Singularities of orbit closures in module varieties (part 3)

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Equations of orbit closures

- Let $M \in \mathrm{rep}_Q(\mathbf{d})$.
- In general, we do not know generators of the ideal

$$I(\overline{\mathcal{O}}_M) \lhd \Bbbk[\mathrm{rep}_Q(\mathbf{d})],$$

- The coordinate ring

$$\Bbbk[\mathrm{rep}_Q(\mathbf{d})] = \Bbbk[x^{\alpha}_{m,n}], \qquad \text{where } \alpha \in Q_1,\ m \leq d_{t\alpha},\ n \leq d_{s\alpha}.$$

- Let $X_\alpha = (x^{\alpha}_{m,n}) \in \mathbb{M}_{d_{t(\alpha)} \times d_{s(\alpha)}}(\Bbbk[\mathrm{rep}_Q(\mathbf{d})])$, for any $\alpha \in Q_1$.
- Given vertices $i, j \in Q_0$ and a linear combination ω of paths in Q from i to j, we define, in a natural way, $X_\omega \in \mathbb{M}_{d_j \times d_i}(\Bbbk[\mathrm{rep}_Q(\mathbf{d})])$.
- If I is a two-sided ideal in $\Bbbk Q$, then the coordinate algebra $\Bbbk[\mathrm{rep}_{Q,I}(\mathbf{d})]$ is the quotient of $\Bbbk[\mathrm{rep}_Q(\mathbf{d})]$ by the ideal generated by entries of X_ω, where ω ranges generators of I (the variety $\mathrm{rep}_{Q,I}(\mathbf{d})$ is not necessarily reduced).
- If $M \in \mathrm{rep}_{Q,I}(\mathbf{d})$ then

$$\overline{\mathcal{O}}_M \subseteq \{L \in \mathrm{rep}_{Q,I}(\mathbf{d}) \mid [Y, L] \geq [Y, M] \text{ for any indecomposable non-projective } Y\}.$$

The right-hand side can be viewed as a (not necessarily reduced) subvariety of $\mathrm{rep}_{Q,I}(\mathbf{d})$.

- We choose a minimal projective presentation

$$P^1 \xrightarrow{p_Y} P^0 \to Y \to 0$$

and consider the induced exact sequence

$$0 \to \mathrm{Hom}_A(Y, L) \to \mathrm{Hom}_A(P^0, L) \xrightarrow{\mathrm{Hom}_A(p_Y, L)} \mathrm{Hom}_A(P^1, L)$$

- $\mathrm{Hom}_A(p_Y, -)$ can be treated as a morphism

$$\mathrm{rep}_{Q,I}(\mathbf{d}) \to \mathbb{M}_{[S_{\mathbf{d}}, \tau Y] \times [Y, S_{\mathbf{d}}]}(\Bbbk),$$

where $S_{\mathbf{d}} = \bigoplus (S_i)^{d_i}$ is the standard semisimple representation with dimension vector $\mathbf{d}$.
- We denote by $J^{Q,I,\mathbf{d}}_{Y,r}$ the ideal in $\Bbbk[rep_{Q,I}(\mathbf{d})]$ generated by the images of the minors of size $1 + r$ in $\Bbbk[\mathbb{M}_{[S_{\mathbf{d}}, \tau Y] \times [Y, S_{\mathbf{d}}]}(\Bbbk)]$. It does not depend on the chosen minimal presentation of Y.
- Finally,

$$I_M := \sum_Y J^{Q,I,\mathbf{d}}_{Y,[Y,S_{\mathbf{d}}]-[Y,M]} \qquad \text{and} \qquad \mathcal{C}_M := \mathrm{Spec}\left(\Bbbk[rep_{Q,I}(\mathbf{d})] \,/\, I_M\right).$$

Example (Dynkin $\mathbb{D}_4$)

Let $A = \Bbbk Q$, where

$$Q: \quad \begin{tikzcd}[row sep=small] & & \bullet \arrow[dl, "\beta"'] \\ \bullet & \bullet \arrow[l, "\alpha"'] & \\ & & \bullet \arrow[ul, "\gamma"] \end{tikzcd}$$

Let $M \in \operatorname{rep}_Q(\mathbf{d})$. Then the ideal I_M is generated by minors of appriopriate size of the following 8 matrices:

$$X_\alpha, \quad X_\beta, \quad X_\gamma, \quad X_{\alpha\beta}, \quad X_{\alpha\gamma}, \quad \begin{bmatrix} X_\beta & X_\gamma \end{bmatrix}, \quad \begin{bmatrix} X_{\alpha\beta} & X_{\alpha\gamma} \end{bmatrix}, \quad \begin{bmatrix} X_\beta & X_\gamma \\ X_{\alpha\beta} & 0 \end{bmatrix}.$$

Corollary

Assume the algebra A is representation-finite, or is tame concealed. If $M \in \operatorname{mod} A$, then

$$\sqrt{I_M} = I(\overline{\mathcal{O}}_M) \qquad \big(\textit{equivalently, } (\mathcal{C}_M)_{red} = \overline{\mathcal{O}}_M\,\big).$$

Theorem (Riedtmann-Z. 2013)

Let Q be a Dynkin quiver of type $\mathbb{A}$, and $M \in \operatorname{rep}(Q)$. Then

$$I_M = I(\overline{\mathcal{O}}_M) \qquad \big(\textit{equivalently, } \mathcal{C}_M = \overline{\mathcal{O}}_M\,\big).$$

Tangent spaces to orbit closures

- Let Q be a quiver, $M \in \operatorname{rep}(Q)$ and $N \in \overline{\mathcal{O}}_M$. Then

$$\mathcal{T}_{N,\mathcal{O}_N} \subseteq \mathcal{T}_{N,\overline{\mathcal{O}}_M} \subseteq \mathcal{T}_{N,\mathcal{C}_M} \subseteq \mathcal{T}_{N,\operatorname{rep}_Q(\mathbf{d})}.$$

- Using the following interpretation of tangent spaces:

$$\begin{aligned}\mathcal{T}_{N,\operatorname{rep}_Q(\mathbf{d})} &\simeq \left\{ \left[\begin{smallmatrix} N & Z \\ 0 & N \end{smallmatrix}\right] = \left(\left[\begin{smallmatrix} N_\alpha & Z_\alpha \\ 0 & N_\alpha \end{smallmatrix}\right] \right)_{\alpha \in Q_1} \in \operatorname{rep}_Q(2 \cdot \mathbf{d}) \right\} \\ &\simeq \left\{ \sigma \colon 0 \to N \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} \left[\begin{smallmatrix} N & Z \\ 0 & N \end{smallmatrix}\right] \xrightarrow{[0\ 1]} N \to 0 \right\}, \\ \mathcal{T}_{N,\mathcal{O}_N} &\simeq \{\sigma \text{ splits }\},\end{aligned}$$

we get a special case of Voigt's theorem:

$$\imath_N \colon \mathcal{T}_{N,\operatorname{rep}_Q(\mathbf{d})} / \mathcal{T}_{N,\mathcal{O}_N} \xrightarrow{\simeq} \operatorname{Ext}^1_Q(N, N).$$

- A natural problem is to describe the subspace

$$\imath_N(\mathcal{T}_{N,\overline{\mathcal{O}}_M} / \mathcal{T}_{N,\mathcal{O}_N}) \subseteq \operatorname{Ext}^1_Q(N, N).$$

- Let $\mathcal{E}_{M,N}$ be a subset of $\mathrm{Ext}^1_Q(N,N)$ consisting of $[\sigma]_\sim$, where $\sigma\colon\ 0 \to N \xrightarrow{f} W \xrightarrow{g} N \to 0$ satisfies one of the following equivalent conditions:
  1. For any $X \in \mathrm{rep}(Q)$ with $[X,M] = [X,N]$ we have $[X,W] = 2\cdot[X,N]$ (equivalently, any homomorphism $X \to N$ factors through g).
  2. For any $Y \in \mathrm{rep}(Q)$ with $[M,Y] = [N,Y]$ we have $[W,Y] = 2\cdot[N,Y]$ (equivalently, any homomorphism $N \to Y$ factors through f).

Theorem (Riedtmann-Z. 2013)

$$\imath_N(\mathcal{T}_{N,\mathcal{C}_M}/\mathcal{T}_{N,\mathcal{O}_N}) = \mathcal{E}_{M,N}.$$

Consequently, if Q is a Dynkin quiver of type $\mathbb{A}$, then

$$\imath_N(\mathcal{T}_{N,\overline{\mathcal{O}}_M}/\mathcal{T}_{N,\mathcal{O}_N}) = \mathcal{E}_{M,N}.$$

Theorem (Bobiński-Z. 2022)

Let Q be a Dynkin quiver of type $\mathbb{D}$. Then

$$\imath_N(\mathcal{T}_{N,\overline{\mathcal{O}}_M}/\mathcal{T}_{N,\mathcal{O}_N}) = \mathcal{E}_{M,N}.$$

Transversal slices

Definition

Let G be an algebraic group acting regularly on a (possibly not reduced) variety $\mathcal{X}$, and $x \in \mathcal{X}$. A **transversal slice** in $\mathcal{X}$ to the orbit $G \cdot x$ at the point x is a subvariety $\mathcal{Y}$ of $\mathcal{X}$ satisfying:

- $x \in \mathcal{Y}$;
- the morphism $\Psi\colon G \times \mathcal{Y} \to \mathcal{X}$, $(g, y) \mapsto g \cdot y$, is smooth;
- $\dim \mathcal{Y}$ is minimal with respect to the above.

- If $\mathcal{Y}$ is a transversal slice in $\mathcal{X}$ at x, then $\operatorname{Sing}(\mathcal{X}, x) = \operatorname{Sing}(\mathcal{Y}, x)$ as we have smooth morphisms:

$$\begin{tikzcd} & (G \times \mathcal{Y}, (1, x)) \arrow[dl, "\Psi"'] \arrow[dr, "projection"] & \\ (\mathcal{X}, x) & & (\mathcal{Y}, x) \end{tikzcd}$$

- Let $\mathcal{X}$ and $\mathcal{X}'$ be G-varieties, $F\colon \mathcal{X} \to \mathcal{X}'$ be a G-equivariant morphism, and $x \in \mathcal{X}$. If $\mathcal{Y}'$ is a transversal slice in $\mathcal{X}'$ at $F(x)$, then $F^{-1}(\mathcal{Y}')$ is a transversal slice in $\mathcal{X}$ at x.
- Let $\mathcal{Y}$ be a transversal slice in $\mathcal{X}$ at x, and consider the following maps:

$$\mu\colon G \to \mathcal{X},\ \mu(g) = g \cdot x, \qquad \text{and} \qquad \mathcal{T}_{1,\mu}\colon \mathcal{T}_{1,G} \to \mathcal{T}_{x,\mathcal{X}}.$$

Then

$$\mathcal{T}_{x,\mathcal{X}} = \mathcal{T}_{x,\mathcal{Y}} \oplus \operatorname{Im}(\mathcal{T}_{1,\mu}),$$

and

$$\dim_x \mathcal{Y} = \dim_x \mathcal{X} - \dim_{\Bbbk} \operatorname{Im}(\mathcal{T}_{1,\mu}).$$

Question

How to construct transversal slices?

- Assume first that x is a smooth point of a G-variety $\mathcal{X}$. We choose a locally closed smooth subvariety $x \in \mathcal{Y} \subseteq \mathcal{X}$ satisfying

$$\mathcal{T}_{x,\mathcal{X}} = \mathcal{T}_{x,\mathcal{Y}} \oplus \mathrm{Im}(\mathcal{T}_{1,\mu}).$$

  Then $\Psi\colon G \times \mathcal{Y} \to \mathcal{X}$ is smooth at (g, x) for any $g \in G$. Replacing $\mathcal{Y}$ by its open neighbourhood of x if necessary, we get a transversal slice in $\mathcal{X}$ at x.
- If $\mathcal{X}$ is an affine G-variety, then there is a smooth affine G-variety $\mathcal{X}'$ together with a G-equivariant closed immersion

$$F\colon \mathcal{X} \to \mathcal{X}'.$$

  Given a point $x \in \mathcal{X}$, we choose a transversal slice $\mathcal{Y}'$ in $\mathcal{X}'$ at $F(x)$. Then $F^{-1}(\mathcal{Y}')$ is a transversal slice in $\mathcal{X}$ at x.
- The above can be applied to orbit closures in $\mathrm{rep}_{Q,I}(\mathbf{d})$, or more generally, to any closed $\mathrm{GL}(\mathbf{d})$-stable subvariety of $\mathrm{rep}_{Q,I}(\mathbf{d})$. Here, we choose smooth variety $\mathcal{X}' = \mathrm{rep}_Q(\mathbf{d})$. Moreover, if the algebra $A = \Bbbk Q/I$ is representation-finite, this can be done simultaneously for all N:

Lemma

Let $A = \Bbbk Q/I$ be a representation-finite algebra. Then there is a bound quiver $(\widehat{Q}, \widehat{I})$ together with an exact functor $F' \colon \operatorname{rep}(\widehat{Q}, \widehat{I}) \to \operatorname{rep}(Q, I)$ and closed immersions

$$F'(\mathbf{m}) \colon \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$$

such that for any $N \in \operatorname{rep}(Q, I)$ there is $\mathbf{m}$ such that

- $N' := F'(\mathbf{m})(0) \simeq N$,
- $\dim_0 \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) = \dim_{N'} \operatorname{rep}_{Q,I}(\phi(\mathbf{m})) - \dim \operatorname{GL}(\phi(\mathbf{m})) * N'$,
- $\operatorname{GL}(\phi(\mathbf{m})) \times \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$, $(g, L) \mapsto g * F'(\mathbf{m})(L)$, *is smooth at* $(1, 0)$.

Example

Let $Q :\ 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$ and $I = \langle \alpha\beta \rangle$. Then we choose

$$\widehat{Q} : \begin{tikzcd}
& 2 & & 4 \arrow[ll, "\gamma_{24}"'] \arrow[dlll, "\gamma_{14}"'] & \\
1 & & 3 \arrow[ll, "\gamma_{13}"] & & 5 \arrow[ll, "\gamma_{35}"] \arrow[ulll, "\gamma_{25}"']
\end{tikzcd}$$

$\widehat{I} = \langle\, \gamma_{14},\ \gamma_{13}\gamma_{35},\ \gamma_{24},\ \gamma_{25} \,\rangle$ and F is given by

$$F\left(\begin{tikzcd}
& V_2 & & V_4 \arrow[ll, "f_{24}"'] \arrow[dlll, "f_{14}"'] & \\
V_1 & & V_3 \arrow[ll, "f_{13}"] & & V_5 \arrow[ll, "f_{35}"] \arrow[ulll, "f_{25}"']
\end{tikzcd} \right) = V_1 \oplus V_2 \xleftarrow{\begin{pmatrix} 0 & f_{13} & f_{14} \\ 1 & 0 & f_{24} \end{pmatrix}} V_2 \oplus V_3 \oplus V_4 \xleftarrow{\begin{pmatrix} 0 & f_{25} \\ 0 & f_{35} \\ 1 & 0 \end{pmatrix}} V_4 \oplus V_5.$$

Lemma

Let $A = \Bbbk Q/I$ be a representation-finite algebra. Then there is a bound quiver $(\widehat{Q}, \widehat{I})$ together with an exact functor $F' \colon \operatorname{rep}(\widehat{Q}, \widehat{I}) \to \operatorname{rep}(Q, I)$ and closed immersions

$$F'(\mathbf{m}) \colon \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$$

such that for any $N \in \operatorname{rep}(Q, I)$ there is $\mathbf{m}$ such that

- $N' := F'(\mathbf{m})(0) \simeq N$,
- $\dim_0 \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) = \dim_{N'} \operatorname{rep}_{Q,I}(\phi(\mathbf{m})) - \dim \operatorname{GL}(\phi(\mathbf{m})) * N'$,
- $\operatorname{GL}(\phi(\mathbf{m})) \times \operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$, $(g, L) \mapsto g * F'(\mathbf{m})(L)$, *is smooth at* $(1, 0)$.

- Unfortunately, we miss a representation-theoretic interpretation of $\operatorname{rep}(\widehat{Q}, \widehat{I})$. For instance, we do not know when the images of two points in $\operatorname{rep}_{\widehat{Q},\widehat{I}}(\mathbf{m})$ belongs to the same orbit in $\operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$.
- Idea is to find a new pair $(\widehat{Q}, \widehat{I})$ which is more closely related to the category $\operatorname{rep}(Q, I)$, and satisfies the above lemma except "closed immersions".

Representation finite standard algebras

- Let $A = \Bbbk Q/I$ be a representation finite algebra.
- Let $\mathrm{ind}(A)$ be a full subcategory of $\mathrm{mod}(A)$ whose objects form a set of representatives of the isomorphism classes of indecomposable A-modules.
- Let (Γ_A, τ) denote the Auslander-Reiten quiver of A. In particular, $(\Gamma_A)_0 = \mathit{Objects}(\mathrm{ind}(A))$.
- The mesh category $\Bbbk(\Gamma_A)$ of Γ_A is a quotient of the path category $\Bbbk[\Gamma_A]$ modulo mesh relations $\sum \beta_i \alpha_i = 0$:

$$\begin{tikzcd}
 & X_1 \arrow[dr, "\beta_1"] & \\
\tau X \arrow[ur, "\alpha_1"] \arrow[r, dotted] \arrow[dr, "\alpha_r"'] & \vdots \arrow[r, dotted] & X \\
 & X_r \arrow[ur, "\beta_r"'] &
\end{tikzcd}$$

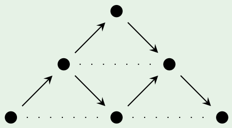
- We assume that the algebra A is **standard**, i.e. there is an equivalence

$$F \colon \Bbbk(\Gamma_A) \to \mathrm{ind}(A).$$

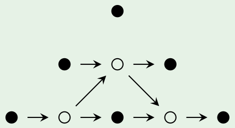
We construct a new translation quiver $(\widehat{\Gamma}_A, \widehat{\tau})$:

Example (equioriented Dynkin $\mathbb{A}_3$)

Γ_A :



$\widehat{\Gamma}_A$:



The vertices of $\widehat{\Gamma}_A$:

- **frozen** (bullet) $\{X \mid X \text{ is a vertex of } \Gamma_A\}$,
- **non-frozen** (circle) $\{X' \mid X \text{ is a non-projective vertex of } \Gamma_A\}$.

The arrows of $\widehat{\Gamma}_A$:

- $\{X' \xrightarrow{\alpha'} Y' \mid X \xrightarrow{\alpha} Y \text{ is an arrow in } \Gamma_A \text{ and } X, Y \text{ are not projective } \}$,
- $\{\tau X \to X',\ X' \to X \mid X \text{ is a non-projective vertex of } \Gamma_A\}$.

The translation $\widehat{\tau}$ of $\widehat{\Gamma}_A$:

- If $\tau^2 X$ exists in Γ_A then $\widehat{\tau}(X') = (\tau X)'$.

Definition

Let A be a standard representation-finite algebra.

- The **regular Nakajima category** $\mathcal{R}_A$ of the algebra A is the mesh category $\Bbbk(\widehat{\Gamma}_A)$.
- The **singular Nakajima category** $\mathcal{S}_A$ of the algebra A is the full subcategory of $\mathcal{R}_A$ whose objects are the frozen vertices.

Theorem (Bobiński-Z.)

Let $A = \Bbbk Q/I$ be a standard representation-finite algebra. Let $(\widetilde{Q}, \widetilde{I})$ be a bound quiver whose path category is equivalent to $(\mathcal{S}_A)^{op}$. Then there is an exact functor

$$\Phi\colon \operatorname{rep}(\widetilde{Q}, \widetilde{I}) \to \operatorname{rep}(Q, I)$$

together with morphism

$$\Phi(\mathbf{m})\colon \operatorname{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m}))$$

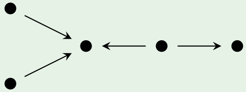
for any $\mathbf{m} \in \mathbb{N}^{(\widetilde{Q})_0}$, *where* $\phi(\mathbf{m}) = \sum_X \ m_X \cdot \mathbf{dim}_A X$ *such that:*

- $\Phi(S_X) \simeq X$, *where* S_X *is the standard simple representation of* $(\widetilde{Q}, \widetilde{I})$ *at the vertex* $X \in (\widetilde{Q})_0 = (\Gamma_A)_0$;
- *The map*
$$\operatorname{Ext}^n_{\widetilde{Q},\widetilde{I}}(S_X, S_Y) \to \operatorname{Ext}^n_{Q,I}(X, Y),$$
induced by Φ, *is bijective for any vertices* X, Y, *and* $n \geq 1$.
- *The induced morphism*
$$\mathrm{GL}(\phi(\mathbf{m})) \times \operatorname{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}) \to \operatorname{rep}_{Q,I}(\phi(\mathbf{m})), \qquad (g, L) \mapsto g * \Phi(\mathbf{m})(L),$$
is smooth at $(1, 0)$ *for any* $\mathbf{m} \in \mathbb{N}^{(\widetilde{Q})_0}$.
- $\operatorname{Sing}(\operatorname{rep}_{Q,I}(\phi(\mathbf{m})), N) = \operatorname{Sing}(\operatorname{rep}_{\widetilde{Q},\widetilde{I}}(\mathbf{m}), 0)$, *where* $N := \Phi(\mathbf{m})(0) \simeq \bigoplus X^{m_X}$.

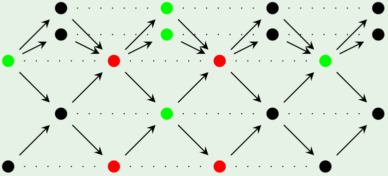
Example (Dynkin $\mathbb{D}_5$)

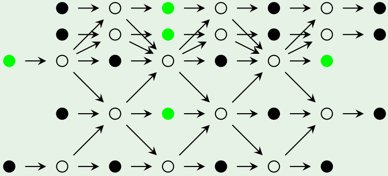
Let $A = \Bbbk Q$, where

Q :



Then Γ_A and $\widehat{\Gamma}_A$ have the form





The representations $N = \bigoplus$ ● and $M = \bigoplus$ ● of Q have the same dimension vector $\mathbf{d} = \begin{smallmatrix}3 & & & \\ & 6 & 4 & 2\\ 3 & & & \end{smallmatrix}$.

We get a codimension 6 degeneration:

$$\mathcal{O}_N \subset \overline{\mathcal{O}}_M$$

(the orbits have dimension 60 and 66, respectively). Using the last thorem one can find that $\mathrm{Sing}(M, N)$ is the type of singularity at 0 of the 6-dimensional hypersurface

$$\mathrm{Spec}\Big(\Bbbk[x_1, x_2, x_3, y_1, y_2, y_3, z] / \left(z^2 + z \cdot (-x_1 y_1 - x_2 y_2 + x_3 y_3) + x_1 y_1 x_2 y_2 \right) \Big).$$