})^{-1}(\alpha) & \text{if } \alpha \in B^n, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Define a sequence of linear maps $\lambda_n : \mathcal{E}^{\otimes n} \to \mathcal{E}^{\otimes n-2}$ of degree $2n$ inductively as follows. There is no map λ_1 , but we formally define the composite $Q\lambda_1 = -\mathrm{id}_\mathcal{E}$. λ_2 is the multiplication of $\mathcal{E}$, i.e. $\lambda_2(\alpha_1, \alpha_2) = \alpha_1 \cdot \alpha_2$. For $n \geq 3$, λ_n is defined by the recursive formula

$$\lambda_n = \sum (-1)^{s+1} \lambda_2[Q\lambda_s \otimes Q\lambda_t]. \tag{4.5.1}$$

where the summation is over $s, t \geq 1$ with $s + t = n$.

Theorem 4.5.2 (Merkulov). *Let $m_i = \Pi\lambda_i$, then $(H\mathcal{E}, \{m_n\}_{n\geq 1})$ is an A_∞-algebra, which is quasi-isomorphic to $\mathcal{E}$ as A_∞-algebra, via the morphism $f = \{f_n\}_{n\geq 1}$ where $f_n = -Q\lambda_n$ for all $n \geq 1$.*

We term the A_∞-model arising in a construction above as *Merkulov model*.

We now look at the (h, r)-degree of the maps λ_n.

Lemma 4.5.3. *Suppose $\alpha_1, \ldots, \alpha_\ell \in \mathcal{E}$ are homogeneous with $\deg(\alpha_i) = (h_i, r_i)$ for all $i = 1, \ldots, \ell$. Then $\deg(\lambda_\ell(\alpha_1, \ldots, \alpha_\ell)) = (\sum_i h_i + 2 - \ell, \sum_i r_i + 2 - \ell)$.*

Proof. Induction on ℓ. Note that λ_n preserves Koszul grading on U, so $\lambda_\ell : \bigotimes_{i=1}^{\ell} \mathcal{E}_{j_i}^{h_i} \to \mathcal{E}_{\sum j_i}^{2-l+\sum h_i}$. Therefore, for all $i = 1, \ldots, \ell$ with α_i having (h,r)-degree $(h_i, r_i) = (h_i, h_i + j_i)$, $\lambda_\ell(\alpha_1 \otimes \cdots \otimes \alpha_n)$ have (h,r)-degree $(2 - \ell + \sum h_i, 2 - \ell + \sum h_i + \sum j_i) = (2 - \ell + \sum h_i, 2 - \ell + \sum r_i)$. □

Lemma 4.5.4. *For $i = 1, \ldots, \ell$, let $\alpha_i \in \operatorname{ext}^{h_i}(\widetilde{\Delta}(x_{i-1}), \widetilde{\Delta}(x_i)\langle j_i \rangle)$ be non-zero homogeneous elements, suppose $\alpha = m_\ell(\alpha_1, \ldots, \alpha_\ell) \neq 0$, then we have $\deg_r(\alpha) \leq \ell$ and*

$$\sum_{i=1}^{\ell} d(x_{i-1}, x_i) - d(x_0, x_\ell) = 2\ell - 4 \tag{4.5.2}$$

Proof. By Lemma 4.5.3, we have $\deg(\alpha) = (\sum_{i=1}^{\ell} h_i + 2 - \ell, \sum_{i=1}^{\ell} r_i + 2 - \ell)$, and so $\alpha \in \operatorname{ext}^{\sum h_i + 2 - \ell}(\widetilde{\Delta}(x_0), \widetilde{\Delta}(x_\ell)\langle \sum j_i \rangle)$. From Theorem 3.3.1, for a rhombal algebra U, we necessarily have

$$\begin{aligned} & 2\left(\sum h_i + 2 - \ell\right) + \sum j_i = d(x_0, x_\ell) \quad \text{and} \quad \sum (h_i + j_i) + 2 - \ell \leq 2 && (4.5.3)\\ \Rightarrow \quad & \sum (2h_i + j_i) + 2(2 - \ell) = d(x_0, x_\ell) \quad \text{and} \quad \sum (h_i + j_i) \leq \ell && (4.5.4)\\ \Rightarrow \quad & \sum_{i=1}^{\ell} d(x_{i-1}, x_i) - d(x_0, x_\ell) = 2\ell - 4 \quad \text{and} \quad \deg_r(\alpha) \leq \ell && (4.5.5) \end{aligned}$$

which proves the assertion. □

Theorem 4.5.5. *Let $\mathcal{X}$ be a Cubist set such that for all $x = (x_1, x_2, x_3) \in \mathcal{X}$, $x_2 \in \{0, 1\}$. Then higher multiplication m_ℓ vanishes for all $\ell > 4$. Furthermore, if x_2 is always zero, then higher multiplication m_ℓ vanishes for all $\ell > 2$.*

Proof. Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathcal{X}$ be two distinct vertices. If $\alpha \in \operatorname{Ext}^\bullet(\Delta(x), \Delta(y))$ non-zero homogeneous, then $\lambda y \cap \mu x \neq \emptyset$ and so $x_1 \geq y_1$ and $x_3 \leq y_3$. Suppose $x^0, \ldots, x^\ell \in \mathcal{X}$ (so writing x_i^j for the i-th coordinate of x^j), with non-zero $\operatorname{Ext}^\bullet(\Delta(x^{i-1}), \Delta(x^i))$ for all $i =$

$1, \ldots, \ell$, so we have

$$\begin{cases} x_1^0 \geq x_1^1 \geq \cdots \geq x_1^\ell \\ x_3^0 \leq x_3^1 \leq \cdots \leq x_3^\ell \end{cases} \tag{4.5.6}$$

$$\Rightarrow \quad \begin{cases} \sum_{i=1}^\ell |(x_1^{i-1} - x_1^i)| = |x_1^0 - x_1^\ell| \\ \sum_{i=1}^\ell |(x_3^{i-1} - x_3^i)| = |x_3^0 - x_3^\ell| \end{cases} \tag{4.5.7}$$

$$\Rightarrow \quad \sum_{i=1}^{\ell} d(x^{i-1}, x^i) - d(x_0, x_\ell) = \left(\sum_{i=1}^{\ell} |x_2^{i-1} - x_2^i| \right) - |x_2^0 - x_2^\ell| \tag{4.5.8}$$

From Lemma 4.5.4, for higher multiplication to be non-vanishing, we then need

$$\left(\sum_{i=1}^{\ell} |x_2^{i-1} - x_2^i| \right) - |x_2^0 - x_2^\ell| = 2\ell - 4.$$

If the 2nd coordinate of all vertices are the same, then left hand side of this equation will be zero, hence the only non-vanishing multiplication is m_2. If 2nd coordinates of all vertices lie in two consecutive integers, then $\sum |x_2^{i-1} - x_2^i| \leq \ell$ and $|x_2^0 - x_2^\ell| \leq 1$, so the left hand side is less than or equal to ℓ, hence $m_\ell \neq 0$ implies $2\ell - 4 \leq \ell$, and so $\ell \leq 4$. □

Remark 4.6. The later Cubist set is a hyperplane Cubist set. We do not know if m_3 and m_4 are non-zero in the case when $x_2 \in \{0, 1\}$ not all equal.

Part II

On simple-minded and mutation theories

Chapter 5

Guide to forthcoming chapters

5.1 Preliminaries

Let A be a finite dimensional k-algebra for a field k. We denote by mod-A (resp. proj-A) the category of all finitely generated (resp. finitely generated projective) right A-modules. We denote by $\underline{\text{mod}}$-A the stable category of mod-A modulo projective modules (*stable module category* of A). This is the category with the same objects as mod-A but with Hom-space $\underline{\text{Hom}}_A(X,Y) := \text{Hom}_{\underline{\text{mod}}\text{-}A}(X,Y)$ given by $\text{Hom}_A(X,Y)$ modulo the morphisms which factor through projective modules. Two algebras are *stably equivalent* if their stable module categories are equivalent. It is well-known that $\underline{\text{mod}}$-A is a triangulated category if and only if A is self-injective. In that case, the stable module category inherits a triangulated structure through a triangulated equivalence $\underline{\text{mod}}\text{-}A \simeq D^b(\text{mod-}A)/K^b(\text{proj-}A)$ [Ric1]. Here $D^b(\text{mod-}A)$ is the bounded derived category of mod-A, and $K^b(\text{proj-}A)$ is the bounded homotopy category of complexes of projective A-modules. The suspension functor in $\underline{\text{mod}}$-A is the inverse *syzygy* (inverse *Heller translate*) Ω^{-1}, see, for example, [Ric1]. In particular, this gives a (canonical) triangulated functor $\eta = \eta_A : D^b(\text{mod-}A) \to \underline{\text{mod}}\text{-}A$.

A k-linear category $\mathcal{C}$ is *locally bounded* if $\text{End}_{\mathcal{C}}(x)$ is local for all $x \in \mathcal{C}$, and for any $x \in \mathcal{C}$, $\sum_{y\in\mathcal{C}} \dim \text{Hom}_{\mathcal{C}}(x,y)$ and $\sum_{y\in\mathcal{C}} \dim \text{Hom}_{\mathcal{C}}(y,x)$ is finite. We denote mod-$\mathcal{C}$ the category of finitely generated right modules of $\mathcal{C}$, that is, the category of k-linear contravariant functors $M : \mathcal{C} \to \text{mod-}k$ which are quotients of finite direct sums of representable functors. We denote by ind-$\mathcal{C}$ the additive full subcategory of mod-$\mathcal{C}$ formed by indecomposable objects. For a finite dimensional basic k-algebra A given by path algebra presentation kQ/I, we can form a locally bounded k-linear category $\mathcal{A}$ with the set of objects being the set Q_0 of vertices of Q, morphisms generated by arrows of Q (whose set we denote Q_1), and relations given by I. Then mod-$\mathcal{A}$

is equivalent to mod-A, and so ind-$\mathcal{A}$ coincides with ind-A, the full subcategory formed by indecomposable A-modules. Similarly, we denote $\underline{\text{ind}}$-A the full subcategory of $\underline{\text{mod}}$-A formed by indecomposable objects of $\underline{\text{mod}}$-A.

5.2 Simple-minded systems

Let k be a field and let $\mathcal{T}$ be a Hom-finite Krull-Schmidt k-linear triangulated category with suspension [1]. For any collections $\mathcal{S}_1, \mathcal{S}_2$ of objects in $\mathcal{T}$, we define a collection of objects

$$\mathcal{S}_1 * \mathcal{S}_2 = \{X \in \mathcal{T} | \exists \text{distinguished triangle } S_1 \to X \to S_2 \to S_1[1] \text{ with } S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}$$

For a collection $\mathcal{S}$ of objects in $\mathcal{T}$, we denote $(\mathcal{S})_0 = \{0\}$. For any $n \in \mathbb{Z}_{\geq 1}$ define inductively $(\mathcal{S})_n = (\mathcal{S})_{n-1} * (\mathcal{S} \cup \{0\})$. Similarly, one can define ${}_n(\mathcal{S}) = (\mathcal{S} \cup \{0\}) * (\mathcal{S})_{n-1}$, but it can be shown that ${}_n(\mathcal{S}) = (\mathcal{S})_n$ for any $n \geq 0$ [Dug2, Lemma 2.2]. For a full subcategory $\mathcal{C}$ of $\mathcal{T}$ (we will always identify $\mathcal{C}$ with the set of its objects), we say $\mathcal{C}$ is *extension closed* if $\mathcal{C} * \mathcal{C} \subset \mathcal{C}$. We define the *filtration closure* (or *extension closure*) of a collection $\mathcal{S}$ of objects of $\mathcal{T}$ as $\mathcal{F}(\mathcal{S}) := (\bigcup_{n \geq 0} (\mathcal{S})_n)$, which is the smallest extension closed full subcategory of $\mathcal{T}$ containing $\mathcal{S}$ [Dug2, Lemma 2.3].

Definition 5.2.1. *An object S in $\mathcal{T}$ is called a* brick *if* $\operatorname{End}_{\mathcal{T}}(S)$ *is a division ring. A collection $\mathcal{S}$ of objects in $\mathcal{T}$ is said to be a* system of (pairwise) orthogonal bricks *if*

$$\operatorname{Hom}_{\mathcal{T}}(S, T) = \begin{cases} 0 & (S \neq T), \\ \textit{division ring} & (S = T). \end{cases} \tag{5.2.1}$$

Such an $\mathcal{S}$ is a simple-minded system *of $\mathcal{T}$ if furthermore $\mathcal{F}(\mathcal{S}) = \mathcal{T}$. We denote by* $\mathsf{sms}(\mathcal{T})$ *the collection of all simple-minded systems of $\mathcal{T}$.*

We note that the fact that $(\mathcal{S})_n$ (in particular, $\mathcal{F}(\mathcal{S})$) being closed under direct summands is non-trivial, and require $\mathcal{S}$ being a system of orthogonal bricks [Dug2, Lemma 2.7]. Our definition presented above is taken from [Dug2], which is different from the original definition of simple-minded system from [KL]. In [KL], simple-minded system is defined for the stable module category of any artinian algebra, which does not necessarily possess any triangulated structure. Nevertheless, in this thesis we are only interested in the stable module category of a self-injective algebra, in which case, the two definitions are equivalent to each other. We will abbreviate simple-minded system by *sms* from now on. We will use the notation $\mathsf{sms}(A)$ instead of $\mathsf{sms}(\underline{\text{mod}}\text{-}A)$. We start our study of simple-minded systems by listing some examples below.

1. $\mathcal{T} = \underline{\text{mod}}\text{-}A$ with A a self-injective k-algebra. The set $\mathcal{S}_A$ of (isoclass representatives of) simple A-modules up to isomorphism is a sms.

2. $\mathcal{T} = \underline{\text{mod}}\text{-}A$ with A a self-injective k-algebra. Let B be another k-algebra, and $\phi : \underline{\text{mod}}\text{-}B \to \underline{\text{mod}}\text{-}A$ be a stable equivalence. Then $\phi(\mathcal{S}_B)$ is an sms of A. We call an sms a *simple-image* if it arises this way. More generally, for any sms $\mathcal{S}$ of B, $\phi(\mathcal{S})$ is also an sms of A. This comes from [KL, Theorem 3.2].

In [KL], sms is invented in order to attempt to give a proper definition for the "generator" of the stable module category which behaves similarly to $\mathcal{S}_A$, hence its name.

If A is a finite dimensional algebra with finite global dimension, then one can obtain sms's of $D^b(\text{mod-}A)$ by considering the (infinite) set of *graded* simple $T(A)$-modules, where $T(A)$ is the trivial extension algebra of A [Dug2], or by using the stable module category of the infinite dimensional repetitive algebra $\hat{A}$. We will study the case when A is representation-finite hereditary in Chapter 7 (Section 7.2).

5.3 Configurations and weakly simple-minded systems

We are primarily interested in the stable module category of a representation-finite self-injective (RFS) algebra. Due to the assumptions needed for the machinery we will be using, we will take the underlying field k to be an algebraically closed field, and assume all the algebras are indecomposable, non-simple, and basic from now on. In particular, the endomorphism ring of a brick is always the underlying field k. RFS algebras were completely classified in the late 80's [Rie1,Rie2,Rie3,Rie4,BLR]. The derived and stable equivalent classification was completed almost two decades later by Asashiba [Asa1] (see Theorem 6.1.4). We will give more details about these results in the next chapter. A combinatorial gadget called *configuration*, which grew out of the Auslander-Reiten (AR) theory around RFS algebras, plays a vital rôle throughout this line of research. We will set up some notations for AR theory first, but will not explain the details of the terms involved. We refer to [ARS, ASS] for a reference on AR theory.

Let Q denote a Dynkin quiver of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ or $\mathbb{E}_8$ (such as Q obtained from Q_Δ in (7.3.1,7.3.2,7.3.3) by removing the barred arrows); and $\mathbb{Z}Q$ the corresponding (stable) *translation quiver* with translation denoted as τ. We will use A_n (resp. D_n, E_n) to denote a Dynkin quiver of type $\mathbb{A}_n$ (resp. $\mathbb{D}_n, \mathbb{E}_n$). Riedtmann showed in [Rie1] that for an RFS algebra over an algebraically closed field, the stable AR-quiver is of the form $\mathbb{Z}Q/\Pi$ for some admissible group Π. Consequently we say such an algebra is of *tree class* Q and has *admissible group* Π.

For a (not necessarily stable) translation quiver Γ, we let $k(\Gamma)$ be its *mesh category*, that is, the path category whose objects are the vertices of Γ; morphisms are generated by arrows of Γ quotiented out by the mesh relations. Each mesh relation is of the following form:

$$\sum_{\alpha} \sigma(\alpha)\alpha = 0$$

where α varies over all arrows ending in a fixed vertex v of Γ, and $\sigma(\alpha)$ is the (unique) arrow on Γ starting in τv and ending at the source of α.

Definition 5.3.1 ([BLR]). *A* configuration *of $\mathbb{Z}Q$ is a subset $\mathcal{C}$ of vertices of $\mathbb{Z}Q$ such that the quiver $\mathbb{Z}Q_{\mathcal{C}}$ is a representable translation quiver. $\mathbb{Z}Q_{\mathcal{C}}$ is constructed by adding one vertex c^* for each $c \in \mathcal{C}$ on $\mathbb{Z}Q$; adding arrows $c \to c^* \to \tau^{-1}c$; and letting the translation of c^* be undefined.*

Here, the following terminology is used: A translation quiver Δ is representable if and only if the mesh category $k(\Delta)$ is an Auslander category, i.e. $k(\Delta)$ is (additively) equivalent to ind-Λ for some locally representation-finite category Λ [BG, 2.3,2.4]. We do not go through the technicalities of these definitions. The idea is that, if A is an RFS algebra with AR-quiver $\Gamma = \mathbb{Z}Q_{\mathcal{C}}/\Pi$ for some set $\mathcal{C} \subset Q_0$, then by considering a locally bounded k-linear category , called the Galois cover of A, the analogue of the Auslander algebra of this category is equivalent to the mesh category $k(\mathbb{Z}Q_{\mathcal{C}})$, where projective vertices correspond to indecomposable projective $\widetilde{A}$-module [BG].

Definition 5.3.2 ([Rie2]). *Let Δ be a stable translation quiver. A* combinatorial configuration *$\mathcal{C}$ is a set of vertices of Δ which satisfy the following conditions:*

1. *For any $e, f \in \mathcal{C}$, $Hom_{k(\Delta)}(e,f) = \begin{cases} 0 & (e \neq f), \\ k & (e = f). \end{cases}$*
2. *For any $e \in \Delta_0$, there exists some $f \in \mathcal{C}$ such that $Hom_{k(\Delta)}(e,f) \neq 0$.*

If $\Delta = \mathbb{Z}Q$ with Q a Dynkin quiver, then we denote the set of combinatorial configurations on Δ as $\mathsf{Conf}(Q)$.

We also note the following fact in [Rie2, Proposition 2.3]: if $\pi : \Delta \to \Gamma$ is a covering of the translation quiver Γ, then $\mathcal{C}$ is a combinatorial configuration of Γ if and only if $\pi^{-1}\mathcal{C}$ is a combinatorial configuration of Δ. When applied to the universal cover of stable AR-quiver of RFS algebra A, this translates to the following statement: $\mathcal{C}$ is a combinatorial configuration of the stable AR-quiver $\mathbb{Z}Q/\Pi$ if and only if $\pi^{-1}\mathcal{C}$ is a Π-stable combinatorial configuration of the universal cover $\mathbb{Z}Q$.

Combinatorial configurations was first defined in [Rie2]. At first a combinatorial configuration is a generalisation of configuration. It is often easier to study and compute than a configuration as it suffices to look 'combinatorially' at sectional paths of the translation quiver $\mathbb{Z}Q$ rather than checking whether $k(\mathbb{Z}Q_{\mathcal{C}})$ can be realised as an Auslander category.

The following notion is closely related to configurations. This was introduced as a weaker version of sms (of stable module category) in [KL], we modify its definition slightly to fit in with the definition of sms's for Hom-finite triangulated categories.

Definition 5.3.3 ([KL], [Pog]). *Let $\mathcal{T}$ be as in Definition 5.2.1. A class $\mathcal{S}$ of indecomposable objects (not isomorphic to* 0*) is called a* weakly simple-minded system *(wsms) if the following two conditions are satisfied:*

1. *(orthogonality) It is a system of pairwise orthogonal bricks.*
2. *(weak generation) For any indecomposable object X not isomorphic to* 0*, there exists some $S, T \in \mathcal{S}$ (depends on X) such that* $\mathrm{Hom}_{\mathcal{T}}(X, S) \neq 0$ *and* $\mathrm{Hom}_{\mathcal{T}}(T, X) \neq 0$.

If moreover $\mathcal{T} = \underline{\mathrm{mod}}\text{-}A$ for some self-injective algebra A, and $\tau S \not\cong S$ for all $S \in \mathcal{S}$, then $\mathcal{S}$ is called a maximal system of orthogonal bricks.

It is easy to see that if $\mathcal{S}$ is an sms of $\mathcal{T}$, then it is also a wsms of $\mathcal{T}$. In the case of $\underline{\mathrm{mod}}\text{-}A$ with A an indecomposable basic non-simple RFS algebras which is not isomorphic to $k[x]/(x^n)$ for any $n \in \mathbb{Z}_{>1}$, then the extra condition for maximal system of orthogonal bricks is automatically satisfied. Although the definitions of wsms and combinatorial configuration are strikingly similar, it is not entirely straightforward that they are the same, for they are defined on rather different categories.

Recall that a *Serre functor* $\mathbb{S}$ on $\mathcal{T}$ is a triangulated auto-equivalence inducing the Serre duality

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \cong D\mathrm{Hom}_{\mathcal{T}}(Y, \mathbb{S}X)$$

for all X, Y in $\mathcal{T}$. Here $D(-)$ is the k-linear dual $\mathrm{Hom}_k(-, k)$. If a Serre functor exists, then it is unique up to natural isomorphism. We note that, in such case either $\mathrm{Hom}_{\mathcal{T}}(X, S) \neq 0$ or $\mathrm{Hom}_{\mathcal{T}}(T, X) \neq 0$ is sufficient as weak generation condition. If $\mathcal{T} = \underline{\mathrm{mod}}\text{-}A$ for a self-injective algebra A, then by *Auslander-Reiten duality* $\underline{\mathrm{Hom}}_A(X, Y) \cong D\mathrm{Ext}^1_A(Y, X)$, the Serre functor is given by $\Omega_A^{-1}\tau \cong \nu_A\Omega_A$ where ν_A is the Nakayama functor (see next section).

One result in Chapter 6 is to establish a connection between configurations and sms's of the stable module category of an RFS algebra.

Theorem 5.3.4 (Theorem 6.1.1). *Let A be an RFS algebra with stable AR-quiver ${}_s\Gamma_A = \mathbb{Z}Q/\Pi$.*

Then the following sets are in bijection:

1. *the set of* Π*-stable configurations of* $\mathbb{Z}Q$*,*
2. *the set of wsms's of* $\underline{\text{mod}}$*-*A*,*
3. *the set of sms's of* $\underline{\text{mod}}$*-*A*.*

Configurations also provide sms's for bounded derived categories of representation-finite hereditary algebra:

Theorem 5.3.5 (Theorem 7.2.1)**.** *Let* Q *be a Dynkin quiver, the following sets are in bijection:*

1. *the set of configurations of* $\mathbb{Z}Q$*,*
2. *the set of sms's of* $D^b(\text{mod-}kQ)$*,*
3. *the set of wsms's of* $D^b(\text{mod-}kQ)$*,*

The relation between these two theorems can be explained by the construction of triangulated orbit categories:

Theorem 5.3.6 (Theorem 7.1.4)**.** *Let* $\mathcal{T} = D^b(\mathcal{H})$ *for some hereditary abelian* k*-category* $\mathcal{H}$ *over algebraically closed field* k*, and* $F : \mathcal{T} \to \mathcal{T}$ *a standard derived equivalence satisfying conditions of [Kel, Thm 1]. Then we have a bijection*

$$\mathsf{sms}(\mathcal{T}/F) \leftrightarrow \mathsf{sms}_F(\mathcal{T}) := \{\mathcal{S} \in \mathsf{sms}(\mathcal{T}) \mid F\mathcal{S} = \mathcal{S}\}$$

This also allows us to obtain sms's of all standard Hom-finite Krull-Schmidt triangulated k-categories which have finitely many indecomposable objects and are 1-Calabi-Yau. The details will be presented in Chapter 7 (Section 7.1).

5.4 Simple-minded collections

One way to attack a problem in the stable module category of a self-injective algebra is to look at the (bounded) derived category, due to the canonical equivalence $D^b(\text{mod-}A)/K^b(\text{proj-}A) \simeq \underline{\text{mod}}\text{-}A$. An sms of the bounded derived category often has infinitely many (indecomposable) objects, and to circumvent this problem, we consider systems of objects behaving like the (finite) set of simple modules called *simple-minded collections.* This notion appears in work of S. Koenig and D. Yang [KY] by generalising conditions originally introduced by Rickard [Ric4] and Al-Nofayee [AN], where they call a (Nakayama-stable) simple-minded collection as *cohomologically Schurian collection.*

Definition 5.4.1 ([KY]). *Let $\mathcal{T}$ be triangulated category, a finite collection $\mathbf{S} = \{X_1, \cdots, X_r\}$ of objects is a* simple-minded collection *(usually abbreviated as smc) if the following conditions are satisfied:*

1. $\mathbf{S}$ *is a system of pairwise orthogonal bricks;*

2. $\mathbf{S}$ generates $\mathcal{T}$, *i.e. the smallest thick subcategory* thick($\mathbf{S}$) *of $\mathcal{T}$ containing $\mathbf{S}$ is $\mathcal{T}$ itself;*

3. $\mathrm{Hom}_{\mathcal{T}}(X, Y[m]) = 0$ *for any $m < 0$, any $X, Y \in \mathbf{S}$.*

Two smc's $\mathbf{S}, \mathbf{S}'$ of $\mathcal{T}$ are equivalent *if their extension closures are the same, i.e. $\mathcal{F}(\mathbf{S}) = \mathcal{F}(\mathbf{S}')$.*

We will only look at smc's of $D^b(\text{mod-}A)$ in this thesis. Denote the set of smc's up to equivalence as $\mathsf{smc}(A)$. For any (finite dimensional) k-algebra A, the set of isoclass representatives of simple A-modules up to isomorphism, considered as stalk complexes concentrated in homological degree 0, is a smc of $D^b(\text{mod-}A)$. We denote this set by $\mathbf{S}_A$.

Recall that for a self-injective algebra A, the *Nakayama functor* $\nu_A = - \otimes_A DA : \text{mod-}A \to \text{mod-}A$ is an exact self-equivalence and therefore induces a self-equivalence of $D^b(\text{mod-}A)$ and of $\underline{\text{mod}}\text{-}A$, which will also be denoted by ν_A. By Rickard [Ric3], if $\phi : D^b(\text{mod-}A) \to D^b(\text{mod-}B)$ is a derived equivalence between two self-injective algebras A and B, then $\phi\nu_A(X) \simeq \nu_B\phi(X)$ for any object $X \in D^b(\text{mod-}A)$. We shall say an smc $X_1, \cdots, X_r$ of $D^b(\text{mod-}A)$ is *Nakayama-stable* if the Nakayama functor ν_A permutes $X_1, \cdots, X_r$. In particular, any derived equivalence $\phi : D^b(\text{mod-}A) \to D^b(\text{mod-}B)$ sends simple modules to a Nakayama-stable smc. We denote ν-$\mathsf{smc}(A)$ as the class of Nakayama-stable smc's of $D^b(\text{mod-}A)$. Denote by $\mathsf{DPic}(A)$ the *derived Picard group* of A; this is the group of automorphisms of $D^b(\text{mod-}A)$ given by tensoring a two-sided complex, so $\mathsf{DPic}(A)$ acts naturally on ν-$\mathsf{smc}(A)$.

It is still unknown whether there is a way to efficiently or systematically look at arbitrary stable equivalences. Instead, one usually concentrates on a particular type of stable equivalence. Let A and B be two algebras. Following Broué [Bro], we say that $\phi : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}B$ is a *stable equivalence of Morita type* (usually we just abbreviate as StM) if there are two left-right projective bimodules ${}_AM_B$ and ${}_BN_A$ such that the following two conditions are satisfied:

1. ${}_AM \otimes_B N_A \simeq {}_AA_A \oplus {}_AP_A, \quad {}_BN \otimes_A M_B \simeq {}_BB_B \oplus {}_BQ_B$,
   where ${}_AP_A$ and ${}_BQ_B$ are some projective bimodules;

2. ϕ is a stable equivalence which lifts to the functor $N \otimes_A -$, that is, the diagram

$$\begin{tikzcd} \text{mod-}A \arrow[r, "N\otimes_A -"] \arrow[d, "\pi_A"'] & \text{mod-}B \arrow[d, "\pi_B"] \\ \underline{\text{mod}}\text{-}A \arrow[r, "\phi"] & \underline{\text{mod}}\text{-}B \end{tikzcd}$$

commutes up to natural isomorphism, where π_A and π_B are the natural quotient functors.

Throughout the study of sms's, we will frequently use the well-known result of Linckelmann [Lin], which asserts that a StM between two self-injective algebras lifts to a Morita equivalence if and only if it sends simple modules to simple modules. We refer to this as Linckelmann's theorem.

It is well-known that any *standard derived equivalence* (i.e. given by tensoring a two-sided complex) between two self-injective algebras always induces a stable equivalence of Morita type. While the converse is known to be false (that is, there are stable equivalences of Morita type which are not induced by standard derived equivalence), it is still interesting and important to ask if a given stable equivalence of Morita type can be obtained from a derived equivalence. If a given StM can be obtained this way, then we say it can be *lifted*, or it is *liftable*. By Linckelmann's theorem, this problem can be reduced to just asking whether a stable auto-equivalence of Morita type can be lifted to a standard derived equivalence. Following convention, we term this problem as the *lifting problem*. For example, the Nakayama functor on $\underline{\mathrm{mod}}$-A for self-injective algebra A, is a StM given by the A-A-bimodule ${}_1A_\nu$ with the right A-action twisted by the Nakayama automorphism, can be lifted to the Nakayama functor on $D^b(\text{mod-}A)$. The Heller translate Ω_A for self-injective algebra A is also a StM, given by the A-A-bimodule $\Omega_{A\otimes A^{\mathrm{op}}}$ which is the kernel of the multiplication map $A \otimes A \to A$ viewed as A-A-bimodule. Ω_A can be lifted to the shift $[-1]$ in $D^b(\text{mod-}A)$. The results of Asashiba [Asa2] and Dugas [Dug1] answer the lifting problem for a class of RFS algebras (called *standard* RFS algebras, see next chapter).

Theorem 5.4.2 (Lifting theorem [Asa2], [Dug1])**.** *Let A be a standard RFS algebra and ϕ : $\underline{\mathrm{mod}}$-$A \to \underline{\mathrm{mod}}$-$A$ be a stable auto-equivalence of Morita type. Then ϕ can be lifted to a standard derived auto-equivalence.*

We will give an alternative proof to Dugas' part of the result using mutations of sms's with the viewpoint of configurations in Section 6.2.

Similar to the situation in the derived category, $\mathsf{StPic}(A)$, the *stable Picard group*, i.e. group of stable self-equivalence *of Morita type* for A, acts naturally on the class of sms's of $\underline{\mathrm{mod}}$-A. The answer to lifting problem will then allow us to reveal the connection between Nakayama-stable smc's of the bounded derived category, and sms's of the stable module category, and algebras which are stably equivalent of Morita type, for an RFS algebra.

Theorem 5.4.3 (Proposition 6.3.1, Theorem 6.3.4)**.** *Let A be an RFS algebra. Then there is a bijection between the follow sets:*

1. *the set of* $\mathsf{DPic}(A)$*-orbits of one-sided tilting complexes of* A*,*

2. *the set of* $\mathsf{DPic}(A)$*-orbits of Nakayama-stable smc's of* $D^b(\text{mod-}A)$*,*

3. *the set of* $\mathsf{StPic}(A)$*-orbits of sms's of* $\underline{\text{mod}}\text{-}A$*,*

4. *the set of isoclass representatives of RFS algebra which are stably equivalent to* A*.*

Remark 5.1. If we take a non-Nakayama-stable smc, then the corresponding induced collection in the stable module categories is in general not an sms. Examples are given in [Dug2, 6.3] and [AN, Section 1].

We will also go through other easy, but interesting, consequences of the answer to lifting problem as applications of sms's in Section 6.3.

5.5 Mutation theories

Mutation technique originates from BGP reflections on quiver representations, and has developed into useful theories, such as APR-tilting modules, mutation of exceptional sequences, Riedtmann-Schofield's mutation of tilting modules, Happel-Unger's tilting quiver, etc. The study of mutations became popular in recent years since the introduction of cluster algebras. However, most of these mutation theories are developed for tilting modules, hence only for algebras of finite global dimension. On the other hand, our interest, at least for this thesis, concerns self-injective algebras, and the only (basic) tilting module for a (basic) self-injective algebra is just the algebra itself. Recent results of Aihara and Iyama [AI] unify the mutation theory developed around algebras of finite global dimension, and the Okuyama-Rickard construction [Oku, Ric2] of tilting complexes (for symmetric algebras). We now review some material from their results, as well as other mutation theories evolved from there [KY, Dug2].

Assume for the moment that $\mathcal{T}$ a Hom-finite Krull-Schmidt triangulated k-linear category. Take X, Y objects in $\mathcal{T}$ and $f : X \to Y$ a morphism in $\mathcal{T}$, f is called *left minimal* if any morphism $g : Y \to Y$ in $\mathcal{T}$ with $gf = f$ is an isomorphism. Dually, f is called *right minimal* if any morphism $h : X \to X$ in $\mathcal{T}$ with $fh = f$ is an isomorphism.

Let $\mathcal{M}$ a full subcategory of $\mathcal{T}$. We say a morphism $f : X \to M$ is a *left* $\mathcal{M}$*-approximation* if M is in $\mathcal{M}$ and $\mathrm{Hom}_{\mathcal{T}}(f, M')$ is surjective for any M' in $\mathcal{M}$. We say that $\mathcal{M}$ is *covariantly finite* if any object in $\mathcal{T}$ has a left $\mathcal{M}$-approximation. Dually, $g : M \to Y$ is a *right* $\mathcal{M}$*-approximation* if M is in $\mathcal{M}$ and $\mathrm{Hom}_{\mathcal{T}}(M', g)$ is surjective for any M' in $\mathcal{M}$; and we say that $\mathcal{M}$ is *contravariantly finite* if any object in $\mathcal{T}$ has a right $\mathcal{M}$-approximation. $\mathcal{M}$ is *functorially finite* if it is both contravariantly and covariantly finite.

In [Dug2, Theorem 3.3], it was proved that the filtration closure of a subset of an sms which is also stable under $\mathbb{S}[1]$, where $\mathbb{S}$ is the Serre functor (if it exists), is always functorially finite, and so the following definition makes sense.

Definition 5.5.1 (Def 4.3 in [Dug2])**.** *Let $\mathcal{T}$ be Hom-finite Krull-Schmidt triangulated k-category with suspension* [1] *and Serre functor $\mathbb{S}$, Suppose $\mathcal{S} = \{X_1, \dots, X_r\}$ a $\mathbb{S}[1]$-stable sms of $\mathcal{T}$, and $\mathcal{X}$ is a $\mathbb{S}[1]$-stable subset of $\mathcal{S}$. The* left sms mutation *of the sms $\mathcal{S}$ with respect to $\mathcal{X}$ is the set $\mu^-_{\mathcal{X}}(\mathcal{S}) = \{Y_1, \dots, Y_r\}$ such that*

1. *$Y_j = X_j[1]$, if $X_j \in \mathcal{X}$*
2. *Otherwise, Y_j is defined by the following distinguished triangle*

$$X_j[-1] \to X \to Y_j \to X_j$$

where the first map is a minimal left $\mathcal{F}(\mathcal{X})$-approximation of $X_j[-1]$.

The right sms mutation *$\mu^+_{\mathcal{X}}(\mathcal{S})$ of $\mathcal{S}$ is defined similarly.*

It has been shown in [Dug2] that the above defined sets $\mu^-_{\mathcal{X}}(\mathcal{S})$ and $\mu^+_{\mathcal{X}}(\mathcal{S})$ are again sms's. We remark that if $\mathcal{T} = \underline{\text{mod}}\text{-}A$ with A self-injective, then the Serre functor is $\nu_A[-1]$. If no non-empty proper subset of a Nakayama-stable set $\mathcal{X}$ is Nakayama-stable, we say that $\mathcal{X}$ is *minimal Nakayama-stable*. If we mutate a sms of $\underline{\text{mod}}\text{-}A$ with respect to a minimal Nakayama-stable subset, then we call the mutation *irreducible*. In particular, when A is furthermore weakly symmetric, we can always mutate at any subset of an sms, and irreducible mutations are those that are performed with respect to an indecomposable module.

We also remark that left and right mutations are inverse to each other in the following sense. Let $\mathcal{S}$ be a $\mathbb{S}[1]$-stable sms, $\mathcal{X}$ a $\mathbb{S}[1]$-stable subset, and $\mathcal{X}[n] = \{X[n] | X \in \mathcal{X}\}$ for any $n \in \mathbb{Z}$. Then we have $\mu^+_{\mathcal{X}[1]}\mu^-_{\mathcal{X}}(\mathcal{S}) = \mathcal{S} = \mu^-_{\mathcal{X}[-1]}\mu^+_{\mathcal{X}}(\mathcal{S})$.

Mutation of sms is designed to keep track of the images of simple modules (which form an sms) under an StM which can be lifted to standard derived equivalence. It is interesting to ask if two sms's are linked by a sequence of mutations. We will give positive answer for $\mathcal{T} = \underline{\text{mod}}\text{-}A$ when A is an RFS algebra in Section 8.1.

The definition of mutation we use is a variation of Dugas' original one by shifting the objects appropriately so that the mutations "align" with the mutations for smc's defined in [KY]. This difference between our "shifted" version and the original is also remarked in [Dug2]. We here show an example:

Example 5.2. Let A be a symmetric Nakayama algebra (see Definition 8.2.3) with 4 simples and Loewy length 5. We display the modules by their Loewy diagrams. The canonical sms is the set of simple A-modules $\mathcal{S}_A = \{1, 2, 3, 4\}$. The left mutation of $\mathcal{S}$ at $\mathcal{X} = \{2, 3\}$ is

$$\mu_{\mathcal{X}}^{-}(\{1,2,3,4\}) = \{\begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}2\\3\\4\\1\end{smallmatrix}, \begin{smallmatrix}3\\4\\1\\2\end{smallmatrix}, 4\}.$$

As mentioned in the previous section, it is interesting to relate observations on $D^b(\text{mod-}A)$ and those on $\underline{\text{mod}}\text{-}A$ for self-injective algebra. When looking at equivalences between derived categories of algebras, the tool we usually use is "projective-minded generator" - tilting complex - an object living in the bounded homotopy category $K^b(\text{proj-}A)$. We give a definition of a generalisation of such notion, and investigate the connection between them and the "simple-minded generators" using mutation theories.

Definition 5.5.2 ([AI]). *Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated k-category, and T an object in $\mathcal{T}$. We call T a* silting object *(resp.* tilting object*) if:*

1. $\text{Hom}_{\mathcal{T}}(T, T[i]) = 0$ *for any* $i > 0$ *(resp.* $i \neq 0$*);*
2. *T generates $\mathcal{T}$.*

In the case of $\mathcal{T} = K^b(\text{proj-}A)$ where A is a finite dimensional algebra, we sometimes call a silting (resp. tilting) object T as silting complex *(resp.* tilting complex*) over A. We denote the set of silting (resp. tilting) complexes over A as $\mathsf{silt}(A)$ (resp. $\mathsf{tilt}(A)$).*

It is easy to see tilting complex in this definition is exactly (one-sided) tilting complex of an algebra classically. Note that in the case of $\mathcal{T} = K^b(\text{proj-}A)$ with A symmetric, Auslander-Reiten duality implies that any silting object is a tilting object.

For convenience, we assume every silting object is basic, i.e. the indecomposable direct summands are pairwise non-isomorphic. In this thesis, we will only look at the case $\mathcal{T} = K^b(\text{proj-}A)$ with A self-injective.

Since we have $\mathcal{T} = K^b(\text{proj-}A)$, for any direct summand M of a silting object T, add$M$ is always functorially finite (shown in [AI]). This allows the notion of mutation for silting objects.

Definition 5.5.3 ([AI]). *Let $T = X_1 \oplus \cdots \oplus X_r$ be a basic silting object (so each of X_i's is indecomposable and they are pairwise non-isomorphic) and write $T = X \oplus M$. A* left silting mutation *of T with respect to X, denoted by $\mu_X^{-}(T) = Y_1 \oplus \cdots \oplus Y_r$ satisfies by definition that the indecomposable summands Y_i are given as follows:*

1. *$Y_i = X_i$ if X_i is not a direct summand of X;*

2. *Otherwise, Y_i is the unique object appearing in the distinguished triangle:*

$$X_i \to X' \to Y_i \to X'[1]$$

where the first map is a minimal left add$M$*-approximation of X_i.*

The right silting mutation $\mu_X^+(T)$ *is defined similarly. A silting mutation with respect to X is called* irreducible *if X is indecomposable.*

This is a generalisation of the techniques invented independently by Okuyama and Rickard, where the case they considered is $\mathcal{T} = K^b(\text{proj-}A)$ with A a symmetric algebra.

Similar to mutation of sms, left and right silting mutations are inverse operations in the following sense. Let $T = X \oplus M$ be a silting complex. Writing $\mu_X^+(T)$ as $Y \oplus M$, and $\mu_X^-(T)$ as $Z \oplus M$, we have $\mu_Y^- \mu_X^+(T) = T = \mu_Z^+ \mu_X^-(T)$.

At this point, one may expect to see the definition of mutation on smc's here. We will omit this as the following results of Koenig and Yang says that one can transfer from smc's to silting complexes.

Theorem 5.5.4 ([KY])**.** *Let A be a finite-dimensional algebra.*

1. Koenig-Yang bijection*: The class of silting complexes of A (up to homotopy equivalence) is in bijection with the class of smc's in $D^b(\text{mod-}A)$ (up to equivalence). Moreover, this bijection respects mutations on respective classes.*

2. *A silting complex T is tilting if and only if it is Nakayama-stable, or equivalently, its corresponding smc $\mathbf{S}$ is Nakayama-stable. Moreover, in this case, $\mathbf{S}$ is given by the preimage of the set of simple modules (regarded as complexes concentrated in degree 0) under the standard derived equivalence induced by T.*

In Section 8.1, we generalise a result of Aihara [Aih1] on tilting-connectedness (Theorem 8.1.3) by considering only mutations with respect to a Nakayama-stable subset. This modification ensures a mutation of tilting complex to also be a tilting complex. Moreover, we can establish analogous notions to those in [Aih1], such as irreducible tilting mutation (Definition 8.1.2) in place of irreducible silting mutation, and obtain a link between mutations of sms's and tilting complexes for RFS algebras by piecing together various results from this thesis.

Proposition 5.5.5 (Proposition 8.1.5)**.** *If A is an RFS algebra, then we have a surjection $f : \mathsf{tilt}(A) \to \mathsf{sms}(A)$ given by $f(T) = \overline{F_T}^{-1}(\mathcal{S}_{E_T})$, where $F_T : D^b(\text{mod-}A) \to D^b(\text{mod-}E_T)$ is the standard derived equivalence induced by the tilting complex T, $E_T = \operatorname{End}(T)$ is the endomorphism ring, and $\overline{F_T} : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}E_T$ is the functor induced by F_T. Moreover,* *if $\mu_X^-(T)$ is an irreducible tilting mutation, then there is a Nakayama-stable subset $\mathcal{X}$ of $f(T)$ such that $f(\mu_X^-(T)) = \mu_{\mathcal{X}}^-(f(T))$.*

Finally, in Section 8.2 to Section 8.2.3, we consider the subset $\mathsf{2tilt}(A)$ of $\mathsf{tilt}(A)$, consisting the *two-term tilting complexes*, i.e. tilting complexes concentrated in homological degree 0 and -1 (up to homotopy equivalence). We ask if any given sms of $\underline{\text{mod}}$-A can be obtained by $\overline{F_T}^{-1}(\mathcal{S}_{E_T})$ for some $T \in \mathsf{2tilt}(A)$. We will show that this is possible for A a self-injective Nakayama algebra (Theorem 8.2.1). Along the way, for a self-injective Nakayama algebra A with n simples, we show connections between various algebraic and combinatorial objects listed in the following, using their mutation theories.

(1) two-term tilting complexes of A,

(2) (rotationally symmetric) triangulations on a punctured convex regular n-gon,

(3) Brauer trees with n edges,

(4) $\tau^{n\mathbb{Z}}$-stable configurations of type $\mathbb{A}$,

(5) simple-minded systems of $\underline{\text{mod}}$-A.

The relation between (4) and (5) is already addressed above (Theorem 5.3.4). The connection between (1), (2), and (3) is shown in Section 8.2.2; the connection of them to (4) and (5) will be presented in Section 8.2.3.

Let A_n^n (resp. A_n^{nk}) be the symmetric Nakayama algebra with n simples and Loewy length $n+1$ (resp. $nk+1$ for any $k > 1$). The relations between all the sets of objects related to our studies are shown in the diagram below.

$$
\begin{tikzcd}[column sep=small]
 & \mathsf{tilt}(A_n^n) \arrow[r, leftrightarrow, "\text{[KY]}"] & \mathsf{smc}(A_n^n) \arrow[r, two heads] \arrow[d, "8.1.5"] & \mathsf{smc}(A_n^n)/\mathsf{DPic}(A_n^n) \arrow[d, leftrightarrow, "6.3.4"] & \\
\mathsf{s}\tau\text{-}\mathsf{tilt}(A_n^n) \arrow[r, leftrightarrow, "\text{[AIR1]}"] \arrow[ddd, leftrightarrow, "\text{[Ada]}"] & \mathsf{2tilt}(A_n^n) \arrow[u, hook] \arrow[r, two heads, "8.2.1"] \arrow[rrrdd, two heads, "\text{[SZI]}"] & \mathsf{sms}(A_n^n) \arrow[r, two heads] \arrow[d, leftrightarrow, "6.1.1"] & \mathsf{sms}(A_n^n)/\mathsf{StPic}(A_n^n) \arrow[d, leftrightarrow, "6.3.1"] & \\
 & & \mathsf{Conf}(\mathbb{Z}A_n) \arrow[r, two heads] & \mathsf{Conf}(\mathbb{Z}A_n)/\mathsf{Aut}(\mathbb{Z}A_n) \arrow[r, leftrightarrow, "\text{[Rie2]}"] & \mathsf{StAlg}(A_n^n) \arrow[d, leftrightarrow, "\text{[GR]}"] \\
\mathcal{T}(n) \times \{\pm 1\} \arrow[rrrr, two heads, "8.2.13"] \arrow[rrrd, two heads] \arrow[dddd, leftrightarrow, "\text{[Ada]}"] & & & & \mathsf{BTree}(n,1) \\
 & & & \mathcal{T}(n)/C_n \arrow[r, leftrightarrow, "8.2.25"] & \mathsf{BTree}(n,k) \arrow[d, leftrightarrow, "\text{[GR]}"] \\
 & & \mathsf{Conf}(\mathbb{Z}A_{nk}/\langle \tau^n \rangle) \arrow[r, two heads] & \frac{\mathsf{Conf}(\mathbb{Z}A_{nk}/\langle \tau^n \rangle)}{\mathsf{Aut}(\mathbb{Z}A_n)} \arrow[r, leftrightarrow, "\text{[Rie2]}"] & \mathsf{StAlg}(A_n^{nk}) \\
 & & & & \\
\mathsf{s}\tau\text{-}\mathsf{tilt}(A_n^{nk}) \arrow[r, leftrightarrow, "\text{[AIR1]}"] & \mathsf{2tilt}(A_n^{nk}) \arrow[r, leftrightarrow, "8.2.1"] \arrow[d, hook] \arrow[rrruuu, two heads, "\text{[SZI]}"] & \mathsf{sms}(A_n^{nk}) \arrow[r, two heads] \arrow[uu, leftrightarrow, "6.1.1"] & \mathsf{sms}(A_n^{nk})/\mathsf{StPic}(A_n^{nk}) \arrow[uu, leftrightarrow, "6.3.1"] & \\
 & \mathsf{tilt}(A_n^{nk}) \arrow[r, leftrightarrow, "\text{[KY]}"] & \mathsf{smc}(A_n^{nk}) \arrow[r, two heads] \arrow[u, two heads, "8.1.5"] & \mathsf{smc}(A_n^{nk})/\mathsf{DPic}(A_n^{nk}) \arrow[u, leftrightarrow, "6.3.4"] &
\end{tikzcd}
$$

Note that the third, fourth, and fifth columns in the lower part of the diagram surject to the corresponding sets in the upper part. This picture can be further connected to other areas of representation theory. For example, in [Ada], the author also showed a bijection between the set of n-part compositions of n with $\mathcal{T}(n)$. This set is of particular interest for Schur algebras. In [AIR1], the set $\mathsf{s}\tau\text{-}\mathsf{tilt}(\Lambda)$ of support τ-tilting modules is shown to correspond to the set of functorially finite torsion classes of mod-Λ. In [BLR], a corollary of the main theorem is that configurations of $\mathbb{Z}A_n$ correspond to (basic) tilting modules of the quiver algebra kA_n. In [Rea], it is also shown that configurations of $\mathbb{Z}A_n$ can be interpreted as *non-crossing partitions* of a convex regular $(n+2)$-gon. It is well-known that non-crossing partitions appear in many other contexts in representation theory. It will be interesting to find implications of the relations established in this thesis on these other areas.

Chapter 6

On simple-minded systems of representation-finite self-injective algebras

6.1 Sms's and configurations

Following Asashiba [Asa1], we abbreviate (indecomposable, basic) representation-finite self-injective algebra (not isomorphic to the underlying field k) by RFS algebra.

We call an RFS algebra A with stable AR-quiver ${}_s\Gamma_A$ *standard* if $\underline{\mathrm{ind}}$-A is equivalent to $k({}_s\Gamma_A)$. Equivalently, the stable Auslander algebra of A is given by the mesh algebra $k({}_s\Gamma_A)$. An RFS algebra which is not standard is called *non-standard*.

Theorem 6.1.1. *Let A be an RFS algebra over an algebraically closed field with stable AR-quiver ${}_s\Gamma_A$. Then there is a bijection:*

$$\{\textit{(Combinatorial) configurations of } {}_s\Gamma_A\} \quad \leftrightarrow \quad \{\textit{sms's of } \underline{\mathrm{mod}}\text{-}A\}$$

Proof. It is shown in [Rie2, Rie3, BLR] that every combinatorial configuration is also a configuration. In particular, the combinatorial configurations of ${}_s\Gamma_A = \mathbb{Z}Q/\Pi$ can be identified with a Π-stable configuration of $\mathbb{Z}Q$.

By the definition of standard RFS algebras, it follows that $\mathcal{S}$ is a wsms of $\underline{\mathrm{mod}}$-A if and only if the positions of elements of $\mathcal{S}$ on ${}_s\Gamma_A$ form a combinatorial configuration of ${}_s\Gamma_A$. Also recall from [KL, Theorem 5.6] that for RFS algebras, wsms's are sms's. The statement for standard RFS algebras now follows.

It is shown in [Rie4] that the non-standard RFS algebras only occur when $Q = D_{3m}$ and $\Pi = \langle \tau^{2m-1} \rangle$. This class of RFS algebras is also studied by Waschbüsch in [Was]. For such an RFS algebra A, $k({}_s\Gamma_A)$ is no longer isomorphic to $\underline{\text{ind}}$-A, so it is unclear if a configuration on ${}_s\Gamma_A$ implies the corresponding set of indecomposable A-modules form a wsms. Note that $\underline{\text{ind}}$-A is equivalent to $k_s\Gamma_A/J$ where $k_s\Gamma_A$ is the path category and J is some ideal defined by modified mesh relations [Rie4]. The ideal J is dependent on the position of simple A-modules in ${}_s\Gamma_A$. Fortunately, covering theory can be used to show this implication. For completeness, we go through the key arguments needed. We note that the following argument works regardless of standardness of A.

Definition 6.1.2 ([Rie1, Rie2])**.** *Let $\pi : \Delta \to \Gamma$ be a covering where Γ is the AR-quiver or the stable AR-quiver of A. A k-linear functor $F : k(\Delta) \to$ ind-A (or $\underline{\text{ind}}$-A) is said to be* well-behaved *if and only if*

1. *For any $e \in \Delta_0$ with $\pi e = e_i$, we have $Fe = M_i$ where M_i is the indecomposable A-module corresponding to e_i;*
2. *For any $e \xrightarrow{\alpha} f$ in Δ_1, $F\alpha$ is an irreducible map.*

By [BG, Example 3.1b], for any RFS algebra A (whenever A is standard or non-standard), there is a well-behaved functor $F : k(\widetilde{\Gamma}_A) \to$ ind-A such that F coincides with π on objects, where $\pi : \widetilde{\Gamma}_A \to \Gamma_A$ is the universal covering of the AR-quiver Γ_A of A. Since an irreducible morphism between non-projective indecomposables remains irreducible under the projection ind-$A \to \underline{\text{ind}}$-$A$, the well-behaved functor $F : k(\widetilde{\Gamma}_A) \to$ ind-A restricts to a well-behaved functor $\overline{F} : k({}_s\widetilde{\Gamma}_A) \to \underline{\text{ind}}$-$A$, where ${}_s\widetilde{\Gamma}_A$ is the stable part of the translation quiver $\widetilde{\Gamma}_A$. Note that the restriction $\pi : {}_s\widetilde{\Gamma}_A \to {}_s\Gamma_A$ is also a covering of the stable AR-quiver ${}_s\Gamma_A$. It follows that there are bijections:

$$\begin{aligned} \bigoplus_{Fh=Ff} \mathrm{Hom}_{k({}_s\widetilde{\Gamma}_A)}(e,h) &\cong \underline{\mathrm{Hom}}_A(Fe,Ff) \\ \bigoplus_{\pi h=\pi f} \mathrm{Hom}_{k({}_s\widetilde{\Gamma}_A)}(e,h) &\cong \mathrm{Hom}_{k({}_s\Gamma_A)}(\pi e,\pi f). \end{aligned}$$

In particular, a configuration of ${}_s\Gamma_A$ (whenever A is standard or non-standard) gives a wsms in $\underline{\text{mod}}$-A, which is what we claimed. □

From now on, we will just say configurations instead of combinatorial configurations. And we denote by $\mathsf{Conf}({}_s\Gamma_A)$ for the set of configurations of ${}_s\Gamma_A$, i.e. Π-stable configurations of $\mathbb{Z}Q$.

One could follows arguments used in [GR, Rie2, Rie4, BLR] to show that every sms of $\underline{\text{mod}}$-A is a simple-image. We will omit the details of this and instead prove the following stronger statement.

Theorem 6.1.3. *Let A be an RFS k-algebra over k algebraically closed. Then every sms $\mathcal{S}$ of A is simple-image under a liftable stable equivalence of Morita type.*

6.1.1 Proof of Theorem 6.1.3

We now recall Asashiba's famous theorem on classification of derived (and stable) equivalent RFS algebras. Before stating his result, we need to define the type of A. If A has stable AR-quiver ${}_s\Gamma_A = \mathbb{Z}Q/\Pi$, one of the theorems in [Rie1] showed Π has the form $\langle \zeta\tau^{-r}\rangle$ where ζ is some automorphism of Q and τ is the translation. We also recall the *Coxeter numbers* of $Q = A_n, D_n, E_6, E_7, E_8$ are $h_Q = n+1, 2n-2, 12, 18, 30$ respectively. The *frequency* of A is defined to be $f_A = r/(h_Q - 1)$ and the *torsion order* t_A of A is defined as the order of ζ. The (RFS) *type* of A is defined as the triple (Q, f_A, t_A).

Theorem 6.1.4 ([Asa1]). *Let A and B be RFS k-algebras for k algebraically closed.*

1. *If A is standard and B is non-standard, then A and B are not stably equivalent, and hence not derived equivalent.*
2. *If both A and B are standard, or both non-standard, the following are equivalent:*
   (a) *A, B are derived equivalent;*
   (b) *A, B are stably equivalent of Morita type;*
   (c) *A, B are stably equivalent;*
   (d) *A, B have the same stable AR-quiver;*
   (e) *A, B have the same type.*
3. *The types of standard RFS algebras are the following:*
   (a) $\{(A_n, s/n, 1) | n, s \in \mathbb{N}\}$,
   (b) $\{(A_{2p+1}, s, 2) | p, s \in \mathbb{N}\}$,
   (c) $\{(D_n, s, 1) | n, s \in \mathbb{N}, n \geq 4\}$,
   (d) $\{(D_{3m}, s/3, 1) | m, s \in \mathbb{N}, m \geq 2, 3 \nmid s\}$,
   (e) $\{(D_n, s, 2) | n, s \in \mathbb{N}, n \geq 4\}$,

(f) $\{(D_4, s, 3)|s \in \mathbb{N}\}$,

(g) $\{(E_n, s, 1)|n = 6, 7, 8; s \in \mathbb{N}\}$,

(h) $\{(E_6, s, 2)|s \in \mathbb{N}\}$.

Non-standard RFS algebras are of type $(D_{3m}, 1/3, 1)$ *for some* $m \geq 2$.

Remark 6.1. (1) The RFS types which correspond to symmetric algebras are $\{(A_n, s/n, 1)|s \in \mathbb{N}, s \mid n\}$, $\{(D_{3m}, 1/3, 1)|m \geq 2\}$, $\{(D_n, 1, 1)|n \in \mathbb{N}, n \geq 4\}$ and $\{(E_n, 1, 1)|n = 6, 7, 8\}$.

(2) It follows immediately from this theorem that the Auslander-Reiten conjecture is true for the class of RFS algebras, i.e. any algebra stably equivalent to A has the same number of (isoclasses of) non-projective simple modules as A. It then follows by works of Martinez-Villa [MV] that the conjecture is valid for all representation-finite algebras. In fact, validity of the Auslander-Reiten conjecture for RFS algebras can be shown using [BLR, Corollary 2.3], and the fact that the number of isoclasses of simple A-modules is equal to $|Q_0|f$ for an RFS algebra A of type (Q, f, t).

As we can see, stable (derived) equivalence classes are determined by the stable AR-quiver and standardness. To distinguish between stably equivalent algebras within a equivalence class, we can use the following observation. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of $\Gamma = {}_s\Gamma_A = \mathbb{Z}Q/\Pi$. We say that they are isomorphic if there is an automorphism f of $\mathbb{Z}Q/\Pi$ such that $f\mathcal{C} = \mathcal{C}'$. The set of isomorphism classes of configurations on Γ is denoted by $\mathsf{Conf}(\Gamma)/\mathsf{Aut}(\Gamma)$. A standard (resp. non-standard) RFS algebra B with stable AR-quiver ${}_s\Gamma_B = \mathbb{Z}Q/\Pi$ is constructed by taking the endomorphism ring of the set of projective vertices associated to $\mathcal{C}$ (see 5.3.1) in the mesh category $k(\Gamma_\mathcal{C})$ (resp. $k\Gamma_\mathcal{C}/J_\mathcal{C}$). In particular, there is a bijection

$$\mathsf{Conf}(\Gamma)/\mathsf{Aut}(\Gamma) \leftrightarrow \mathsf{StAlg}(A) := \{\text{isoclasses of RFS algebras stably equivalent to } A\}. \quad (6.1.1)$$

Note that under this construction of $B \in \mathsf{StAlg}(A)$, the configuration $\mathcal{C}$ can be identified with $\{\mathrm{rad}(P)|P$ an (isoclass representative of) indecomposable projective B-module$\}$ of $\underline{\mathrm{ind}}$-B. In particular, applying Ω^{-1} on the set we can obtain $\mathcal{S}_B$, which can be identified with a configuration isomorphic to $\mathcal{C}$, as Ω^{-1} is a stable self-equivalence (of Morita type) inducing a quiver automorphism on ${}_s\Gamma_B \cong \mathbb{Z}Q/\Pi$.

Suppose further that B be an RFS algebra stably equivalent to A but not isomorphic to A. From Asashiba's theorem, we obtain a derived equivalence $\phi : D^b(\text{mod-}B) \to D^b(\text{mod-}A)$, and let $\overline{\phi}$ be the induced functor on the stable categories. So $\overline{\phi}$ is a stable equivalence of Morita type sending $\mathcal{S}_B$ to a sms $\mathcal{S} \in \mathsf{sms}(A)$ with $\mathcal{S}$ corresponding to a configuration isomorphic to $\mathcal{C}$.

Since composition of (resp. liftable) stable equivalences of Morita type is also a (resp. liftable) stable equivalence, it remains to show that for f sending $\mathcal{C}$ to an isomorphic configuration, we can find a stable *self*-equivalence of Morita type which gives the corresponding sms.

The standard case.

In short, this follows directly from lifting theorem 5.4.2 when A is a standard RFS algebra. For the convenience of the reader, we give a brief review of the main steps.

First recall that for the Dynkin graph $\mathbb{D}_n$ with $n > 4$, there is an order 2 graph automorphism fixing all but two vertices. Following [Rie3], we call the two vertices as *high vertices*. For clarity, the high vertices are those labelled 0 and 1 in the diagram 7.3.2.

Recall the following structure theorem for the group of stable self-equivalences of standard RFS algebras given by Asashiba [Asa1].

Theorem 6.1.5 ([Asa2])**.** *Let A be a standard RFS algebra, and $\mathsf{StSE}(A)$ be the group of all stable self-equivalences of A up to natural isomorphism. If A is not of type $(D_{3m}, s/3, 1)$ with $m \geq 2, 3 \nmid s$, then*

$$\mathsf{StSE}(A) = \mathsf{Pic}'(A)\langle[\Omega_A]\rangle.$$

If A is of type $(D_{3m}, s/3, 1)$ with $m \geq 2, 3 \nmid s$, then

$$\mathsf{StSE}(A) = (\mathsf{Pic}'(A)\langle[\Omega_A]\rangle) \cup (\mathsf{Pic}'(A)\langle[\Omega_A]\rangle)[H],$$

where H is a stable self-equivalence of A induced from an automorphism of the quiver D_{3m} by swapping the two high vertices; it satisfies $[H]^2 \in \mathsf{Pic}'(A)$.

Remark 6.2. $\mathsf{Pic}'(A)$ is the image of the Picard group of A in $\mathsf{StSE}(A)$.

Clearly elements in $\mathsf{Pic}'(A)$ and the Heller functor Ω_A can be lifted to standard derived equivalences. In [Dug1], Dugas used the mutation theory of tilting complexes to prove that H is also liftable. In the next part (subsection 6.2), we will use another result of Dugas in [Dug2] concerning mutation of sms's to give an alternative proof for lifting H to a standard derived equivalence. We also note that, as every element of $\mathsf{StSE}(A)$ is liftable, they are all of Morita type, and so $\mathsf{StSE}(A) = \mathsf{StPic}(A)$.

On the level of configurations, combining the liftability of $\mathsf{StPic}(A)$ with Asashiba's construction of derived equivalences, we have that every automorphism on ${}_s\Gamma_A$ sending a configuration to another can be realised by a liftable StM. The statement of Theorem 6.1.3 in the standard case now follows.

The non-standard case.

Now we prove Theorem 6.1.3 in the non-standard case. We fix A from now on as the representative of a non-standard RFS algebra with m simples. The quiver and relations are given in the next section. This algebra is denoted as $B(T_m, S, 1)$ in [Was].

We now fix A_s the standard counterpart of A, that is, the standard RFS algebra such that $\mathcal{S}_{A_s}$ and $\mathcal{S}_A$ are the same set when regarded as a $\tau^{(2m-1)\mathbb{Z}}$-stable configuration of $\mathbb{Z}D_{3m}$. This is the RFS algebra denoted as $B(T_m, S, 0)$ in [Was]. First we recall some facts:

1. (standard-non-standard correspondence): There is a bijection ind-$A \leftrightarrow$ ind-A_s between the set of indecomposable objects and irreducible morphisms, which is compatible with the position on the stable AR-quiver $\Gamma = \mathbb{Z}D_{3m}/\langle\tau^{2m-1}\rangle$. In particular, when A is the representative of non-standard RFS algebra, whose quiver is given in Figure 6.1, then Waschbüsch [Was] described the AR-quiver of A using that of A_s by replacing every part of the Loewy diagram:

$$\begin{array}{c} 1_{m-1}\\ v_1\\ v_1\\ 1_1\end{array} \quad \text{to} \quad \begin{tikzcd}[row sep=small] & 1_{m-1} \\ & v_1 \arrow[dd, dash] \\ v_1 \arrow[ur, dash] \arrow[dr, dash] & \\ & 1_1 \end{tikzcd} \tag{6.1.2}$$

2. There is one-to-one correspondence between the following three sets:

$$\mathsf{sms}(A) \leftrightarrow \mathsf{Conf}(\Gamma) \leftrightarrow \mathsf{sms}(A_s)$$

   where the first is the set of sms's of A, the second is the set of configurations of Γ, and the third is sms's of A_s.

Lemma 6.1.6. *Every stable self-equivalence $\phi_s \in \mathsf{StPic}(A_s)$ has a non-standard counterpart $\phi \in \mathsf{StPic}(A)$ such that, if ϕ_s maps the set $\mathcal{S}_{A_s}$ of simple A_s-modules to $\mathcal{S}_s$, then $\phi(\mathcal{S}_A) = \mathcal{S}$ where $\mathcal{S}$ corresponds to $\mathcal{S}_s$ in the above correspondence. Moreover, ϕ is a liftable stable equivalence of Morita type.*

Proof. By Asashiba's description, $\mathsf{StPic}(A_s) = \mathsf{Pic}'(A_s)\langle\Omega\rangle[H]$. If $\phi_s \in \mathsf{Pic}'(A_s)$, then it must permute the $m-1$ simple modules on the mouth of the stable tube and fixes the remaining one in a high vertex. It follows from the description of the stable AR-quiver of A_s that ϕ_s fixes $\mathcal{S}_{A_s}$ and induces the identity map $\mathsf{Conf}(\Gamma) \to \mathsf{Conf}(\Gamma)$. Therefore we can simply pick the (liftable StM) identity functor for ϕ. If $\phi_s = \Omega^n_{A_s}$ for some $n \in \mathbb{Z}$, then by standard-non-standard correspondence, picking ϕ to be the Heller shift Ω^n_A of A will do the trick. This is obviously a liftable StM. For $\phi_s = H$, Proposition 6.2.1 has already done the job. $\square$

This finishes the proof of Theorem 6.1.3.

6.2 Lifting a stable self-equivalence to a standard derived equivalence

In this section, we give a new proof for Dugas' result in [Dug1, Section 5] as promised in the proof of Theorem 6.1.3. Our proof here is carried out in the same spirit as his by using mutation theory, but with the point of view focussing on configurations, so that we only need to observe on the effects of mutations on configurations. Another advantage of our proof is that one only needs to substitute suitable modules to find the same result for non-standard algebra, which is needed in the proof of Lemma 6.1.6. We keep all notations used in the previous sections throughout. Let us state our result explicitly first.

Proposition 6.2.1. *(1) For a standard RFS algebra of type $(D_{3m}, s/3, 1)$ with $m \geq 2$ and $3 \nmid s \in \mathbb{N}$, the stable self-equivalence H in Theorem 6.1.5 can be lifted to a derived equivalence.*

(2) For a non-standard RFS algebra A of type $(D_{3m}, 1/3, 1)$ with $m \geq 2$, there is a (standard) derived self-equivalence of A which induces H on the $\underline{\text{mod}}$-A, with the same effect as the functor H in (1) on the stable AR-quiver of A.

By [Asa2, Prop 3.3], if F, F' are two stable self-equivalences on a standard RFS algebra A with $F(M) \cong F(M')$ for each indecomposable module M, then $F = GF'$ for some Morita equivalence G. Therefore, stable self-equivalences are uniquely determined by their effect on the stable AR-quiver ${}_s\Gamma_A$ up to Morita equivalences, and we can fix a representative algebra A for each derived/stable equivalence class. Moreover, if A is standard and we can find a liftable stable self-equivalence H' which has the same effect as H on the vertices of ${}_s\Gamma_A$, then H is liftable, as a composition of a liftable equivalence with non-liftable equivalence is non-liftable.

For non-standard RFS algebras, we can also fix a representative for derived/stable equivalence class, as we can conjugate the lift of H by the derived equivalence given in Asashiba's construction to obtain the corresponding liftable functor in another derived equivalent algebra.

We assume from now on that A is the algebra given in [Was] and in [Asa2, Appendix 2]. The quivers are given in Figures 6.1 and 6.2 (depending on s). The following shows the relations of the standard algebras:

(i) $\alpha_m^{(i)} \cdots \alpha_2^{(i)} \alpha_1^{(i)} = \beta_{i+1}\beta_i$ for each $i = 1, \ldots, s$;

(ii) $\alpha_1^{(i+2)}\alpha_m^{(i)} = 0$ for each $i = 1, \dots, s$;

(iii) $\alpha_j^{(i+3)} \cdots \alpha_1^{(i+3)}\beta_{i+2}\alpha_m^{(i)} \cdots \alpha_j^{(i)} = 0$ for each $i = 1, \dots, s$ and for each $j = 1, \dots, m$.

Note that the index i is taken modulo s with representatives in $\{1, \dots, s\}$. When $s = 1$, we ignore the superscripts in the relations. The relations for the non-standard algebras are the same except for (ii), which we replace by $\alpha_1\alpha_m = \alpha_1\beta\alpha_m$. In particular, the indecomposable A-modules can be obtained from a corresponding one in the standard counterpart by (6.1.2).

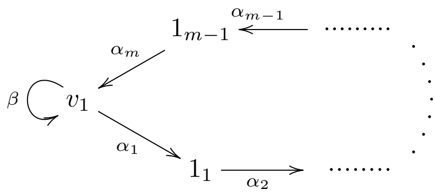


Figure 6.1: Quiver $Q(D_{3m}, 1/3)$ of an RFS algebra of type $(D_{3m}, 1/3, 1)$.

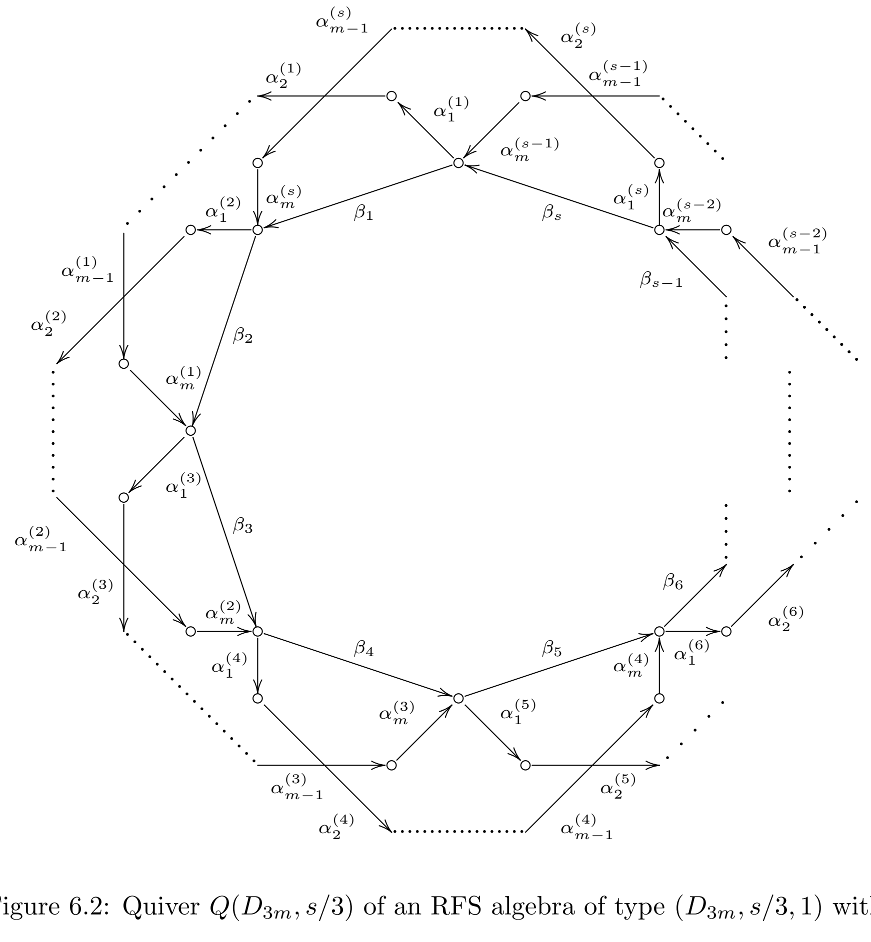


Figure 6.2: Quiver $Q(D_{3m}, s/3)$ of an RFS algebra of type $(D_{3m}, s/3, 1)$ with $3 \nmid s$.

When $s = 1$, the stable AR-quiver ${}_s\Gamma_A = \mathbb{Z}D_{3m}/\langle\tau^{(2m-1)}\rangle$ is given by connecting $(2m-1)$ copies of D_{3m}. The position of the indecomposable A-modules on ${}_s\Gamma_A$ can be found in Waschbüsch [Was]. The $m - 1$ simple modules lie on the mouth (boundary) of the stable tube; and the remaining one lies in a high vertex. When $s > 1$ and s not multiple of 3, the stable AR-quiver is ${}_s\Gamma_A = \mathbb{Z}D_{3m}/\langle\tau^{(2m-1)s}\rangle$, and its configurations are $\tau^{(2m-1)\mathbb{Z}}$-stable. In particular, there is a bijection between sms's of an RFS algebra of type $(D_{3m}, s/3, 1)$ and sms's of an RFS algebra of type $(D_{3m}, 1/3, 1)$, given by $(x, y) \mapsto (x \bmod 2m-1, y)$ for the vertices in the corresponding configurations. Explicit calculations demonstrate the following observation on the indecomposable A-modules.

1. The vertices in the inner cycle $\beta_s \cdots \beta_1$ correspond to simple modules in the high vertices of the stable AR-quiver, namely $\{((2m-1)i, n) | i = 0, \ldots, s-1\}$. We label these vertices by $v_1, \cdots, v_s$, which can be thought of as ramification of the vertex v_1 in the $s = 1$ case, see Figure 6.1.

2. Let $i \in \{1, \cdots, s\}$, and consider vertices on the path $\alpha^{(i)}_{m-1} \cdots \alpha^{(i)}_2$. There are $m-1$ such vertices for each i, and we label these by $i_1, \cdots, i_{m-1}$. When A is standard, the corresponding indecomposable projective modules are uniserial of length $m+2$, with composition factors
$$i_j, \ldots, i_{m-1}, v_{i+2}, v_{i+3}, (i+3)_1, \ldots, (i+3)_j.$$
When A is non-standard (hence $s = 1$), these projective modules are not uniserial, one can replace the Loewy diagram as described in (6.1.2). Regardless of standardness, the corresponding $m-1$ simple modules lie on the mouth of i-th copy of stable AR-quiver, they have coordinates $((2m-1)i - j, 1)$ with $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, m-1\}$.

3. The Nakayama functor permutes the simple A-modules by $v_i \mapsto v_{i+3}$ and $i_j \mapsto (i+3)_j$, for all $i \in \{1, \cdots, s\}$ and all $j \in \{1, \cdots, m-1\}$.

We now mutate the sms $\mathcal{S}$ of simple A-modules at the Nakayama-stable subset $\mathcal{X} = \{1_1, \cdots, s_1\}$, which yields the set consisting of the following indecomposable modules:

(1) uniserial module with Loewy diagram $\substack{v_i \\ i_1}$ for each $i = 1, \ldots, s$;

(2) simple module corresponding to i_j for each $i = 1, \ldots, s$ and $j = 2, \ldots, m-1$;

(3) $P_{i_1}/\operatorname{soc} P_{i_1}$ where P_{i_1} is the projective indecomposable module with top i_1 for each $i = 1, \ldots, s$.

On the level of configurations, the mutated set $\mu^-_{\mathcal{X}}(\mathcal{S}_A)$ is $\{((2m-1)(i-1)-1, n-1), ((2m-1)i - m, 1), ((2m-1)i - j, 1) | i = 1, \ldots, s; j = 2, \ldots, m-1\}$, which is the same as applying $\tau \circ H$ on the configuration corresponding to $\mathcal{S}_A$. According to Dugas [Dug2, Section 5], $\mu^-_{\mathcal{X}}(\mathcal{S}_A)$ can be realised by a (standard) derived equivalence $\phi : D^b(\text{mod-}B) \to D^b(\text{mod-}A)$ for some standard RFS algebra B of the same type. Then ϕ induces a StM $\overline{\phi} : \underline{\text{mod}}\text{-}B \to \underline{\text{mod}}\text{-}A$ such that $\overline{\phi}$ sends simple B-modules to $\mu_{\mathcal{X}}^{-}(\mathcal{S}_A)$, which coincides with the image of $\tau \circ H$ on $\mathcal{S}_A$.

Clearly, the configuration corresponding to $\mu_{\mathcal{X}}^{-}(\mathcal{S}_A)$ is isomorphic to that of $\mathcal{S}_A$ (via $\tau \circ H$), so by (6.1.1), B is isomorphic to A. It follows that H is a stable self-equivalence which can be lifted to a derived equivalence. This completes the proof.

6.3 Some consequences and connection with Nakayama-stable smc's

The proof of Theorem 6.1.3 actually gives us the following:

Proposition 6.3.1. *Let A be an RFS algebra. There are bijections between the following sets:*

$$\mathsf{sms}(A)/\mathsf{StPic}(A) \longleftrightarrow \mathsf{Conf}({}_s\Gamma_A)/\mathsf{Aut}({}_s\Gamma_A) \longleftrightarrow \mathsf{StAlg}(A)$$

Using these bijections, we can pick out the RFS algebras for which the transitivity problem raised in [KL] has a positive answer. That is, we can decide whether given two sms's of an algebra there always is a stable self-equivalence sending the first sms to the second one.

Proposition 6.3.2. *If A is an RFS algebra of RFS type in the following list, then for any pair of sms's $\mathcal{S}, \mathcal{S}'$ of A, there is a stable self-equivalence $\phi : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A$ such that $\phi(\mathcal{S}) = \mathcal{S}'$. The list consists of $\{(A_2, s/2, 1)|s \geq 1\}$, $\{(A_n, s/n, 1)|n \geq 1, \gcd(s,n) = 1\}$, $\{(A_3, s, 2)|s \geq 1\}$, $\{(D_6, s/3, 1)|s \geq 1, 3 \nmid s\}$, $\{(D_4, s, 3)|s \geq 1\}$.*

Proof. By Proposition 6.3.1, A is an RFS algebra satisfying the condition stated if and only if $\mathsf{Conf}({}_s\Gamma_A)/\mathsf{Aut}({}_s\Gamma_A)$ is of size 1. So we can check this case-by-case.

For E_n cases, one can count explicitly from the list of configurations in [BLR] that the number of $\mathsf{Aut}({}_s\Gamma_A)$-orbits is always greater than 1.

Now consider class $(A_n, s/n, 1)$, ${}_s\Gamma_A = \mathbb{Z}A_n/\langle\tau^s\rangle$. Note that configurations of $\mathbb{Z}A_n$ are $\tau^{n\mathbb{Z}}$-stable, so any configuration of $(A_n, s/n, 1)$ is $\tau^{d\mathbb{Z}}$-stable with $d = \gcd(s, n)$. Let $s = ld$ and $n = md$. The above implies configurations of $(A_n, l/m, 1)$ are the same as configurations of $(A_n, 1/m, 1)$. But the number of the configurations of $(A_n, 1/m, 1)$ is equal to the number of Brauer trees with d edges and multiplicity m, which is equal to 1 if and only if the pair $(d, m) = (2, 1)$ or $d = 1$. Therefore, $(d, m) = (2, 1)$ gives $\{(A_2, 1, 1)\}$, and $d = 1$ yields the family $\{(A_m, 1/m, 1)\}$.

Let $n = 2p+1$. For the class $(A_n, s, 2)$, ${}_s\Gamma_A = \mathbb{Z}A_n/\langle\zeta\tau^{sn}\rangle$. A configuration of $(A_n, s, 2)$ is $\tau^{n\mathbb{Z}}$-stable as it is also a configuration of $\mathbb{Z}A_n$. So we only need to consider the case $s = 1$. Recall from [Rie4, Lemma 2.5] that there is a map which takes configurations of $\mathbb{Z}A_n$ to configurations of $\mathbb{Z}A_{n+1}$, so the numbers of orbits of $(A_n, 1, 2)$-configurations form an increasing sequence. Therefore, we can just count the orbits explicitly. $(A_3, 1, 2)$ has one orbit of configurations given by the representative $\{(0, 1), (1, 2), (2, 3)\}$, whereas $(A_5, 1, 2)$ has two orbits. This completes the A_n cases.

Note that a configuration of $\mathbb{Z}D_n$ is $\tau^{(2n-3)\mathbb{Z}}$-stable, so similar to the A_n case we can reduce to the cases $(D_n, 1, 1), (D_n, 1, 2), (D_4, 1, 3)$, and $(D_{3m}, 1/3, 1)$. We make full use of the main theorem in [Rie3] combining with our result in the A_n cases. Part (a) of the theorem implies that $(D_n, 1, 1)$ and $(D_n, 1, 2)$ with $n \geq 5$ all have more than one orbit. Part (c) of the theorem implies that $(D_4, 1, 1)$ and $(D_4, 1, 2)$ has two orbits, with representatives $\{(0, 1), (1, 1), (3, 3), (3, 4)\}$ and $\{(0, 2), (3, 3), (3, 4), (4, 1)\}$. Only the latter one is stable under the order 3 automorphism of $\mathbb{Z}D_4$. This implies that $\{(D_4, s, 3)|s \geq 1\}$ is on our required list. Finally, for $(D_{3m}, 1/3, 1)$ case, we use the description of this class of algebras from [Was], which says that such class of algebra can be constructed via Brauer tree with m edges and multiplicity 1 with a chosen extremal vertex. Therefore, the only m with a single isomorphism class of stably equivalent algebra is when $m = 2$, hence giving us $\{(D_6, s/3, 1)|s \geq 1, 3 \nmid s\}$. □

The following are two easy consequences of Theorem 6.1.3. The first one is Nakayama-stability of sms's of RFS algebras, which is not apparent from the definition of sms's. The second one is the answer to a question posed in [KL, Section 6]: Is the cardinality of each sms over an artin algebra A equal to the number of non-isomorphic non-projective simple A-modules? A positive answer to this question implies the validity of the Auslander-Reiten conjecture. On the other hand, if every sms is a simple-image, then the validity of Auslander-Reiten conjecture implies a positive answer to the question.

Proposition 6.3.3. *(1) Any sms of an RFS algebra is Nakayama-stable.*

(2) Every sms of an RFS algebra A has the same cardinality.

Proof. (1) Since the set of isoclass representatives of simple modules is Nakayama-stable, the statement follows from the fact that Nakayama functor commutes with derived equivalences and Theorem 6.1.3.

(2) Straightforward from Theorem 6.1.3. □

Finally, we establish a connection between Nakayama-stable simple-minded collections (smc's) and sms's for RFS algebra A, that is, every sms $\mathcal{S}$ of A can be "*lifted*" to a Nakayama-stable smc $\mathbf{S}$ of $D^b(\text{mod-}A)$, i.e. $\eta_A(\mathbf{S}) = \mathcal{S}$.

Theorem 6.3.4. *Let A be an RFS algebra over k algebraically closed. Then every sms $\mathcal{S}$ of A lifts to a Nakayama-stable smc of $D^b(\text{mod-}A)$. In particular, there is a bijection*

$$\mathsf{tilt}(A)/\mathsf{DPic}(A) \leftrightarrow \nu\text{-}\mathsf{smc}(A)/\mathsf{DPic}(A) \leftrightarrow \mathsf{sms}(A)/\mathsf{StPic}(A)$$

between the set of $\mathsf{DPic}(A)$-orbits of tilting complexes of A, the set of $\mathsf{DPic}(A)$-orbits of Nakayama-stable smc's of A, and the set of $\mathsf{StPic}(A)$-orbits of sms's of A.

Proof. The first bijection is immediate from Theorem 5.5.4(2). Consider the map $\mathbf{S} \mapsto \eta_A(\mathbf{S})$ defined on $\nu\text{-}\mathsf{smc}(A)$. This is well-defined as every Nakayama-stable smc is the image of simple modules of some derived equivalent (self-injective) algebra, by Theorem 5.5.4(2). So such a smc projects to a sms which is image of simple modules under some liftable stable equivalence of Morita type.

This induces a map on $\nu\text{-}\mathsf{smc}(A)/\mathsf{DPic}(A) \to \mathsf{sms}(A)/\mathsf{StPic}(A)$. The well-definedness of this map follows from the fact that every standard derived (self-)equivalence restricts to a stable (self-)equivalence of Morita type. Injectivity follows from Linckelmann's theorem. Surjectivity follows from Theorem 6.1.3. $\square$

Chapter 7

More examples of simple-minded systems

In this chapter, we calculate some examples of simple-minded systems not written down in the literature. We first relate the simple-minded systems of a derived category of a hereditary abelian category to its triangulated orbit category.

7.1 Sms's of triangulated orbit categories

We work in the setup given in [Kel]. Let $\mathcal{T}$ be $D^b(\mathcal{H})$ where $\mathcal{H}$ is a hereditary abelian k-category for k an algebraically closed field, and $F : \mathcal{T} \to \mathcal{T}$ be a standard derived auto-equivalence satisfying conditions of [Kel, Thm 1]. Then we obtain a triangulated orbit category $\mathcal{T}/F$ along with the canonical triangulated projection $\pi : \mathcal{T} \to \mathcal{T}/F$. Here $\mathcal{T}/F$ is the *orbit category* of $\mathcal{T}$, which has the same class of objects as $\mathcal{T}$, and morphisms between $X, Y \in \mathcal{T}/F$ are defined by

$$\mathrm{Hom}_{\mathcal{T}/F}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, F^pY), \tag{7.1.1}$$

where composition $g \circ f$ of morphisms f, g in $\mathcal{T}/F$ is given by the composition $(\sum_{p\in\mathbb{Z}} F^p g) \circ f$ in $\mathcal{T}$. We say that a class of objects $\mathcal{S}$ in $\mathcal{T}$ is *F-stable* if $F^p\mathcal{S} \cong \mathcal{S}$ for any $p \in \mathbb{Z}$.

We denote $F^{\mathbb{Z}}$ as the group with elements F^p for $p \in \mathbb{Z}$. For an F-stable class $\mathcal{S}$ of objects in $\mathcal{T}$, define $\mathcal{S}/F$ as the set of representatives of $F^{\mathbb{Z}}$-orbits of $\mathcal{S}$. For a class $\pi\mathcal{S}$ of objects in $\mathcal{T}/F$, define $F^{\mathbb{Z}}\mathcal{S}$ to be the class of objects consisting of F^pS for all $p \in \mathbb{Z}$ and $S \in \mathcal{S}$.

Lemma 7.1.1. *Let $G : \mathcal{U} \to \mathcal{V}$ be a triangulated functor between triangulated categories $\mathcal{U}$*

and $\mathcal{V}$, and $\mathcal{S}$ be a set of objects in $\mathcal{U}$. Then $G((\mathcal{S})_n) \subset (G\mathcal{S})_n$ for all $n \geq 0$. In particular, $G(\mathcal{F}(\mathcal{S})) \subset \mathcal{F}(G\mathcal{S})$.

Proof. We prove by induction on n. The statement for $n = 0$ and $n = 1$ is trivial. Assume now that $n > 1$, and take $U \in (\mathcal{S})_n$. Then we obtain a triangle $X \to U \to Y \to X[1]$ in $\mathcal{U}$, where $X \in \mathcal{S}$ and $Y \in (\mathcal{S})_{n-1}$. Applying G to this triangle we obtain a triangle $GX \to GU \to GY \to GX[1]$ in $\mathcal{V}$. Since $GY \in G((\mathcal{S})_{n-1})$, it is also in $(G\mathcal{S})_{n-1}$ by induction hypothesis. Also, GX is clearly in $G\mathcal{S}$. It follows that $GU \in (G\mathcal{S})_n$. $\square$

Corollary 7.1.2. *Let $\mathcal{T}$ be a triangulated category with auto-equivalence F satisfying conditions of [Kel, Thm 1].*

(1) Suppose $\mathcal{S}$ is an F-stable sms of $\mathcal{T}$. Then $\pi(\mathcal{S}/F)$ is an sms of $\mathcal{T}/F$.

(2) Suppose $\mathcal{S}$ is a set of objects in $\mathcal{T}$. Then $(F^{\mathbb{Z}}\mathcal{S})_n$ is F-stable for all $n \geq 0$.

Proof. (1) Orthogonality of $\pi(\mathcal{S}/F)$ follows from the construction of the morphism space in the orbit category (7.1.1). By Lemma 7.1.1, $\pi\mathcal{F}(\mathcal{S}) \subset \mathcal{F}(\pi(\mathcal{S}/F))$. Since the functor π is surjective on the class of objects, so $\pi(\mathcal{F}(\mathcal{S})) = \pi/F$. The claim follows.

(2) By Lemma 7.1.1, applying F to $(F^{\mathbb{Z}})_n$ yields a subclass of $(F(F^{\mathbb{Z}}\mathcal{S}))_n = (F^{\mathbb{Z}}\mathcal{S})_n$. This proves the claim. $\square$

Lemma 7.1.3. *Let $\pi\mathcal{S}$ be an sms in the orbit category $\mathcal{T}/F$ for some class of objects $\mathcal{S}$ in $\mathcal{T}$. Then $F^{\mathbb{Z}}\mathcal{S}$ is a sms in $\mathcal{T}$.*

Proof. Note that for all $S \in \mathcal{S}$, as an object in $\mathcal{T}$, $F^pS \notin \mathcal{S}$ for all $p \in \mathbb{Z}$. Otherwise, $\pi F^pS \cong \pi S$, violating orthogonality of $\pi\mathcal{S}$. Orthogonality of $F^{\mathbb{Z}}\mathcal{S}$ is inherited from orthogonality of $\pi\mathcal{S}$.

Suppose $Y \in \mathcal{T}$ with $\pi Y \in (\pi\mathcal{S})_n$ in $\mathcal{T}/F$. We claim that $Y \in (F^{\mathbb{Z}}\mathcal{S})_n$ by induction on $n \geq 0$. For $n = 0$, this is trivial. For $n > 0$, we obtain a triangle $\pi X \xrightarrow{f} \pi Y \to E \to \pi X[1]$ in $\mathcal{T}/F$, with $\pi X \in \pi\mathcal{S} \cup \{0\}$ and $E \in (\pi\mathcal{S})_{n-1}$. By the induction hypothesis we can assume $\pi X \neq 0$. Since $f \in \mathrm{Hom}_{\mathcal{T}/F}(\pi X, \pi Y) \cong \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, F^pY)$, $f = (f_p)_{p\in\mathbb{Z}}$ with $f_p : X \to F^pY$ zero for all but finitely many $p \in \mathbb{Z}$. Pick a p with $f_p \neq 0$, we obtain the triangle $X \xrightarrow{f_p} F^pY \to \mathrm{cone}(f_p) \to X[1]$ in $\mathcal{T}$. Project this triangle onto the orbit category and we obtain a morphism of triangles:

$$\begin{tikzcd}
\pi X \arrow[r, "\pi(f_p)"] \arrow[d, equal] & \pi F^pY \arrow[r] \arrow[d, "g"', "\cong"] & \pi(\mathrm{cone}(f_p)) \arrow[r] \arrow[d, dashed, "\exists h"] & \pi X[1] \arrow[d, equal] \\
\pi X \arrow[r, "f"] & \pi Y \arrow[r] & E \arrow[r] & \pi X[1]
\end{tikzcd}$$

with $g = \pi(id_Y : Y \to F^{-p}(F^pY))$. So $gf = \pi(f_p)$, and h is an isomorphism, and we get $\pi(\text{cone}(f_p)) \cong E \in (\pi\mathcal{S})_{n-1}$. By the induction hypothesis, $\text{cone}(f_p) \in (F^{\mathbb{Z}}\mathcal{S})_{n-1}$, and so we obtain $F^pY \in (F^{\mathbb{Z}}\mathcal{S})_n$. By the previous lemma **??**, we get $Y \in (F^{\mathbb{Z}}\mathcal{S})_n$. $\square$

Combining the Corollary 7.1.2 and Lemma 7.1.3, we get:

Theorem 7.1.4. *Let $\mathcal{T} = D^b(\mathcal{H})$ for some hereditary abelian k-category $\mathcal{H}$ over an algebraically closed field k, and $F : \mathcal{T} \to \mathcal{T}$ a standard derived equivalence satisfying conditions of [Kel, Thm 1]. Then we have a bijection*

$$\begin{array}{rcl} \mathsf{sms}(\mathcal{T}/F) & \leftrightarrow & \mathsf{sms}_F(\mathcal{T}) := \{\mathcal{S} \in \mathsf{sms}(\mathcal{T}) \,|\, \mathcal{S} \textit{ is } F\textit{-stable}\} \\ \pi(\mathcal{S}/F) & \leftarrow\!\shortmid & \mathcal{S} \\ \mathcal{S}' & \mapsto & F^{\mathbb{Z}}\mathcal{S}' \end{array}$$

In what follows, we look at an application of this theorem.

7.2 Sms's of bounded derived categories of representation-finite hereditary algebras

Let A be a basic indecomposable representation-finite hereditary algebra over an algebraically closed field k. Gabriel's theorem says that A can be given as the path algebra of simply-laced Dynkin quiver. A well-known theorem of Happel [Hap] says that the bounded derived category $D^b(\text{mod-}A)$ is triangulated equivalent to the stable module category $\underline{\text{mod}}\text{-}\hat{A}$ of the repetitive algebra $\hat{A}$. Recall the repetitive algebra $\hat{A}$ is the locally bounded k-algebra with underlying vector space $(\oplus_{i\in\mathbb{Z}}A) \oplus (\oplus_{i\in\mathbb{Z}}DA)$ with elements denoted by $(a_i, \phi_i)_i$ where $a_i \in A$, $\phi_i \in DA$ with all but finitely many a_i, ϕ_i zero. (Note that $DA = \text{Hom}_k(A, k)$.) The multiplication is defined by

$$(a_i, \phi_i)_i \cdot (b_i, \psi_i)_i = (a_ib_i, a_{i+1}\psi_i + \phi_ib_i)_i.$$

Alternatively, one could think of $\hat{A}$ as the doubly infinite matrix algebra

$$\begin{pmatrix} \ddots & & & & 0 \\ \ddots & A_{i-1} & & & \\ & DA_{i-1} & A_i & & \\ & & DA_i & A_{i+1} & \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

An $\hat{A}$-module can be written as $M = (M_i, f_i)_i$ where $M_i \in \text{mod-}A$, all but finitely many zero, and $f_i \in \text{Hom}_A(DA \otimes_A M_i, M_{i+1})$ such that $(1 \otimes f_i)f_{i+1} = 0$ for all $i \in \mathbb{Z}$. The set of simple $\hat{A}$-modules $\mathcal{S}_{\hat{A}}$ is $\{S_{i,j} | i \in \mathbb{Z}, j = 1, \ldots, n\}$ where n is the number of isoclasses of simple A-modules, and $S_{i,j} = (M_a, 0)_a$ with $M_i = S_j$ a simple A-module and $M_a = 0$ for all $a \neq i$. Clearly, this is a sms of $\underline{\text{mod}}\text{-}\hat{A} \simeq D^b(\text{mod-}A)$. Moreover, by [Asa1, Thm 5.2], B is derived equivalent to A if and only if $\hat{B}$ is (derived and) stably equivalent to $\hat{A}$, we can pin down a large class of sms's of $D^b(\text{mod-}A)$. In fact, these are all the sms's.

Theorem 7.2.1. *Let Q be a Dynkin quiver. Then there are the following bijections:*

$$\{\textit{configurations of } \mathbb{Z}Q\} \leftrightarrow \{\textit{wsms's of } D^b(\text{mod-}kQ)\} \leftrightarrow \{\textit{sms's of } D^b(\text{mod-}kQ)\}.$$

Proof. The main theorem of [BLR] implies that $\mathsf{Conf}(Q)$ bijects with isoclasses of (basic) tilting modules over kQ, so taking A as the endomorphism ring of the tilting module corresponding to $\mathcal{C} \in \mathsf{Conf}(Q)$, we obtain a stable equivalence $\phi : \underline{\text{mod}}\text{-}\hat{A} \to \underline{\text{mod}}\text{-}\hat{kQ}$. From [Asa1, Lemma 6.1], $\hat{A}$ is isomorphic to the Galois covering of an RFS algebra corresponding to $\mathcal{C}$ (under (6.1.1)). In particular, vertices in $\mathcal{C}$ correspond to the radicals of projective indecomposable of $\hat{A}$. This shows that a configuration $\mathcal{C}$ induces an sms $\mathcal{S}_{\mathcal{C}} = \phi(\mathcal{S}_{\hat{A}})$ in $\underline{\text{mod}}\text{-}\hat{kQ} \simeq D^b(\text{mod-}kQ)$. Since $k(\mathbb{Z}Q) \simeq \text{ind-}D^b(\text{mod-}kQ)$, the definition of configuration coincides with that of weakly sms of $D^b(\text{mod-}kQ)$. If there is some sms of $D^b(\text{mod-}A)$ which is not induced by a configuration, then it is not a weakly sms, which is a contradiction as all sms's are weakly sms's. The claim follows. $\square$

Remark 7.1. Important examples of $\mathcal{T}$ includes the stable category $\underline{\text{MCM}}^{\mathbb{Z}}(R)$ of graded maximal Cohen-Macaulay modules of Kleinian singularity R, and the stable category of graded matrix factorisations (see next section).

As an application for the classification of sms's for $D^b(\text{mod-}kQ)$ and Theorem 7.1.4, we obtain the following result.

Corollary 7.2.2. *The sms's of a triangulated orbit category $D^b(\text{mod-}kQ)/F$ of $D^b(\text{mod-}kQ)$ for Dynkin quiver Q can be identified with the configurations of $\mathbb{Z}Q$ which are stable under the induced action of F on the AR-quiver $\mathbb{Z}Q$ of $D^b(\text{mod-}kQ)$.*

The special case of above corollary is when $\underline{\text{mod}}\text{-}A$ for A a standard RFS algebra. Suppose A is of tree class Q, by [Ami, Thm 7.0.5], $\underline{\text{mod}}\text{-}A \simeq D^b(\text{mod-}kQ)/F$ for some auto-equivalence F of $D^b(\text{mod-}kQ)$, so $\mathsf{sms}(\underline{\text{mod}}\text{-}A) = \mathsf{sms}_F(D^b(\text{mod-}kQ))$. Let ϕ denote the automorphism induced on the AR-quiver $\Gamma_Q = \mathbb{Z}Q$ of $D^b(\text{mod-}kQ)$, then we have k-linear equivalences $\underline{\text{ind}}\text{-}A \simeq k(\mathbb{Z}Q/\langle\phi\rangle) \simeq \text{ind-}(D^b(\text{mod-}kQ)/F)$ (cf. [Ami, Thm 5.1.1]). So $\langle\phi\rangle$ is isomorphic to the admissible group Π acting on $\mathbb{Z}Q$, where the stable AR-quiver of A is $\mathbb{Z}Q/\Pi$. In particular, sms's of A are given by Π-stable configurations of $\mathbb{Z}Q$. Note that this proof of classification of sms's of A by-passes the use of [KL, Thm 5.6] in the previous chapter.

The relation between RFS algebras and $D^b(\text{mod-}kQ)$ also gives us an alternative method to classify sms's of $D^b(\text{mod-}kQ)$. As mentioned [Dug2, Section 6], for a finite dimensional algebra A with finite global dimension, the sms's of $D^b(\text{mod-}A)$ can be obtained from the set of *graded* simple $T(A)$-modules, where $T(A) = A \ltimes DA$ is the trivial extension algebra of A, using the triangulated equivalence $D^b(\text{mod-}A) \simeq \underline{\text{gr}}\text{-}T(A)$. In the case when A is in derived equivalence to kQ for Q Dynkin, the trivial extension algebras $T(kQ)$ are in fact the RFS algebras of type $(Q, 1, 1)$, presented as the "representative" of its derived equivalence class in [Asa2, Appendix 2]. We note that for any finite dimensional algebra B, $T(B)$ is isomorphic to the orbit algebra $\hat{B}/\langle \nu_{\hat{B}} \rangle$ where $\nu_{\hat{B}}$ is the Nakayama automorphism of $\hat{B}$ (see, for example, [Sko, 2.6]). Therefore, we can in fact use our classification of sms's for RFS algebras to obtain sms's of $D^b(\text{mod-}kQ)$ using Theorem 7.1.4.

Another important special case of Corollary 7.2.2 is the (2-)cluster categories of Dynkin type, which are of the form $D^b(\text{mod-}kQ)/\tau[-1]$.

7.3 Sms's of finite 1-Calabi Yau triangulated categories

The following examples are not written down in the literature, and provide examples of sms's in a triangulated category which is not a stable module category of self-injective algebra.

Let k be an algebraically closed field, and $\mathcal{T}$ be a triangulated category with the following properties

- k-linear Hom-finite Krull-Schmidt;
- *1-Calabi-Yau*, that is, there is a natural isomorphism $\text{Hom}_{\mathcal{T}}(X, Y) \cong D\text{Hom}_{\mathcal{T}}(Y, X[1])$ bifunctorial for all $X, Y \in \mathcal{T}$;
- (additively) finite, that is, it has finitely many indecomposable modules.

Then by [Ami, Thm 9.3.4], $\mathcal{T}$ is triangulated equivalent to the category proj-$P^f(\Delta)$ of projective modules over the deformed preprojective algebra of generalised Dynkin type Δ, whose suspension functor is the Nakayama automorphism. Our aim is to deduce all the sms's (if they exist) of such category.

We start by recalling the definition of $P^f(\Delta)$ from [BES]. Let Δ be a generalized Dynkin graph of type $\mathbb{A}_n$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_n$ $(n = 6, 7, 8)$, or $\mathbb{L}_n$. Let Q_Δ be the following associated quiver:

$$\Delta = \mathbb{A}_n\ (n \geq 1): \qquad \begin{tikzcd} 0 \arrow[r, shift left, "a_0"] & 1 \arrow[l, shift left, "\overline{a}_0"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, dotted, no head] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a}_{n-2}"] \end{tikzcd} \tag{7.3.1}$$

$$\Delta = \mathbb{D}_n\ (n \geq 4): \qquad \begin{tikzcd} 0 \arrow[dr, shift left, "a_0"] & & & & \\ & 2 \arrow[ul, shift left, "\overline{a}_0"] \arrow[dl, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[l, shift left, "\overline{a}_2"] \arrow[r, dotted, no head] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a}_{n-2}"] \\ 1 \arrow[ur, shift left, "a_1"] & & & & \end{tikzcd} \tag{7.3.2}$$

$$\Delta = \mathbb{E}_n\ (n = 6, 7, 8): \qquad \begin{tikzcd} & & 0 \arrow[d, shift left, "a_0"] & & & \\ 1 \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & 3 \arrow[u, shift left, "\overline{a}_0"] \arrow[l, shift left, "\overline{a}_2"] \arrow[r, shift left, "a_3"] & 4 \arrow[l, shift left, "\overline{a}_3"] \arrow[r, dotted, no head] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1 \arrow[l, shift left, "\overline{a}_{n-2}"] \end{tikzcd} \tag{7.3.3}$$

$$\Delta = \mathbb{L}_n\ (n \geq 1): \qquad \begin{tikzcd} 0 \arrow[loop left, "\epsilon=\overline{\epsilon}"] \arrow[r, shift left, "a_0"] & 1 \arrow[l, shift left, "\overline{a}_0"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, dotted, no head] & n-2 \arrow[r, shift left, "a_{n-2}"] & n-1\,. \arrow[l, shift left, "\overline{a}_{n-2}"] \end{tikzcd} \tag{7.3.4}$$

The *preprojective algebra* $P(\Delta)$ associated to the graph Δ is the quotient of the path algebra kQ_Δ by the relations:

$$\sum_{i \xrightarrow{a} j} a\overline{a}, \quad \text{for each vertex } i \text{ of } Q_\Delta.$$

Now define a local algebra $R(\Delta)$ as follows:

$$\begin{aligned} R(\mathbb{A}_n) &= k; \\ R(\mathbb{D}_n) &= k\langle x, y\rangle/(x^2, y^2, (x+y)^{n-2}); \\ R(\mathbb{E}_n) &= k\langle x, y\rangle/(x^2, y^3, (x+y)^{n-3}); \\ R(\mathbb{L}_n) &= k[x]/(x^{2n}). \end{aligned}$$

Pick any $f \in \operatorname{rad}^2 R(\Delta)$. The *deformed preprojective algebra* $P^f(\Delta)$ (of generalised Dynkin type Δ) is the quotient of the path algebra kQ_Δ by the relations:

$$\sum_{sa=i} a\overline{a}, \quad \text{for each non exceptional vertex } i \text{ of } Q,$$

and

$$\begin{array}{lll} a_0\overline{a}_0 & \text{for} & \Delta = \mathbb{A}_n; \\ \overline{a}_0 a_0 + \overline{a}_1 a_1 + a_2\overline{a}_2 + f(\overline{a}_0 a_0, \overline{a}_1 a_1), \text{ and } (\overline{a}_0 a_0 + \overline{a}_1 a_1)^{n-2} & \text{for} & \Delta = \mathbb{D}_n; \\ \overline{a}_0 a_0 + \overline{a}_2 a_2 + a_3\overline{a}_3 + f(\overline{a}_0 a_0, \overline{a}_2 a_2), \text{ and } (\overline{a}_0 a_0 + \overline{a}_2 a_2)^{n-3} & \text{for} & \Delta = \mathbb{E}_n; \\ \epsilon^2 + a_0\overline{a}_0 + \epsilon f(\epsilon), \text{ and } \epsilon^{2n} & \text{for} & \Delta = \mathbb{L}_n, \end{array}$$

where we regard f as a function of x (resp. x, y) for $\Delta = \mathbb{L}_n$ (resp. $\Delta = \mathbb{D}_n, \mathbb{E}_n$). Note that if f is zero, we get the (non-deformed) preprojective algebra $P(\Delta)$.

We note that a finite 1-Calabi-Yau triangulated category $\mathcal{T}$ is *standard*, i.e. $\mathcal{T}$ is triangulated equivalent to a stable category of a Frobenius category, if and only if, $\mathcal{T} \simeq$ proj-$P(\Delta)$.

Theorem 7.3.1. *Let $\mathcal{T}$ = proj-$P^f(\Delta)$, there does not exists any sms unless $\Delta = \mathbb{A}_n$. In case $\Delta = \mathbb{A}_n$ and $n \geq 2$, the $\{P_0\}$ and $\{P_{n-1}\}$ are the only sms's, where P_i is the projective indecomposable module corresponding to vertex i of Q_Δ.*

Proof. By definition, $\dim \mathrm{Hom}_{\mathcal{T}}(X, Y)$ is an entry of the Cartan matrix of $P^f(Q)$ for any indecomposable objects X, Y. By the defining relations of $P^f(\Delta)$, there is no zero entry in its Cartan matrix, so any sms of $\mathcal{T}$ must have only one indecomposable module X with $\dim \mathrm{End}_{\mathcal{T}}(X) = 1$, and so we can pick out any diagonal entry of the Cartan matrix of $P^f(Q)$ to check orthogonality. Moreover, recall from [BES, Thm 1.1(i)] that Cartan matrix of $P^f(Q)$ is the same as that of the non-deformed version $P(Q)$, so it suffices to check the Cartan matrices of $P(Q)$.

(1) $\Delta = \mathbb{A}_n (n \geq 1)$: The i-th diagonal entry (i.e. $\dim \mathrm{Hom}_{P(Q)}(P_i, P_i)$) of the Cartan matrix is i if $i \leq (n+1)/2$, or $n - i + 1$ otherwise.

(2) $\Delta = \mathbb{D}_n (n \geq 4)$: the Cartan matrix is given in [ES, 3.4]:

$$\begin{pmatrix} m+1 & u & n-2 & n-3 & \cdots & 3 & 2 & 1 \\ u & m+1 & n-2 & n-3 & \cdots & 3 & 2 & 1 \\ n-2 & n-2 & 2(n-2) & 2(n-3) & \cdots & 6 & 4 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 3 & 3 & 6 & 6 & \cdots & 6 & 4 & 2 \\ 2 & 2 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 1 & 1 & 2 & 2 & \cdots & 2 & 2 & 2 \end{pmatrix}$$

where $u = m$ if n even, or $m + 1$ otherwise; and m is the supremum of the τ-period of indecomposable $P(\Delta)$-modules (which always exists and finite). In particular, none of the diagonal entries takes value 1.

(3) $\Delta = \mathbb{E}_6$: the Cartan matrix is given by:

$$\begin{pmatrix} 4 & 2 & 4 & 6 & 4 & 2 \\ 2 & 2 & 3 & 4 & 3 & 2 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 6 & 4 & 8 & 12 & 8 & 4 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 2 & 2 & 3 & 4 & 3 & 2 \end{pmatrix}$$

For the remaining types, [BES, Prop 2.1] tells us that the Nakayama permutation is identity, so then any diagonal entry of the Cartan matrix would be even (hence greater than 1).

Now we look at $\Delta = \mathbb{A}_n$ with $n \geq 1$. Note that any deformation of $P(\Delta)$ is just $P(\Delta)$ itself, so we only need to check that P_0 generates proj-$P(\Delta)$ by extension. (Applying the suspension factor, equivalently Nakayama automorphism, the claim implies P_{n-1} also generates proj-$P(\Delta)$.) From the defining relation of $P(\mathbb{A}_n)$, we get the following distinguished triangles of proj-$P(\Delta)$:

$$\begin{array}{c}
P_0 \xrightarrow{a_0} P_1 \xrightarrow{\overline{a}_0} P_0 \longrightarrow P_0[1] \\
P_i \xrightarrow{(\overline{a}_{i-1}, a_i)^T} P_{i-1} \oplus P_{i+1} \xrightarrow{(a_{i-1} \ \overline{a}_i)} P_i \longrightarrow P_i[1] \\
P_{n-1} \xrightarrow{\overline{a}_{n-2}} P_{n-2} \xrightarrow{a_{n-2}} P_{n-1} \longrightarrow P_{n-1}[1]
\end{array}$$

where $i = 2, \ldots, n-2$. Now the claim follows. □

Remark 7.2. The "standard" part of this theorem can be proved as an immediate consequence of Theorem 7.1.4. Our proof here accounts for non-standard triangulated categories as well.

Let k be algebraically closed field, $S = k[x, y]$, G a finite subgroup of $\mathrm{SL}_2(k)$, and $R = S^G$ the invariant ring. Further assume that order of G is invertible in k. It is well-known that (see, for example, [AIR2]) the stable category $\underline{\mathrm{MCM}}(R)$ of maximal Cohen-Macaulay R-module is triangulated equivalent to proj-$P(\Delta)$ for some Dynkin type Δ. Here Δ is uniquely determined by G using McKay correspondence. We immediately obtain classification of sms's for these categories using above result. We note that this category is a generalisation of the stable module category of finite dimensional self-injective algebra. Since the only simple R-modules (the trivial module k) has Krull dimension 1, whereas the Krull dimension of R is 2, so k is not maximal Cohen-Macaulay. From this perspective, non-existence of sms's is somewhat natural, and the surprising element of our result is that we do have sms's in the type $\mathbb{A}_n$ case.

Corollary 7.3.2. *Retaining the above set-up of R, and let $\mathcal{T}$ be any of the following triangulated* *categories:*

- *1-cluster category* $\mathcal{C}_1(kQ) = D^b(\text{mod-}kQ)/\tau$ *for* Q *Dynkin quiver of type* Δ*;*
- *stable category* $\underline{\mathrm{MCM}}(R)$ *of maximal Cohen-Macaulay* R*-modules;*
- *stable category of matrix factorisation of* $k[X,Y,Z]$ *with respect to some* $f \in k[X,Y,Z]$*, where* f *is uniquely determined by* $k[X,Y,Z]/(f) \cong R$*.*

Then $\mathcal{T}$ *has no sms unless* G *is of type* $\mathbb{A}_n$ *under McKay correspondence; in which case, the indecomposable* R*-module corresponding to either of the vertices at the end of the Dynkin diagram* $\mathbb{A}_n$ *under McKay correspondence is an sms of* $\mathcal{T}$*.*

Proof. The triangulated equivalence of these categories are well-known [Kel, Ami, LW]. These are all triangulated equivalent to proj-$P(\Delta)$ from [Ami, Thm 9.3.4] and/or [Kel]. $\square$

Chapter 8

Connection between mutation theories for representation-finite self-injective algebras

In this chapter, we first discuss connections between mutations of tilting complexes and sms's, and how to use these concepts for sms's. The first result is Theorem 8.1.3, which states that the homotopy category $\mathcal{T} = K^b(\mathrm{proj}A)$ is strongly tilting-connected when A is an RFS algebra. In the second part of this chapter, we consider the connection between two-term tilting complexes and sms's for a self-injective Nakayama algebra, and established Theorem 8.2.1 that the naturally defined map from two-term tilting complexes to sms's is always a surjection. We also determined the precise condition for this surjection to be injective.

As in Chapter 6, all the algebras are assumed to be finite dimensional over algebraically closed field, indecomposable, basic, non-simple, and self-injective. In particular, they can be presented by quivers and relation. We work with right modules and write morphisms on the left, composing them from right to left. Likewise, paths in a quiver Q will be composed from right to left, and we often identify them with morphisms between projective right modules over (a quotient of) the path algebra.

8.1 Connection with tilting complexes

The first connection we consider here comes from the aforementioned result of Dugas [Dug2], which opens up a new and efficient way to study (and compute) simple-image sms's of Morita type and their liftability, as demonstrated in the previous section.

We have seen how mutation of sms and Nakayama-stable smc are connected. We remind the reader of the main result of [KY], which in particular gives a bijection between smc and silting objects as well as compatibility of the respective mutations. Since we have already established a connection between sms and smc, we can now exploit the connection with silting / tilting objects.

First we briefly recall some information on silting theory developed by Aihara and Iyama [AI]. We use $\mathcal{T}$ to denote the (triangulated) homotopy category $K^b(\text{proj-}A)$ of bounded complexes of projective A-modules; the suspension functor in this category is denoted by [1], and by $[n]$ we mean $[1]^n$.

Note that tilting objects in $\mathcal{T}$ (i.e. one-sided tilting complexes) are exactly the silting objects that are stable under Nakayama functor (see, for example, the discussion after Theorem 3.5 of [KY]). As we have hinted throughout the whole thesis, Nakayama-stability plays a vital role in the study of sms's, at least for sms's which are liftable and simple-image of Morita type. For convenience, we denote the Nakayama functor $\nu = \nu_A$ when the algebra A under consideration is clear, and we assume every tilting object is basic, i.e. its indecomposable summands are pairwise non-isomorphic.

Lemma 8.1.1. *Let $A, \mathcal{T}$ be as above and $\mathcal{C}$ a full subcategory of $\mathcal{T}$ with $\nu\mathcal{C} = \mathcal{C}$. If $X \in \mathcal{T}$ and $f : X \to Y$ is a (minimal) left $\mathcal{C}$-approximation of X, then $\nu_A(f) : \nu X \to \nu Y$ is a (minimal) left $\mathcal{C}$-approximation of νX. In particular, if $\nu X \cong X$ and f is a minimal left $\mathcal{C}$-approximation of νX, then $\nu Y \cong Y$.*

Proof. Since A is self-injective, so $\nu\mathcal{T} = \mathcal{T}$, and $Hom_{\mathcal{T}}(X, Y) \simeq Hom_{\mathcal{T}}(\nu X, \nu Y)$. As $\nu X \in \mathcal{C}$, to see νf is a $\mathcal{C}$-approximation, we need to show that $Hom_{\mathcal{T}}(\nu f, X')$ is surjective for all $X' \in \mathcal{C}$. Since $\nu\mathcal{C} = \mathcal{C}$, every object in $X' \in \mathcal{C}$ is of the form νZ for some $Z \in \mathcal{C}$. Also $Hom_{\mathcal{T}}(\nu X, \nu Z) \simeq Hom_{\mathcal{T}}(X, Z)$, so every map $\nu X \to \nu Z$ can be written as νh for some $h : X \to Z$. Since f is an approximation, $h = fg$ for some $g \in Hom_{\mathcal{T}}(Y, Z) \simeq Hom_{\mathcal{T}}(\nu Y, \nu Z)$. As ν is an auto-equivalence of $\mathcal{C}$, $\nu h = \nu(fg) = (\nu f)(\nu g)$. Hence $\nu f : \nu X \to \nu Y$ is a $\mathcal{C}$-approximation. For minimality we proceed similarly. i.e. for $g : \nu X \to \nu X$, $g = \nu h$ for some $h : X \to X$, the condition $g(\nu f) = \nu f$ can now be rewritten as $\nu(hf) = (\nu h)(\nu f) = \nu f$ which implies $hf = f$. By minimality of f, h is an isomorphism, hence so is νh. □

By this lemma, a mutation of a tilting object (i.e. a Nakayama-stable silting object) is a tilting object if we mutate at a Nakayama-stable summand. For convenience, we say that a Nakayama-stable mutation of a tilting complex is a tilting mutation. An irreducible silting mutation mutates with respect to an indecomposable summand. By thinking of this as mutating with respect to a "minimal" Nakayama-stable summand, we can make sense of "irreducibility" for tilting mutation for general self-injective algebras (rather than just weakly symmetric algebras).

Definition 8.1.2. *(Compare to [Aih1]) (1) Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic tilting object in $\mathcal{T} = K^b(\mathrm{proj}A)$. If X is a Nakayama-stable summand of T such that for any non-zero Nakayama-stable summand Y of X, we have $Y = X$, then we call X a* minimal Nakayama-stable summand*. A (left)* tilting mutation *$\mu_X^-(T)$ is said to be* irreducible *if X is minimal. Similarly for right tilting mutation.*

(2) Let T, U be basic tilting objects in $\mathcal{T}$. We say that U is connected *(respectively,* left-connected*) to T if U can be obtained from T by iterative irreducible (respectively, left) tilting mutations.*

(3) $\mathcal{T}$ is tilting-connected *if all its basic tilting objects are connected to each other. We say that $\mathcal{T}$ is* strongly tilting-connected *if for any basic tilting objects T, U with $Hom_{\mathcal{T}}(T, U[i]) = 0$ for all $i > 0$, U is left-connected to T.*

Remark 8.1. (1) Note that the irreducible tilting mutation just defined is different from an irreducible silting mutation when A is self-injective non-weakly symmetric, even though it is itself a silting mutation as well. We will emphasise irreducible *tilting* mutation throughout to distinguish between our notion and irreducible silting mutation.

(2) We can define the analogous notion of (left or right) irreducible sms mutation similar to irreducible tilting mutation above. More precisely, for a Nakayama-stable sms $\mathcal{S} = \{X_1, \ldots, X_r\}$ as in Definition 5.5.1, its irreducible mutation means that we mutate at a Nakayama-stable subset $\mathcal{X} = \{X_{i_1}, \ldots, X_{i_m}\}$ which is minimal in the obvious sense.

(3) For any tilting complex T, there exists a tilting complex P (e.g. $A[l]$ for $l >> 0$) such that $Hom_{\mathcal{T}}(T, P[i]) = 0$ for all $i > 0$.

(4) Strongly tilting-connected implies tilting-connected. This follows from (3) and the fact that left and right mutations are inverse operations to each other.

We can now reformulate a question asked in [AI] and [Aih1, Question 3.2]: Is $\mathcal{T} = K^b(\mathrm{proj}A)$ tilting-connected for self-injective algebra A? By reproving the Nakayama-stable analogue of the results in [AI] and [Aih1], we can answer this question positively for RFS algebras A. These proofs are not directly related to the simple-minded theories and are really about modifying the proofs of Aihara and of Aihara and Iyama in an appropriate way.

Theorem 8.1.3. *Let A be an RFS algebra. Then the homotopy category $\mathcal{T} = K^b(\mathrm{proj}A)$ is* *strongly tilting-connected.*

The proof will occupy a separate subsection below.

Recall the silting quiver as defined in [AI] and [Aih1]. Again we can define a "Nakayama-stable version" and the sms's version of this combinatorial gadget.

Definition 8.1.4. *(Compare to [Aih1, AI]) Let A be a self-injective algebra.*

(1) Let $\mathsf{tilt}(A)$ be the class of all tilting objects in $\mathcal{T} = K^b(\mathrm{proj}A)$ up to shift and homotopy equivalence. The exchange quiver of $\mathsf{tilt}(A)$ is a quiver $Q_{\mathrm{tilt}}(A)$ such that the set of vertices is the class of basic tilting objects of $\mathcal{T}$; and for T, U tilting objects, $T \to U$ is an arrow in the quiver if U is an irreducible left tilting mutation of T.

(2) The exchange quiver of $\mathsf{sms}(A)$ is a quiver $Q_{\mathrm{sms}}(A)$ such that the set of vertices is $\mathsf{sms}(A)$; and for two sms's $\mathcal{S}, \mathcal{S}'$, $\mathcal{S} \to \mathcal{S}'$ is an arrow in the quiver if $\mathcal{S}'$ is an irreducible left mutation of $\mathcal{S}$.

Remark 8.2. (1) Long before the work of [AI], the term tilting quiver has been used for a graph whose vertices are tilting modules over a finite dimensional algebra. The tilting quiver here is a specialisation of the silting quiver of [AI], whose vertices are objects in a triangulated category.

(2) Combinatorially (i.e. ignoring the "labeling" of the vertices), $Q_{\mathrm{tilt}}(A) = Q_{\mathrm{tilt}}(B)$ (respectively $Q_{\mathrm{sms}}(A) = Q_{\mathrm{sms}}(B)$) if A and B are derived (resp. stably) equivalent.

Proposition 8.1.5. *Suppose A is an RFS algebra. Then there is a surjective quiver morphism $Q_{\mathrm{tilt}}(A) \to Q_{\mathrm{sms}}(A)$. In particular, every sms of A can be obtained by iterative left irreducible mutation starting from the simple A-modules.*

Proof. Define a map

$$\begin{array}{rcl} f : \mathsf{tilt}(A) & \to & \mathsf{sms}(A) \\ T & \mapsto & \phi^{-1}(\mathcal{S}_A) \end{array} \tag{8.1.1}$$

where ϕ is the induced stable equivalence of Morita type associated to the derived equivalence F given by T. Let $\mathbf{S}$ be the image of the collection of stalk complexes of isoclass representatives of simple A-modules under F. Then by Koenig-Yang's result (Theorem 5.5.4) $\mathbf{S}$ is a Nakayama-stable smc, and this assignment $T \mapsto \mathbf{S}$ defines a mutation-respecting bijection between $\mathsf{tilt}(A)$ and ν-$\mathsf{smc}(A)$. Note that $f(T) = \phi^{-1}(\mathcal{S}_A) = \eta_A(\mathbf{S})$. So it follows from [Dug2, Proposition 5.4] that f is a mutation-respecting map. Hence this induces a quiver morphism $Q_{\mathrm{tilt}}(A) \to Q_{\mathrm{sms}}(A)$. Now surjectivity on the set of vertices follows from the proof of Theorem 6.3.4, which asserts that every sms of A is liftable simple-image. For the last statement, let $\mathcal{S}$ be an sms of

A, then $\mathcal{S}$ is liftable to a Nakayama-stable smc $\mathbf{S}$, which corresponds to a tilting object T. By Theorem 8.1.3, we can obtain T by iterative tilting mutations starting from A. The bijection in [KY] then implies that $\mathbf{S}$ can be obtained by iterative smc mutations starting from simple A-modules. The statement now follows from the surjective quiver morphism. □

Remark 8.3. By symmetry from Theorem 8.1.3, the result holds if we replace iterative left irreducible mutation by iterative right irreducible mutation.

This result can also be compared with Theorem 6.3.4, where we formed the quotient of the class of all Nakayama-stable smc's (respectively sms's) by the derived (respectively stable) Picard group, obtaining an injection regardless of representation-finiteness. On the other hand, these quivers enable us to visualise how we can "track" simple-image sms's of Morita type, and they contain more structure than the sets considered in Theorem 6.3.4. Yet it is still unclear how these links between smc's (hence tilting complexes) and sms's can be used to extract information about derived and/or stable Picard groups.

8.1.1 Proof of Theorem 8.1.3 à la Aihara

We use the notation $\mathcal{T} = K^b(\mathrm{proj}A)$ with A an RFS algebra over a field. The term tilting object refers to objects in $\mathcal{T}$, that is, to complexes. Recall the following notation from [AI] and [Aih1].

Definition 8.1.6. *Let T, U be objects of $\mathcal{T}$, write $T \geq U$ if $Hom_{\mathcal{T}}(T, U[i]) = 0$ for all $i > 0$.*

Note this defines a partial order on the class of silting (and hence, tilting) objects of $\mathcal{T}$. Applying Lemma 8.1.1 to [AI, Prop 2.24] yields:

Proposition 8.1.7. *Let T be a tilting complex of a self-injective algebra, and $U_0 \cong \nu U_0 \in \mathcal{T}$ such that $T \geq U_0$, then there are triangles*

$$\begin{aligned}
&U_1 \xrightarrow{g_1} T_0 \xrightarrow{f_0} U_0 \longrightarrow U_1[1],\\
&\cdots,\\
&U_\ell \xrightarrow{g_\ell} T_{\ell-1} \xrightarrow{f_{\ell-1}} U_{\ell-1} \longrightarrow U_\ell[1],\\
&0 \xrightarrow{g_{\ell+1}} T_\ell \xrightarrow{f_\ell} U_\ell \longrightarrow 0,
\end{aligned}$$

for some $\ell \geq 0$ such that f_i is a minimal right add$T$*-approximation, g_{i+1} belongs to the Jacobson radical $J_{\mathcal{T}}$, $\nu U_i = U_i$ and $\nu T_i = T_i$, for any $0 \leq i \leq \ell$.*

Proof. The only difference of the proof here and the one in [AI] is to use Lemma 8.1.1 on the triangles in the proof. More precisely, following the proof in [AI] we have a triangle $U_1 \xrightarrow{g_1}$

$T_0 \xrightarrow{f_0} U_0 \to U_1[1]$ with f_0 a minimal right add$T$-approximation of U_0. Apply the Nakayama functor to this triangle yields another triangle

$$\nu U_1 \xrightarrow{\nu g_1} \nu T_0 \xrightarrow{\nu f_0} \nu U_0 \to \nu U_1[1],$$

where $\nu T_0 \cong T_0$ and νf_0 is a minimal right add$T$-approximation by Lemma 8.1.1. Let $\theta : \nu U_0 \to U_0$ be an isomorphism. Then both f_0 and $\theta \circ \nu f_0$ are minimal right add$T$-approximation of U_0. As $\theta \circ \nu f_0$ is a right add$T$-approximation, there is $\phi : \nu T_0 \to T_0$ with $f_0 \circ \phi = \theta \circ \nu f_0$. Minimality of $\theta \circ \nu f_0$ implies that ϕ is an isomorphism. We then obtain a morphism of triangles:

$$\begin{tikzcd}
\nu U_1 \arrow[r, "\nu g_1"] \arrow[d, dotted] & \nu T_0 \arrow[r, "\nu f_0"] \arrow[d, "\phi", "\cong"'] & \nu U_0 \arrow[r] \arrow[d, "\theta", "\cong"'] & \nu U_1[1] \arrow[d, dotted] \\
U_1 \arrow[r, "g_1"] & T_0 \arrow[r, "f_0"] & U_0 \arrow[r] & \nu U_1[1].
\end{tikzcd}$$

By the axioms of triangulated category, $\nu U_1 \cong U_1$. Now the proof continues as in [AI]. $\square$

This can be used to deduce the Nakayama-stable analogue of [AI, Theorem 2.35, Prop 2.36]:

Theorem 8.1.8. *Let T, U be tilting objects of a self-injective algebra. Then*

1. *If $T > U$, then there exists an irreducible left tilting mutation P of T such that $T > P \geq U$.*
2. *The following are equivalent:*
   (a) *U is an irreducible left tilting mutation of T;*
   (b) *T is an irreducible right tilting mutation of U;*
   (c) *$T > U$ and there is no P tilting such that $T > P > U$.*

Proof. Proof of (1) is the same as the proof of [Aih1, Prop 2.12], except that now we take a ν-stable summand of T_ℓ instead of an indecomposable summand. Proof of (2) is the same as the proof of [AI, Theorem 2.35], without any change. $\square$

We modify the proof of Aihara in [Aih1] to show that any tilting object of an RFS algebra can be obtained through iterative irreducible tilting mutation.

The proof of Theorem 8.1.3 is based on the following key proposition:

Proposition 8.1.9. *[Aih1, Prop 5.1] $\mathcal{T}$ is tilting-connected if, for any algebra B derived equivalent to A, the following conditions are satisfied:*

(A1) *Let T be a basic tilting object in $K^b(\mathrm{proj} B)$ with $B[-1] \geq T \geq B$. Then T is tilting-connected to both $B[-1]$ and B.*

(A2) *Let P be a basic tilting object in $K^b(\text{proj}B)$ with $B[-\ell] \geq P \geq B$ for a positive integer ℓ. Then there exists a basic tilting object T in $K^b(\text{proj}B)$ satisfying $B[-1] \geq T \geq B$ such that $T[-\ell+1] \geq P \geq T$.*

The proof of this follows almost word-to-word as the proof in [Aih1, Prop 5.2]. Since we are only interested in tilting-connectedness rather than silting-connectedness, the original condition (A3), which says that any silting object is (silting-)connected to a tilting object, is discarded; and then replace (silting-)connectedness by tilting-connectedness everywhere in the proof.

(A2) is known to be true for RFS algebras from [Aih1, Lemma 5.4]. Therefore, what is left is to look carefully at the arguments and results that are used by Aihara in the proof of (A1).

Lemma 8.1.10. *[Aih1, Lemma 5.3] Condition (A1) holds for all RFS algebras A.*

Proof. The original proof relies on [Aih1, Prop 2.9] and [Aih1, Theorem 3.5]. Proposition 2.9 is true regardless of what kind of algebra A is. We are left to show the analogue of [Aih1, Theorem 3.5] is true, i.e. the following: □

Theorem 8.1.11. *[Aih1, Theorem 3.5] Let T, U be basic tilting objects in $\mathcal{T}$ with $T \geq U$. If there exist only finitely many tilting objects P such that $T \geq P \geq U$, then U is left-connected to T.*

Proof. If $U \in \text{add}T$, then we have $U \cong T$. So suppose $U \notin \text{add}T$. Theorem 8.1.8 provides a sequence:

$$T = T_0 > T_1 > T_2 > \cdots$$

such that each T_{i+1} is an irreducible left tilting mutation of T_i, and $T_i \geq U$, for all $i \geq 0$. If U is not left-connected to T, then this sequence is infinitely long, contradicting the condition that there are only finitely many tilting objects P with $T \geq P \geq U$. Therefore, U is isomorphic to T_i for some $i \geq 0$. □

8.2 Connection with two-term tilting complexes

For an RFS algebra A, let $\mathsf{2tilt}(A)$ denote the set of two-term tilting complexes concentrated in degree 0 and -1 up to homotopy equivalence. While from Proposition 8.1.5 we see that sms's give some mutation-respecting control to $\mathsf{tilt}(A)$, it is unknown if this is true for $\mathsf{2tilt}(A)$. We are interested in the composition $\mathfrak{F}$ of the natural injection $\mathsf{2tilt}(A) \hookrightarrow \mathsf{tilt}(A)$ and the natural surjection $\mathsf{tilt}(A) \to \mathsf{sms}(A)$ given in Theorem 8.1.1. We investigate the case when A is Nakayama, i.e. uniserial. We denote by A_n^ℓ as the self-injective Nakayama algebra with n simples and of

Loewy length $\ell + 1$. For a tilting complex T, we will denote E_T the endomorphism algebra of T, $F_T : D^b(\text{mod-}A) \to D^b(\text{mod-}E_T)$ the corresponding (standard) derived equivalence given in [Ric3] and $\overline{F_T} : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}E_T$ be the functor induced on the stable categories. The composition we are interested is therefore the map $T \mapsto \overline{F_T}^{-1}(\mathcal{S}_{E_T})$. The following is our main result.

Theorem 8.2.1. *For the self-injective Nakayama algebra A_n^ℓ with $\ell \nmid n$ (resp. $\ell | n$, the map $\mathfrak{F} : \mathsf{2tilt}(A_n^\ell) \to \mathsf{sms}(A_n^\ell)$ given by $T \mapsto \overline{F_T}^{-1}(\mathcal{S}_{E_T})$ is a bijection (resp. non-injective surjection). Moreover, for a minimal Nakayama-stable summand X of $T \in \mathsf{2tilt}(A_n^\ell)$, $\mathfrak{F}(\mu_X(T)) = \mu_S^-(\mathfrak{F}(T))$ for some (unique) $S \in \mathfrak{F}(T)$. In particular, when $\ell \neq \gcd(n, \ell)$, the exchange quiver of $\mathsf{2tilt}(A_n^\ell)$ embeds into that of $\mathsf{sms}(A_n^\ell)$.*

A corollary of the main theorem is that every simple-minded system can be obtained by derived equivalence given by two-term tilting complex, which is not apparent from the definition of $\mathfrak{F}$, and can be seen to be false for general RFS algebras.

Corollary 8.2.2. *Let A be a self-injective Nakayama algebra and $F : D^b(\text{mod-}A) \to D^b(\text{mod-}B)$ be a derived equivalence. Then there is a two-term tilting complex T concentrated in degree 0 and -1 which gives rise to the following diagram.*

$$\begin{array}{ccccc} \underline{\text{mod}}\text{-}E_T & \xrightarrow{\overline{F_T}} & \underline{\text{mod}}\text{-}A & \xleftarrow{\overline{F}} & \underline{\text{mod}}\text{-}B \\ \{\textit{simple } E_T\textit{-modules}\} & \longmapsto & \mathcal{S} & \longleftarrow\!\shortmid & \{\textit{simple } B\textit{-modules}\} \end{array}$$

where $\overline{F_T}$ and $\overline{F}$ are the induced stable equivalences and $n \in \mathbb{Z}$. In particular, $B \cong E_T$.

Proof. This follows from combining Theorem 8.2.1, Theorem 6.1.1, and Theorem 6.3.4. $\square$

8.2.1 Reminders on self-injective Nakayama algebras

Definition 8.2.3 (see for example [GR, ARS, ASS])**.** 1. *A basic indecomposable* self-injective Nakayama algebra A_n^ℓ *with n simples and Loewy length $\ell + 1$ is given by the path algebra kQ/I with quiver*

$$Q : \quad \begin{tikzcd} & 1 \arrow[r, "\alpha_1"] & n \arrow[dr, "\alpha_n"] & \\ 2 \arrow[ur, "\alpha_2"] \arrow[rrr, dotted, dash, bend right] & & & n-1 \end{tikzcd} \tag{8.2.1}$$

and relation ideal $I = \text{rad}^{\ell+1}(kQ)$.

2. *A* Brauer graph *G is a finite undirected connected graph (possibly with loops and multiple edges) with the following data. To each vertex we assign a cyclic ordering of edges incident to it, and a positive integer called* multiplicity*.*

3. *A* Brauer tree *is a Brauer graph which is a tree, having at most one vertex with multiplicity greater than one. If there is such vertex, it is called* exceptional vertex*, otherwise we say the Brauer tree has trivial multiplicity. Traditionally, we choose the counter-clockwise direction as the cyclic ordering of edges; and denote the Brauer tree as (G, v, m) for a tree G with exceptional multiplicity m at the exceptional vertex v. For simplicity, we usually just use G as the notation for this triple.*

4. *A finite dimensional algebra A is a* Brauer tree algebra *associated to a given Brauer tree (G, v, m), if there is a one-to-one correspondence between the edges j of G and the simple A-modules S_j in such a way that the projective cover P_j of S_j has the following description. We have $P_j/\operatorname{rad} P_j \cong \operatorname{soc} P_j \cong S_j$, and the heart $\operatorname{rad} P_j/\operatorname{soc} P_j$ is a direct sum of two (possibly zero) uniserial modules U_j and W_j corresponding to the two vertices u and w at the end of the edge j. If the edges around u are cyclically ordered $j, j_1, j_2, \ldots, j_r, j$ and the multiplicity of the vertex u is m_u, then the corresponding uniserial module U_j has composition factors (from the top) $S_{j_1}, S_{j_2}, \ldots, S_{j_r},\ S_j, S_{j_1}, \ldots, S_{j_r},\ S_j, \ldots, S_{j_r}$ so that $S_{j_1}, \ldots, S_{j_r}$ appear m_u times and S_j appears $m_u - 1$ times. We denote the basic algebra associated to a Brauer tree G with e edges and exceptional multiplicity m as $B^G_{e,m}$.*

A *Brauer star*, which we usually denote by $\star$, is a Brauer tree where the underlying graph is a star, with exceptional vertex at the centre. The corresponding Brauer tree algebra is called Brauer star algebra. Note that the class of Brauer star algebras coincides with the class of symmetric Nakayama algebras, i.e. $B^\star_{e,m} = A^{em}_e$ for any $e, m \geq 1$, and so we fix the quiver and relation presentation for Brauer star algebra with the one given by Nakayama algebra.

Let n be the number of simple modules of a self-injective Nakayama algebra. We denote by $\bar{i}$ the positive integer in $\{1, \ldots, n\}$ with $i \equiv \bar{i} \mod n$ for any $i \in \mathbb{Z}$. Now the radical of a projective indecomposable P_i of a self-injective Nakayama algebra has projective cover $P_{\overline{i+1}}$. An indecomposable A^ℓ_n-module is uniserial, hence, hence uniquely determined by its socle S_i and Loewy length l, and so we denote it by $M_{i,l}$ with $i \in \{1, \ldots, n\}$ and $1 \leq l \leq \ell + 1$. For any i and any $l \leq \ell$, the Auslander-Reiten translate $\tau \cong \nu_{A^\ell_n}\Omega^2$ sends $M_{i,l}$ to $M_{\overline{i+1},l}$. The Heller translate Ω, which is the inverse of suspension functor in the triangulated category $\underline{\mathrm{mod}}$-A^ℓ_n, sends $M_{i,l}$ to $M_{\overline{i-l+1},\ell+1-l}$; and inverse Heller translate Ω^{-1} sends $M_{i,l}$ to $M_{\overline{i-l},\ell+1-l}$. The Nakayama functor $\nu_{A^\ell_n}$ sends $M_{i,l}$ to $M_{\overline{i+e},l}$ where $e = \gcd(n, \ell)$.

From now on, we label the vertices of the stable AR-quiver of A^ℓ_n using the pair appearing in the subscript of an indecomposable A^ℓ_n-module, unless otherwise specified. Thus the simple A^ℓ_n-modules lie on the bottom rim of the stable AR-quiver, and radical of projective indecomposable A^ℓ_n-modules lie on the top rim of the stable AR-quiver. Note that the stable AR-quiver ${}_s\Gamma_{A^\ell_n}$

is isomorphic to the stable tube $\mathbb{Z}A_\ell/\langle\tau^n\rangle$, so by Theorem 6.1.1, we can identify $\mathsf{sms}(A_n^\ell)$ with the set $\mathsf{Conf}(\mathbb{Z}A_\ell/\langle\tau^n\rangle)$ of $\tau^{n\mathbb{Z}}$-stable configurations of $\mathbb{Z}A_\ell$, induced by $M_{(i,j)} \mapsto (i,j)$.

Example 8.4. The following is the stable AR-quiver of $B^\star_{3,2} = A_3^6$ (we omit the symbol M):

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & & & & & (3,6) \arrow[dr] & & (2,6) \arrow[dr] & & (1,6) \arrow[dr] & & (3,6) \\
 & & & & (3,5) \arrow[ur] \arrow[dr] & & (2,5) \arrow[ur] \arrow[dr] & & (1,5) \arrow[ur] \arrow[dr] & & (3,5) \arrow[ur] & \\
 & & & (3,4) \arrow[ur] \arrow[dr] & & (2,4) \arrow[ur] \arrow[dr] & & (1,4) \arrow[ur] \arrow[dr] & & (3,4) \arrow[ur] & & \\
 & & (3,3) \arrow[ur] \arrow[dr] & & (2,3) \arrow[ur] \arrow[dr] & & (1,3) \arrow[ur] \arrow[dr] & & (3,3) \arrow[ur] & & & \\
 & (3,2) \arrow[ur] \arrow[dr] & & (2,2) \arrow[ur] \arrow[dr] & & (1,2) \arrow[ur] \arrow[dr] & & (3,2) \arrow[ur] & & & & \\
(3,1) \arrow[ur] & & (2,1) \arrow[ur] & & (1,1) \arrow[ur] & & (3,1) \arrow[ur] & & & & &
\end{tikzcd}$$

$\{(1,1),(2,1),(3,1)\}$ is the set of simple $B^\star_{3,2}$-modules. Another example of a simple-minded system (i.e. configuration) is $\mathcal{S} = \{(1,1),(2,3),(3,5)\}$. The (unique) Brauer tree algebra B such that $\mathcal{S}$ is a B-simple-image is the one associated to the graph (tree) of a line with exceptional vertex at the end of the line, i.e. $\bullet \text{---} \circ \text{---} \circ \text{---} \circ$.

We will use this coordination throughout the remaining of this chapter. Note that our coordination is slightly different from the conventional one, where the "x-axis" goes from left to right, the transformation from our coordination to the conventional one is $(x,y) \mapsto (em - x, y)$.

In the following, we play with this combinatorics to give us some tools which will be useful later in the proof of the theorem 8.2.1 in the symmetric case. Recall that an *extremal vertex* of a Brauer tree G is a vertex of valency 1; we call the edge connected to an extremal vertex as *leaf*.

Lemma 8.2.4. *Let G be a Brauer tree and S a simple $B^G_{e,m}$-module. Then S lies on the rim of the stable AR-quiver if, and only if, the edge in G which corresponds to S is a leaf attached to a non-exceptional extremal vertex.*

Proof. By the construction of Brauer tree algebras, an edge is a leaf attached to a non-exceptional extremal vertex if and only if the corresponding indecomposable projective module is uniserial. Also note that for any simple $B^G_{e,m}$-module S, whose projective cover is P, the almost split sequence starting at $\Omega(S)$ is:

$$0 \to \Omega(S) \to P \oplus \operatorname{rad}(P)/\operatorname{soc}(P) \to \Omega^{-1}(S) \to 0$$

This says that P is uniserial if and only if, $\Omega(S)$ and $\Omega^{-1}(S)$ are on a rim of the stable AR-quiver, say located (without loss of generality) at $(1, em), (e, em)$ respectively, which in turns is equivalent to S located at $(1,1)$, i.e. another rim of the stable AR-quiver. $\square$

Let A be a finite dimensional algebra and $P = \varepsilon A$ a projective A-module. For an A-module M, we denote by $O_\varepsilon M$ (resp. $O^\varepsilon M$) the maximal (resp. minimal) submodule N such that $N\varepsilon = 0$ (resp. $(M/N)\varepsilon = 0$). Let $\mathcal{C}_\varepsilon$ be the full subcategory of mod-A consisting of modules M with $O_\varepsilon M = 0$ and $O^\varepsilon M = M$. It is well-known that the functor $- \otimes_{\varepsilon A\varepsilon} \varepsilon A$, which is the left adjoint of the Schur functor $- \otimes_A A\varepsilon$, induces an equivalence between mod-$\varepsilon A\varepsilon$ and $\mathcal{C}_\varepsilon$ [Aus]. Clearly, the projective objects of $\mathcal{C}_\varepsilon$ are the objects in $\mathrm{add}(\varepsilon A)$.

For each pair M, N of A-modules, the functor $- \otimes_A A\varepsilon$ induces a bijection between the set of morphisms $M \to N$ factoring through objects of $\mathrm{add}(\varepsilon A)$ and the set of $\varepsilon A\varepsilon$-module morphisms $M\varepsilon \to N\varepsilon$ factoring through $\mathrm{add}(\varepsilon A\varepsilon)$. Therefore, we have an equivalence between the stable category $\underline{\mathcal{C}_\varepsilon}$ of $\mathcal{C}_\varepsilon$ (i.e. the category with the same class of objects, but with hom-spaces quotiented out by morphisms which factor through objects in $\mathrm{add}(\varepsilon A)$) and $\underline{\mathrm{mod}}$-$\varepsilon A\varepsilon$.

When A is a Brauer tree algebra, given a special choice of ε, the following lemma gives a description of $\underline{\mathrm{mod}}$-$\varepsilon A\varepsilon \simeq \underline{\mathcal{C}_\varepsilon}$ in terms of AR-theory of $\underline{\mathrm{mod}}$-A.

Lemma 8.2.5. *Let $A = B^G_{e+1,m}$ be a Brauer tree algebra, ε_{e+1} be the primitive idempotent corresponding to a leaf attached to a non-exceptional extremal vertex, and $\varepsilon = 1-\varepsilon_{e+1}$. Position the stable AR-quiver ${}_s\Gamma_A$ of A so that the simple module S_{e+1} corresponding to ε_{e+1} is located at $(e+1, 1)$. Then we have the following.*

(i) $\varepsilon A\varepsilon$ is isomorphic to the Brauer tree algebra $B^{G'}_{e,m}$, where G' is obtained from G by removing the leaf corresponding to S_{e+1}.

(ii) A non-projective indecomposable A-module in $\mathcal{C}_\varepsilon$ is located at (x, y) if, and only if, $x \neq e+1$ and $x \neq \overline{y-1}$.

(iii) There is a map $\omega^{(m)}_e : {}_s\Gamma_{\varepsilon A\varepsilon} \to {}_s\Gamma_A$, given by $(x, y) \mapsto (x, y+\ell)$ where $\ell \in \{0, \dots, m\}$ is determined by the condition $-\ell e \leq x - y < (1-\ell)e$, mapping the vertices of the stable AR-quiver of $\varepsilon A\varepsilon$ to those of A. Moreover, if a vertex (x, y) of ${}_s\Gamma_{\varepsilon A\varepsilon}$ represents the (isoclass of) indecomposable $\varepsilon A\varepsilon$-module N, then $\omega^{(m)}_e(x, y)$ represents the (isoclass of) indecomposable A-module M such that $M\varepsilon \cong N$.

Proof. (i) This is clear from the construction of Brauer tree algebras.

(ii) Recall that a hook [Erd, II.5.2] is a non-projective uniserial module $\varepsilon_i A/\alpha A$ for some arrow α of the quiver of A. In our case, these are all the modules lying on the rim of the AR-quiver of A. Let M_j be an indecomposable $B^G_{e+1,m}$-module positioned at $(e+1, j)$ with $j \in \{1, \dots, em\}$.

We claim that the simple S_{e+1} is a composition factor of $\operatorname{soc} M$. The claim can be seen from the algorithm of "adding and removing hooks" in [BC, Section 2.3 and Theorem 3.5]. Applying the algorithm, one can observe that as we move from $(e+1, j)$ to $(e+1, j+1)$, adding or removing a hook leaves the composition factor in the S_{e+1} in the socle untouched. Let N_j be an indecomposable module positioned at (x, j) with $x = \overline{j-1}$. We have $N_j = \Omega^{-1}(M_{em+1-j})$, which implies S_{e+1} is a composition factor of the top of N_j. Since $\operatorname{soc}(M_j)$ (resp. top of N_j) contains S_{e+1}, we have $O_\varepsilon M_j \neq 0$ (resp. $O^\varepsilon N_j \neq N_j$).

Note that $M_j = N_j$ if and only if $j \cong 1 \bmod e$, so simple counting shows that there are precisely $2(e+1)m - m$ distinct modules of the form M_j and N_j. Also, by (i), $\underline{\mathcal{C}_\varepsilon}$ is equivalent to $\underline{\text{mod}}$-$B^{G'}_{e,m}$, which consists of $e^2 m = (e+1)^2 m - (2(e+1)em - m)$ indecomposable modules up to isomorphism. In other words, $\{M_j, N_j | j = 1, \ldots, em\}$ is the set of isoclasses of the indecomposable modules that do not lie in $\mathcal{C}_\varepsilon$.

(ii) Let $M \xrightarrow{f} N$ be an irreducible map between non-projective modules in mod-A such that M, N cannot be annihilated by ε, and $M\varepsilon \ncong N\varepsilon$. We claim that $f \otimes_A A\varepsilon$ is also irreducible in mod-$\varepsilon A\varepsilon$. Since A is symmetric, we can always assume f is right almost split. For any $X \in \mathcal{C}_\varepsilon$ and $h : X \to N$ a non-retraction, h always factors through f using the fact that $\mathcal{C}_\varepsilon$ is a full subcategory of mod-A and the right almost splitness of f. The last condition implies that $f \otimes_A A\varepsilon$ is not a retraction. This shows that f is right almost split in $\mathcal{C}_\varepsilon$, hence in mod-$\varepsilon A\varepsilon$.

By a similar argument, one can also shows that an AR-sequence $0 \to M \to X \to N \to 0$ in mod-A with $M, X, N \in \mathcal{C}_\varepsilon$ implies that the induced sequence is also an AR-sequence in mod-$\varepsilon A\varepsilon$.

Note that the simple A-module S_{e+1} has no self-extension, so the condition $O_\varepsilon M = 0$ (resp. $O^\varepsilon M = 0$) is equivalent to saying $\operatorname{soc} M$ (resp. the top of M) does not contain S_{e+1}. Moreover, the only indecomposable module annihilated by ε is S_{e+1}. Since S_{e+1} is a hook, the indecomposable A-module N_j with $j \ncong 1 \bmod e$ can be viewed as adding the hook S_{e+1} on a module X_j in $\mathcal{C}_\varepsilon$ with $X_j\varepsilon \cong N_j\varepsilon$. In particular, by [Erd, II.6.2] there is an irreducible map $X_j \to N_j$. This implies that X_j is the module positioned at $(\overline{j}, j)$ (with $j > 1$). By a dual argument, the module Y_j in $\mathcal{C}_\varepsilon$ positioned at $(1, j)$ with $j \ncong 1 \bmod e$ satisfies $Y_j\varepsilon \cong M_j\varepsilon$.

We now view the stable AR-quivers ${}_s\Gamma_A, {}_s\Gamma_{\varepsilon A\varepsilon}$ as quivers with vertices labelled by isoclass representatives of indecomposable modules and arrows labelled by irreducible maps. Using the above facts, we can describe the combinatorial effect of applying $- \otimes_A A\varepsilon$ to ${}_s\Gamma_A$ and its relation with ${}_s\Gamma_{\varepsilon A\varepsilon}$ as follows. We first remove the "diagonal" going into S_{e+1} (modules positioned at $(\overline{y-1}, y)$ and their connecting arrows), and the diagonal coming out of S_{e+1} (modules positioned at $(e+1, y)$ and their connecting arrows). We then connect the remaining part in the obvious way. Visualisation of this process is given in Figure 8.1. It is easy to see the inverse of this operation is described by the map $\omega_e^{(m)}$. □

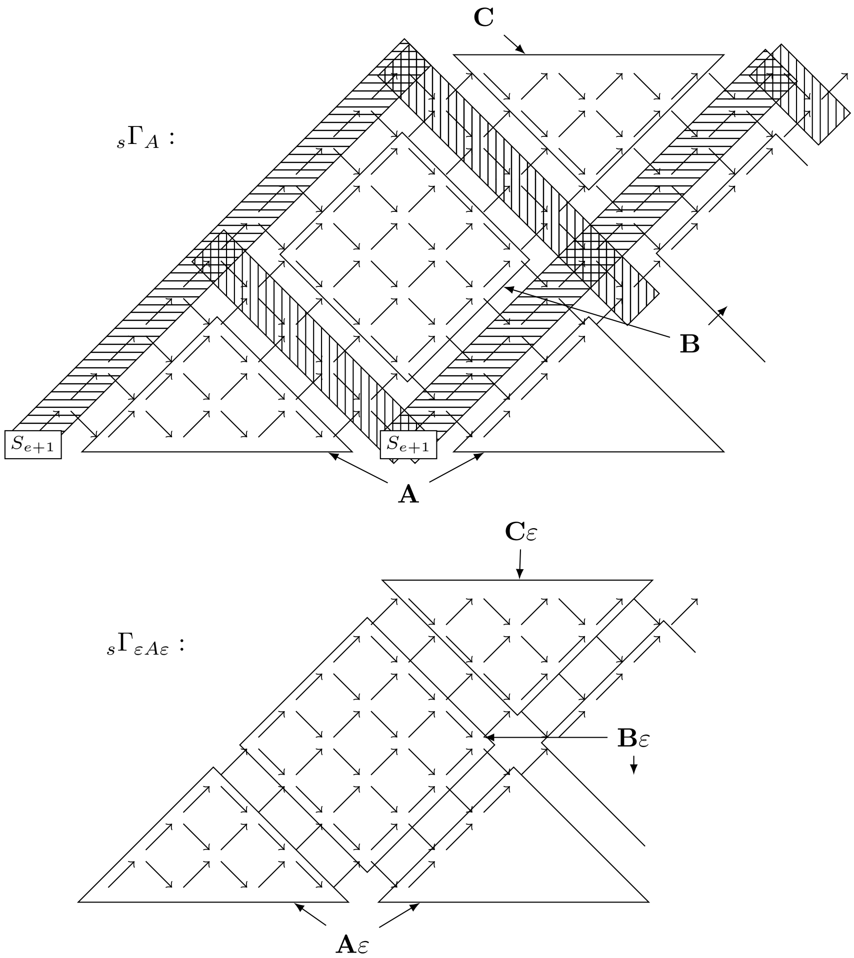


Figure 8.1: Example of deleting diagonals on ${}_s\Gamma_A$ to obtain ${}_s\Gamma_{\varepsilon A\varepsilon}$ for $A = B^G_{5,2}$.

Suppose we have a configuration $\mathcal{C}$ of $\mathbb{Z}A_{em}/\langle\tau^e\rangle$. By Theorem 6.1.3 and Theorem 6.1.1, $\mathcal{C}$ corresponds to the images of simple $B^G_{e,m}$-modules under some stable equivalence $F : \underline{\text{mod}}\text{-}B^G_{e,m} \to \underline{\text{mod}}\text{-}B^\star_{e,m}$ for some Brauer tree G. We can then speak of identifying a configuration with a Brauer tree (or the set of simple modules of the associated Brauer tree algebra).

Lemma 8.2.6. *If $\mathcal{C}$ is a configuration of $\mathbb{Z}A_{em}/\langle\tau^e\rangle$, then $\mathcal{C}^+ := \omega_e^{(m)}\mathcal{C} \cup \{(e+1,1)\}$ is a configuration of $\mathbb{Z}A_{(e+1)m}/\langle\tau^{e+1}\rangle$ where $\omega_e^{(m)}$ is given as in Lemma 8.2.5. Conversely, any configuration of $\mathbb{Z}A_{(e+1)m}/\langle\tau^{e+1}\rangle$ is of the form $\Omega^n\mathcal{C}^+$ for some configuration $\mathcal{C}$ of $\mathbb{Z}A_{em}/\langle\tau^e\rangle$ and some $n \in \mathbb{Z}$. Moreover, by identifying $\mathcal{C}^+$ as the set of simple $B^G_{e+1,m}$-modules with the simple module S positioned at $(e+1,1)$, $\mathcal{C}$ can then be identified as the set of simple $B^{G'}_{e,m}$-module, where G' is obtained from G by removing the leaf corresponding to S.*

Proof. The first part is extracted from [Rie4, Lemma 2.5]. We explain the reasoning for the remaining statements.

Note that any configuration $\mathcal{D}$ of $\mathbb{Z}A_{(e+1)m}/\langle\tau^{e+1}\rangle$ can be adjusted to a configuration containing $\{(e+1,1)\}$ by applying Ω^{-n} for some n. More explicitly, suppose (x,y) is vertex in $\mathcal{C}$ lying on the rim (such a vertex always exists by Lemma 8.2.4 and the fact that the configuration can be identified as the set of simple $B^{G}_{e+1,m}$-modules for some Brauer tree G). Apply τ^{-x} (i.e. $n=2x$) if $y=1$, or $\tau^{-x}\Omega^{-1}$ (i.e. $n=2x+1$) if $y=em$. This is the n required in the statement of the lemma.

We now assume $\mathcal{D}$ contains $(e+1,1)$. Retaining the notations of Lemma 8.2.5, we identify $\mathcal{D}$ with the set $\mathcal{S}_A$ of simple A-modules for some Brauer tree algebra $A=B^{G}_{e+1,m}$. Under such identification, $(e+1,1)$ corresponds to the simple module S_{e+1}. It is easy to see that for any simple module S not isomorphic to S_{e+1}, $S\varepsilon$ is a simple $B^{G'}_{e,m}$-module. Moreover, viewing G' as a subtree of G, S and $S\varepsilon$ correspond to the same edge. Hence, $(\mathcal{S}_A)\varepsilon\setminus\{0\}$ is the set $\mathcal{S}_{\varepsilon A\varepsilon}$ with $\varepsilon A\varepsilon\cong B^{G'}_{e,m}$. Identify $\mathcal{S}_{\varepsilon A\varepsilon}$ with a configuration $\mathcal{C}$ of $\mathbb{Z}A_{em}/\langle\tau^{e}\rangle$, then by Lemma 8.2.5 we have $\mathcal{D}=\mathcal{C}^{+}$. □

We call the process of removing a leaf *not attached to the exceptional vertex* as *cutting off a leaf*. Let $\mathcal{D}$ be a configuration of $\mathbb{Z}A_{(e+1)m}/\langle\tau^{e+1}\rangle$ with $(x,1)$ (resp. (x,em)) in $\mathcal{D}$. By the above lemma, we obtain a configuration $\mathcal{C}$ of $\mathbb{Z}A_{em}/\langle\tau^{e}\rangle$ with $\mathcal{C}^{+}=\tau^{-x}\mathcal{D}$ (resp. $\mathcal{C}^{+}=\tau^{-x}\Omega^{-1}\mathcal{D}$). We say that the configuration $\mathcal{C}$ (resp. $\Omega\mathcal{C}$) of $\mathbb{Z}A_{em}/\langle\tau^{e}\rangle$ is obtained by cutting off the leaf corresponding to (x,y).

Consider the following two configurations of $\mathbb{Z}A_{hm}/\langle\tau^{h}\rangle$: $\mathcal{C}^{-}_{h,m}=\{(i,1)|i=1,\ldots,h\}$ and $\mathcal{C}^{+}_{h,m}=\{(i,hm)|i=1,\ldots,h\}=\Omega\mathcal{C}^{-}_{h,m}$. They can be identified with the set $\mathcal{S}_A$ of simple modules of a Brauer star algebra A, and the set $\Omega\mathcal{S}_A$, respectively. Moreover, by Theorem 6.1.5, the two simple-minded systems of A corresponding to these configurations are the only simple-image simple-minded systems obtained by a applying stable self-equivalence to $\mathcal{S}_A$.

Corollary 8.2.7 (Tree Pruning Lemma)**.** *Suppose G is a Brauer tree with e edges and multiplicity m, where the valency of the exceptional vertex is h. Let $\mathcal{C}$ be the configuration of $\mathbb{Z}A_{em}/\langle\tau^{e}\rangle$ identified with the set of simple $B^{G}_{e,m}$-modules. Then the configuration obtained from $\mathcal{C}$ after repeatedly cutting off leaves is either $\mathcal{C}^{-}_{h,m}$ or $\mathcal{C}^{+}_{h,m}$. Moreover, if $m>1$, then the resulting configuration depends only on the τ-orbit $\mathcal{C}$ lies in.*

Proof. By repeatedly cutting off leaves, we will end up with a Brauer star with h edges. By Lemma 8.2.6, the final configuration obtained is either $\mathcal{C}^{-}_{h,m}$ or $\mathcal{C}^{+}_{h,m}$. We call this process (Brauer) *tree pruning*.

Let $\mathcal{C}$ be a configuration of $\mathbb{Z}A_{em}/\langle\tau^e\rangle$, identified with the set of simple $B^G_{e,m}$-modules and the set of edges of G. Using Lemma 8.2.6, one can observe that for all $(x,y) \in \mathcal{C}$, the value of y lies in either $\{1,\dots,e\}$ or $\{em-m+1,\dots,em\}$. Note that these two intervals are disjoint if and only if $m > 1$. We then say that $(x,y) \in \mathcal{C}$ is in the bottom interval (respectively, in the top interval). Observe that the vertices of the configuration obtained by cutting off a leaf lies in the same interval. In particular, if $(x,y) \in \mathcal{C}$ corresponds to an edge of G attached to the exceptional vertex and lies in the top (resp. bottom) interval, then after tree pruning $\mathcal{C}$ gives $\mathcal{C}^+_{h,m}$ (resp. $\mathcal{C}^-_{h,m}$), regardless of the order of cutting off leaves.

Note that stable self-equivalences of a Brauer tree algebra are generated by the Heller shift Ω (up to Morita equivalence) by Theorem 6.1.5. The automorphism induced by Ω on $\mathbb{Z}A_{km}/\langle\tau^k\rangle$ sends (i,j) to $(\overline{i-j+1}, km+1-j)$. This swaps the interval for which a vertex in a configuration lies in. On the other hand, $\Omega^2 \cong \tau$ leaves the interval for which a vertex lies in unchanged. This says that the result of tree pruning depends only on the τ-orbit of the configuration. $\square$

By the virtue of this result, for a Brauer tree with non-trivial exceptional vertex, we can classify the sms's of the associated Brauer tree algebra into two types as follows. We say that a configuration $\mathcal{C}$ of $\mathbb{Z}A_{em}/\langle\tau^e\rangle$ (with $m > 1$) is of "*bottom-type*" (resp. "*top-type*") if the resulting configuration after tree pruning is $\mathcal{C}^-_{h,m}$ (resp. $\mathcal{C}^+_{\ell,m}$). The two types distinguish a configuration from which τ-orbit it lies in. We denote $\mathsf{sms}_-(B^G_{e,m})$ (resp. $\mathsf{sms}_+(B^G_{e,m})$) for the set of sms's such that their corresponding configurations can be truncated to $\mathcal{C}^-_{h,m}$ (resp. $\mathcal{C}^+_{h,m}$) for some $h \in \{1,\dots,e\}$. Tree pruning lemma can now be restated in terms of simple-minded systems as follows.

Corollary 8.2.8. *For the Brauer tree algebra $A = B^G_{e,m}$ with non-trivial multiplicity $m > 1$, a simple-minded system of A lies in either $\mathsf{sms}_-(A)$ or $\mathsf{sms}_+(A)$. Moreover, there is a one-to-one correspondence between $\mathsf{sms}_-(A)$ and $\mathsf{sms}_+(A)$ given by Heller translate Ω.*

For $m = 1$ case, tree pruning will not give us a well-defined type as every vertex of the Brauer tree is (non-)exceptional. This undermines the non-bijection nature of the map $\mathfrak{F}$ in Theorem 8.2.1 in the case of A^n_n. However, comparing tree pruning on $B^\star_{e,m}$ with $m > 1$ and the corresponding procedure on $B^\star_{e,1}$ gives us the following relation between the configurations of their stable AR-quiver.

Proposition 8.2.9. *Let $\mathcal{C}$ be a configuration of $\mathbb{Z}A_{em}/\langle\tau^e\rangle$. Then for each $(x,y) \in \mathcal{C}$, either $(x,y) = (x,\tilde{y})$ or $(x, e(m-1)+\tilde{y})$ for a unique $\tilde{y} \in \{1,\dots,e\}$, and $\widetilde{\mathcal{C}} := \{(x,\tilde{y}) | (x,y) \in \mathcal{C}\}$ is a configuration of $\mathbb{Z}A_e/\langle\tau^e\rangle$. Moreover, the assignment induces a surjection from $\mathsf{Conf}(\mathbb{Z}A_{em}/\langle\tau^e\rangle)$ onto $\mathsf{Conf}(\mathbb{Z}A_e/\langle\tau^e\rangle)$.*

8.2.2 Mutation theories for self-injective Nakyama algebras

Combinatorial description of two-term tilting complexes. Given a tilting complex T, let E_T denote the derived equivalent algebra $\mathrm{End}_{\mathcal{T}}(T)$, and $F_T : D^b(\text{mod-}A) \to D^b(\text{mod-}E_T)$ be the associated derived equivalence. By a result of Rickard [Ric1], any algebra derived equivalent to a Brauer tree algebra is also a Brauer tree algebra, hence, we sometimes call a tilting complex over $B^\star_{e,m}$ to be "star-to-tree tilting complex" if A is the Brauer star algebra $B^\star_{e,m}$. However, there are infinitely many tilting complexes (even up to shifts and homotopy equivalences), and we should restrict to a much more refined subclass when studying the homological theories around these algebras. Our choice in the current article is the set of two-term tilting complexes. The main reason comes from the fact that every derived equivalence between representation-finite self-injective algebras given by a tilting complex is a composition of derived equivalences given by two-term tilting complexes as shown by Abe and Hoshino [AH]. In [AIR1], it is shown that the set of two-term tilting complexes of symmetric algebra is in bijection with the set of functorially finite torsion classes of its module category, emphasising the importance of two-term tilting complexes in the study of homological behaviour of a symmetric algebra.

We will use the combinatorial description of two-term tilting complexes from a mixture of results from [SZI, RS, Ada]. In [Ada], combinatorial descriptions are given to the so-called support τ-tilting modules; since this is not our main interest, we will not go through the definitions of τ-tilting theory. Instead we just recall the result from [AIR1], which says that the set of two-term silting complexes over a finite dimensional algebra A is in order-preserving correspondence to the set of support τ-tilting A-module, so that we can translate the results from [Ada] for our needs. As we have mentioned in a previous chapter, a silting complex is tilting if and only if it is Nakayama-stable; this translates into the following result in τ-tilting theory: For a finite dimensional algebra A, there is a mutation-respecting correspondence between $\mathsf{2tilt}(A)$ and the set of Nakayama-stable support τ-tilting A-modules. This is also implicit in a work of Mizuno [Miz].

For a self-injective Nakayama algebra A^ℓ_n, the result of [Ada] gives us a combinatorial description of two-term tilting complexes over A^ℓ_n via triangulations on a punctured regular convex n-gon.

Definition 8.2.10 (cf. [Ada])**.** *Let $i, j \in \{1, \dots, n\}$, and $\mathcal{G}_n$ be a punctured regular convex n-gon (*punctured n-disc*) with vertices labelled by $\{1, \dots, n\}$ with counter-clockwise ordering.*

(1) An inner arc *$\langle j, i\rangle$ in $\mathcal{G}_n$ is a path from the vertex i to the vertex j homotopic to the boundary path $i, \overline{i+1}, \cdots, \overline{i+l} = j$ such that $1 < l \leq n$. Then we call i (respectively, j) a* initial *(respectively,* terminal*) point and $\ell(\langle i, j\rangle) := l$ the* length *of the inner arc.*

(2) A projective arc $\langle \bullet, j\rangle$ *in* $\mathcal{G}_n$ *is a path from the puncture to the vertex* j*. Then we call* j *a terminal point.*

(3) An admissible arc *is an inner arc or a projective arc. We denote by* $\mathcal{A}(n)$ *the set of admissible arcs in* $\mathcal{G}_n$*.*

(3) Two admissible arcs in $\mathcal{G}_n$ *are called* compatible *if they do not intersect in* $\mathcal{G}_n$ *(except at their initial and terminal points).*

(4) A triangulation *of* $\mathcal{G}_n$ *is a maximal set of distinct pairwise compatible admissible arcs. We denote by* $\mathcal{T}(n)$ *the set of triangulations of* $\mathcal{G}_n$*, and by* $\mathcal{T}(n;l)$ *the subset of* $\mathcal{T}(n)$ *consisting of triangulations such that the length of every inner arc has length at most* $l \leq n$*.*

Remark 8.5. The original notation used by Adachi is $\langle i, j\rangle$ instead of $\langle j, i\rangle$. This is due to the different vertex labelling and direction of composition of arrows on the quiver we use, so that the new notation still matches up with the terms appearing in two-term tilting complexes, as we will see in the following theorem.

We also note that $\mathcal{T}(n)$ admits a mutation theory, namely, for a given triangulation $X \in \mathcal{T}(n)$ and an admissible arc $a \in X$, the (irreducible) mutation of X with respect to a is a unique triangulation $\mu_a(X) \in \mathcal{T}(n)$ obtained by replacing the arc a by another (unique) admissible arc. This gives a partial order structure on $\mathcal{T}(n)$ with the triangulation $\{\langle \bullet, i\rangle | i = 1, \ldots, n\}$ being the unique maximal one. Also recall from the previous section that the set of tilting complexes (up to shifts and homotopy equivalences) admits a partial ordering given by $T \geq U$ if and only if $\mathrm{Hom}(T, U[> 0]) = 0$, which is compatible with its mutation theory (Theorem 8.1.8). We restate the theorem of Adachi using two-term silting complexes instead of support τ-tilting modules.

Theorem 8.2.11 ([Ada]). *Let* $n, \ell \in \mathbb{N}$*,*

1. *The map* $\langle j, i\rangle \mapsto (P_{\overline{j-1}} \to P_i)$ *and* $\langle \bullet, i\rangle \mapsto P_i$ *the stalk complex concentrated in degree 0 induces a map* ϕ_- *from* $\mathcal{T}(n; \min\{\ell, n\})$ *to the subset* $\mathsf{2silt}_-(A_n^\ell)$ *of two-term silting complexes of* A_n^ℓ*, which is order-preserving when* $n \leq \ell$*, i.e.* $\phi_-(\mu_a(X)) = \mu^-_{\phi_-(a)}(\phi_-(X))$ *if* $X \geq \mu_a(X)$*.*

   Dually, the map $\langle j, i\rangle \mapsto (P_j \to P_{\overline{i+1}})$ *and* $\langle \bullet, i\rangle \mapsto P_i$ *the stalk complex concentrated in degree* -1 *induces a map* ϕ_+ *from* $\mathcal{T}(n; \min\{\ell, n\})$ *to the subset* $\mathsf{2silt}_+(A_n^\ell)$ *of two-term silting complexes of* A_n^ℓ*, which is anti-order-preserving when* $n \leq \ell$*, i.e.* $\phi_+(\mu_a(X)) = \mu^+_{\phi_+(a)}(\phi_+(X))$ *if* $X \geq \mu_a(X)$*.*

In particular, there is a bijection between $\mathsf{2silt}_-(A_n^\ell)$ *and* $\mathsf{2silt}_+(A_n^\ell)$.

2. $\mathsf{2silt}(A_n^\ell) = \mathsf{2silt}_-(A_n^\ell) \sqcup \mathsf{2silt}_+(A_n^\ell)$.

3. *For a two-term silting complex* T *of* A_n^ℓ, $T \in \mathsf{2silt}_-(A_n^\ell)$ *(resp.* $T \in \mathsf{2silt}_+(A_n^\ell)$*) if and only if all its indecomposable stalk complexes are concentrated in degree 0 (resp.* -1*).*

We now refine this result on tilting complexes. Note that the Nakayama permutation of A_n^ℓ is a product of e disjoint n/e-cycles, so the effect of applying Nakayama functor on a silting complex now manifests as turning the punctured n-disc anti-clockwise by $2\pi/(n/e)$. Recall that two-term tilting complexes of self-injective algebras are just Nakayama-stable silting complexes, so they correspond to triangulations of punctured n-disc with a $2\pi/(n/e)$-rotation symmetry. Such triangulations will then be in correspondence with triangulations on a punctured e-disc by identifying the punctured point of n-disc with punctured point of e-disc, and vertex i with $ke+i$ for all $k = 1, \dots, n/e-1$ and $i = 1, \dots, e$.

Example 8.6. Let $n = 12, \ell = 16$, Figure 8.2 shows a triangulation of a punctured 12-disc on the left, which is $\pi/3$-rotationally symmetric. This triangulation can be identified with a triangulation of a 4-disc shown on the right.

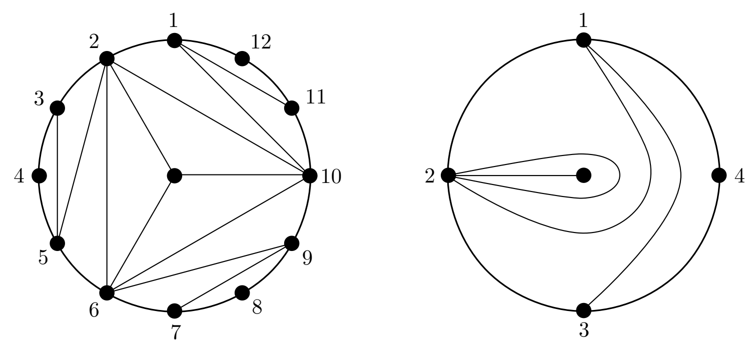


Figure 8.2: Identifying a rotationally symmetric triangulation of a 12-disc and a triangulation of a 4-disc

Summarising, we have the following result:

Theorem 8.2.12. *For any* $n, \ell \in \mathbb{Z}$ *and* $e = \gcd(n, \ell)$*, there are bijections:*

$$\begin{array}{ccccc} \mathsf{2tilt}_-(A_n^\ell) & \leftrightarrow & \mathcal{T}(e) & \leftrightarrow & \mathsf{2tilt}_-(A_e^\ell) \\ \mathsf{2tilt}_+(A_n^\ell) & \leftrightarrow & \mathcal{T}(e) & \leftrightarrow & \mathsf{2tilt}_+(A_e^\ell) \end{array} \tag{8.2.2}$$

where those in top row are order-preserving and those in the bottom row are anti-order-preserving respectively. In particular, we have mutation-preserving bijections $\mathsf{2tilt}_\pm(A_n^\ell) \leftrightarrow \mathsf{2tilt}_\pm(A_e^\ell)$.

Remark 8.7. The reader should be careful when considering the case $n > \ell$. Note that rotational symmetry restricts the lengths of admissible arcs to be less than $e \le \ell$, hence the assignment from a rotationally symmetric triangulation in $\mathcal{T}(n;\ell)$ to a triangulation in $\mathcal{T}(e)$ as before is still well-defined and (anti-)order-preserving.

This result is a refinement of the covering theory for derived categories of representation-finite self-injective algebras used in [Asa1]. Readers familiar with covering theory in [Asa1] would naturally expect such a result as a consequence of [CKL, 5.11]. We note that it is not clear from the proofs of [Asa1] whether all (two-term) tilting complexes of A_n^ℓ can be obtained by using covering theory of the (two-term) tilting complexes of A_e^ℓ; our result gives an affirmative answer.

Constructing Brauer trees from a two-term tilting complex. Given a two-term tilting complex T of $B_{e,m}^\star = A_e^{em}$, there is a simple construction to determine the Brauer tree G associated to E_T using results of Schaps and Zakay-Illouz [SZI,RS], which we will go through in the next section. Here, we use the Schaps-Zakay-Illouz construction to obtain G directly from the triangulation of a punctured e-disc.

Consider a triangulation $X \in \mathcal{T}(e)$. For each vertex i on the punctured disc, we distinguish some sets of inner arcs in X as follows

$$\begin{aligned} A_i^-(X) &= \{\langle j,i\rangle | j \in \{1,\dots,e\}\},\\ A_i^+(X) &= \{\langle i,k\rangle | k \in \{1,\dots,e\}\}. \end{aligned} \tag{8.2.3}$$

Note that X is the disjoint union of projective arcs and arcs in $A_i^-(X)$ (resp. $A_i^+(X)$) over all i. One can now construct the Brauer tree G associated to the endomorphism ring of $\phi_-(X)$ or $\phi_+(X)$ using the results in [SZI] as follows.

Proposition 8.2.13. *Let X be a triangulation of a punctured disc. Construct a pair of Brauer trees G_X^- and G_X^+ as follows.*

(1) Let $\{v_0, v_1, \dots, v_e\}$ be vertices of $G_\pm$. For each projective arc $\langle\bullet, i\rangle \in X$, connect v_0 and v_i by an edge.

(2) For each $i \in \{1,\dots,e\}$ and each arc in $\langle j,i\rangle \in A_i^-(X)$ (resp. $\langle i,k\rangle \in A_i^+(X)$), connect the vertices v_i and $v_{\overline{j-1}}$ of G_X^- (resp. v_i and $v_{\overline{k+1}}$ of G_X^+) by an edge.

Then G_X^- (resp. G_X^+) with exceptional vertex v_0 and multiplicity m is precisely the Brauer tree G such that $B_{e,m}^G \cong \mathrm{End}_{K^b(\text{proj-}B_{e,m}^\star)}(\phi_-(X))$ (resp. $\mathrm{End}_{K^b(\text{proj-}B_{e,m}^\star)}(\phi_+(X))$).

Proof. We prove the minus part of the proposition; the plus part can be done dually.

Let $\{v_0,\dots,v_n\}$ be the set of vertices of the Brauer tree G. For any projective arc $\langle\bullet, j\rangle \in X$, we put an edge connecting v_0 and v_j. For any inner arc $\langle i,k\rangle$, we put an edge connecting v_{i-1}

and v_k. Therefore, for each arc $a \in A_i^-(X)$, $\phi_-(a)$ has degree 0 component P_i, and any other arc a attached to i not in $A_i^-(X)$ will be sent to a complex with degree -1 component P_{i-1} under ϕ_-. According to Theorem 3 of [SZI], the counter-clockwise ordering of edges around each vertices can then be chosen to be compatible with the cyclic ordering on $\{1, \ldots, e\}$, i.e. $E_{i_1}, \ldots, E_{i_r}$ is the counter-clockwise ordering of edges around v_k, connected with $v_{i_1}, \ldots, v_{i_r}$ respectively, if and only if $i_1 < i_2 < \cdots < i_r$ in $\{1, \ldots, e\}$. By the main theorem of [SZI], the tree constructed this way is then the Brauer tree G with exceptional vertex v_0 of multiplicity m. $\square$

Let $\mathsf{BrTree}(e, m)$ be the set of Brauer tree with e edges and multiplicity m. This proposition says that we obtain a pair of well-defined maps $\psi_\pm : \mathcal{T}(e) \to \mathsf{BrTree}(e, m)$ given by $X \mapsto G_X^\pm$ for any $m > 1$.

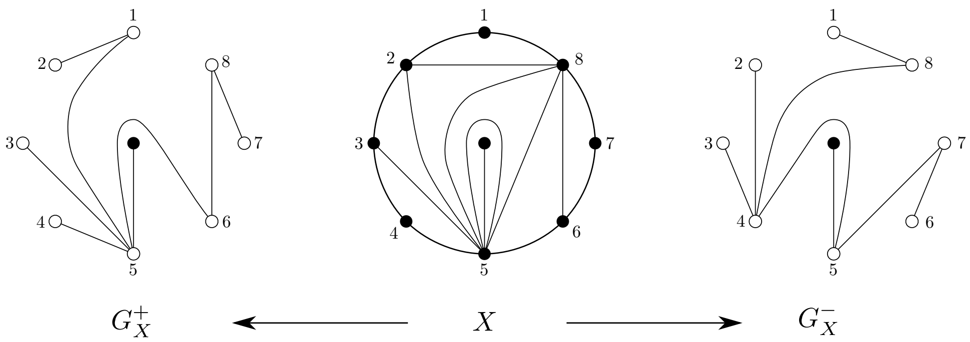


Figure 8.3: Brauer trees from a triangulation of punctured disc

Corollary 8.2.14. *Let T be a two-term tilting complexes in $\mathsf{2tilt}_-(A_n^\ell)$, and G be $\psi_-\phi_-^{-1}(T)$. Then each minimal Nakayama-stable summand M of T corresponds to an edge of G. Moreover, under this correspondence, each minimal Nakayama-stable summands concentrated in degree 0 corresponds to an edge emanating from the exceptional vertex of G.*

Proof. Immediate from combining Proposition 8.2.13 with Theorem 8.2.11 and Theorem 8.2.12. $\square$

Remark 8.8. An analogous statement holds for any two-term tilting complex in $\mathsf{2tilt}_+(A_n^\ell)$.

Some properties of mutations. Mutation of tilting complexes can be reformulated as mutation on the class of derived equivalent algebras: for a summand P of an algebra A, let T be the tilting mutation $\mu_X^-(A)$, then we can define the left algebra mutation as the algebra E_T. On the class of Brauer tree algebras, this gives a mutation on the Brauer trees. This mutation has been given in several papers already [KZ, Kau, Aih2]. We recommend [Aih2] for the most concise and precise description that is sufficient for our needs.

Definition 8.2.15 (Mutation of Brauer tree)**.** *Let G be a Brauer tree, and i be an edge of G. The* left mutation *of G at i, denoted $\mu_i^-(G)$, can be constructed as follows. Suppose the vertices attached to i are u and v, with j and k being the previous edges in the cyclic ordering around u and v respectively. The mutated tree is given by removing the edge i from G, and replacing with an edge i' connected to j and k. In particular, if (without loss of generality) u is only of valency one (i.e. an extremal vertex), then i' is attached to u again.*

Similarly, define the right mutation $\mu_i^+(G)$ by removing i and connecting i' to the next *edges in the cyclic ordering around u and v. The two mutations can be visualised as in Figure 8.4.*

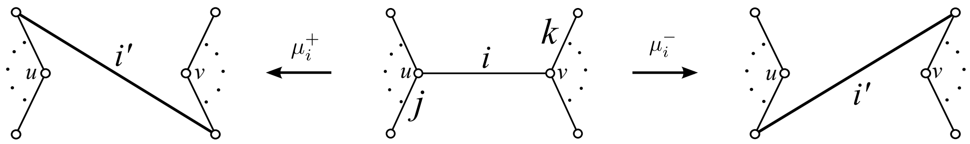


Figure 8.4: Mutation of Brauer tree.

Needless to say, mutation of Brauer trees is compatible with mutations of (two-term) tilting complexes and triangulations on punctured discs. In this subsection, we will only consider the case when the algebra is $B^{\star}_{e,m}$, so if T is a star-to-tree tilting complex which takes the Brauer star to a Brauer tree G, and letting i be an edge corresponding to a summand X of T, then $\mu_X^{\pm}(T)$ is a star-to-tree tilting complex which takes the Brauer star to the mutated Brauer tree $\mu_i^{\pm}(G)$.

From Theorem 8.1.3 (or [Aih1, Thm 3.5]), we know that every tilting complex of A_n^{ℓ} can be obtained by a sequence of irreducible left (or right) mutations starting from A_n^{ℓ}. Our aim now is to find some "canonical sequence" to obtain any given two-term tilting complex. We start with some sufficient criteria for a mutated tilting complex to be two-term.

Lemma 8.2.16. *Let T be a tilting complex concentrated in non-positive (resp. non-negative) homological degrees, and X be a minimal Nakayama-stable summand of T. We have the following:*

(1) If $\mu_X^-(T)$ (resp. $\mu_X^+(T)$) is two-term, then so is T.

(2) If T is two-term and X is a direct sum of stalk complexes concentrated in homological degree 0, then $\mu_X^-(T)$ (resp. $\mu_X^+(T)$) is two-term concentrated in homological degree -1 (resp. $+1$) and 0.

Proof. We prove the minus version of the statements; plus version can be done analogously.

(1): Recall from [AI] that there is a partial order on the tilting complexes defined by $T \geq U$ if $\mathrm{Hom}_{\mathcal{T}}(T, U[i]) = 0$ for all $i > 0$. As T and $\mu_X^-(T)$ are both concentrated in non-positive degrees, it follows from [Aih1, 2.9] that, $A \geq \mu_X^-(T) \geq A[1]$ and $A \geq T \geq A[l]$ for some $l \geq 1$. We also have $T \gneqq \mu_X^-(T)$ from [AI, 2.35] (for self-injective version see [CKL, 5.11]). These combine to give $A \geq T \gneqq \mu_X^-(T) \geq A[1]$, and so $l = 1$, which means T is two-term by [Aih1, 2.9].

(2): This is easy to see from the definition of mutation, for $T = X \oplus M$ with $\mu_X^-(T) = Y \oplus M$, then Y is the cone of a morphism from stalk complex concentrated in degree 0 to a two-term complex concentrated in degree 0 and -1, so Y is two-term as well. $\square$

Proposition 8.2.17. *Suppose $T \in \mathsf{2tilt}_-(A_n^\ell)$ (resp. $T \in \mathsf{2tilt}_+(A_n^\ell)$). Then T can be obtained by h irreducible left (resp. right) mutations starting from A_n^ℓ (resp. $A_n^\ell[1]$) for some $h < e = \gcd(n, \ell)$.*

Proof. Again, we only prove the minus-version of the statement.

Let U be the unique two-term tilting complex in $\mathsf{2tilt}_-(A_e^{em})$ with any $m > 1$ corresponding to T under the correspondence of Theorem 8.2.12. Then $E_T \cong B_{e,m}^G$ with $G = \psi_-\phi_-^{-1}(T)$ with valency of exceptional vertex being $h < e$.

By Lemma 8.2.16 and Corollary 8.2.14, it suffices to prove the following combinatorial result: The Brauer tree G with valency $e - h$ at the exceptional vertex can be obtained by h left mutations at edges attached to the exceptional vertex. Thanks to [AI, Prop 2.33], this can be proved by finding an algorithm to obtain the Brauer star from the Brauer tree using right mutations, such that after each mutation, the valency of exceptional vertex is increased by 1.

Given an edge i of G connected to the exceptional vertex, we can define a branch of G connected to i as the following subtree of G. If u is the exceptional vertex itself, or u is a non-exceptional vertex for which there is a path from u to the exceptional vertex, ending at edge i, then u is in the branch of i. The edges of the branch connected to i are all the edges of G for which both ends are as described before. In particular, a branch is a Brauer tree where the exceptional vertex is of valency 1.

Our algorithm is to repeat the following recursively on every branch of the tree G: take a branch of the tree G connected to $i = i_0$ which contains more than one edge, let the non-exceptional end of i_0 be u and suppose $i_0, i_1, \ldots, i_n$ (note $n \geq 1$ always) is the cyclic ordering of edges around u, we then right mutate at i_n. According to the mutation rule of Brauer trees, i_n is then removed and replaced by an edge with one end connected to the exceptional vertex. Therefore the valency of the exceptional vertex has increased precisely by 1. $\square$

Example 8.9. Let $T = \bigoplus_{i=1}^{6} T_i \in 2\mathsf{tilt}(B^{\star}_{e,m})$ be given by

$$\begin{array}{rlrlrl} T_1 = (0 & \to P_2), & T_2 = (P_3 & \to P_2), & T_3 = (\ 0 & \to P_4), \\ T_4 = (P_1 & \to P_4), & T_5 = (P_1 & \to P_5), & T_6 = (P_1 & \to P_6). \end{array}$$

Use Proposition 8.2.13 to obtain a Brauer tree and apply the proof of Theorem 8.2.17 to deduce $T = \mu^-_{P_5}\mu^-_{P_6}\mu^-_{P_1}\mu^-_{P_3}(A)$. The details of this computation can be found in Figure 8.5.

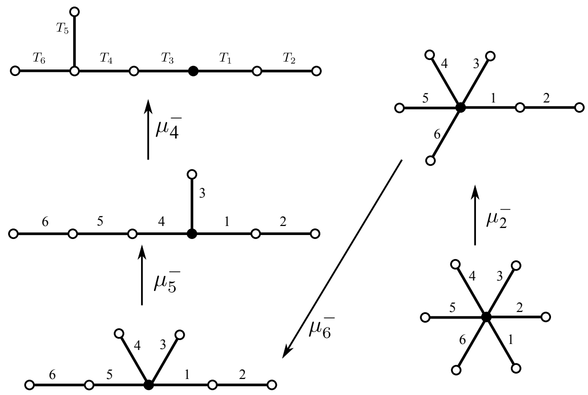


Figure 8.5: An example for Proposition 8.2.17

Mutation of simple-minded systems. Having known how to obtain the sequence of mutation to reach any given two-term tilting complex, we need some observations on the effect of mutation on simple-minded systems. We will only consider the case $A_e^{em} = B^{\star}_{e,m}$, analogous results can be obtained by covering theory.

Lemma 8.2.18. *Let $\mathcal{S}$ be the set of simple $B^G_{e,m}$-modules, and S_i be a simple $B^G_{e,m}$-module corresponding to an edge i of G. Then an irreducible left mutation $\mu^-_{S_i}(\mathcal{S})$ replaces exactly two (indecomposable) modules in $\mathcal{S}$ if i is a leaf, or replaces exactly three modules in $\mathcal{S}$ otherwise. In particular, at most three (indecomposable) modules in any simple-minded system of any Brauer tree algebra will be replaced in performing an irreducible left mutation.*

Proof. The first statement follows from straightforward calculation using the definition of mutation, or alternatively, implicitly implied by a result of Okuyama in his unpublished preprint [Oku, Lemma 2.1], which also appears in the proof in [Aih2, Lemma 3.4].

Now suppose $\mathcal{S}$ is an arbitrary simple-minded system of an arbitrary Brauer tree algebra. We know from Theorem 6.1.3 that there is a stable equivalence ϕ making $\mathcal{S}$ a simple-image sms. The last statement now follows from the fact that $\mu^-_X(\mathcal{S}) = \phi^{-1}(\mu^-_{\phi X}(\phi\mathcal{S}))$. □

Remark 8.10. Dually, the same result holds for irreducible right mutation. We also remark that, in the notation of Figure 8.4, the modules that are replaced after mutating at S_i are precisely the simple modules S_j and/or S_k.

Recall from Proposition 8.2.9 that associated each $\mathcal{S} \in \mathsf{sms}(B^G_{e,m})$ with $m > 1$, there is a sms $\widetilde{\mathcal{S}} \in \mathsf{sms}(B^G_{e,1})$; the following observation connects the mutation action between them.

Lemma 8.2.19. *Suppose $\mathcal{S}$ is a simple-minded system of $B^G_{e,m}$ for some Brauer tree G with $m > 1$. Then for any $X \in \mathcal{S}$, $\widetilde{\mu^\pm_X(\mathcal{S})} = \mu^\pm_{\widetilde{X}}(\widetilde{\mathcal{S}})$.*

Proof. If $\mathcal{S}$ is the image of simple $B^H_{e,m}$-modules, then $\widetilde{\mathcal{S}}$ is the image of simple $B^H_{e,1}$-modules. Mutating at $X \in \mathcal{S}$ corresponding to an edge i in H implies that we have $\mu^\pm_X(\mathcal{S})$ as image of simple $B^{\mu^\pm_i(H)}_{e,m}$-modules. So $\widetilde{\mu^\pm_X(\mathcal{S})}$ is the image of simple $B^{\mu^\pm_i(H)}_{e,1}$-modules given by $\mu^\pm_{\widetilde{X}}(\widetilde{\mathcal{S}})$. $\square$

The following observation is crucial for the proof of the main theorem. Also recall from a previous section that the type of a configuration indicates the rim where the simple module lies after tree pruning (cf. Lemma 8.2.7).

Proposition 8.2.20. *Let $\mathcal{S}$ is a simple-minded system of $B^\star_{e,m}$ which is a $B^G_{e,m}$-simple-image for some Brauer tree G, with $m > 1$ and the valency of exceptional vertex of G being $h > 1$. If $X \in \mathcal{S}$ is the image of a simple $B^G_{e,m}$-module corresponding to an edge attached to the exceptional vertex, then $\mathcal{S}$, $\mu^+_X(\mathcal{S})$, and $\mu^-_X(\mathcal{S})$ are all of the same type.*

Proof. We will use the labelling for edges as in Definition 8.2.15 and suppose X corresponds to an edge i attached to the exceptional vertex v. We show the proof for left mutation; right mutation is done analogously. It follows from Lemma 8.2.18 that the effect of irreducible mutation on the configuration is to replace at most three of the vertices. After tree pruning, the effect on the configuration $\mathcal{C}_h$ corresponding to $\mathcal{S}$ is either $\mathcal{C}^-_{h,m}$ (bottom-type) or $\mathcal{C}^+_{h,m}$ (top-type). After the mutation at X, all but one module corresponding to the edges attached to v remain unchanged in $\mu^-_X(\mathcal{S})$. So after pruning the mutated tree, we are left with a configuration $\mathcal{C}_{h-1}$ of $\mathbb{Z}A_{(h-1)m}$ such that the $h-2$ vertices in $\mathcal{C}_{h-1}$ lie in the same rim as their corresponding vertices in $\omega^m_e(\mathcal{C}_{h-1}) \subset \mathcal{C}_h$. When $h > 2$, this forces $\mathcal{C}_{h-1}$ and $\mathcal{C}_h$ to be of the same type, and hence $\mathcal{S}$ and $\mu^-_X(\mathcal{S})$. For $h = 2$, apply tree pruning on G until reaching a Brauer star with 2 edges. Now considering an irreducible (left or right) mutation of the set of simple $B^\star_{2,m}$-modules $\{(1,1),(2,1)\}$, one will obtain either $\{(2,2m),(1,1)\}$ or $\{(1,2m),(2,1)\}$. Hence the three sms's are of the same type. $\square$

8.2.3 Proof of Theorem 8.2.1

This entire section is devoted to proving the main theorem 8.2.1. For convenience, we denote A the Brauer star algebra $B^\star_{e,m}$ with multiplicity $m > 1$ throughout this section. As usual, for any algebra Λ, we denote by $\mathcal{S}_\Lambda$ the set of simple Λ-modules, and we will always identify $\mathsf{sms}(A)$ with $\mathsf{Conf}(\mathbb{Z}A_{em}/\langle\tau^e\rangle)$ implicitly.

Our plan to prove Theorem 8.2.1 is to first show that it holds (i.e. $\mathfrak{F}$ is a bijection) for the case $A = B^\star_{e,m}$, then extend to A^ℓ_n with $\ell \nmid n$. Afterwards, we show $\mathfrak{F}$ is surjective non-injective for other cases of A^ℓ_n.

We will achieve the first goal by showing this:

Theorem 8.2.21. *Restricting $\mathfrak{F} : \mathsf{2tilt}(A) \to \mathsf{sms}(A)$ to the disjoint subsets $\mathsf{2tilt}_\pm(A)$, we have bijections $\mathfrak{F}_\pm : \mathsf{2tilt}_\pm(A) \to \mathsf{sms}_\pm(A)$. Moreover, $\mathfrak{F}_\pm$ are mutation-preserving, that is, for all $T \in \mathsf{2tilt}_-(A)$ and $T' \in \mathsf{2tilt}_+(A)$, with indecomposable pretilting summand X and X' respectively, such that $\mu^-_X(T)$ and $\mu^+_{X'}(T')$ are two-term tilting complexes, then*

$$\mathfrak{F}_-(\mu^-_X(T)) = \mu^-_{\widetilde{X}}(\mathfrak{F}_-T) \quad \textit{and} \quad \mathfrak{F}_+(\mu^+_{X'}(T')) = \mu^+_{\widetilde{X'}}(\mathfrak{F}_+T')$$

for some indecomposable A-modules $\widetilde{X}$ and $\widetilde{X'}$.

Proof. The mutation preserving property is inherited from the composition of mutation preserving maps $\mathsf{2tilt}(A) \to \mathsf{tilt}(A)$ and $\mathsf{tilt}(A) \to \mathsf{sms}(A)$. This is by combining the following lemmas 8.2.22, 8.2.26, 8.2.27.

□

Lemma 8.2.22. $\mathfrak{F}_\pm : \mathsf{2tilt}_\pm(A) \to \mathsf{sms}_\pm(A)$ *are well-defined.*

Proof. For each $T \in \mathsf{2tilt}_-(A)$, by Theorem 8.2.17, T can be obtained by iterative irreducible left mutation with respect to stalk complexes starting from A. Since stalk complexes correspond to edges attached to the exceptional vertex in G where $E_T \cong B^G_{e,m}$, and mutation is preserved when restricting a standard derived equivalence to stable equivalence (Proposition 8.1.5), we can repeatedly apply Proposition 8.2.20 and $\mathcal{S} = \mathfrak{F}(T)$ has the same type as $\mathfrak{F}(A) = \mathcal{S}_A \in \mathsf{sms}_-(A)$, so $\mathcal{S} \in \mathsf{sms}_-(A)$.

If $T \in \mathsf{2tilt}_+(A)$, then T can be obtained by iterative irreducible right mutation with respect to stalk complexes (which are concentrated in degree -1) starting from $A[1]$. So $U = T[-1]$ can be obtained by iterative irreducible right mutation with respect to stalk complexes concentrated in degree 0. Again, applying Proposition 8.2.20 repeatedly, then $\mathcal{S} = \overline{F_U}^{-1}(\mathcal{S}_{E_U})$ has the same type as $\mathfrak{F}(A)$. Since $E_U \cong B^G_{e,m} \cong E_T$ for some Brauer tree G, so $\mathcal{S}_{E_U} = \mathcal{S}_{E_T}$. So we have

$$\begin{aligned}
\mathfrak{F}(T) &= \overline{F_T}^{-1}(\mathcal{S}_{E_T}) \\
&= \overline{F_{U[1]}}^{-1}(\mathcal{S}_{E_U}) \\
&= \overline{F_U \circ [-1]}^{-1}(\mathcal{S}_{E_U}) \\
&= \Omega^{-1} \circ \overline{F_U}^{-1}(\mathcal{S}_{E_U}) \\
&= \Omega^{-1}\mathcal{S}
\end{aligned}$$

Note that the third equality follows from the fact that standard derived equivalence $F_T = F_{U[1]}$ is naturally isomorphic to the composition $F_U \circ [-1]$, and the fourth equality follows from the fact that the quotient functor $\eta_A : D^b(\text{mod-}A) \to \underline{\text{mod}}\text{-}A$ is triangulated and so the inverse suspension functor $[-1]$ restricts to Ω. As $\mathcal{S}$ is of bottom-type, using Corollary 8.2.8, $\mathfrak{F}(T) = \Omega^{-1}(\mathcal{S})$ is of top-type. $\square$

Recall the following result implicit from [SZI, RS].

Proposition 8.2.23. *For each Brauer tree G with e edges and multiplicity $m > 1$, there is a pair of two-term tilting complexes $T_\pm \in \mathsf{2tilt}_\pm(A)$ such that $E_{T_\pm} \cong B^G_{e,m}$. In particular, $\psi_\pm\phi_\pm^{-1} : \mathsf{2tilt}_\pm(A) \to \mathsf{BrTree}(e,m)$ are surjective.*

Proof. As remarked in [RS, Example 1], one can put extra combinatorial data on G resulting in so called Brauer tree with completely folded pointing, and then use the main theorem of [SZI] to construct a two-term tilting complex concentrated in degree 0 and 1 with stalk summands concentrated in degree 0. So we can just shift this complex to obtain $T_+ \in \mathsf{2tilt}(A)$. To get T_- one uses the dual pointing of the completely folded pointing (see [RS, Example1]) and apply Schaps-Zakay-Illouz correspondence, but without shifting this time. $\square$

Lemma 8.2.24. *Suppose $T_\pm \in \mathsf{2tilt}_\pm(A)$ with $G_\pm = \psi_\pm\phi_\pm^{-1}(T)$. Every $T'_\pm \in \mathsf{2tilt}_\pm(A)$ with $\psi_\pm\phi_\pm^{-1}(T') = G_\pm$ is obtained by cyclically permuting the labels of projective indecomposable modules in the components of $T_\pm$.*

Proof. The statement follows by observing closely the construction and proof of the main theorem of [SZI]. To be slightly more precise, two-term tilting complex T with $E_T \cong B^G_{e,m}$ corresponds to a completely folded pointing (or its dual) Brauer tree with a choice of non-exceptional vertex. Changing this choice corresponds to cyclically permuting the labels of projective indecomposable modules in the components of T. $\square$

Let C_e be the cyclic group of order e. We note that its generator acts on a triangulation of the punctured e-disc by rotating by 1 unit. We obtain the following interesting bijection which is not relevant to the proof of the main theorem.

Corollary 8.2.25. *$\psi_\pm$ induces two different bijections between the set $\mathcal{T}(e)/C_e$ of orbits of triangulations of a punctured e-disc and the set $\mathsf{BrTree}(e, m)$ of Brauer trees with $m > 1$.*

Proof. Surjectivity follows from Proposition 8.2.23. It follows from Lemma 8.2.24 that and T, T' with $E_T \cong B^G_{e,m} \cong E_{T'}$, then $\phi_\pm^{-1}(T)$ differs from $\phi_\pm^{-1}(T')$ by a rotation of the punctured disc. The two bijections are different as we can see from Figure 8.3 that $\psi_-(X)$ and $\psi_+(X)$ are not the same Brauer tree in general. $\square$

Lemma 8.2.26. *If $\mathcal{S} \in \mathsf{sms}(A)$ is in the image of $\mathfrak{F}_-$ (or $\mathfrak{F}_+$), then $\tau^n\mathcal{S}$ for any $n \in \mathbb{Z}$ is also in the image of $\mathfrak{F}_-$ (resp. $\mathfrak{F}_+$). In particular, $\mathfrak{F}_\pm$ are surjective.*

Proof. Again, we prove only for $\mathcal{S} \in \mathsf{sms}_-(A)$, the other case is analogous. The first statement follows from Lemma 8.2.24, as changing labelling of projective modules in T corresponds to changing the x-coordinate of the configuration corresponding to $\mathfrak{F}(T)$.

Since every sms of A is simple-image, we have $\mathcal{S} = \phi(\mathcal{S}_{B^G_{e,m}})$ for some Brauer tree G and stable equivalence $F : \underline{\mathrm{mod}}\text{-}B^G_{e,m} \to \underline{\mathrm{mod}}\text{-}A$. By Proposition 8.2.23, we can find a $T' \in \mathsf{2tilt}_-(A)$ with $E_{T'} \cong B^G_{e,m}$. Let $\mathcal{S}' = \mathfrak{F}(T')$, so $\mathcal{S}'$ can be obtained from $\mathcal{S}$ by a stable auto-equivalence. Recall from [Asa2] that any stable auto-equivalence of a self-injective Nakayama algebra is generated by the its Picard group and Ω, hence $\mathcal{S}' = \Omega^h\mathcal{S}$ for some h. Since $\mathcal{S}'$ and $\mathcal{S}$ are of the same type by Lemma 8.2.22, $\mathcal{S}' = \tau^h\mathcal{S}$. Surjectivity of $\mathfrak{F}_-$ now follows. $\square$

Lemma 8.2.27. *$\mathfrak{F}_\pm : \mathsf{2tilt}_\pm(A) \to \mathsf{sms}_\pm(A)$ are injective.*

Proof. We prove only for the minus version. Suppose $T, T' \in \mathsf{2tilt}_-(A)$ with $\mathcal{S} = \mathfrak{F}(T) = \mathfrak{F}(T') = \mathcal{S}'$. This implies $E_T \cong E_{T'}$, or equivalently $\psi_-\phi_-^{-1}(T) = G = \psi_-\phi_-^{-1}(T')$. By Lemma 8.2.24, we have T is given by permuting labels of components of T'. As in Lemma 8.2.26, this implies $\mathcal{S} = \tau^h\mathcal{S}'$ for some h. Hence $\mathcal{S}'$ is τ^h-stable. But this means that the corresponding permutation on the labels on projective modules in T' only permute the summands of T'. Hence $T = T'$. $\square$

We now extend Theorem 8.2.21 to some of the self-injective Nakayama algebras.

Theorem 8.2.28. *For fixed $n, \ell \in \mathbb{Z}$ with $n \nmid \ell$, restricting $\mathfrak{F} : \mathsf{2tilt}(A_n^\ell) \to \mathsf{sms}(A_n^\ell)$ to the disjoint subsets $\mathsf{2tilt}_\pm(A_n^\ell)$, we have bijections $\mathfrak{F}_\pm : \mathsf{2tilt}_\pm(A_n^\ell) \to \mathsf{sms}_\pm(A_n^\ell)$. Moreover, $\mathfrak{F}_\pm$ are mutation preserving.*

Proof. Let e be the gcd of n and ℓ. Recall that there is a one-to-one correspondence between $\mathsf{sms}(A_n^\ell)$ and $\mathsf{sms}(A_e^\ell)$, as they can both be identified as $\tau^{e\mathbb{Z}}$-stable configurations of $\mathbb{Z}A_\ell$. Note that A_e^ℓ is just the Brauer star algebra $B_{e,m}^\star$ with $m = \ell/e > 1$. This correspondence respects mutations, in the sense that an irreducible mutation of sms in $\mathsf{sms}(A_e^\ell)$ corresponds to an irreducible (Nakayama-stable) mutation of sms in $\mathsf{sms}(A_e^\ell)$. Also, the bijections $\mathsf{2tilt}_\pm(A_n^\ell) \leftrightarrow \mathsf{2tilt}_\pm(A_e^{em})$ are also mutation preserving (Theorem 8.2.12). Therefore, we have a composition of mutation preserving maps

$$\mathsf{2tilt}_\pm(A_n^\ell) \to \mathsf{2tilt}_\pm(A_e^{em}) \to \mathsf{sms}_\pm(A_e^{em}) \to \mathsf{sms}_\pm(A_n^\ell),$$

and all the maps are bijective. The theorem follows. □

Finally, we prove Theorem 8.2.1 for the remaining cases $A_{\ell k}^\ell$. Note that putting put $k = 1$, the algebra is just a multiplicity-free Brauer star algebra.

Theorem 8.2.29. *For any fixed $e, \ell \in \mathbb{Z}$ with $\ell | e$. $\mathfrak{F} : \mathsf{2tilt}(A_e^\ell) \to \mathsf{sms}(A_e^\ell)$ is surjective non-injective and preserves mutations.*

Proof. We have

$$\mathsf{2tilt}(A_e^\ell) \leftrightarrow \mathsf{2tilt}(A_e^{em}) \leftrightarrow \mathsf{sms}(A_e^{em}) \twoheadrightarrow \mathsf{sms}(A_e^e) \leftrightarrow \mathsf{sms}(A_e^\ell) \tag{8.2.4}$$

with all the left-to-right maps being mutation-preserving by Theorem 8.2.12, Theorem 8.2.21, Lemma 8.2.19. It is easy to see that the canonical stalk tilting complex A_e^ℓ maps to $\mathcal{S}_{A_e^\ell}$ along the composition of maps in (8.2.4). Since all self-injective Nakayama algebras are (strongly left) tilting-connected and the composition (8.2.4) respects mutation, this implies that the composition (8.2.4) is precisely $\mathfrak{F}$. Now it remains to show that $\mathsf{sms}(A_e^{em}) \twoheadrightarrow \mathsf{sms}(A_e^e)$ is not injective. Since $\mathsf{sms}(A_e^{em})$ bijects with $\mathsf{2tilt}(A_e^{em})$, by [Ada, Cor 2.24], $|\mathsf{sms}(A_e^{em})| = \binom{2e}{e}$. On the other hand, $\mathsf{sms}(A_e^e)$ bijects with the set of τ^e-stable configurations of $\mathbb{Z}A_e$, which is the same as the set of configurations of $\mathbb{Z}A_e$. Hence $\mathsf{sms}(A_e^e)$ bijects with non-crossing partitions of type $\mathbb{A}_e$, for which the cardinality is well-known, namely the Catalan number $\frac{1}{e+1}\binom{2e}{e}$ (see, for example, [Rea, CS]). This completes the proof. □

List of Symbols

Part I

Part II

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