Singularities of orbit closures in module varieties (part 2)

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Desingularizations

- Let A be a representations finite algebra and X be a direct sum of the indecomposable A-modules, so $E = \mathrm{End}_A(X)$ is the Auslander algebra of A.
- We choose a module $M \in \mathrm{mod}\, A$ and put $c = [X, M]$ and $d = \dim_{\Bbbk} M$.
- Recall that existence of a short exact sequence of the form

$$0 \to Z \to Z \oplus M \to N \to 0$$

  for some module Z is equivalent to existence of a right E-submodule $U \subseteq \mathrm{Hom}_A(X, N)$ with $\mathbf{dim}_E(U) = \mathbf{dim}_E \mathrm{Hom}_A(X, M)$.
- Inside the $\mathrm{GL}_d(\Bbbk)$-variety

$$\mathrm{mod}_A(d) \times \mathrm{Grass}(\mathrm{Hom}_{\Bbbk}(X, \Bbbk^d), c)$$

  consider a $\mathrm{GL}_d(\Bbbk)$-orbit

$$\mathcal{O}_{M_X} := \{(M', \mathrm{Hom}_A(X, M'))|\ M' \in \mathcal{O}_M\}.$$

Theorem (Z. 2002)

- *The orbit closure $\overline{\mathcal{O}}_{M_X}$ is a nonsingular variety consisting of the pairs (N, V), such that $N \in \overline{\mathcal{O}}_M$ and V is a right E-submodule of $\mathrm{Hom}_A(X, N)$ with $\mathbf{dim}_E(V) = \mathbf{dim}_E \mathrm{Hom}_A(X, M)$.*
- *Projection onto the first factor leads to map*

$$p_M \colon \overline{\mathcal{O}}_{M_X} \to \overline{\mathcal{O}}_M,$$

  which is projective and birational (hence p_M is a desingularization).
- *The fibre $p_M^{-1}(N)$ is a coonected set for any $N \in \overline{\mathcal{O}}_M$.*

Unibranch varieties

- An irreducible variety $\mathcal{X}$ is said to be unibranch if the normalization map $\widetilde{\mathcal{X}} \to \mathcal{X}$ is bijective. Since any normalization map is closed, the above condition implies that $\widetilde{\mathcal{X}} \to \mathcal{X}$ is a homeomorphism. Hence unibranch varieties are topologically like normal varieties.

Example

The curve

$$\{(x, y) \in \mathbb{k}^2 \mid x^2 = y^3\}$$

is unibranch, and the curve

$$\{(x, y) \in \mathbb{k}^2 \mid x^2 = y^2 + y^3\}$$

is unibranch only if $char(\mathbb{k}) = 2$.

- Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper birational morphism of irreducible varieties. If $\mathcal{Y}$ is a unibranch variety and the fibres of f are connected then $\mathcal{X}$ is a unibranch variety as well.

Corollary

Let A be a representation finite algebra and $M \in \operatorname{mod} A$. Then $\overline{\mathcal{O}}_M$ is a unibranch variety.

- The above corollary generalizes to an arbitrary algebra A and a module M such that there are, up to isomorphism, only finitely many indecomposable modules cogenerated by M (i.e. modules L such that there is a monomorph $L \to M^h$ for some $h > 0$).

Example

Let $A = \mathbb{k}[X, Y]/(XY, X^2 - Y^2)$,

$$M = {}_AA : \quad \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \end{bmatrix} \circlearrowright \mathbb{k}^4 \circlearrowleft \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \end{bmatrix}$$

$$N : \quad \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \circlearrowright \mathbb{k}^4 \circlearrowleft \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{bmatrix}$$

Then $\mathrm{Sing}(M, N) = \mathrm{Sing}\left(\{(x, y) \in \mathbb{k}^2 \mid xy = 0\}, 0\right)$.
Thus $\overline{\mathcal{O}}_M$ is not unibranch at N.
Observe that $\dim \overline{\mathcal{O}}_M = 16 - \dim_{\mathbb{k}} \mathrm{End}_A(A) = 16 - 4 = 12$.

- A classification of affine varieties of dimension at most 4, which appear as orbit closures of representations or modules is known ([Rochman, 2008]), and it follows that all of them are normal and Cohen-Macaulay. In particular, they are unibranch.
- It is an open problem to find the smallest $u \in [5, 12]$ such that there is an orbit closure $\overline{\mathcal{O}}_M$ of dimennsion u which is not unibranch.

Orbit closure, which is not a Cohen-Macaulay variety

Example

Let $Q = 1 \overset{\alpha}{\underset{\beta}{\leftleftarrows}} 2$, $\mathbf{d} = (3,3)$ and choose two scalars $\lambda \neq \mu$.

$$M: \quad \Bbbk^3 \overset{\begin{bmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{bmatrix}}{\underset{\begin{bmatrix} 1&0&0\\ 0&0&0\\ 0&0&1 \end{bmatrix}}{\longleftleftarrows}} \Bbbk^3 \qquad\qquad N: \quad \Bbbk^3 \overset{\begin{bmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{bmatrix}}{\underset{\begin{bmatrix} 0&0&0\\ \lambda&0&0\\ 0&\mu&0 \end{bmatrix}}{\longleftleftarrows}} \Bbbk^3$$

Then $M = P_2 \oplus I_1$ degenerates to $N = S_1 \oplus S_2 \oplus U_\lambda \oplus U_\mu$ and

$$\mathrm{Sing}(M,N) = \mathrm{Sing}(\{(x_1,x_2,y_1,y_2) \in \Bbbk^4 \mid x_i y_j = 0\}, 0).$$

In particular, $\overline{\mathcal{O}}_M$ is neither Cohen-Macaulay nor unibranch at N.
Observe that $\dim \overline{\mathcal{O}}_M = \dim \mathrm{GL}(\mathbf{d}) - \dim_{\Bbbk} \mathrm{End}_Q(M) = 18 - 4 = 14$.

- It is an open problem to find the smallest $u \in [5,14]$ such that there is an orbit closure $\overline{\mathcal{O}}_M$ of dimennsion u which is not Cohen-Macaulay.

The invariance of geometric properties under tilting functors

- Let $A = \Bbbk Q/I$ be a finite dimensional algebra and T be a tilting A-module.
- Let $B = \mathrm{End}_A(T)^{op} = \Bbbk Q'/I'$. Then we have an equivalence

$$F = \mathrm{Hom}_A({}_AT_B, -) \colon \mathcal{T} \to \mathcal{Y},$$

where $\mathcal{T} \subset \mathrm{mod}\, A$ is the subcategory consisting of modules L with $\mathrm{Ext}^1_A(T, L) = 0$ and $\mathcal{Y} \subset \mathrm{mod}\, B$ is the subcategory consisting of modules L with $\mathrm{Tor}^B_1({}_AT_B, L) = 0$.

- Given a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, there is $\mathbf{e} \in \mathbb{N}^{Q'_0}$ such that

$$\mathbf{dim}_B\, F(L) = \mathbf{e} \qquad \text{for any } L \in \mathcal{T} \text{ with } \mathbf{dim}_A\, L = \mathbf{d}.$$

Hence F induces a correspondence between the set of $\mathrm{GL}(\mathbf{d})$-orbits in

$$\mathcal{T}(\mathbf{d}) \subseteq_{open} \mathrm{rep}_{Q,I}(\mathbf{d})$$

and the set of $\mathrm{GL}(\mathbf{e})$-orbits in

$$\mathcal{Y}(\mathbf{e}) \subseteq_{open} \mathrm{rep}_{Q',I'}(\mathbf{e}).$$

Theorem (Bongartz, 1994)

The above correspondence induces a bijection between $\mathrm{GL}(\mathbf{d})$*-stable subsets of* $\mathcal{T}(\mathbf{d})$ *and* $\mathrm{GL}(\mathbf{e})$*-stable subsets of* $\mathcal{Y}(\mathbf{e})$*, preserving and reflecting closures, inclusions, codimensions and types of singularities occuring in orbit closures.*

Hom-controlled exact functors

- Let $F\colon \operatorname{mod} B \to \operatorname{mod} A$ be an exact functor. Then
$$M \leq_{deg} N \quad \Longrightarrow \quad FM \leq_{deg} FN.$$

- Given a short exact sequence $\sigma : 0 \to U \to W \to V \to 0$ in $\operatorname{mod} B$ and $X \in \operatorname{mod} B$ we have the induced exact sequences
$$0 \to \operatorname{Hom}_B(V, X) \to \operatorname{Hom}_B(W, X) \to \operatorname{Hom}_B(U, X) \to \mathbb{k}^{\delta_\sigma(X)} \to 0,$$
$$0 \to \operatorname{Hom}_B(X, U) \to \operatorname{Hom}_B(X, W) \to \operatorname{Hom}_B(X, V) \to \mathbb{k}^{\delta'_\sigma(X)} \to 0.$$
We call the functor F **hom-controlled** if
$$\delta_{F\sigma}(FX) = \delta_\sigma(X) \qquad \text{and} \qquad \delta'_{F\sigma}(FX) = \delta'_\sigma(X),$$
for any short exact sequence σ in $\operatorname{mod} B$ and $X \in \operatorname{mod} B$.
- Equivalently, F is hom-controlled iff there is a bilinear form $\xi : K_0(B) \times K_0(B) \to \mathbb{Z}$ on the Grothendieck group $K_0(B)$ of the category $\operatorname{mod} B$, such that
$$[FX, FY] - [X, Y] = \xi(\mathbf{dim}\, X, \mathbf{dim}\, Y), \qquad \text{for any } X, Y \in \operatorname{mod} B.$$

Theorem (Z. 2002)

Let $F : \operatorname{mod} B \to \operatorname{mod} A$ be a hom-controlled exact functor, where the algebra B is finite dimensional. If $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$ for modules $M, N \in \operatorname{mod} B$, then
$$\operatorname{Sing}(FM, FN) = \operatorname{Sing}(M, N).$$

Example

Let $\quad Q = 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \quad$ and $\quad Q' = \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet.$

Then the following functor $\mathcal{F} : \mathrm{rep}(Q) \to \mathrm{rep}(Q')$ is hom-controlled:

$$\mathcal{F}(V_1 \xrightarrow{V_\alpha} V_2 \xleftarrow{V_\beta} V_3) \quad = \quad V_2 \xleftarrow{(\,V_\alpha \;\; 1\,)} V_1 \oplus V_2 \xleftarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & V_\beta \end{smallmatrix}\right)} V_1 \oplus V_3 \xleftarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} V_1,$$

Theorem (Bobiński - Z. 2001)

Let Q be a Dynkin quiver of type $\mathbb{A}_{p+q+1}$ with p arrows in one direction and q arrows in the other. Then there is a hom-controlled exact functor

$$\mathrm{rep}(Q) \to \mathrm{rep}(Q'),$$

where Q' is an equioriented Dynkin quiver of type $\mathbb{A}_{p+2q+1}$.

Theorem (Lakshmibai-Magyar, 1998)

Let M be a representation of an equioriented Dynkin quiver of type $\mathbb{A}$. Then $\overline{\mathcal{O}}_M$ is a normal Cohen-Macaulay variety with rational singularities.

Corollary

Let M be a representation of a Dynkin quiver of type $\mathbb{A}$. Then $\overline{\mathcal{O}}_M$ is a normal and Cohen-Macaulay variety. It has rational singularities if $\mathrm{char}(\Bbbk) = 0$.

Theorem (Bobiński - Z. 2002)

Let M be a representation of a Dynkin quiver of type $\mathbb{D}$. Then $\overline{\mathcal{O}}_M$ is a normal and Cohen-Macaulay variety. It has rational singularities if $char(\Bbbk) = 0$.

- The above theorem was proved by using Brion's geometric results on spherical varieties, and again applying hom-controlled functors.
- Question about Dynkin quivers of type $\mathbb{E}$ remains open.

Theorem (Magyar, 2002)

Let M be a nilpotent representation of a cyclic quiver. Then $\overline{\mathcal{O}}_M$ is a normal Cohen-Macaulay variety with rational singularities.

- Here, immersion to affine flag varieties were used.

Theorem (Skowroński - Z. 2003)

Let M be a module over a Brauer tree algebra. Then $\overline{\mathcal{O}}_M$ is a normal and Cohen-Macaulay variety. It has rational singularities if $char(\Bbbk) = 0$.

- The above is a consequence of mentioned Magyar's theorem and Rickard's results on derived equivalences for selfinjective algebras.

Singularities in codimension 1

Theorem (Z. 2005)

Assume that M degenerates to N and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 1$. Then $\overline{\mathcal{O}}_M$ is nonsingular at any point of $\mathcal{O}_N$.

- Suppose the contrary, then one can conclude existence of a short exact sequence of the form

$$0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus Y \xrightarrow{(f\ h)} Z \to 0,$$

  for some indecomposable module Z and module Y such that $\dim \mathcal{O}_Y - \dim \mathcal{O}_Z = 1$.
- A contradiction is obtained by considering free resolutions of $\mathrm{End}(Y)$ treated as a bimodule over the algebra generated by endomorphisms gh and gfh.

Corollary

Let M be a module or a representation such that $\overline{\mathcal{O}}_M$ has only finitely many orbits. Then $\overline{\mathcal{O}}_M$ is regular in codimension 1.

- Considered example of $M = {}_AA$, where $A = \Bbbk[X, Y]/(XY, X^2 - Y^2)$, shows that orbit closures can be singular in codimension 1.

Singularities in codimension 2

Theorem (Z. 2005)

Let Q be a Dynkin quiver. Then $\overline{\mathcal{O}}_M$ is regular in codimension 2.

Example

Let $A = \Bbbk[X]/(X^{n+1})$, $n \geq 1$.It follows from the exact sequence

$$0 \to \Bbbk[X]/(X^n) \xrightarrow{\cdot \overline{X}} \Bbbk[X]/(X^{n+1}) \to \Bbbk[X]/(X) \to 0$$

that $M = \Bbbk[X]/(X^{n+1})$ degenerates to $N = \Bbbk[X]/(X^n) \oplus \Bbbk[X]/(X)$. Moreover, $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$ and

$$\operatorname{Sing}(M, N) = \operatorname{Sing}(\{(x, y, z) \in \Bbbk^3 |\ x^{n+1} + yz = 0\}, 0)$$

is the Kleinian singularity of type $\mathbb{A}_n$.

- The above theorem can not be generalized for representation-directed algebras, as Kleinian singularity of type $\mathbb{A}_1$ can appear in orbit closures for modules over such algebras.

Example

Let $Q = 1 \leftleftarrows 2$ and P_n be the preprojective representation

$$\Bbbk^n \xleftleftarrows[\begin{bmatrix} 0 & \cdots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}]{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \end{bmatrix}} \Bbbk^{n-1}$$

It follows from the exact sequence

$$0 \to P_1 \to P_2 \oplus P_{n+2} \to P_{n+3} \to 0$$

that $M = P_2 \oplus P_{n+2}$ degenerates to $N = P_1 \oplus P_{n+3}$, $n \geq 1$. Moreover, $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$ and

$$\mathrm{Sing}(M, N) = \mathrm{Sing}(\{(s^n, s^{n-1}t, \cdots, t^n) \in k^{n+1} |\ s, t \in k\}, 0).$$

Theorem (Z. 2007)

Let Q be an extended Dynkin quiver. Assume that M degenerates to N and $\dim \mathcal{O}_M - \dim \mathcal{O}_N = 2$. Then $\mathrm{Sing}(M, N)$ is one of the types listed in the last two examples.